Tables of Phase of a Semi-Infinite Unit Attenuation Slope

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(Manuscript received February 24, 1956)

Five and seven place tables of the integral

$$B(x_c) = \frac{1}{\pi} \int_{x=0}^{x=x_c} \log \left| \frac{1+x}{1-x} \right| \frac{dx}{x}$$

which gives the phase associated with a semi-infinite unit slope of attenuation, are now available in monograph form. The usefulness of this integral and its tabulation are discussed.

H. W. Bode[1] has shown that on the imaginary axis, the values of the imaginary part of certain functions of a complex variable may be obtained from the corresponding values of the real part, and vice versa. This theorem was immediately recognized as a powerful tool in the communications and network fields. The most generally useful function which was given by Bode for use in applying this theorem to the solution of communications problems, is the phase associated with a semi-infinite unit slope of attenuation. This is given by the integral[2]

$$B(x_c) = \frac{1}{\pi} \int_{x=0}^{x=x_c} \log \left| \frac{1+x}{1-x} \right| \frac{dx}{x} \qquad (1)$$

where: $B(x_c)$ is the phase in radians at frequency f_c,

$$x = \frac{f}{f_0}, \; x_c = \frac{f_c}{f_0} < 1.0$$

and f_0 = the frequency at which the semi-infinite unit slope begins

The usefulness of Integral (1) is illustrated by some of the communication problems which stimulated its accurate tabulation.

[1] Bode, H. W., Network Analysis and Feedback Amplifier Design, D. Van Nostrand Co., Inc., New York, 1945, Chap. XIV.
[2] Ibid: Chap. XV, pp. 342–343.

When the development program on deep sea repeatered submarine telephone cable systems was reactivated at the close of World War II, one of the first problems to present itself was the determination of the delay distortion of a transatlantic repeatered cable system. The only means then known of obtaining an answer to this problem was by computing the minimum phase of the system from its predictable attenuation characteristic, using Bode's straight line approximation method,[3] and then determining the delay distortion from the non-linear portion of this minimum phase. However, the non-linear phase is such a small part of the total phase, that a five figure accuracy tabulation of Integral (1) was needed for a satisfactory determination of the non-linearity. The necessary table was therefore compiled. A numerical computation was used to evaluate the integral because of the simplicity of its integrand. The minimum phase of the projected transatlantic repeatered telephone cables was then computed using this table and the anticipated delay distortion was determined from the non-linear portion of this minimum phase.

About this time the delay equalization of coaxial cable systems for television transmission became a pressing problem. Bode's technique proved to be the simplest means for determining the delay to be equalized and so the existing phase table was immediately put to use in the coaxial cable delay equalization program.

The increasing use of the tables led to a decision to publish them in The Bell System Technical Journal.[4] In order to make the tables more generally useful, the published paper included a tabulation of the phase in radians as well as in degrees. The radian tables can, for example, be used to determine the reactance characteristic associated with a given resistance characteristic of a minimum reactance impedance function.

Because of the demand for higher accuracy which occasionally arose after the publication of the five place tables, it was decided to undertake the computation of seven-place tables. These tables were also computed numerically using intervals selected to give at least ±1 accuracy in the final figure. The complete tables require forty-nine pages for tabulation. Since it is probable that only a fraction of the Journal readers would need these tables, it did not seem desirable to publish the actual tables in the Journal. They are therefore being published in original monograph form as Bell System Monograph 2550[5] entitled "Tables of Phase of a Semi-Infinite Unit Attenuation Slope." The phase is tabulated in the

[3] Ibid: Chap. XV.

[4] Thomas, D. E., Tables of Phase Associated with a Semi-Infinite Unit Slope of Attenuation, B.S.T.J., **26,** pp. 870–899, Oct., 1947.

[5] This Monograph will be available about June 15, 1956.

monograph both in degrees and radians for values of f greater than f_0 as well as for f less than f_0. The tabular intervals are 0(0.001) 0.600 (0.0005) 0.9000 (0.0001) 0.9940 (0.00005) 0.99800 (0.00001) 1.00000. These intervals were selected to permit linear interpolation for intermediate values of the phase to an accuracy of the same order as the accuracy of the tabulated values, i.e., ± 1 in the last place. The original JOURNAL article discussed the construction of the tables and the errors involved in the numerical evaluation of Integral (1), described and illustrated the use of the tables, and gave five-place tabulations of the integral. This entire article is therefore included in Monograph 2550 for completeness along with the newer seven-place tables.

B. A. Kingsbury[6] has pointed out that the Integral (1) which is tabulated in the phase tables in question is useful in other than the communications and network fields. A bibliography covering other possible fields of interest is given in an article by Murakami and Corrington.[7]

ACKNOWLEDGMENT

The author is indebted to R. W. Hamming of the Mathematical Research Department who supervised the computation of the seven place tables, to Miss R. A. Weiss who planned, programmed, ran, and checked the IBM computations of the tables and to Miss J. D. Goeltz who computed the ten-figure accuracy check points required for the construction of the tables. He also wishes to acknowledge the support and encouragement given to the project by R. L. Dietzold and P. H. Richardson, and the continued interest and helpful comments of B. A. Kingsbury.

[6] Kingsbury, B. A., private communication.
[7] Murakami, T., and Corrington, M. S., Relation Between Amplitude and Phase in Electrical Networks, R.C.A. Review, **9**, pp. 602–631, Dec., 1948.