Analysis of the Single Tapered Mode Coupler

By W. H. LOUISELL

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Broadband directional couplers in which the phase constants and coupling coefficients vary with distance along two coupled transmission lines as suggested in the two preceding papers are analyzed. A criterion is given for the allowed variations in the line parameters for a given bandwidth. Parameters which describe the performance of such couplers are given. It is found that such couplers give much greater bandwidth than conventional couplers of the distributed coupling type but they must be longer than conventional couplers.

I. INTRODUCTION

Conventional directional couplers[1] of the distributed coupling type may be thought of as two coupled uniform ideal lossless transmission lines. Such a system can in general support two forward normal modes of propagation. In order to effect power transfer between two lines of such a system, both forward normal modes must be present. Since the two normal modes in general travel with a different phase velocity, they can interfere with one another to set up a standing or "beat" wave pattern. Power transfer is thus effected by the interference of these two normal modes and for this reason such couplers are called mode interference directional couplers. If for example a 100 per cent coupler is desired, that is, complete power transfer from one line to the other, the coupler is ended after half a "beat" wave length. Similarly, if a three db coupler is desired, that is, a 50–50 power division between the two lines, the coupler is ended after one-quarter of a beat wave length. Since such couplers depend on interference of normal modes to effect power transfer they are innately narrow band.

J. S. Cook[2] and A. G. Fox[3] in the two preceding articles have proposed a new principle for broad-banding directional couplers in which the phase constant difference between the two lines and the coupling coefficient vary with distance along the lines. Strictly speaking, the concept of normal modes is not applicable to non-uniform coupled lines, but if the phase constant difference and the coupling coefficient vary slowly enough with distance, it is useful to introduce the idea of "quasi-normal modes." A quasi-normal mode is a field configuration at a given point in the lines when the line parameters vary slowly compared with the local beat wave length. It differs only slightly from the field configuration of a normal mode that would have existed at the same point if the line parameters were constant at their local values. Since the line parameters vary, the quasi-normal modes are actually tapered, so that the same quasi-normal mode may correspond to all the power in line 1 at the beginning of the coupler and practically all the power in line 2 at the end. The quasi-normal modes in non-uniform lines are coupled, but the coupling is small if the variation of line parameters is gradual. Hence directional couplers may be made in which essentially all the power remains in one of the quasi-normal modes while passing from one line to the other. Such couplers will be called single tapered mode couplers.

A study will be made of the restrictions that must be placed on the phase constant difference and the coupling coefficient variation in order that power transfer may be effected between two lines in a single tapered mode coupler. It is found that the phase constant difference and the coupling coefficient must vary slowly compared with the local beat wave length. In general, the coupler must be several beat wave lengths long to effect complete power transfer, in contrast to the mode interference coupler which must be only a half beat wave length long. Also it is found that the greater the length of the single tapered mode coupler, the greater the bandwidth.

Several classes of couplers with different assumed variations of phase constants and coupling coefficient will be studied. The bandwidths of a few couplers having the same length, maximum phase constant difference, and maximum coupling coefficient will be compared.

II. NORMAL MODES OF UNIFORM COUPLED TRANSMISSION LINES

If two ideal uniform lossless transmission lines are coupled together, it is to be expected that the system can support four normal modes — two in the forward and two in the backward direction. In the present work, the two backward normal modes will be disregarded.* We proceed to find the two forward normal modes for such a system. S. E. Miller[1] has

* The labor of including the backward modes in our calculation is not expected to be compatible with the additional obtainable physical insight.

shown that the wave amplitudes for the two coupled lines may be written in the form

$$\begin{aligned}\frac{dE_1}{dz} &= -j(\beta_1 + c)E_1 + jcE_2 \\ \frac{dE_2}{dz} &= jcE_1 - j(\beta_2 + c)E_2\end{aligned} \tag{1}$$

in which

$E_{1,2}(z)$ = wave amplitudes in lines 1 and 2, respectively

$\beta_{1,2}$ = uncoupled phase constants of lines 1 and 2, respectively

c = mutual and self-coupling coefficient between the lines

Thus the backward waves are disregarded. Further, it is assumed that the lines are lossless, and that the mutual and self-coupling coefficients are identical. Energy conservation requires that c be real. Also it is assumed that the characteristic impedances are normalized so that the power in either line is equal to the square of the wave amplitude.

Although it is easy to solve (1) directly, we shall begin by making a transformation which will be useful in what follows. Taking out a common phase factor and introducing the normal coordinates $w_1(z)$ and $w_2(z)$, we let

$$\begin{aligned}E_1(z) &= e^{-j(\beta+c)z}[\cos \tfrac{1}{2}\, \xi\, w_1(z) - \sin \tfrac{1}{2}\, \xi\, w_2(z)] \\ E_2(z) &= e^{-j(\beta+c)z}[\sin \tfrac{1}{2}\, \xi\, w_1(z) + \cos \tfrac{1}{2}\, \xi\, w_2(z)]\end{aligned} \tag{2}$$

where we define*

$$\begin{aligned}\beta &= \tfrac{1}{2}(\beta_1 + \beta_2) \\ \varphi &= \tfrac{1}{2}(\beta_2 - \beta_1) \\ \Gamma &= \sqrt{\varphi^2 + c^2}\end{aligned} \tag{3}$$

$$\begin{aligned}\cos\frac{\xi}{2} &= \sqrt{\frac{\Gamma + \varphi}{2\Gamma}} \\ \sin\frac{\xi}{2} &= \sqrt{\frac{\Gamma - \varphi}{2\Gamma}} \\ \cot\xi &= \frac{\varphi}{c}\end{aligned} \tag{4}$$

* For convenience to the reader, it may be noted that ξ in the present paper is equal to 2ϑ in the preceding paper.

Substituting (2) in (1), we find that the normal coordinates satisfy the uncoupled equations:

$$\begin{aligned}\frac{dw_1}{dz} - j\Gamma w_1 &= 0\\ \frac{dw_2}{dz} + j\Gamma w_2 &= 0\end{aligned} \tag{5}$$

The normal mode solutions may be written down immediately as

$$\begin{aligned}w_1(z) &= w_1(0)e^{j\Gamma z}\\ w_2(z) &= w_2(0)e^{-j\Gamma z}\end{aligned} \tag{6}$$

where $w_1(0)$ and $w_2(0)$ are arbitrary constants. $w_1(z)$ is called the fast normal mode and $w_2(z)$ is called the slow normal mode. The voltages in the two lines are given by substituting equations (6) into equations (2). $|w_1(0)|^2$ represents the amount of power excited in the fast normal mode and $|w_2(0)|^2$ represents the amount of power excited in the slow normal mode. The voltages are normalized so that $|w_1(0)|^2 + |w_2(0)|^2 = 1$. The voltage amplitudes in the two lines for the *fast normal mode* are

$$\begin{aligned}E_1'(z) &= \cos \tfrac{1}{2}\, \xi\, e^{-j(\beta+c-\Gamma)z}\\ E_2'(z) &= \sin \tfrac{1}{2}\, \xi\, e^{-j(\beta+c-\Gamma)z}\end{aligned} \tag{7}$$

while the voltage amplitudes for the *slow normal mode* are

$$\begin{aligned}E_1''(z) &= -\sin \tfrac{1}{2}\, \xi\, e^{-j(\beta+c+\Gamma)z}\\ E_2''(z) &= \cos \tfrac{1}{2}\, \xi\, e^{-j(\beta+c+\Gamma)z}\end{aligned} \tag{8}$$

(The fast normal mode has the same phase in each line and is called the in-phase normal mode, while the slow mode is called the out-of-phase normal mode.*)

III. MODE INTERFERENCE DIRECTIONAL COUPLERS

The two coupled uniform transmission lines† treated above can be used as a directional coupler.[1] From equations (2) and (6) it is seen that the power in line 1 is given by

$$\begin{aligned}P_1(z) &= |E_1(z)|^2\\ &= |\, w_1(0)\,|^2 \cos^2 \tfrac{1}{2}\xi + |\, w_2(0)\,|^2 \sin^2 \tfrac{1}{2}\xi\\ &\qquad - \sin \xi Re(w_1(0)w_2^*(0)e^{2j\Gamma z})\end{aligned} \tag{9}$$

* There may be cases in which the slow mode is the in-phase mode.

† The present work is assumed to be applicable to coupled wave guides, coupled helices, etc.

and the power in line 2 is given by

$$P_2(z) = |E_2(z)|^2$$
$$= |w_1(0)|^2 \sin^2 \tfrac{1}{2}\xi + |w_2(0)|^2 \cos^2 \tfrac{1}{2}\xi$$
$$+ \sin \xi Re(w_1(0) w_2^*(0) e^{2j\Gamma z}) \quad (10)$$

where Γ is defined in (3).

From (9) and (10) it is seen that if only one of the normal modes is present at $z = 0$, then *there is no power transfer between the two lines.* In order for the two coupled lines to act as a directional coupler, both normal modes must be excited at $z = 0$. Power transfer between the lines is effected by interference of the two normal modes. Since directional couplers[1] using distributed coupling utilize the interference of two normal modes, they will be called *mode interference directional couplers.* Both normal modes must be excited to effect power transfer between the two elements of the coupler.

The beat wave length of the coupler is defined as the minimum distance between two points along the lines at which the power in a given line has its maximum value. For example, if $\beta_1 = \beta_2$, $w_1(0) = -w_2(0) = 1/\sqrt{2}$, the beat wave length is given by $\lambda_{b_0} = \pi/c$ in the example treated.*

The transfer loss τ may be defined as the ratio of the power in line 2 at $z = \ell$, where ℓ is the length of the coupler, to the amount of power excited in line 1 at $z = 0$, assuming no excitation of line 2 at $z = 0$. Thus,

$$\tau = \frac{P_2(\ell)}{P_1(0)} = \text{Transfer Loss} \quad (11)$$

If, for example, $P_1(0) = 1$ (all the power initially in line 1) and $\tau = 1$ then all the power is transferred from line 1 to line 2. This will be called a 100 per cent or zero-db coupler. If $P_1(0) = 1$ and $\tau = \tfrac{1}{2}$, half the power is transferred from line 1 to line 2. This will be called a 50 per cent, or 3 db coupler. τ can thus have any value from 0 to 1 and serves as a parameter to describe "conventional" mode interference couplers.

If $\beta_1 = \beta_2$, $P_1(0) = 1$, $w_1(0) = 1/\sqrt{2} = -w_2(0)$, then from (9) and (10), it is seen that

$$P_1(z) = \cos^2 cz$$

* Reasons could be given for calling $\lambda_{b_0} = 2\pi/c$, but in this paper one half of this value has been chosen as representing the beat wave length in order to conform with the preceding papers.

and

$$P_2(z) = \sin^2 cz$$

so that $\lambda_{b_0} = \pi/c$. The transfer loss is given by

$$\tau = \sin^2 \frac{\pi \ell}{\lambda_{b_0}}$$

To get a 3 db coupler in this case, the coupler is made ¼ of a beat wave length long while a 100 per cent coupler is made ½ of a beat wave length long.

In general, when $\beta_1 \neq \beta_2$ if both modes are equally excited, so that $w_1(0) = -w_2(0) = 1/\sqrt{2}$, then from (9) and (10), it is seen that the power in the two lines becomes

$$2P_1(z) = 1 + \frac{1}{\sqrt{\left(\frac{\varphi}{2c}\right)^2 + 1}} \cos 2\Gamma z$$

and (12)

$$2P_2(z) = 1 - \frac{1}{\sqrt{\left(\frac{\varphi}{2c}\right)^2 + 1}} \cos 2\Gamma z$$

Miller[1] has plotted the transfer loss for several values of $\varphi/2c$. However, it can be seen from (12) that if $(\varphi/2c) \gg 1$, there is practically no power transfer, while if $(\varphi/2c) \ll 1$, there is practically complete periodic power transfer between the lines.

Since "conventional" couplers depend on the interference of two normal modes, it is to be expected that they will be frequency sensitive. This can be seen from the fact that in general, the beat wave length will depend on frequency, and since for a desired transfer loss the coupler must be made some definite fraction of a beat wave length long, such couplers are innately frequency-sensitive. By adding more coupling elements and by means of an ingenious variation of the strength of the coupling, Mumford[1] and Miller[1] have shown that the bandwidth may be increased, although there is a fundamental limit to the bandwidth obtained by such schemes. The proposals of Cook[2] and Fox[3] will now be shown to yield couplers which are at least an order of magnitude more broadband than mode interference couplers.

IV. QUASI-NORMAL MODES IN TAPERED COUPLED LINES

The problem we should now like to consider is this: Can a coupler in which, as before, power is injected into one transmission line only, but in which only one of the quasi-normal modes is excited throughout, and in which the propagation "constants" of the two lines β_1 and β_2, as well as the coupling coefficient c, vary with distance along the lines? It will be shown in the following that the answer to this question is in the affirmative, approximately, provided that the variations of the line parameters are sufficiently gradual.

In order to see what restrictions must be placed on the variations of β_1, β_2 and c, consider the following. For symmetry, assume the variation of β_1 and β_2 with z can be expressed by

$$\begin{aligned} \beta_1 &= \beta - \varphi(z) \\ \beta_2 &= \beta + \varphi(z) \\ c &= c(z) \end{aligned} \tag{13}$$

where β = constant and $\varphi(0) \geqq 0$. The equations for the wave amplitudes in the two lines are given by (1) with β_1, β_2 and c given by (13). In analogy with the transformation used to reduce (1) to normal form for the uniform coupled lines, we shall now introduce *local normal coordinates* $w_1(z)$ and $w_2(z)$. The local normal coordinates are related to $E_1(z)$ and $E_2(z)$ by

$$\begin{aligned} E_1(z) &= \exp\left(-j\left[\beta z + \int_0^z c(\zeta)\,d\zeta\right]\right)\{\cos \tfrac{1}{2}\xi(z)w_1(z) \\ &\qquad - \sin \tfrac{1}{2}\xi(z)w_2(z)\} \\ E_2(z) &= \exp\left(-j\left[\beta z + \int_0^z c(\zeta)\,d\zeta\right]\right)\{\sin \tfrac{1}{2}\xi(z)w_1(z) \\ &\qquad + \cos \tfrac{1}{2}\xi(z)w_2(z)\} \end{aligned} \tag{14}$$

where all symbols are defined by (3) and (4) but where it is to be understood that Γ, φ, c and ξ are now functions of z. Then, substituting (14) into equations (1) where c, β_1 and β_2 have the form of (13), we find after simple manipulation that w_1 and w_2 must satisfy

$$\begin{aligned} \frac{dw_1}{dz} - j\Gamma(z)w_1 &= \frac{1}{2}\frac{d\xi}{dz}w_2 \\ \frac{dw_2}{dz} + j\Gamma(z)w_2 &= -\frac{1}{2}\frac{d\xi}{dz}w_1 \end{aligned} \tag{15}$$

Equations (15) are coupled in general through the terms proportional to $d\xi/dz$. They reduce to the uncoupled (5) when $d\xi/dz = 0$, i.e., when ξ is constant, or equivalently when $\cot \xi = \varphi/c$ is constant. Such will be the case with uniform lines in which $\varphi(z)$ is everywhere proportional to $c(z)$. However, the condition φ/c = constant leads only to mode interference couplers which are of no interest in the present discussion.

It is clear, then, that there will be some coupling between the quasi-normal modes in a tapered mode coupler. Such coupling between quasi-normal modes will be called "hypercoupling" to distinguish it from ordinary electromagnetic coupling between two transmission lines (as represented by the parameter c). A "hypercoupling coefficient" $\eta(z)$ may be defined by

$$\eta(z) = \frac{1}{2\Gamma(z)} \frac{d\xi}{dz} \equiv \frac{1}{2} \frac{d\xi}{d\rho} \tag{16}$$

which gives a measure of the strength of the coupling between the quasi-normal modes.

Now if $\varphi(z)$ and $c(z)$ vary slowly compared to the local beat wavelength $\lambda_b(z)$, where

$$\lambda_b(z) = \frac{\pi}{\Gamma(z)}$$

then $\eta(z) \ll 1$ for all z, and the quasi-normal modes have very little hypercoupling. We can then write down approximate solutions of (15) which proceed essentially in powers of the hypercoupling coefficient. Thus the *In-Phase Quasi-Normal Mode* is given approximately by

$$\begin{aligned} w_1(z) \cong e^{j\rho(z)} \Bigg(& w_1(0) + \frac{1}{2} w_2(0) \int_0^z \frac{d\xi}{dz'} e^{-2j\rho(z')} \, dz' \\ & - \frac{1}{4} w_1(0) \int_0^z \frac{d\xi}{dz'} e^{-2j\rho(z')} \int_0^{z'} \frac{d\xi}{dz''} e^{2j\rho(z'')} \, dz'' \, dz' \Bigg) \end{aligned} \tag{17}$$

and the *Out-of-Phase Quasi-Normal Mode* by

$$\begin{aligned} w_2(z) \cong e^{-j\rho(z)} \Bigg(& w_2(0) - \frac{1}{2} w_1(0) \int_0^z \frac{d\xi}{dz'} e^{2j\rho(z')} \, dz' \\ & - \frac{1}{4} w_2(0) \int_0^z \frac{d\xi}{dz'} e^{2j\rho(z')} \int_0^{z'} \frac{d\xi}{dz''} e^{-2j\rho(z'')} \, dz'' \, dz' \Bigg) \end{aligned} \tag{18}$$

where

$$\rho(z) = \int_0^z \Gamma(\zeta) \, d\zeta \tag{19}$$

If $d\xi/dz = 0$, it is seen that these become the ordinary normal modes of (6).

V. SINGLE TAPERED MODE DIRECTIONAL COUPLERS

The power in the two tapered lines [by (14)] is

$$\begin{aligned} P_1(z) &= |E_1(z)|^2 = \cos^2\frac{\xi(z)}{2}\,|w_1(z)|^2 + \sin^2\frac{\xi(z)}{2}\,|w_2(z)|^2 \\ &\qquad - \sin\xi(z)Re(w_1(z)w_2^*(z)) \\ P_2(z) &= |E_2(z)|^2 = \sin^2\frac{\xi(z)}{2}\,|w_1(z)|^2 + \cos^2\frac{\xi(z)}{2}\,w_2(z)\,|^2 \\ &\qquad + \sin\xi(z)Re(w_1(z)w_2^*(z)) \end{aligned} \tag{20}$$

If only one quasi-normal mode is excited by putting power in only one line provided $c(z)$ and $\varphi(z)$ are chosen so that $\eta(z) \ll 1$ for all z, $\xi(0) = 0$, and $w_1(0) = 1$, $w_2(0) = 0$, then it is seen that the power in the two lines is approximate [by (17)–(20)]

$$\begin{aligned} P_1(z) &\cong \cos^2\tfrac{1}{2}\xi(z)\{1 + \nu(z)\} + \sin^2\tfrac{1}{2}\xi(z)\mu(z) + \sin\xi(z)\delta(z) \\ P_2(z) &\cong \sin^2\tfrac{1}{2}\xi(z)\{1 + \nu(z)\} + \cos^2\tfrac{1}{2}\xi(z)\mu(z) - \sin\xi(z)\delta(z) \end{aligned} \tag{21}$$

where

$$\begin{aligned} \mu(z) &= \frac{1}{4}\left|\int_0^z \frac{d\xi}{dz'}\,e^{2j\rho(z')}dz'\right|^2 \\ \nu(z) &= -\tfrac{1}{2}Re\left(\int_0^z \frac{d\xi}{dz'}\,e^{-2j\rho(z')}\int_0^{z'} \frac{d\xi}{dz''}\,e^{2j\rho(z'')}dz''dz'\right) \\ \delta(z) &= \tfrac{1}{2}Re\left(e^{2j\rho(z)}\int_0^z \frac{d\xi}{dz'}\,e^{-2j\rho(z')}dz'\right) \end{aligned} \tag{22}$$

Since power must be conserved, we must have $\mu(z) + \nu(z) = 0$. This requirement may easily be verified in the specific examples treated. If, furthermore, $\varphi(z)$ and $c(z)$ are chosen so that $\xi(\ell) = \pi$ (a 100 per cent coupler), it is seen from (21) that power can be transferred almost completely from line 1 in the in-phase quasi-normal mode to line 2 in the same quasi-normal mode by exciting power in one line only. In fact, by (17)–(22) we have for a 100 per cent coupler

$$\begin{aligned} P_1(\ell) &= |w_2(\ell)|^2 = \mu(\ell) \\ P_2(\ell) &= |w_1(\ell)|^2 = 1 + \nu(\ell) \end{aligned} \tag{23}$$

where μ and ν are given in equation (22). Thus, $\mu(\ell)$ gives a measure of the error involved in making a complete power transfer coupler of length ℓ in which only the w_1-mode is excited if $\varphi(z)$ and $c(z)$ are selected so that: (1) $\eta(z) \ll 1$, all z, and (2) $\xi(0) = 0$ and $\xi(\ell) = \pi$. Since $\mu(\ell) = | w_2(\ell) |^2$, $\mu(\ell)$ also gives a measure of the power in the initially non-excited mode that is present at the end of the coupler. It is therefore appropriate to call it the "mode crosstalk."

For other power divisions, although $\mu(\ell)$ (with appropriate boundary condition on $\xi(\ell)$) gives the power in the non-excited mode present at the end of the coupler, it does not necessarily give the error in making the desired power of division. For example, for a 3 db coupler ($\xi(\ell) = \pi/2$), we find

$$
\begin{aligned}
P_1(\ell) &\cong \tfrac{1}{2}(1 + \nu(\ell)) + \tfrac{1}{2}\mu(\ell) + \frac{1}{\sqrt{2}}\,\delta(\ell) \\
P_2(\ell) &\cong \tfrac{1}{2}(1 + \nu(\ell)) + \tfrac{1}{2}\mu(\ell) - \frac{1}{\sqrt{2}}\,\delta(\ell)
\end{aligned}
\tag{24}
$$

Since $\mu(\ell) + \nu(\ell) = 0$ (energy conservation, $\delta(\ell)$ would be the more appropriate parameter for describing the performance of a 3 db coupler, although $\mu(\ell)$ is the mode crosstalk. $\delta(\ell)$ might be called the "interference error power" in this case. Although no actual examples will be worked out for 3 db couplers, by comparing $\mu(\ell)$ and $\delta(\ell)$ in equations (22), it is seen that the extension from the 100 per cent coupler cases considered is very easy.

Another case of interest might be that of sampling only a very small amount of power. In this case $\xi(\ell)$ is very small and

$$
\begin{aligned}
P_1(\ell) &\cong 1 + \nu(\ell) + \xi(\ell)\delta(\ell) \\
P_2(\ell) &\cong \mu(\ell) - \xi(\ell)\delta(\ell)
\end{aligned}
\tag{25}
$$

In this case, $\mu(\ell)$, $\delta(\ell)$, and $\nu(\ell)$ are all needed to describe the coupler performance.

VI. FREQUENCY SENSTIVITITY OF SINGLE TAPERED MODE COUPLERS

The "mode crosstalk" is a parameter which measures the "goodness" or quality of a 100 per cent coupler. Since, in general φ and c depend on frequency, $\mu(\ell)$ will also depend on frequency. Thus if a 100 per cent coupler is required with a "mode cross-talk" less than or equal to ε (say $\varepsilon \leqq 0.01$), the frequency range over which $\mu(\ell,\omega) \leqq \varepsilon$ determines the bandwidth. Since, in general, the frequency dependence of φ and c is not precisely known, we can also obtain a semi-quantitative estimate of the performance if we keep ω fixed and vary ℓ.

A 100 per cent coupler has strictly zero "mode crosstalk" only if it is infinitely long, except at isolated frequencies. Of course such a coupler would be flat over an infinite bandwidth. As a general principle, we infer that the longer the coupler the greater the bandwidth. This is obviously not the case for mode interference couplers where any addition to the optimum length causes increasingly serious deterioration of performance. To give a plausibility argument, the "mode crosstalk" may be integrated by parts to yield

$$\mu(\ell) \cong \frac{1}{4}\left|\frac{1}{2j}\left(\left(\frac{d\xi}{d\rho}\right)_{\rho(\ell)} - \left(\frac{d\xi}{d\rho}\right)_{\rho(0)}\right) - \left(\frac{1}{2j}\right)^2\left(\left(\frac{d^2\xi}{d\rho^2}\right)_{\rho(\ell)} - \left(\frac{d^2\xi}{d\rho^2}\right)_{\rho(0)}\right) + \cdots\right|^2 \tag{26}$$

If the series in (26) converges, then $\mu(\ell) \equiv 0$ if all derivatives, $d^n\xi/d\rho^n$, vanish at both ends of the coupler. Since $\mu(\ell)$ is identically zero, such a coupler would be flat for infinite bandwidth. Thus, to increase the bandwidth we make ξ vary slowly at the ends. However, the coupler must be made longer, in general, so that the weak hypercoupling approximation is not violated.

VII. COMPARISON OF BANDWIDTHS FOR SEVERAL CLASSES OF TAPERED MODE COUPLERS

In general, a compromise must be made between bandwidth and length of tapered mode couplers. Several classes of tapered mode couplers will be considered to illustrate this. Perhaps other variations of $\varphi(z)$ and $c(z)$ will eventually prove better than those considered here but until they are discovered we must be content with what we have. The ones illustrated here are chosen primarily for mathematical simplicity, but it will be shown that they should be useful for bandwidths of the order of 3:1, although physical length of no more than about three minimum local beat wave lengths are required.

Two classes of couplers will be considered which will be called, respectively, uniform single tapered mode couplers and constant local beat wave length couplers.

Class 1. Uniform Single Tapered Mode Couplers

This class is characterized by a constant taper (constant hypercoupling coefficient); i.e.,

$$\eta(z) = \frac{1}{2}\frac{d\xi}{d\rho} = p \tag{27}$$

where p is a constant. For couplers to satisfy (27), it can be shown that $\varphi(z)$ and $c(z)$ must be related by

$$\frac{\varphi(z)}{c(z)} = \frac{1 - 2\sigma(z)}{2\sqrt{\sigma(z) - \sigma^2(z)}} \tag{28}$$

where

$$\sigma(z) = \frac{\int_0^z c(\zeta)\, d\zeta}{\int_0^\ell c(\zeta)\, d\zeta} \tag{29}$$

and

$$p = -\frac{1}{\int_0^\ell c(\zeta)\, d\zeta} \tag{30}$$

In order that the weak hypercoupling requirement is satisfied, $p \ll 1$. For a 100 per cent coupler $\xi(\ell) = \pi$, p must satisfy (30).

Table I — Mode Crosstalk for Two Special Cases of Uniform Tapered Mode Couplers

	$\varphi(z)$	$c(z)$	$\sigma(z)$	$\xi(z)$	$\frac{1}{2}\left\lvert\frac{d\xi}{d\rho}\right\rvert$	Crosstalk	Range of Validity*
a)	$\frac{c(1-2\sigma)}{2\sqrt{\sigma-\sigma^2}}$	$c = \frac{\pi}{\lambda_{b0}}$	$\frac{z}{\ell}$		$+\frac{\lambda_{b0}}{\ell\pi}$	$\frac{\sin^2 \pi\left(\frac{\pi\ell}{2\lambda_{b0}}\right)}{4\left(\frac{\pi\ell}{2\lambda_{b0}}\right)^2}$	$\frac{\ell}{\lambda_{b0}} \geqq 1$
b)	$\frac{\pi}{\lambda_{b0}} \cos \xi(z)$	$\frac{\pi}{\lambda_{b0}} \sin \xi(z)$	$\sin^2 \frac{\pi z}{2\ell}$	$\frac{\pi z}{\ell}$	$\frac{\lambda_{b0}}{2\ell}$	$\frac{\sin^2\left(\frac{\pi\ell}{\lambda_{b0}}\right)}{4\left(\frac{\ell}{\lambda_{b0}}\right)^2}$	$\frac{\ell}{\lambda_{b0}} \geqq \frac{\pi}{2}$

* The criterion for establishing the range of validity is somewhat arbitrary, but it is taken the same for all cases considered.

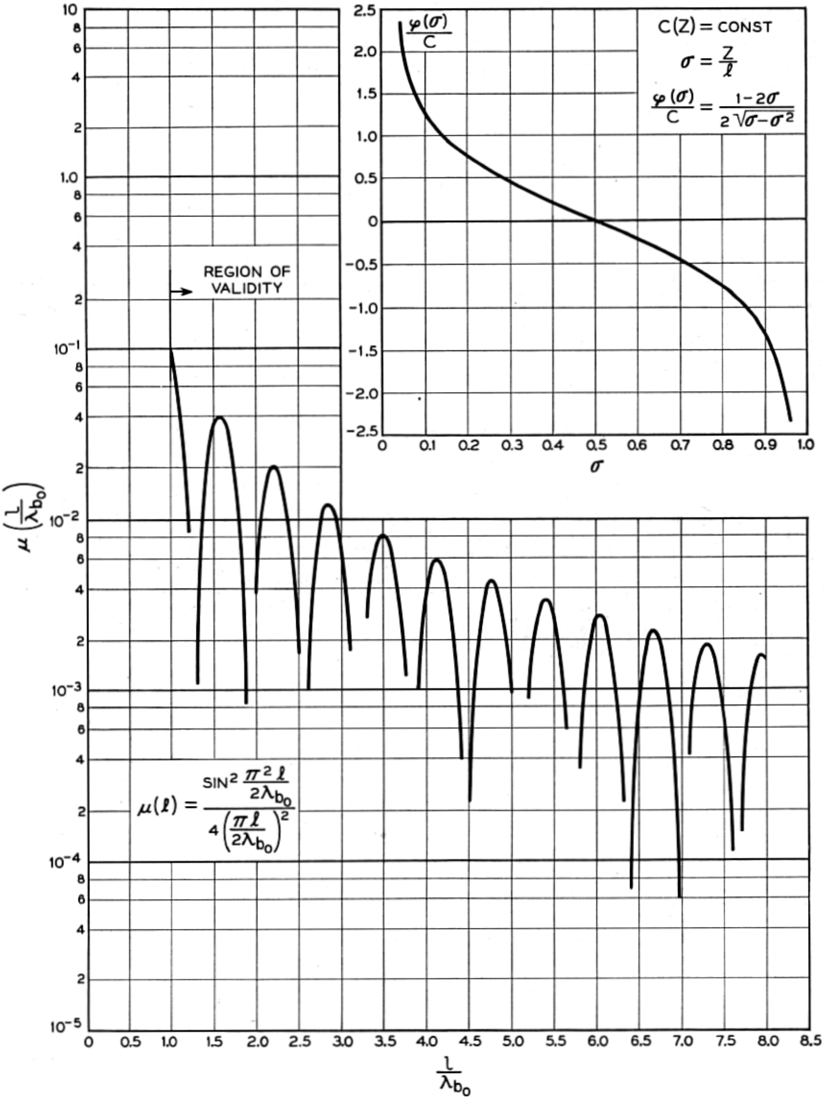
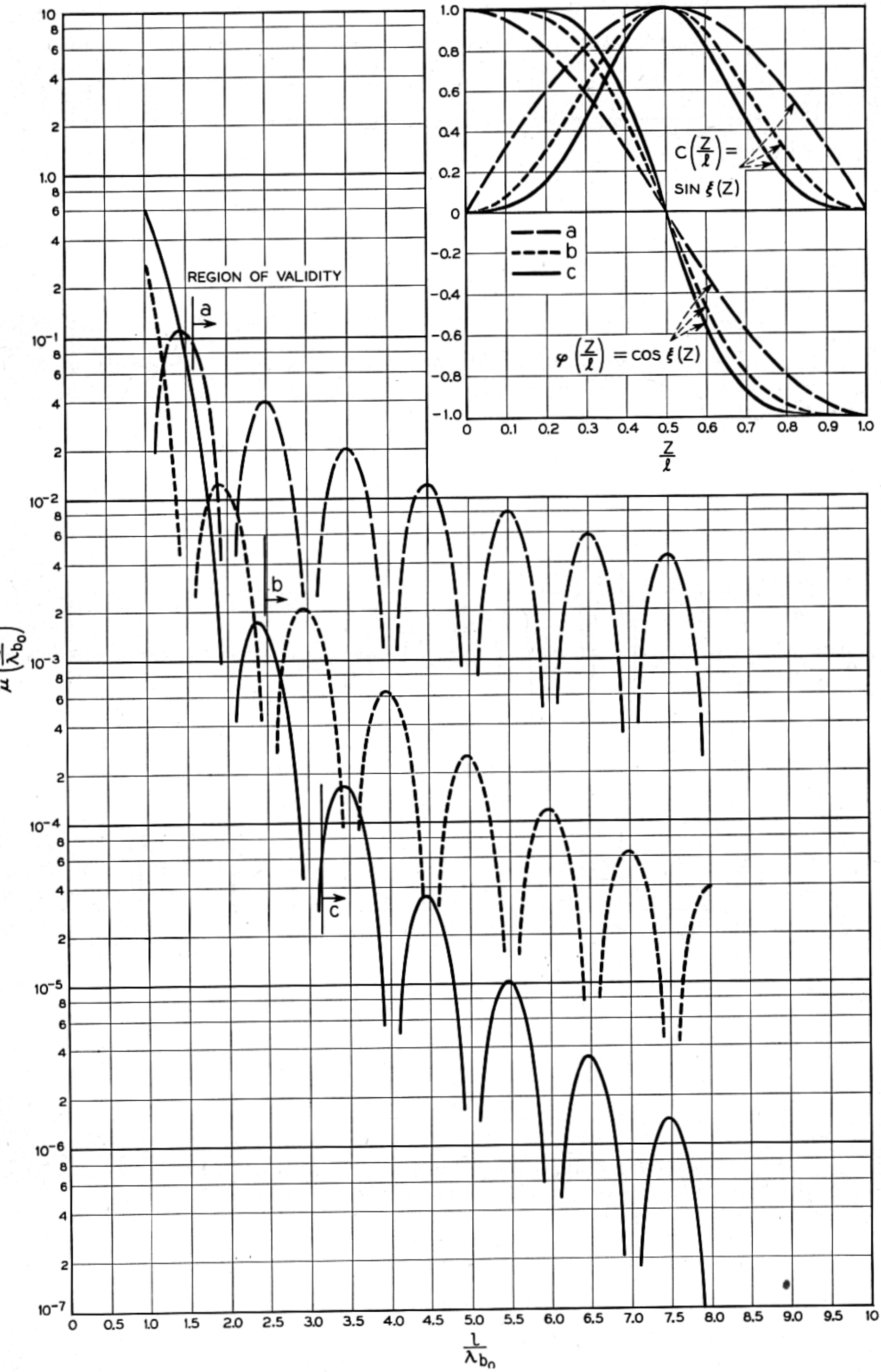


Fig. 1 — Mode crosstalk, $\mu(\ell/\lambda_{b_0})$, for a uniform tapered mode coupler as a function of coupler wavelength in minimum local beat wavelength units (ℓ/λ_{b_0}). Insert — Phase "constant" variation $\varphi(z/\ell)$ as a function of distance along coupler (z/ℓ) in units of coupler length for constant coupling coefficient $c(z)$.



REGION OF VALIDITY
a
b
c
$C\left(\frac{Z}{\ell}\right) = \sin \xi(Z)$
$\varphi\left(\frac{Z}{\ell}\right) = \cos \xi(Z)$
$\frac{Z}{\ell}$
$\mu\left(\frac{l}{\lambda_{b_0}}\right)$
$\frac{l}{\lambda_{b_0}}$

Fig. 2 — Mode crosstalk, $\mu(\ell/\lambda_{b_0})$ for three constant local beat wavelength couplers as a function of the coupler length in minimum local beat wavelength units (ℓ/λ_{b_0}). Cot $\xi(z)$ = ratio of phase "constant" variation to coupling coefficient variation. Insert — Phase "constant" variations $\varphi(z/\ell)$ and coupling coefficient variations $c(z/\ell)$ as functions of distance along the couplers in units of coupler length.

$$\text{Curve (a): } \xi(z) = \frac{\pi z}{\ell}$$

$$\text{Curve (b): } \xi(z) = \pi \sin^2 \frac{\pi z}{2\ell}$$

$$\text{Curve (c): } \xi(z) = \pi \left[\frac{z}{\ell} - \frac{1}{2\pi} \sin \frac{2\pi z}{\ell} \right]$$

The crosstalk for these three cases is given in Table 2.

$$P_1(\ell) = \frac{1}{1 + y^2} \sin^2 \frac{\pi}{2} \sqrt{1 + y^2}$$

$$P_2(\ell) = \cos^2 \frac{\pi}{2} \sqrt{1 + y^2} + \frac{y^2}{1 + y^2} \sin^2 \frac{\pi}{2} \sqrt{1 + y^2}$$

where $y = \ell/\lambda_{b_0}$.

The mode crosstalk for this class of couplers is seen to be

$$\mu(\ell) = \frac{\sin^2\left[\pi \int_0^\ell c(\zeta)\, d\zeta\right]}{\left[\int_0^\ell c(\zeta)\, d\zeta\right]^2} \tag{31}$$

Two special cases are considered:

$$\text{a)} \qquad c(z) = c \qquad \text{(Treated by Cook)} \tag{32}$$

$$\text{b)} \qquad \varphi(z) = \frac{\pi}{\lambda_{b_0}} \cos \xi(z) \qquad \text{(Treated by Fox)}$$

$$c(z) = \frac{\pi}{\lambda_{b_0}} \sin \xi(z) \tag{33}$$

$$\xi(z) = \frac{\pi z}{\ell}$$

The results for these two cases are summarized in Table I. The variations

TABLE II — MODE CROSSTALK FOR THREE SPECIAL CASES OF CONSTANT LOCAL BEAT WAVELENGTH COUPLERS

$\xi(z)$	Mode Crosstalk	Range of Validity*
a) $\dfrac{\pi z}{l}$	$\dfrac{\sin^2\left(\dfrac{\pi\ell}{\lambda_{b_0}}\right)}{4\left(\dfrac{\ell}{\lambda_{b_0}}\right)^2}$	$\dfrac{\ell}{\lambda_{b_0}} \geqq \dfrac{\pi}{2}$
b) $\pi \sin^2 \dfrac{\pi z}{2\ell}$	$\pi^2 \dfrac{\cos^2\left(\dfrac{\pi\ell}{\lambda_{b_0}}\right)}{64\left(\dfrac{\ell}{\lambda_{b_0}}\right)^4}$	$\dfrac{\ell}{\lambda_{b_0}} \geqq \dfrac{\pi^2}{4}$
c) $\pi\left[\dfrac{z}{\ell} - \dfrac{1}{2\pi}\sin^2\dfrac{\pi z}{2\ell}\right]$	$\dfrac{\sin^2\left(\dfrac{\pi\ell}{\lambda_{b_0}}\right)}{4\left(\dfrac{\ell}{\lambda_{b_0}}\right)^6}$	$\dfrac{\ell}{\lambda_{b_0}} \geqq \pi$

* The criterion for establishing the range of validity is somewhat arbitrary, but it is taken the same for all cases considered.

of $\varphi(z)$, $c(z)$ and the mode crosstalk are plotted in Figs. 1 and 2 for these cases.*

Class 2. Constant Local Beat Wavelength Couplers

This class is characterized by

$$\Gamma(z) = \sqrt{\varphi^2(z) + c^2(z)} = \pi/\lambda_{b_0} = \text{constant}. \tag{34}$$

For couplers to have constant local beat wavelengths, the phase constant difference and coupling coefficient must satisfy the following:

$$\begin{aligned} \varphi(z) &= \frac{\pi}{\lambda_{b_0}} \cos \xi(z) \\ c(z) &= \frac{\pi}{\lambda_{b_0}} \sin \xi(z) \\ \Gamma(z) &= \frac{\pi}{\lambda_{b_0}} \equiv \Gamma. \end{aligned} \tag{35}$$

The mode crosstalk is seen to be [by (22)]

$$\mu(\ell) = \frac{1}{4}\left| \int_0^\ell \frac{d\xi}{dz'} e^{2j\Gamma z'} dz' \right| \tag{36}$$

Several examples of this class will be considered†

$$\text{a)} \qquad \xi(z) = \frac{\pi z}{\ell} \tag{37}$$

It is seen that this is also a member of Class 1 so it will not be discussed further.

$$\text{b)} \qquad \xi(z) = \pi \sin^2 \frac{\pi z}{2\ell} \tag{38}$$

$$\text{c)} \qquad \xi(z) = \pi \left[\frac{z}{\ell} - \frac{1}{2\pi} \sin 2 \frac{\pi z}{\ell} \right] \tag{39}$$

The mode crosstalk and region of validity for these cases are given in Table II and the variations $\varphi(z)$, $c(z)$ and the crosstalk are plotted in Fig. 2.

As can be seen from Table II, for Case a, $d\xi/d\rho \neq 0$ at either end of the coupler, while for Case b, $d\xi/d\rho = 0$ at both ends of the coupler and

* It may be noted in case (b) in which $\Gamma = \pi/\lambda_{b_0}$ and $d\xi/dz = \pi/\ell$ that (15) may be solved exactly. There are pure normal modes which are elliptically polarized in the normal coordinates $w_1(z)$ and $w_2(z)$. If power is injected in line 1 only, it is found that the power in the two lines at $z = \ell$ is

† It may easily be verified directly in these cases that $\mu(\ell) + \nu(\ell) = 0$ in agreement with energy conservation. See remarks following (22).

for Case c, $d\xi/d\rho$ and $d^2\xi/d\rho^2$ are zero at both ends of the coupler. Comparison of the mode crosstalk for these three cases illustrates the general remarks made concerning (26), viz., as higher order derivatives of the taper vanish at the ends of the coupler, the mode crosstalk becomes less. Again by comparing the range of validity in the three cases of Table II, it is seen the coupler length must be increased to satisfy the weak hypercoupling requirement, so that the process cannot be carried infinitely far in this direction. Presumably at very short wavelengths where physical size is not as important, this principle can be used most advantageously.

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2. J. S. Cook, page 807 of this issue.
3. A. G. Fox, page 823 of this issue.