Statistical Techniques for Reducing the Experiment Time in Reliability Studies

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Given two or more processes, the units from which fail in accordance with an exponential or delayed exponential law, the problem is to select the particular process with the smallest failure rate. It is assumed that there is a common guarantee period of zero or positive duration during which no failures occur. This guarantee period may be known or unknown. It is desired to accomplish the above goal in as short a time as possible without invalidating certain predetermined probability specifications. Three statistical techniques are considered for reducing the average experiment time needed to reach a decision.

1. One technique is to increase the initial number of units put on test. This technique will substantially shorten the average experiment time. Its effect on the probability of a correct selection is generally negligible and in some cases there is no effect.

2. Another technique is to replace each failure immediately by a new unit from the same process. This replacement technique adds to the bookkeeping of the test, but if any of the population variances is large (say in comparison with the guarantee period) then this technique will result in a substantial saving in the average experiment time.

3. A third technique is to use an appropriate sequential procedure. In many problems the sequential procedure results in a smaller average experiment time than the best non-sequential procedure regardless of the true failure rates. The amount of saving depends principally on the "distance" between the smallest and second smallest failure rates.

For the special case of two processes, tables are given to show the probability of a correct selection and the average experiment time for each of three types of procedures.

Numerical estimates of the relative efficiency of the procedures are given by computing the ratio of the average experiment time for two procedures of different type with the same initial sample size and satisfying the same probability specification.

INTRODUCTION

This paper is concerned with a study of the advantages and disadvantages of three statistical techniques for reducing the average duration of life tests. These techniques are:

1. Increasing the initial number of units on test.
2. Using a replacement technique.
3. Using a sequential procedure.

To show the advantages of each of these techniques, we shall consider the problem of deciding which of two processes has the smaller failure rate. Three different types of procedures for making this decision will be considered. They are:

R_1, A nonsequential, nonreplacement type of procedure
R_2, A nonsequential, replacement type of procedure
R_3, A sequential, replacement type of procedure

Within each type we will consider different values of n, the initial number of units on test for each process. The effect of replacement is shown by comparing the average experiment time for procedures of type 1 and 2 with the same value of n and comparable probabilities of a correct selection. The effect of using a sequential rule is shown by comparing the average experiment time for procedures of type 2 and 3 with the same value of n and comparable probabilities of a correct selection.

ASSUMPTIONS

1. It is assumed that failure is clearly defined and that failures are recognized without any chance of error.

2. The lifetime of individual units from either population is assumed to follow an exponential density of the form

$$\begin{aligned} f(x;\theta,g) &= \frac{1}{\theta}e^{-(x-g)/\theta} && \text{for } x \geqq g \\ f(x;\theta,g) &= 0 && \text{for } x < g \end{aligned} \tag{1}$$

where the location parameter $g \geqq 0$ represents the common guarantee period and the scale parameter $\theta > 0$ represents the *unknown* parameter which distinguishes the two different processes. Let $\theta_1 \geqq \theta_2$ denote the *ordered* values of the unknown parameter θ for the two processes; then the *ordered* failure rates are given by

$$\lambda_1 = 1/(\theta_1 + g) \leqq \lambda_2 = 1/(\theta_2 + g) \tag{2}$$

3. It is not known which process has the parameter θ_1 and which has the parameter θ_2.

4. The parameter g is assumed to be the same for both processes. It may be known or unknown.

5. The initial number n of units put on test is the same for both processes.

6. All units have independent lifetimes, i.e., the test environment is not such that the failure of one unit results in the failure of other units on test.

7. Replacements used in the test are assumed to come from the same population as the units they replace. If the replacement units have to sit on a shelf before being used then it is assumed that the replacements are not affected by shelf-aging.

CONCLUSIONS

1. Increasing the initial sample size n has at most a negligible effect on the probability of a correct selection. It has a substantial effect on the average experiment time for all three types of procedures. If the value of n is doubled, then the average time is reduced to a value less than or equal to half of its original value.

2. The technique of replacement always reduces the average experiment time. This reduction is substantial when $g = 0$ or when the population variance of either process is large compared to the value of g. This decrease in average experiment time must always be weighed against the disadvantage of an increase in bookkeeping and the necessity of having the replacement units available for use.

3. The sequential procedure enables the experimenter to make rational decisions as the evidence builds up without waiting for a predetermined number of failures. It has a shorter average experiment time than non-sequential procedures satisfying the same specification. This reduction brought about by the sequential procedure increases as the ratio α of the two failure rates increases. In addition the sequential procedure always terminates with a decision that is clearly convincing on the basis of the observed results, i.e., the à posteriori probability of a correct selection is always large at the termination of the experiment.

SPECIFICATION OF THE TEST

Each of the three types of procedures is set up so as to satisfy the same specification described below. Let α denote the true value of the ratio θ_1/θ_2 which by definition must be greater than, or equal to, one. It turns out that in each type of procedure the probability of a correct selection depends on θ_1 and θ_2 only through their ratio α.

1. The experimenter is asked to specify the smallest value of α (say it is $\alpha^* > 1$) that is worth detecting. Then the interval $(1, \alpha^*)$ represents a zone of indifference such that if the true ratio α lies therein then we would still like to make a correct selection, but the loss due to a wrong selection in this case is negligible.

2. The experimenter is also asked to specify the minimum value $P^* > \frac{1}{2}$ that he desires for the probability of a correct selection whenever $\alpha \geqq \alpha^*$. In each type of procedure the rules are set up so that the probability of a correct selection for $\alpha = \alpha^*$ is as close to P^* as possible without being less than P^*.

The two constants $\alpha^* > 1$ and $\frac{1}{2} < P^* < 1$ are the only quantities specified by the experimenter. Together they make up the *specification* of the test procedure.

EFFICIENCY

If two procedures of different type have the same value of n and satisfy the same specification then we shall regard them as comparable and their relative efficiency will be measured by the ratio of their average experiment times. This ratio is a function of the true α but we shall consider it only for selected values of α, namely, $\alpha = 1$, $\alpha = \alpha^*$ and $\alpha = \infty$.

PROCEDURES OF TYPE R_1 — NONSEQUENTIAL, NONREPLACEMENT

"The same number n of units are put on test for each of the two processes. Experimentation is continued until either one of the two samples produces a predetermined number r $(r \leqq n)$ of failures. Experimentation is then stopped and the process with fewer than r failures is chosen to be the better one."

TABLE I — PROBABILITY OF A CORRECT SELECTION — PROCEDURE TYPE R_1

($\alpha = 2$, any $g \geqq 0$, to be used to obtain r for $\alpha^* = 2$)

n	$r = 1$	$r = 2$	$r = 3$	$r = 4$
1	0.667	—	—	—
2	0.667	0.733	—	—
3	0.667	0.738	0.774	—
4	0.667	0.739	0.784	0.802
10	0.667	0.741	0.789	0.825
20	0.667	0.741	0.790	0.826
∞	0.667	0.741	0.790	0.827

Note: The value for $r = 0$ is obviously 0.500 for any n.

We shall assume that the number n of units put on test is determined by non-statistical considerations such as the availability of units, the availability of sockets, etc. Then the only unspecified number in the above procedure is the integer r. This can be determined from a table of probabilities of a correct selection to satisfy any given specification (α^*, P^*). If, for example, $\alpha^* = 2$ then we can enter Table I. If n is given to be 4 and we wish to meet the specification $\alpha^* = 2$, $P^* = 0.800$ then we would enter Table I with $n = 4$ and select $r = 4$, it being the smallest value for which $P \geqq P^*$.

The table above shows that for the given specification we would also have selected $r = 4$ for any value of n. In fact, we note that the probability of a correct selection depends only slightly on n. The given value of n and the selected value of r then determine a particular procedure of type R_1, say, $R_1(n, r)$.

The average experiment time for each of several procedures $R_1(n, r)$ is given in Table II for the three critical values of the true ratio α, namely, $\alpha = 1$, $\alpha = \alpha^*$ and $\alpha = \infty$. Each of the entries has to be multiplied by θ_2, the smaller of the two θ values, and added to the common guarantee period g. For $n = \infty$ the entry should be zero $(+g)$ but it was found convenient to put in place of zero the leading term in the asymptotic expansion of the expectation in powers of $1/n$. Hence the entry for $n = \infty$ can be used for any large n, say, $n \geqq 25$ when $r \leqq 4$.

We note in Table II the undesirable feature that for each procedure the average experiment time increases with α for fixed θ_2. For the sequential procedure we shall see later that the average experiment time is greater at $\alpha = \alpha^*$ than at either $\alpha = 1$ or $\alpha = \infty$. This is intuitively more desirable since it means that the procedure spends more time when the choice is more difficult to make and less time when we are indifferent or when the choice is easy to make.

PROCEDURES OF TYPE R_2 — NONSEQUENTIAL, REPLACEMENT

"Such procedures are carried out exactly as for procedures of R_1 except that failures are immediately replaced by new units from the same population."

To determine the appropriate value of r for the specification $\alpha^* = 2$, $P^* = 0.800$ when $g = 0$ we use the last row of Table I, i.e., the row marked $n = \infty$, and select $r = 4$. The probability of a correct selection for procedures of type R_2 is exactly the same for all values of n and depends only on r. Furthermore, it agrees with the probability for procedures of type R_1 with $n = \infty$ so that it is not necessary to prepare a separate table.

TABLE II — AVERAGE EXPERIMENT TIME — PROCEDURE TYPE R_1

(Multiply entry by θ_2 and add g)

n	$r = 1$			$r = 2$			$r = 3$			$r = 4$		
	$\alpha = 1$	$\alpha = 2$	$\alpha = \infty$	$\alpha = 1$	$\alpha = 2$	$\alpha = \infty$	$\alpha = 1$	$\alpha = 2$	$\alpha = \infty$	$\alpha = 1$	$\alpha = 2$	$\alpha = \infty$
1	0.500	0.667	1.000	—	—	—	—	—	—	—	—	—
2	0.250	0.333	0.500	0.917	1.200	1.500	—	—	—	—	—	—
3	0.167	0.222	0.333	0.517	0.675	0.833	1.217	1.572	1.833	—	—	—
4	0.125	0.167	0.250	0.363	0.474	0.583	0.735	0.944	1.083	1.449	1.854	2.083
10	0.050	0.067	0.100	0.132	0.172	0.211	0.231	0.297	0.336	0.347	0.439	0.479
20	0.025	0.033	0.050	0.064	0.084	0.103	0.109	0.139	0.158	0.157	0.200	0.217
∞	$0.500/n$	$0.667/n$	$1.000/n$	$1.250/n$	$1.630/n$	$2.000/n$	$2.063/n$	$2.642/n$	$3.000/n$	$2.906/n$	$3.669/n$	$4.000/n$

TABLE III — VALUE OF r REQUIRED TO MEET THE SPECIFICATION (α^*, P^*) FOR PROCEDURES OF TYPE R_2 ($g = 0$)

P^*	α^*												
	1.05	1.10	1.15	1.20	1.25	1.30	1.35	1.40	1.45	1.50	2.00	2.50	3.00
0.50	0	0	0	0	0	0	0	0	0	0	0	0	0
0.55	14	4	2	2	1	1	1	1	1	1	1	1	1
0.60	55	15	7	5	3	3	2	2	2	1	1	1	1
0.65	126	33	16	10	7	5	4	3	3	3	1	1	1
0.70	232	61	29	17	12	9	7	6	5	4	2	1	1
0.75	383	101	47	28	19	14	11	9	7	6	3	2	1
0.80	596	157	73	43	29	21	17	13	11	9	4	2	2
0.85	903	238	111	65	44	32	25	20	16	14	5	3	3
0.90	1381	363	169	100	67	49	37	30	25	21	8	5	4
0.95	2274	597	278	164	110	80	61	49	40	34	12	7	5
0.99	4549	1193	556	327	219	160	122	98	80	68	24	14	10

It is also unnecessary to prepare a separate table for the average experiment time for procedures of type R_2 since for $g = 0$ the exact values can be obtained by substituting the appropriate value of n in the expressions appearing in Table II in the row marked $n = \infty$. For example, for $n = 2$, $r = 1$ and $\alpha = 1$ the exact value for $g = 0$ is $0.500\ \theta_2/2 = 0.250\ \theta_2$, and for $n = 3$, $r = 4$, $\alpha = \infty$ the exact value for $g = 0$ is $4.000\ \theta_2/3 = 1.333\ \theta_2$. It should be noted that for procedures of type R_2 we need not restrict our attention to the cases $r \leqq n$ but can also consider $r > n$.

Table III shows the value of r required to meet the specification (α^*, P^*) with a procedure of type R_2 for various selected values of α^* and P^*.

PROCEDURES OF TYPE R_3 — SEQUENTIAL, REPLACEMENT

Let $D(t)$ denote the absolute difference between the number of failures produced by the two processes at any time t. The sequential procedure is as follows:

"Stop the test as soon as the inequality

$$D(t) \geqq \frac{ln\ [P^*/(1 - P^*)]}{ln\ \alpha^*} \tag{3}$$

is satisfied. Then select the population with the smaller number of failures as the better one."

To get the best results we will choose (α^*, P^*) so that the right hand member of the inequality (3) is an integer. Otherwise we would be operating with a higher value of P^* (or a smaller value of α^*) than was specified.

TABLE IV — AVERAGE EXPERIMENT TIME AND PROBABILITY OF A CORRECT SELECTION — PROCEDURE TYPE R_3

($\alpha^* = 2$, $P^* = 0.800$, $g = 0$)

(Multiply each average time entry by θ_2)

n	$\alpha = 1$	$\alpha = 2$	$\alpha = \infty$
1	2.000	2.400	2.000
2	1.000	1.200	1.000
3	0.667	0.800	0.667
4	0.500	0.600	0.500
10	0.200	0.240	0.200
20	0.100	0.120	0.100
∞	$2.000/n$	$2.400/n$	$2.000/n$
Probability.....	0.500	0.800	1.000

For example, we might choose $\alpha^* = 2$ and $P^* = 0.800$. For procedures of type R_3 the probability of a correct selection is again completely independent of n; here it depends only on the true value of the ratio α. The average experiment time depends strongly on n and only to a limited extent on the true value of the ratio α. Table IV gives these quantities for $\alpha = 1$, $\alpha = 2$, and $\alpha = \infty$ for the particular specification $\alpha^* = 2$, $P^* = 0.800$ and for the particular value $g = 0$.

EFFICIENCY

We are now in a position to compare the efficiency of two different types of procedures using the same value of n. The efficiency of R_1 relative to R_2 is the reciprocal of the ratio of their average experiment time. This is given in Table V for $\alpha^* = 2$, $P^* = 0.800$, $r = 4$ and $n = 4, 10, 20$ and ∞. By Table I the value $P^* = 0.800$ is not attained for $n < 4$.

In comparing the sequential and the nonsequential procedures it was found that the slight excesses in the last column of Table I over 0.800

TABLE V — EFFICIENCY OF TYPE R_1 RELATIVE TO TYPE R_2

($\alpha^* = 2$, $P^* = 0.800$, $r = 4$, $g = 0$)

n	$\alpha = 1$	$\alpha = 2$	$\alpha = \infty$
4	0.501	0.495	0.480
10	0.837	0.836	0.835
20	0.925	0.917	0.922
∞	1.000	1.000	1.000

TABLE VI — EFFICIENCY OF ADJUSTED R_1 RELATIVE TO R_3
($\alpha^* = 2$, $P^* = 0.800$, $g = 0$)

n	$\alpha = 1$	$\alpha = 2$	$\alpha = \infty$
4	0.615	0.575	0.419
10	0.754	0.708	0.528
20	0.818	0.768	0.573
∞	0.873	0.822	0.612

had an effect on the efficiency. To make the procedures more comparable the values for $r = 3$ and $r = 4$ in Table I were averaged with values p and $1 - p$ computed so as to give a probability of *exactly* 0.800 at $\alpha = \alpha^*$. The corresponding values for the average experiment time were then averaged with the same values p and $1 - p$. The nonsequential procedures so altered will be called "adjusted procedures." The efficiency of the adjusted R_1 relative to R_3 is given in Table VI.

In Table VI the last row gives the efficiency of the adjusted procedure R_2 relative to R_3. Thus we can separate out the advantage due to the replacement feature and the advantage due to the sequential feature. Table VII gives these results in terms of percentage reduction of average experiment time.

We note that the reduction due to the replacement feature alone is greatest for small n and essentially constant with α while the reduction

TABLE VII — PER CENT REDUCTION IN AVERAGE EXPERIMENT TIME DUE TO STATISTICAL TECHNIQUES
($\alpha^* = 2$, $P^* = 0.800$, $g = 0$)

α	n	Reduction due to *Replacement* Feature Alone	Reduction due to *Sequential* Feature Alone	Reduction due to both Replacement and Sequential Features
1	4	29.5	12.7	38.5
	10	13.7	12.7	24.6
	20	6.3	12.7	18.2
	∞	0.0	12.7	12.7
2	4	30.1	17.8	42.5
	10	13.9	17.8	29.2
	20	6.6	17.8	23.2
	∞	0.0	17.8	17.8
∞	4	31.5	38.8	58.1
	10	13.6	38.8	47.2
	20	6.3	38.8	42.7
	∞	0.0	38.8	38.8

due to the sequential feature alone is greatest for large α and is independent of n. Hence if the initial sample size per process n is large we can disregard the replacement technique. On the other hand the true value of α is not known and hence the advantage of sequential experimentation should not be disregarded.

The formulas used to compute the accompanying tables are given in Addendum 2.

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ADDENDUM 1

In this addendum we shall consider the more general problem of selecting the best of k exponential populations treated on a higher mathematical level. For $k = 2$ this reduces to the problem discussed above.

DEFINITIONS AND ASSUMPTIONS

There are given k populations Π_i $(i = 1, 2, \cdots, k)$ such that the lifetimes of units taken from any of these populations are independent chance variables with the exponential density (1) with a common (known or unknown) location parameter $g \geqq 0$. The distributions for the k populations are identical except for the unknown scale parameter $\theta > 0$ which may be different for the k different populations. We shall consider three different cases with regard to g.

Case 1: The parameter g has the value zero $(g = 0)$.

Case 2: The parameter g has a positive, known value $(g > 0)$.

Case 3: The parameter g is unknown $(g \geqq 0)$.

Let the *ordered values* of the k scale parameters be denoted by

$$\theta_1 \geqq \theta_2 \geqq \cdots \geqq \theta_k \tag{4}$$

where equal values may be regarded as ordered in any arbitrary manner. At any time t each population has a certain number of failures associated with it. Let the *ordered values* of these integers be denoted by $r_i = r_i(t)$ so that

$$r_1 \leqq r_2 \leqq \cdots \leqq r_k \tag{5}$$

For each unit the life beyond its guarantee period will be referred to as its Poisson life. Let $L_i(t)$ denote the total amount of Poisson life observed up to time t in the population with r_i failures ($i = 1, 2, \cdots, k$). If two or more of the r_i are equal, say $r_i = r_{i+1} = \cdots = r_{i+j}$, then we shall assign r_i and L_i to the population with the largest Poisson life, r_{i+1} and L_{i+1} to the population with the next largest, $\cdots$, r_{i+j} and L_{i+j} to the population with the smallest Poisson life. If there are two or more *equal pairs* (r_i, L_i) then these should be ordered by a random device giving equal probability to each ordering. Then the subscripts in (5) as well as those in (4) are in one-to-one correspondence with the k given populations. It should be noted that $L_i(t) \geqq 0$ for all i and any time $t \geqq 0$. The complete set of quantities $L_i(t)$ $(i = 1, 2, \cdots, k)$ need not be ordered. Let $\alpha = \theta_1/\theta_2$ so that, since the θ_i are ordered, $\alpha \geqq 1$.

We shall further assume that:

1. The initial number n of units put on test is the same and the starting time is the same for each of the k populations.

2. Each replacement is assumed to be a new unit from the same population as the failure that it replaces.

3. Failures are assumed to be clearly recognizable without any chance of error.

SPECIFICATIONS FOR CASE 1: $g = 0$

Before experimentation starts the experimenter is asked to specify two constants α^* and P^* such that $\alpha^* > 1$ and $\frac{1}{2} < P^* < 1$. The procedure $R_3 = R_3(n)$, which is defined in terms of the specified α^* and P^*, has the property that it will correctly select the population with the largest scale parameter with probability at least P^* whenever $\alpha \geqq \alpha^*$. The initial number n of units put on test may either be fixed by nonstatistical considerations or may be determined by placing some restriction on the average experiment time function.

Rule R_3:

"Continue experimentation with replacement until the inequality

$$\sum_{i=2}^{k} \alpha^{*-(r_i - r_1)} \leqq (1 - P^*)/P^* \tag{6}$$

is satisfied. Then stop and select the population with the smallest number of failures as the one having the largest scale parameter."

Remarks

1. Since $P^* > \frac{1}{2}$ then $(1 - P^*)/P^* < 1$ and hence no two populations can have the same value r_1 at stopping time.
2. For $k = 2$ the inequality (6) reduces to the inequality (3).
3. The procedure R_3 terminates only at a failure time, never between failures, since the left member of (6) depends on t only through the quantities $r_i(t)$.
4. After experimentation is completed one can make, at the $100P$ per cent confidence level, the *confidence statement*

$$\theta_s \leqq \theta_1 \leqq \alpha^* \theta_s \quad (\text{or} \quad \theta_1/\alpha^* \leqq \theta_s \leqq \theta_1) \tag{7}$$

where θ_s is the scale parameter of the selected population.

Numerical Illustrations

Suppose the preassigned constants are $P^* = 0.95$ and $\alpha^* = 19^{1/4} = 2.088$ so that $(1 - P^*)/P^* = \frac{1}{19}$. Then for $k = 2$ the procedure is to stop when $r_2 - r_1 \geqq 4$. For $k = 3$ it is easy to check that the procedure reduces to the simple form: "Stop when $r_2 - r_1 \geqq 5$". For $k > 3$ either calculations can be carried out as experimentation progresses or a table of stopping values can be constructed before experimentation starts. For $k = 4$ and $k = 5$ see Table VIII.

In the above form the proposed rule is to stop when, for at least one

TABLE VIII — SEQUENTIAL RULE FOR $P^* = 0.95$, $\alpha^* = 19^{1/4}$

$k = 4$

$r_2 - r_1$	$r_3 - r_1$	$r_4 - r_1$	
5	5	9	
5	6	6	
6	6	6	*

$k = 5$

$r_2 - r_1$	$r_3 - r_1$	$r_4 - r_1$	$r_5 - r_1$	
5	5	9	10	
5	5	10	10	*
5	6	6	8	
5	6	7	7	
5	7	7	7	*
6	6	6	6	

* Starred rows can be omitted without affecting the test since every integer in these rows is at least as great as the corresponding integer in the previous row. They are shown here to illustrate a systematic method which insures that all the necessary rows are included.

row (say row j) in the table, the observed row vector $(r_2 - r_1, r_3 - r_1, \cdots, r_k - r_1)$ is such that *each component* is at least as large as the corresponding component of row j.

Properties of R_3 for $k = 2$ and $g = 0$

For $k = 2$ and $g = 0$ the procedure R_3 is an example of a Sequential Probability Ratio test as defined by A. Wald in his book.[5] The Average Sample Number (ASN) function and the Operating Characteristics (OC) function for R_3 can be obtained from the general formulae given by Wald. Both of these functions depend on θ_1 and θ_2 only through their ratio α. In our problem there is no excess over the boundary and hence Wald's approximation formulas are *exact*. When our problem is put into the Wald framework, the symmetry of our problem implies equal probabilities of type 1 and type 2 errors. The OC function takes on complementary values for any point $\alpha = \theta_1/\theta_2$ and its reciprocal θ_2/θ_1. We shall therefore compute it only for $\alpha \geqq 1$ and denote it by $P(\alpha)$. For $\alpha > 1$ the quantity $P(\alpha)$ denotes the probability of a correct selection for the true ratio α.

The equation determining Wald's h function[5] is

$$\frac{(\alpha^*)^h}{1 + \alpha} + \frac{\alpha(\alpha^*)^{-h}}{1 + \alpha} = 1 \tag{8}$$

for which the non-zero solution in h is easily computed to be

$$h(\alpha) = \frac{\ln \alpha}{\ln \alpha^*} \tag{9}$$

Hence we obtain from Wald's formula (3:43) in Reference 5

$$P(\alpha) = \frac{\alpha^s}{\alpha^s + 1} \tag{10}$$

where s is the smallest integer greater than or equal to

$$S = \ln [P^*/(1 - P^*)]/\ln \alpha^* \tag{11}$$

In particular, for $\alpha = 1^+$, α^* and ∞ we have

$$P(1^+) = \tfrac{1}{2}, \qquad P(\alpha^*) \geqq P^*, \qquad P(\infty) = 1 \tag{12}$$

We have written $P(1^+)$ above for $\lim P(x)$ as $x \to 1$ from the right. The procedure becomes more efficient if we choose P and α^* so that S is an integer. Then $s = S$ and $P(\alpha^*) = P^*$.

Letting F denote the total number of observed failures required to

terminate the experiment we obtain for the ASN function

$$E(F;\alpha) = s\left(\frac{\alpha+1}{\alpha-1}\right)\left(\frac{\alpha^s-1}{\alpha^s+1}\right) \quad \text{for } \alpha > 1 \quad (13)$$

and, in particular, for $\alpha = 1, \infty$

$$E(F;1) = s^2 \quad \text{and} \quad E(F;\infty) = s \quad (14)$$

It is interesting to note that for $s = 1$ we obtain

$$E(F;\alpha) = 1 \qquad \textit{for all } \alpha \geqq 1 \quad (15)$$

and that this result is exact since for $s = 1$ the right-hand member S of (3) is at most one and hence the procedure terminates with certainty immediately after the first failure.

As a result of the exponential assumption, the assumption of replacement and the assumption that $g = 0$ it follows that the intervals between failures are independently and identically distributed. For a single population the time interval between failures is an exponential chance variable. Hence, for two populations, the time interval is the minimum of two exponentials which is again exponential. Letting τ denote the (chance) duration of a typical interval and letting T denote the (chance) total time needed to terminate the procedure, we have

$$E(T;\alpha,\theta_2) = E(F;\alpha)E(\tau;\alpha,\theta_2) = E(F;\alpha)\left(\frac{\theta_2}{n}\right)\left(\frac{\alpha}{1+\alpha}\right) \quad (16)$$

Hence we obtain from (13) and (14)

$$E(T;\alpha,\theta_2) = \frac{\theta_2}{n}\frac{s\alpha}{\alpha-1}\frac{\alpha^s-1}{\alpha^s+1} \qquad \text{for } \alpha > 1 \quad (17)$$

$$E(T;1,\theta_2) = \frac{\theta_2 s^2}{2n} \quad \text{and} \quad E(T;\infty,\theta_2) = \frac{\theta_2 s}{n} \quad (18)$$

For the numerical illustration treated above with $k = 2$ we have

$$P(\alpha) = \frac{\alpha^4}{1+\alpha^4} \quad (19)$$

$$P(1^+) = \tfrac{1}{2}; \qquad P(2.088) = 0.95; \qquad P(\infty) = 1 \quad (20)$$

$$E(F;\alpha) = 4\frac{\alpha+1}{\alpha-1}\frac{\alpha^4-1}{\alpha^4+1} = 4\frac{(\alpha+1)^2(\alpha^2+1)}{\alpha^4+1} \quad (21)$$

$$E(F;1) = 16.0; \qquad E(F;2.088) = 10.2; \qquad E(F;\infty) = 4 \quad (22)$$

$$E(T; 1, \theta_2) = \frac{8\theta_2}{n}; \qquad E(T; 2.088, \theta_2) = \frac{6.9\theta_2}{n}; \qquad E(T; \infty, \theta_2) = \frac{4\theta_2}{n} \tag{23}$$

For $k > 2$ the proposed procedure is an application of a general sequential rule for selecting the best of k populations which is treated in [1]. Proof that the probability specification is met and bounds on the probability of a correct decision can be found there.

CASE 2: COMMON KNOWN $g > 0$

In order to obtain the properties of the sequential procedure R_3 for this case it will be convenient to consider other sequential procedures. Let $\beta = 1/\theta_2 - 1/\theta_1$ so that, since the θ_i are ordered, $\beta \geqq 0$. Let us assume that the experimenter can specify three constants α^*, β^* and P^* such that $\alpha^* > 1$, $\beta^* > 0$ and $\frac{1}{2} < P^* < 1$ and a procedure is desired which will select the population with the largest scale parameter with probability at least P^* whenever we have both

$$\alpha \geqq \alpha^* \quad \text{and} \quad \beta \geqq \beta^*$$

The following procedure meets this specification.

Rule R_3':

"Continue experimentation with replacement until the inequality

$$\sum_{i=2}^{k} \alpha^{*-(r_i - r_1)} e^{-\beta^*(L_1 - L_i)} \leqq (1 - P^*)/P^* \tag{24}$$

is satisfied. Then stop and select the population with the smallest number of failures as the one having the largest scale parameter. If, at stopping time, two or more populations have the same value r_1 then select that particular one of these with the largest Poisson life L_1."

Remarks

1. For $k = 2$ the inequality reduces to

$$(r_2 - r_1) \ln \alpha^* + (L_1 - L_2)\beta^* \geqq \ln [P^*/(1 - P^*)] \tag{25}$$

If $g = 0$ then $L_1 \equiv L_i$ for all t and the procedure R_3' reduces to R_3.

2. The procedure R_3' may terminate not only at failures but also between failures.

3. The same inequality (24) can also be used if experimentation is carried on *without replacement*, one advantage of the latter being that there is less bookkeeping involved. In this case there is a possibility that the units will all fail before the inequality is satisfied so that the procedure is not yet completely defined for this case. One possibility in such a situation is to continue experimentation with new units from each population until the inequality is satisfied. Such a procedure will terminate in a finite time with probability one, i.e., $\text{Prob}\{T > T_0\} \to 0$ as $T_0 \to \infty$, and the probability specification will be satisfied.

4. A procedure R_3' $(n_1, n_2, \cdots, n_k, t_1, t_2, \cdots, t_k)$ using the same inequality (24) but based on different initial sample sizes and/or on different starting times for the initial samples also satisfies the above probability specification. In the case of different starting times it is required that the experimenter wait at least g units of time after the last initial sample is put on test before reaching any decision.

5. One disadvantage of R_3' is that there is some (however remote) possibility of terminating while $r_1 = r_2$. This can be avoided by adding the condition $r_2 > r_1$ to (24) but, of course, the average experiment time is increased. Another way of avoiding this is to use the procedure R_3 which depends only on the number of failures; the effect of using R_3 when $g > 0$ will be considered below.

6. The terms of the sum in (24) represent likelihood ratios. If at any time each term is less than unity then we shall regard the decision to select the population with r_1 failures and L_1 units of Poisson life as optimal. Since $(1 - P^*)/P^* < 1$ then each term must be less than unity at termination.

Properties of Procedure R_3' for $k = 2$

The OC and ASN functions for R_3' will be approximated by comparing R_3' with another procedure R_3'' defined below. We shall assume that P^* is close to unity and that g is small enough (compared to θ_2) so that the probability of obtaining two failures within g units of time is small enough to be negligible. Then we can write approximately at termination

$$L_i \cong nT - r_i g \qquad (i = 1, 2, \cdots, k) \tag{26}$$

and

$$L_1 - L_i \cong (r_i - r_1)g \qquad (i = 2, 3, \cdots, k) \tag{27}$$

Substituting this in (24) and letting

$$\delta^* = \alpha^* e^{\beta^* g} \tag{28}$$

suggests a new rule, say R_3'', which we now define.

Rule R_3''

"Continue experimentation with replacement until the inequality

$$\sum_{i=2}^{k} \delta^{*-(r_i-r_1)} \leqq (1 - P^*)/P^* \tag{29}$$

is satisfied. Then stop and select the population with r_1 failures as the one with the largest scale parameter."

For rule R_3'' the experimenter need only specify P^* and the smallest value δ^* of the *single* parameter

$$\delta = \frac{\theta_1}{\theta_2} e^{g((1/\theta_2)-(1/\theta_1))} = \alpha e^{g\beta} \tag{30}$$

that he desires to detect with probability at least P^*.

We shall approximate the OC and ASN function of R_3'' for $k = 2$ by computing them under the assumption that (27) holds at termination. The results will be considered as an approximation for the OC and ASN functions respectively of R_3' for $k = 2$. The similarity of (29) and (6) immediately suggests that we might replace α^* by δ^* and α by δ in the formulae for (6). To use the resulting expressions for R_3' we would compute δ^* as a function of α^* and β^* by (28) and δ as a function of α and β by (30).

The similarity of (29) and (6) shows that Z_n (defined in Reference 5, page 170) under (27) with $g > 0$ is the same function of δ^* and δ as it is of α^* and α when $g = 0$. To complete the justification of the above result it is sufficient to show that the individual increment z of Z_n is the same function of δ^* and δ under (27) with $g > 0$ as it is of α^* and α when $g = 0$. To keep the increments independent it is necessary to associate each failure with the Poisson life that follows rather than with the Poisson life that precedes the failure. Neglecting the probability that any two failures occur within g units of time we have two values for z, namely

$$z = \log \frac{\frac{n}{\theta_1} e^{-(nt-g)/\theta_1} e^{-nt/\theta_2}}{\frac{n}{\theta_2} e^{-(nt-g)/\theta_2} e^{-nt/\theta_1}} = -\log \delta \tag{31}$$

and, interchanging θ_1 and θ_2, gives $z = \log \delta$. Moreover

$$\text{Prob}\{z = -\log \delta\} = \frac{\int_g^\infty \int_g^\infty \frac{n}{\theta_2} e^{-(nx-g)/\theta_2} e^{-ny/\theta_1}\, dx\, dy}{\frac{\theta_2}{n} e^{-g[\theta_2(n-1)+\theta_1 n]/\theta_1\theta_2} + \frac{\theta_1}{n} e^{-g[\theta_2 n+\theta_1(n-1)]/\theta_1\theta_2}} \tag{32}$$

$$= \frac{\delta}{1+\delta}$$

Thus the OC and ASN functions under (27) with $g > 0$ bear the same relation to δ^* and δ as they do to α^* and α when $g = 0$. Hence, letting w denote the smallest integer greater than or equal to

$$W = \frac{\ln [P^*/(1-P^*)]}{\ln \delta^*} = \frac{\ln [P^*/(1-P^*)]}{g\beta^* + \ln \alpha^*} \tag{33}$$

we can write (omitting P^* in the rule description)

$$P\{\delta;\, R_3'\,(\alpha^*, \beta^*)\} \cong P\{\delta;\, R_3''(\delta^*)\} \cong \frac{\delta^w}{\delta^w + 1} \tag{34}$$

$$E\{F;\, R_3'\,(\alpha^*, \beta^*)\} \cong E\{F;\, R_3''(\delta^*)\}$$

$$\cong \begin{cases} w\left(\frac{\delta+1}{\delta-1}\right)\left(\frac{\delta^w-1}{\delta^w+1}\right) & \text{for } \delta > 1 \\ w^2 & \text{for } \delta = 1 \end{cases} \tag{35}$$

We can approximate the average time between failures by

$$E\{\tau;\, \theta_1, \theta_2, g\} \cong \frac{(g+\theta_1)(g+\theta_2)}{n(\theta_1+\theta_2+2g)} \leqq g + \frac{\theta_2}{n}\left(\frac{\alpha}{1+\alpha}\right) \tag{36}$$

and the average experiment time by

$$E\{T;\, R_3'(\alpha^*, \beta^*)\} \cong E\{F;\, R_3'(\alpha^*, \beta^*)\}\, \frac{(g+\theta_1)(g+\theta_2)}{n(\theta_1+\theta_2+2g)} \tag{37}$$

Since $\delta \geqq 1$ then $\delta^w/(1+\delta^w)$ is an increasing function of w and by (33) it is a non-increasing function of δ^*. By (28) $\delta^* \geqq \alpha^*$ and hence, if we disregard the approximation (34),

$$P\{\delta;\, R_3''(\alpha^*)\} \cong \frac{[P^*/(1-P^*)]^{\ln \delta/\ln \alpha^*}}{1 + [P^*/(1-P^*)]^{\ln \delta/\ln \alpha^*}} \geqq P\{\delta; R_3''(\delta^*)\} \tag{38}$$

Clearly the rules $R_3(\alpha^*, P^*)$ and $R_3''\,(\alpha^*, P^*)$ are equivalent so that for $g > 0$ we have

$$P\{\delta;\, R_3(\alpha^*)\} \equiv P\{\delta;\, R_3''(\alpha^*)\} \tag{39}$$

and hence, in particular, letting $\delta = \delta^*$ in (38) we have

$$P\{\delta^*; R_3(\alpha^*)\} \geqq P\{\delta^*; R_3''(\delta^*)\} \geqq P^* \tag{40}$$

since the right member of (34) reduces to P^* when W is an integer and $\delta = \delta^*$. The error in the approximations above can be disregarded when g is small compared to θ_2. Thus we have shown that for small values of g/θ_2 the probability specification based on (α^*, β^*, P^*) is satisfied in the sense of (40) if we use the procedure $R_3(\alpha^*, P^*)$, i.e., if we proceed as if $g = 0$.

It would be desirable to show that we can proceed as if $g = 0$ for all values of g and P^*. It can be shown that *for sufficiently large* n the rule $R_3(\alpha^*, P^*)$ meets it specification for all g. One effect of increasing n is to decrease the average time $E(\tau)$ between failures and to approach the corresponding problem *without replacement* since $g/E(\tau)$ becomes large. Hence we need only show that $R_3(\alpha^*, P^*)$ meets its specification for the corresponding problem without replacement. If we disregard the information furnished by Poisson life and rely solely on the counting of failures then the problem reduces to testing in a *single* binomial whether $\theta = \theta_1$ for population Π_1 and $\theta = \theta_2$ for population Π_2 or vice versa. Letting p denote the probability that the next failure arises from Π_1 then we have formally

$$H_0{:}p = \frac{1}{1+\alpha} \qquad \text{versus} \qquad H_1{:}p = \frac{\alpha}{1+\alpha}$$

For preassigned constants $\alpha^* > 1$ and P^* $(\frac{1}{2} < P^* < 1)$ the appropriate sequential likelihood test to meet the specification:

"Probability of a Correct Selection $\geqq P^*$ whenever $\alpha \geqq \alpha^*$" (41)

then turns out to be precisely the procedure $R_3(\alpha^*, P^*)$. Hence we may proceed as if $g = 0$ when n is sufficiently large.

The specifications of the problem may be given in a different form. Suppose $\theta_1^* > \theta_2^*$ are specified and it is desired to have a probability of a correct selection of at least P^* whenever $\theta_1 \geqq \theta_1^* > \theta_2^* \geqq \theta_2$. Then we can form the following sequential likelihood procedure R_3^* which is more efficient than $R_3(\alpha^*, P^*)$.

Rule R_3^*:

"Continue experimentation without replacement until a time t is reached at which the inequality

$$\sum_{i=2}^{k} \left[\frac{e^{t/\theta_2^*} - 1}{e^{t/\theta_1^*} - 1}\right]^{-(r_i - r_1)} \leqq \frac{1 - P^*}{P^*} \tag{42}$$

is satisfied. Then stop and select the population with r_1 failures as the population with $\theta = \theta_1$".

It can be easily shown that the greatest lower bound of the bracketed quantity in (42) is θ_1^*/θ_2^*. Hence for $\theta_1^*/\theta_2^* = \alpha^*$ and $P^* > 1/2$ the time required by $R_3^*(\theta_1^*, \theta_2^*, P^*)$ will always be less than the time required by $R_3(\alpha^*, P^*)$.

Another type of problem is one in which we are given that $\theta = \theta_1^*$ for one population and $\theta = \theta_2^*$ for the $k - 1$ others where $\theta_1^* > \theta_2^*$ are specified. The problem is to select the population with $\theta = \theta_1^*$. Then (42) can again be used. In this case the parameter space is discrete with k points only one of which is correct. If Rule R_3^* is used then the probability of selecting the correct point is at least P^*.

Equilibrium Approach When Failures Are Replaced

Consider first the case in which all items on test are from the same exponential population with parameters (θ, g). Let T_{nj} denote the length of the time interval between the j^{th} and the $j + 1^{\text{st}}$ failures, $(j = 0, 1, \cdots)$, where n is the number of items on test and the 0^{th} failure denotes the starting time. As time increases to infinity the expected number of failures per unit time clearly approaches $n/(\theta + g)$ which is called the equilibrium failure rate. The inverse of this is the expected time between failures at equilibrium, say $E(T_{n\infty})$. The question as to how the quantities $E(T_{nj})$ approach $E(T_{n\infty})$ is of considerable interest in its own right. The following results hold for any fixed integer $n \geqq 1$ unless explicitly stated otherwise. It is easy to see that

$$E(T_{n1}) \leqq E(T_{n\infty}) \leqq E(T_{n0}) \tag{43}$$

since the exact values are respectively

$$\frac{\theta}{n-1}\left(1 - \frac{e^{-(n-1)g/\theta}}{n}\right) \leqq \frac{g+\theta}{n} \leqq g + \frac{\theta}{n} \tag{44}$$

In fact, since all units are new at starting time and since at the time of the first failure all units (except the replacement) have passed their guarantee period with probability one then

$$E(T_{n1}) \leqq E(T_{nj}) \leqq E(T_{n0}) \qquad (j \geqq 0) \tag{45}$$

If we compare the case $g > 0$ with the special case $g = 0$ we obtain

$$E(T_{nj}) \geqq \frac{\theta}{n} \qquad (j = 1, 2, \cdots) \tag{46}$$

and if we compare it with the non-replacement case (g/θ is large) we obtain

$$E(T_{nj}) \leqq \frac{\theta}{n-j} \qquad (j = 1, 2, \cdots, n-1). \quad (47)$$

These comparisons show that the difference in (46) is small when g/θ is small and for $j < n$ the difference in (47) is small when g/θ is large.

It is possible to compute $E(T_{nj})$ exactly for $g \geqq 0$ but the computation is extremely tedious for $j \geqq 2$. The results for $j = 1$ and 0 are given in (44). For $j = 2$

$$E(T_{n2}) = \frac{\theta}{n-2}\left[1 - \frac{(n+2)(n-1)}{n^2} e^{-(n-2)g/\theta} + \frac{n-2}{n-1} e^{-(n-1)g/\theta} - \frac{n-2}{n^2(n-1)} e^{-2(n-1)g/\theta}\right] \qquad (n > 2) \quad (48)$$

and

$$E(T_{22}) = g - \frac{\theta}{4}[1 - 4e^{-g/\theta} + e^{-2g/\theta}] \quad (49)$$

For the case of two populations with a common guarantee period g we can write similar inequalities. We shall use different symbols a, b for the initial sample size from the populations with scale parameters θ_1, θ_2 respectively even though our principal interest is in the case $a = b = n$ say. Let $T_{a,b,j}$ denote the interval between the j^{th} and $j+1^{\text{st}}$ failures in this case and let $\lambda_i = 1/\theta_i$ $(i = 1, 2)$. We then have for all values of a and b

$$[a\lambda_1 + b\lambda_2]^{-1} \leqq E(T_{a,b,j}) \leqq E(T_{a,b,0}) = g + [a\lambda_1 + b\lambda_2]^{-1} \quad (j = 0, 1, 2, \cdots, \infty) \quad (50)$$

$$E(T_{a,b,\infty}) = \frac{(\theta_1 + g)(\theta_2 + g)}{a(\theta_2 + g) + b(\theta_1 + g)} \quad (51)$$

The result for $E(T_{a,b,1})$ corresponding to that in (43) does not hold if the ratio θ_1/θ_2 is too large; in particular it can be shown that

$$E(T_{a,b,1}) = \left(\frac{a\lambda_1}{a\lambda_1 + b\lambda_2}\right)\left(\frac{1}{(a-1)\lambda_1 + b\lambda_2}\right)\left[1 - \frac{\lambda_1 e^{-g[(a-1)\lambda_1 + b\lambda_2]}}{a\lambda_1 + b\lambda_2}\right] + \left(\frac{b\lambda_2}{a\lambda_1 + b\lambda_2}\right)\left(\frac{1}{a\lambda_1 + (b-1)\lambda_2}\right)\left[1 - \frac{\lambda_2 e^{-g[a\lambda_1 + (b-1)\lambda_2]}}{a\lambda_1 + b\lambda_2}\right] \quad (52)$$

is larger than $E(T_{a,b,\infty})$ for $a = b = 1$ when $g/\theta_1 = 0.01$ and $g/\theta_2 = 0.10$

so that $\theta_1/\theta_2 = 10$. The expression (52) reduces to that in (44) if we set $\theta_1 = \theta_2 = \theta$ and replace a and b by $n/2$ in the resulting expression.

Corresponding exact expressions for $E(T_{a,b,j})$ for $j > 1$ are extremely tedious to derive and unwieldy although the integrations involved are elementary. If we let $g \to \infty$ then we obtain expressions for the non-replacement case which are relatively simple. They are best expressed as a recursion formula.

$$E(T_{a,b,j}) = \frac{a\lambda_1}{a\lambda_1 + b\lambda_2} ET_{a-1,b,j-1} + \frac{b\lambda_2}{a\lambda_1 + b\lambda_2} ET_{a,b-1,j-1} \qquad (j \geqq 1) \tag{53}$$

$$E(T_{a,b,1}) = \frac{a\lambda_1}{a\lambda_1 + b\lambda_2} \frac{1}{(a-1)\lambda_1 + b\lambda_2} + \frac{b\lambda_2}{a\lambda_1 + b\lambda_2} \frac{1}{a\lambda_1 + (b-1)\lambda_2} \qquad (a, b \geqq 1) \tag{54}$$

$$E(T_{a,0,j}) \leqq g + \theta_1/a \qquad \text{for } j \geqq a \text{ and } j = 0 \tag{55}$$

$$E(T_{a,0,j}) = \theta_1/(a-j) \qquad \text{for } 1 \leqq j \leqq a-1 \tag{56}$$

Results similar to (55) and (56) hold for the case $a = 0$. The above results for $g = \infty$ provide useful approximations for $E(T_{a,b,j})$ when g is large. Upper bounds are given by

$$E(T_{a,b,j}) \leqq [a\lambda_1 + (b-j)\lambda_2]^{-1} \qquad (j = 1, 2, \cdots, b) \tag{57}$$

$$E(T_{a,b,j+b}) \leqq [(a-j)\lambda_1]^{-1} \qquad (j = 1, 2, \cdots, a-1). \tag{58}$$

Duration of the Experiment

For the sequential rule R_3' with $k = 2$ we can now write down approximations as well as upper and lower bounds to the expected duration $E(T)$ of the experiment. From (50)

$$g + \frac{E(F;\delta)}{n(\lambda_1 + \lambda_2)} \leqq E(T) = \sum_{j=0}^{c-1} E(T_{n,n,j}) + [E(F;\delta) - c]E(T_{n,n,c}) \tag{59}$$

where c is the largest integer less than or equal to $E(F;\delta)$. The right expression of (59) can be approximated by (53) and (54) if g is large. If $c < 2n$ then the upper bounds are given by (57) and (58). A simpler

upper bound, which holds for all values of c is given by

$$E(T) \leqq E(F; \delta)E(T_{n,n,0}) = E(F; \delta)\left(g + \frac{\theta_1}{n}\right) \tag{60}$$

CASE 3: COMMON UNKNOWN LOCATION PARAMETER $g \geqq 0$

In this case the more conservative procedure is to proceed under the assumption that $g = 0$. By the discussion above the probability requirement will in most problems be satisfied for all $g \geqq 0$. The OC and ASN functions, which are now functions of the true value of g, were already obtained above. Of course, we need not consider values of g greater than the smallest observed lifetime of all units tested to failure.

ADDENDUM 2

For completeness it would be appropriate to state explicitly some of the formulas used in computing the tables in the early part of the paper. For the nonsequential, nonreplacement rule R_1 with $k = 2$ the probability of a correct selection is

$$P(\alpha; R_1) = \int_0^\infty \int_0^x f_r(y, \theta_2) f_r(x, \theta_1)\, dy\, dx \tag{61}$$

where

$$f_r(x, \theta) = \frac{r}{\theta} C_r^n (1 - e^{-x/\theta})^{r-1} e^{-x(n-r+1)/\theta}. \qquad (r \leqq n) \tag{62}$$

and C_r^n is the usual combinatorial symbol. This can also be expressed in the form

$$P(\alpha; R_1) = 1 - (rC_r^n)^2 \sum_{j=1}^{r} \frac{(-1)^{j-1}}{n - r + j} C_{j-1}^{r-1}\{B[r, n - r + 1 + \alpha(n - r + j)]\}^{-1} \tag{63}$$

where $B[x, y]$ is the complete Beta function. Equation (66) holds for any $g \geqq 0$.

For the rule R_1 the expected duration of the experiment for $k = 2$ is given by

$$E(T) = \int_0^\infty x\{f_r(x, \theta_1)[1 - F_r(x, \theta_2)] + f_r(x, \theta_2)[1 - F_r(x, \theta_1)]\}\, dx \tag{64}$$

where $f_r(x, \theta)$ is the density in (62) and $F_r(x, \theta)$ is its c.d.f. This can

also be expressed in the form

$$\theta_1 r(C_r^n)^2 \sum_{i=1}^{r} \sum_{j=1}^{r} \frac{(-1)^{i+j} C_{i-1}^{r-1} C_{j-1}^{r-1}}{(n-r+j)[i+n-r+\alpha(j+n-r)]^2} \tag{65}$$

plus another similar expression in which θ_1, α are replaced by θ_2, α^{-1} respectively. For $g > 0$ we need only add g to this result. This result was used to compute $E(T)$ in table 1A for $\alpha = 1$ and $\alpha = 2$. For $\alpha = \infty$ the expression simplifies to

$$E(T) = \theta_2 r C_r^n \sum_{j=1}^{r} C_{j-1}^{r-1} \frac{(-1)^{r-j}}{(n-j+1)^2} \tag{66}$$

which can be shown to be equivalent to

$$E(T) = \theta_2 \sum_{j=1}^{r} \frac{1}{n-j+1} \tag{67}$$

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