Wave Coupling by Warped Normal Modes

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(Manuscript received February 23, 1955)

It has been shown by J. S. Cook that wave power may be transferred from one to another of two coupled waveguides through a variation of their phase constants. It is now clear that this is but one example of a new principle of coupling which is here called "normal mode warping." Wave power inserted at one end of a coupled waveguide system may be made to appear at the other end with any desired power distribution by gradual warping of the normal mode field patterns along the coupler. In general, variation of both the coupling coefficient and phase constants are required. Much wider bands are theoretically possible than with any other distributed type of coupler. This principle may be applied to dielectric waveguides, birefringent media, and waveguides containing ferrite, to obtain both reciprocal and nonreciprocal couplers.

INTRODUCTION

It is now well known that complete transfer of power can be effected from one to another of two waveguides provided there is distributed coupling between the waveguides and provided the phase velocities are equal.[1, 2] A good illustrative analog is a pair of coupled pendulums having the same period. The periods correspond to the wavelengths in the waveguides; passage of time for the pendulums corresponds to distance along the waveguides; the energies in the pendulums at a particular time correspond to the wave powers in the two waveguides at a particularly point along the waveguides. As energy is interchanged between pendulums with increasing time, power is interchanged between waveguides with distance.

To obtain a complete interchange of power for the waveguides requires a particular length which is determined by the coupling. As long as the

[1] S. E. Miller, Coupled Wave Theory and Waveguide Applications, B.S.T.J., **33,** p. 661, May, 1954. See this paper for additional bibliography on wave coupling.
[2] B. M. Oliver, Directional Electromagnetic Couplers, Proc. I.R.E., **42,** p. 1686, Nov., 1954.

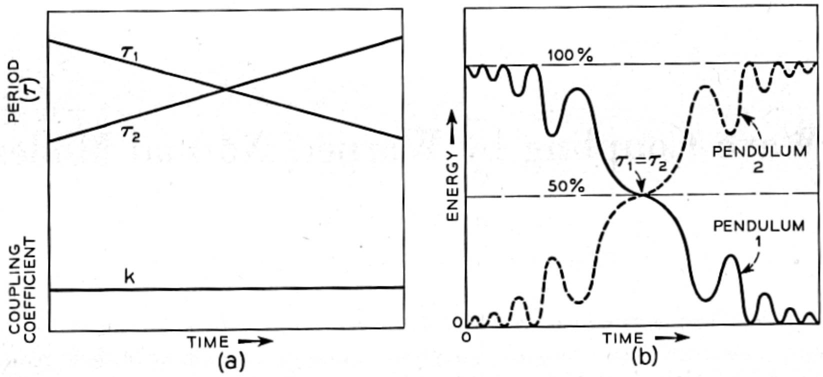

Fig. 1 — Transfer of energy from one to another of two coupled pendulums whose periods are varying with time.

coupling is constant, the power transfer should be independent of frequency. However, in practice, the coupling does vary with frequency, and hence the power division is a slowly varying function of frequency.

Recently a new and rather surprising method of transferring power has been described by J. S. Cook. He pointed out that if the phase constants of two waveguides are grossly *unequal* at one end, but are continuously varied so that they become *equal* in the middle of the coupling region and again grossly *unequal* in the opposite sense at the other end, complete power transfer should take place. Moreover this transfer is independent of the size of the coupling coefficient so that a coupler built in this way should be very broadband indeed. The conclusion is an unexpected one since we know that a uniform coupler having unequal phase constants can never give complete power transfer.

Nevertheless, the principle may be demonstrated using a pair of coupled pendulums whose lengths are continuously varied so that the one which is initially shorter finally becomes the longer of the two. In Fig. 1 is shown a typical result when the longer pendulum is initially excited. Fig. 1(a) shows how the periods (τ) and the coupling (k) vary with time. There is a fluctuation of energy which is quite small at first, but which increases until at the time the periods become equal, the energy is equally divided and the pendulums are in phase. (When the shorter pendulum is initially excited, they are 180 degrees out of phase.) With increasing time the energy is finally transferred to the other pendulum with small residual fluctuations which gradually diminish. It appears then that while the transfer of energy is almost complete, it will not be complete unless the difference between the periods approaches infinity at the beginning and end of the process.

In looking for applications for Cook's coupling scheme, the writer discovered that by varying both the coupling coefficient and the phase constants of the waveguides simultaneously, the residual power fluctuations may be substantially eliminated. The design of such a coupler may appear to be complicated, but it turns out that the requirements are very simply expressed by a new principle of broad-band coupling which is here called "normal mode warping". Using this principle it should be possible to build wave couplers providing any desired degree of power division over very large bandwidths, limited only by the bandwidth capacity of the waveguides themselves. It may be applied equally well to non-reciprocal and reciprocal structures, and in a wide variety of ways. In this paper the principle of normal mode warping will be developed in terms of physical concepts and with the aid of some rather straightforward analysis.

NORMAL MODES VERSUS COUPLED MODES

Before attempting a discussion of normal mode warping it will be necessary to clarify somewhat the meaning of "normal modes." In a uniform metal waveguide, the usual understanding is quite adequate. Normal modes are characterized by unique distributions of transverse electric and magnetic field components, which distributions are independent of frequency or position along the waveguide. Also for a given frequency, the normal modes have unique phase velocities (except for certain pairs which are said to be degenerate because they have the same phase velocity). This description automatically requires that the normal modes be orthogonal, which means that the flow of energy in one mode does not contribute to energy flow in any other mode. If this were not so, the excitation of one normal mode would result in a transverse field pattern which would change with distance along the waveguide. These statements are equally true whether we are talking about modes in one waveguide or in a system of several waveguides.

When two identical dominant mode waveguides are uniformly coupled throughout a certain portion of their length, a field excitation corresponding to a dominant wave in one of the waveguides is no longer a normal mode. Such modes are now coupled, and power will cycle back and forth between them as stated earlier. However, there will be a new pair of normal modes which are orthogonal to each other, and for which the power is equally divided between the waveguides. One of these may be called the even mode because the electric field will have the same direction in both waveguides. The other may be called the odd mode because the field will be oppositely directed in the two waveguides. While the two coupled modes will have the same phase velocities, the normal modes will have different phase velocities. The behavior of such a coupler can be explained either in terms of coupled modes, or in terms of normal modes, and these two concepts are completely equivalent.[3] According to the normal mode explanation, if all of the power is initially introduced into one of the waveguides, the subsequent transfer of power to the other waveguide is due to the excitation of both normal modes and to the interference between them which is a result of their unequal phase velocities. On the other hand, if all of the power were introduced into both waveguides with the proper phase and amplitude to correspond with one of the normal modes, it would travel through the coupler with the phase velocity of that mode and would emerge at the far end in that mode.

When the two coupled waveguides have different phase velocities, we will again have a pair of normal modes which are orthogonal to one another, but they will no longer have equal amounts of energy in the two waveguides. Instead, one will have more of its energy in one waveguide, and the other will have more of its energy in the other waveguide, the unbalance depending upon the difference in the phase velocities of the coupled modes and upon the magnitude of the coupling coefficient.

Finally, if the coupled waveguides are not uniform, but are "warped" so that their phase constants and coupling coefficient vary along the coupling region, the normal mode concept clearly requires re-examination. It is no longer possible to define a normal mode in terms of a characteristic and invariant field distribution in any cross sectional plane. The field distribution will change along the coupler. However, we will show that if the warping is sufficiently gradual, the normal modes will have transverse field patterns at any cross section which are approximately the same as the normal mode patterns for a uniform coupler having the same cross sectional properties. We will call the normal modes for the equivalent uniform coupler the "local normal modes" to distinguish them from the true normal modes for the warped coupler.

TWISTED WAVEGUIDE AS PROTOTYPE BROADBAND COUPLER

Perhaps the most easily understood example of normal mode warping is a twisted birefringent waveguide. Fig. 2 shows in cross-section a long

[3] A. G., Fox, Miller, S. E., Weiss, M. T. Behavior and Applications of Ferrites in the Microwave Range, B.S.T.J., **34**, p. 16, Jan., 1955.

dielectric strip which is twisting in a clockwise direction with propagation into the paper. We may think of this as being either an unshielded dielectric waveguide, or as a dielectric fin inside of a circular waveguide sheath. In either case we know from experiment that if we launch a linearly polarized wave with its electric polarization either parallel or perpendicular to the dielectric fin, it will propagate along the twist with its polarization remaining, to a first approximation, either parallel or perpendicular to the fin at all points. Thus, the polarization will rotate with rotation of the fin. We know experimentally that if the twist is performed too rapidly, some depolarization will result. On the other hand, if the twist is long and gradual, the polarization will remain quite linear at all points.

Now if the twist in the waveguide is just 90°, as suggested in Fig. 2, a vertically polarized wave introduced at one end will emerge as a horizontally polarized wave at the other end. Thus, we find that the twist section, by some coupling mechanism not yet defined, can transfer 100 per cent of the power in a vertically polarized wave to a horizontally polarized wave. Moreover, we know that this transfer is not frequency sensitive. We suspect, therefore, that the twist represents a preferred way of effecting a 100 per cent power transfer between the two modes. Let us arbitrarily identify the vertically and horizontally polarized modes as the modes between which coupling takes place, and ask how the coupling coefficient and phase constants for these modes vary along the twist.

We will define the phase constants and coupling coefficient at any point along the twist as being the same as those of a uniform (nontwisted) dielectric strip having the same cross sectional geometry. Let us now determine these parameters for the uniform birefringent waveguide indicated schematically in Fig. 3. If a vertically polarized wave

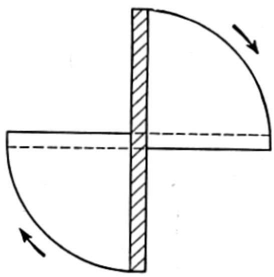

Fig. 2 — End view of a twisted dielectric waveguide.

with amplitude e_0 is launched on this waveguide, there will be a partial transfer of power from the vertical to the horizontal polarization and back again, just as though these two polarizations were coupled waveguides with unequal phase velocities. We can analyze the situation by breaking the input polarization into components along the cross sectional axes (A and B) of the waveguide, allowing for the difference in the phase constants β_a and β_b, and determining the vertically and horizontally polarized components which result at some other cross section. The result is:

$$e_v = e_0\, \varepsilon^{-j\beta_0 z} [(\cos^2 \theta)\, \varepsilon^{-j\Delta z} + (\sin^2 \theta)\, \varepsilon^{+j\Delta z}] \tag{1}$$

$$e_h = -je_0\, \varepsilon^{-j\beta_0 z} [\sin 2\theta \sin \Delta z] \tag{2}$$

where

$$\beta_0 = \frac{\beta_a + \beta_b}{2} \tag{3}$$

$$\Delta = \frac{\beta_a - \beta_b}{2} \tag{4}$$

(1) and (2) must now be compared with the equations for a pair of coupled waveguides.

The basic equations governing the amplitudes of waves in coupled waveguides are:

$$\frac{de_1}{dz} = -j\beta_1 e_1 + k_{21} e_2 \tag{5}$$

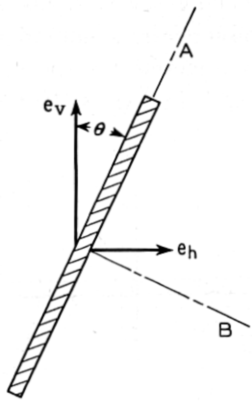

Fig. 3 — A uniform birefringent medium producing coupling between vertical and horizontal polarizations.

$$\frac{de_2}{dz} = -j\beta_2 e_2 + k_{12} e_1 \tag{6}$$

This has already been treated by S. E. Miller[1] using slightly different notation. For simplicity we are assuming that there is no attenuation. e_1 and e_2 are the amplitudes of the forward travelling coupled waves. These waves have phase constants β_1 and β_2 and field distributions which are perturbed by the presence of the coupling. Hence they are truly the coupled modes and not the modes which would exist in the absence of coupling. There must also be a pair of backward traveling waves in the two waveguides since power leaking through the coupling aperture will set up both forward and backward traveling components. However, if the coupling per unit length is small so that the coupler would have to be many wavelengths long to produce a complete power transfer, then the backward traveling components will interfere with one another, so that the backward waves may be safely neglected. This greatly simplifies the analysis and the weak coupling assumption will be made here and throughout the rest of this paper. Characteristic impedances are normalized so that the power in either mode is equal to the square of the amplitude (e). Propagation takes place in the $+ z$ direction. Power conservation requires that

$$|k_{12}| = k = |k_{21}| \tag{7}$$

$$k_{12}k_{21} = -k^2 \tag{8}$$

The solution of (5) and (6) then gives,

$$e_1 = \varepsilon^{-j\beta_0 z}\left[A_1 \varepsilon^{-j(\sqrt{\delta^2+k^2})z} + B_1 \varepsilon^{+j(\sqrt{\delta^2+k^2})z}\right] \tag{9}$$

$$e_2 = \varepsilon^{-j\beta_0 z}\left[A_2 \varepsilon^{-j(\sqrt{\delta^2+k^2})z} + B_2 \varepsilon^{+j(\sqrt{\delta^2+k^2})z}\right] \tag{10}$$

where

$$\beta_0 = \frac{\beta_1 + \beta_2}{2} \tag{11}$$

$$\delta = \frac{\beta_1 - \beta_2}{2} \tag{12}$$

and the coefficients A and B have the following relation:

$$\frac{A_2}{A_1} = j\,\frac{\delta - \sqrt{\delta^2 + k^2}}{k_{21}} \tag{13}$$

$$\frac{B_2}{B_1} = j\,\frac{k_{12}}{\delta - \sqrt{\delta^2 + k^2}} \tag{14}$$

The amplitudes of these two waves will oscillate with distance along the structure, and a beat wavelength may be defined as

$$\lambda_b = \frac{\pi}{\sqrt{\delta^2 + k^2}}$$

in which interval there will occur one complete power transfer cycle. If all of the power is initially in mode 1, then

$$A_1 = \left(\frac{1}{2} + \frac{\delta}{2\sqrt{\delta^2 + k^2}}\right) e_0 \tag{15}$$

$$B_1 = \left(\frac{1}{2} - \frac{\delta}{2\sqrt{\delta^2 + k^2}}\right) e_0 \tag{16}$$

By comparing (1) and (2), which give the amplitudes of vertically and horizontally polarized modes on the birefringent waveguide, with (9) and (10) for a pair of coupled modes, we can see that they are of the same form. They may be made identical if we make the following substitutions:

$$\beta_1 = \beta_a \cos^2 \theta + \beta_b \sin^2 \theta \tag{17}$$

$$\beta_2 = \beta_a \sin^2 \theta + \beta_b \cos^2 \theta \tag{18}$$

$$k = \frac{\beta_a - \beta_b}{2} \sin 2\theta \tag{19}$$

It follows also that

$$\frac{\beta_1 + \beta_2}{2} = \beta_0 = \frac{\beta_a + \beta_b}{2} \tag{20}$$

$$\delta = \frac{\beta_a - \beta_b}{2} \cos 2\theta \tag{21}$$

Thus, the behavior of the uniform birefringent medium can be described in terms of (9) and (10) for coupled transmission modes. Conversely, any pair of coupled waveguides can be described in terms of parameters for an equivalent birefringent medium.

We shall find it convenient to make use of this analogy since warped mode structures can be easily visualized in terms of an equivalent birefringent twist.

We are now in position to interpret the birefringent twist of Fig. 2 in terms of coupling between vertical and horizontal polarizations. Although θ is now variable, we use it to define at each point along the twist section a set of equivalent coupled line parameters corresponding to a

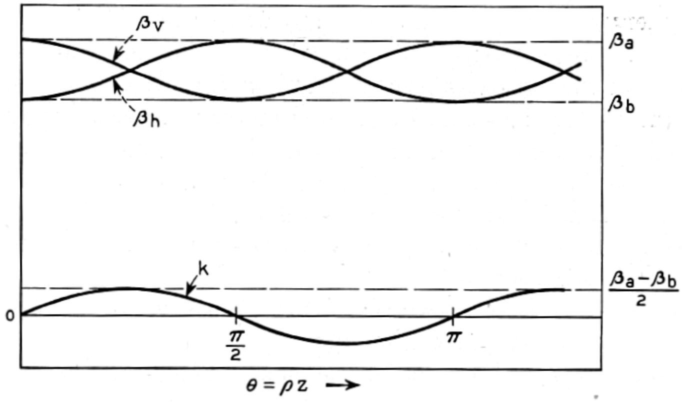

Fig. 4 — Coupling coefficient and phase constants for vertical and horizontal polarizations in a twisted birefringent medium.

non-twisted section having the angle θ. Fig. 4 shows how the local coupling coefficients and phase constants [(17), (18), (19)] vary with distance along a uniform birefringent twist section (θ is directly proportional to z). We conclude that if we should build a pair of coupled waveguides [1 and 2] where the phase constants and coupling coefficient vary with z as shown in Fig. 4, the power division between the two waveguides will be as shown in Fig. 5, since this is the way the twist prototype behaves. θ is now simply a parameter relating k, β_1 and β_2 as a function of length, but it is still convenient to think of it as the angle of the equivalent twist section. Fig. 5 shows that if the coupler is made of length $\theta = \pi/2$, a complete power transfer will take place, and this will occur smoothly and without the fluctuations of Fig. 1(b). A 3-db coupler will be pro-

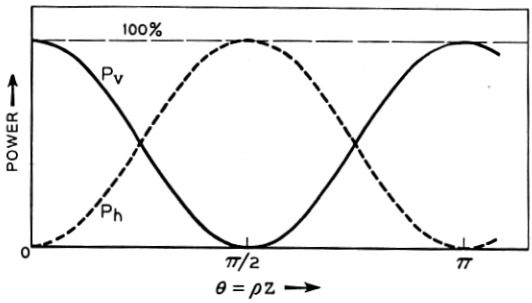

Fig. 5 — Power transfer between vertical and horizontal polarizations in a twisted birefringent medium.

vided by length $\theta = \pi/4$; and in fact any desired division is obtainable by using the proper length. All of these should be broadband because we know that the twist prototype is broadband.

It might appear that since the desired power transfer is obtained for only a particular length, it would therefore be as frequency sensitive as matched velocity couplers. Actually, the electrical length is not important. It is simpler to think in terms of the twist prototype where we see that the only requirement for the desired transfer is that the total twist angle θ be chosen correctly.

At this point we have shown that we can think about coupled waveguides in terms of a twist medium, and when we do so, we discover that this medium holds the secret of how to make a broad-band power transfer from one waveguide to another. Specifically, we conclude that both the coupling coefficient and the phase constants should be varied.

One way in which these design requirements may be met is suggested in Fig. 6 for a pair of coupled rectangular waveguides providing complete power transfer ($\theta = \pi/2$). The top wall has been removed to show a divided aperture so tapered as to give a coupling coefficient which varies as the sine of the distance from one end. At the same time the dividing partition is warped so as to produce the cosinusoidal cross-over of phase

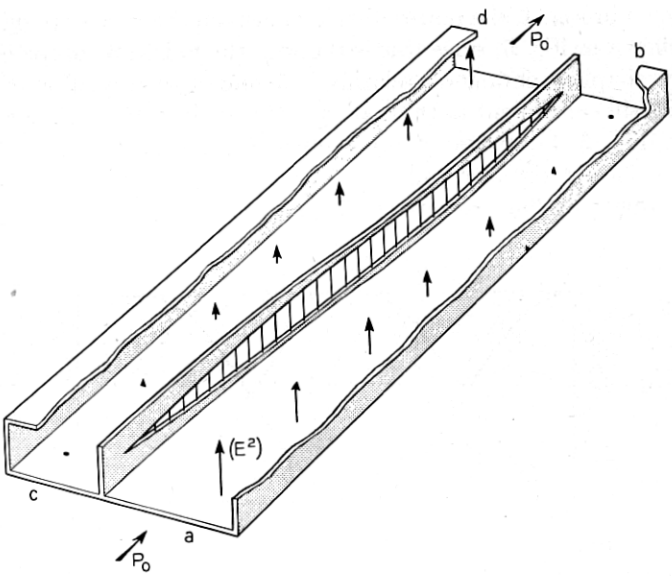

Fig. 6 — A broadband 100 per cent power transfer coupler using mode warping.

constants. The phase constants could also be adjusted by the insertion of a variable amount of dielectric loading. The vertical vectors represent the square of the electric field present in the two waveguides at various cross sections along the structure when all of the power is initially inserted at a. A complete transfer takes place with all of the field appearing at d and none at b.

The phase relations between the field vectors on the two sides of the coupling aperture are of interest. We know that in the case of a coupler employing uniform waveguides, the induced wave in one waveguide is always 90 degrees out of phase with the driving wave in the other waveguide ($k_{12} = k_{21} = -jk$). This is also true for the two coupled modes (e_v and e_h) in a uniform birefringent medium [see (1) and (2)] and hence $k_{vh} = k_{hv} = -jk$. However, we know that if we launch a linearly polarized wave on the twist medium with polarization paralled to one of the principal cross sectional axes, then the wave will remain linearly polarized. Consequently for this medium the vertically and horizontally polarized modes (e_v and e_h) will have a zero or 180-degree phase relation at all points along the medium. It follows that the coupler of Fig. 6 which was derived from the twist medium must also have a zero or 180-degree phase relation between the field components on opposite sides of the coupling partition at every cross section. This situation is illustrated in Fig. 7 where the transverse electric field is plotted for a series of cross sections of the coupler. The input end is shown at the top, and the output end at the bottom. The left hand column represents the field configurations when the wave is initially launched in the smaller of the two waveguides. The right hand column is for a wave launched in the larger of the two waveguides.

It may be seen that when the wave is launched in the larger waveguide terminal, it emerges at the other end of the coupler from the larger waveguide terminal. Moreover, at the center cross section where the two waveguides have the same phase constant, the energy is equally divided and we recognize this as the even symmetric normal mode for the local cross section. In fact, the wave travels throughout the length of the coupler in the local normal mode which has the higher phase constant.

Conversely, if the wave is launched in one of the smaller waveguide terminals it appears at the center cross section in the odd symmetric mode and emerges from the smaller waveguide at the far end. Thus it travels throughout the coupler in the local normal mode having the lower phase constant.

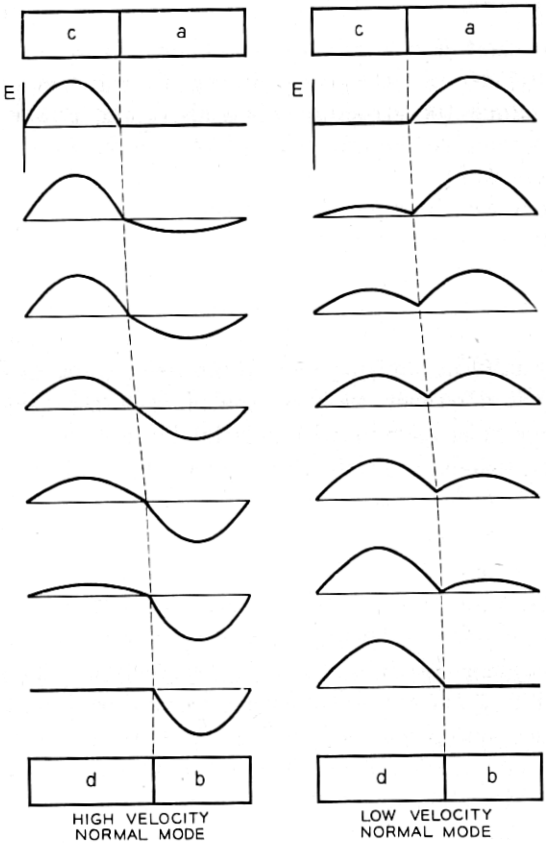

Fig. 7 — Warping of odd and even symmetric modes in the coupler of Fig. 6.

PRINCIPLE OF BROADBAND COUPLING BY NORMAL MODE WARPING

We have discussed a twisted birefringent medium and a rectangular waveguide coupler, both of which are examples of normal mode warping. We will now attempt a statement of the principle which is basic to all such couplers. We assume a waveguide system having two modes of propagation which are to be coupled so as to effect transfer of power. There are then two normal modes for this system, which are the dual of the coupled modes, and which may vary in field pattern from point to point along the structure depending upon the phase constants and coupling coefficients of the coupled modes. Provided these parameters vary slowly and smoothly along the structure, then if all of the power is injected in one of the normal modes at one end, it will remain in one of the normal modes at successive cross sections and will emerge in one of the normal modes at the far end. This situation will be independent of frequency.

On the other hand, if both normal modes are excited at one end, power will be transmitted through the structure in both normal modes and emerge in both modes. This is what happens in conventional matched β couplers, and interference between the modes can vary as frequency changes.

The objective of a broadband design should then be: (1) to adjust parameters at the ends of the structure so that the normal modes at those points are identical with the desired input and output field excitations, and (2) to smoothly vary the parameters along the structure so that the normal modes are transmuted or "warped" from the one to the other set of field excitations. In this way the power will at all points be in only one of the normal modes, and interference between modes is avoided.

EXAMPLES OF MODE WARPING

We have already seen how this objective was achieved in the birefringent twist. Power was injected in one of the normal modes. This mode was smoothly warped from vertical polarization at one end to horizontal polarization at the other end by twisting. If we had chosen to excite this medium with a wave polarized at 45° to the birefringent axes, both of the normal modes of the medium would have been excited equally. As a result, equal amounts of power would have propagated down the twist in the two normal modes and would have arrived at the far end with relative phases which would depend upon the phase velocities of the two modes and the total distance travelled along the twist. The output polarization would, in general, be elliptical and would be frequency dependent.

In the case of the coupled waveguides of Figure 6, the normal modes at the ends corresponded to dominant wave excitation of the separate waveguide terminals, and they are smoothly warped from one set of terminals to the other. Excitation of one of the waveguide terminals would cause power to flow through the system in only one of the normal modes. If the waveguide structure had been cut at the center, either half of it would constitute a 3 db coupler. With excitation of one of the end terminals, the power at the center cross-section will exist entirely in one of the normal modes, which requires equal voltages in the two waveguides. Thus, the three db power division should be very broadband. On the other hand, if both of the waveguide terminals at one end were excited power would flow in both of the normal modes. The relative phases of these modes arriving at the center would vary with frequency, and hence the power division would be frequency dependent.

In Fig. 8 is shown another example of mode warping which is interesting because it illustrates some departures from the type of warping used in the previous examples. This device is a broadband converter from linear to circular polarization. It comprises a long section of round waveguide containing a slender axially magnetized rod of ferrite. Throughout a certain portion of its length the waveguide is gradually flattened to produce an elliptical cross-section. The rod of ferrite is also tapered to pointed ends. Transverse cross-section views are shown below the longitudinal cross section, and at the bottom is shown the way in which the magnitude of the coupling coefficient and the phase constants for vertically polarized and horizontally polarized waves vary along the length. We may identify three different parts of the transducer. At either end is a region in which the normal modes undergo no warping, and which functions solely as a taper section for matching the normal modes into the central region. Between these two matching regions is the region where mode warping occurs. At the left of this central region the elliptical cross section produces a linear birefringence where vertically polarized waves have a larger phase constant than horizontally polarized waves. The absence of any ferrite at this point means that the coupling coefficient between the two polarizations is zero. These relations are shown by the β

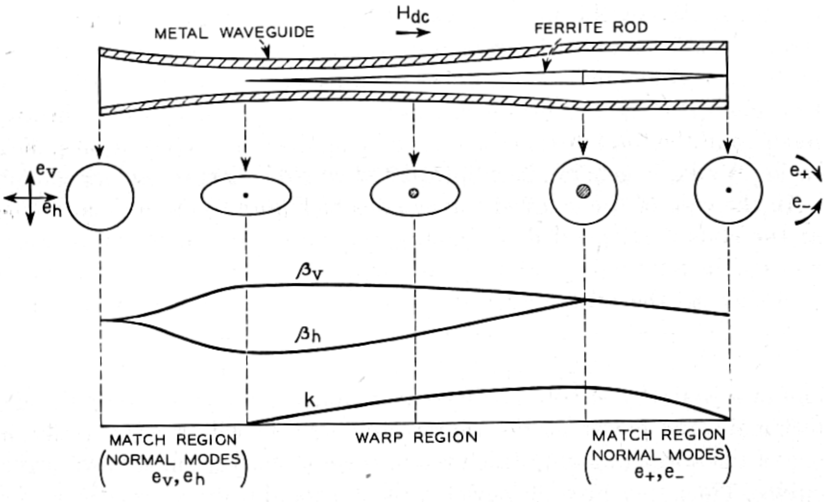

Fig. 8 — Broadband linear-to-circular polarization converter.

and k curves at the bottom of the figure. Proceeding toward the right, the gradual insertion of ferrite and decrease in ellipticity of the waveguide sheath cause the difference between the phase constants to decrease and the coupling coefficient to increase. At the right hand end of the central region, the sheath is round and the linear birefringence is zero. At the same time the presence of a maximum amount of ferrite causes the coupling to be a maximum. We see that the programming of the coupling coefficient and phase constant difference corresponds to 45 degrees of the prototype twist medium, and a linearly polarized wave at the left should have half its power transferred to horizontal polarization at the right. Unlike the birefringent twist, however, where the coupling coefficient is imaginary, the coupling coefficient due to the ferrite is real. As a result there will be a 90-degree phase relation between the horizontal and vertical polarizations, yielding circular polarization, which we already know is the normal mode for the longitudinally magnetized ferrite. Because a vertically polarized wave at the left will have the higher phase constant, it will be warped into the negative circularly polarized wave at the right (counterclockwise) since this wave sees the higher permeability, and hence phase constant, for the usual case where the applied magnetic field is less than that required for gyromagnetic resonance.

Unlike the couplers described previously, this coupler does not keep β_0 constant. The phase constant difference and the coupling coefficient still vary as the cosine and sine of distance respectively from the left hand end of the central section. But the increase in dielectric loading due to the ferrite rod causes both β_v and β_h to be larger than they would be if the dielectric constant of the ferrite were unity. However, examination of (9) and (10) show that within the bracketed factor which controls the amplitude of the wave, only δ and k appear. Provided these parameters vary in the proper way, the power transfer should take place regardless of β_0. Change in β_0 will cause the velocity of a normal mode to change from point to point along the structure, but will not change the field distribution for the normal mode. What this means physically can be visualized in the case of the birefringent twist by assuming that the dielectric strip either changes in dielectric constant or in total cross section from point to point without changing birefringence or rate of twist. Clearly, the rate at which the polarization of the wave is altered as a function of distance is not affected.

Another peculiarity of this coupler is that because of the non-reciprocal nature of the ferrite, the signs of the coupling coefficients will be opposite for opposite directions of propagation. Thus, for propagation from left to right, $k_{vh} = +k$ and $k_{hv} = -k$, while for propagation from right to left, $k_{vh} = -k$ and $k_{hv} = +k$. It is simpler to see what this means, however, by considering that a counterclockwise polarized wave sent in from the right will have positive polarization relative to the ferrite magnetization, and will therefore have a lower phase constant than a clockwise polarized wave. As a result, this wave will be warped into the horizontally polarized wave at the left hand end, because this wave has the lower phase constant in the squashed region. Thus the circular to linear polarization conversion properties of this section are nonreciprocal and it is possible to combine it with reciprocally birefringent elements to obtain a circulator.*

In Fig. 9 is shown another example of mode warping where ferrite is employed in rectangular waveguide to produce a brcadband circulator.

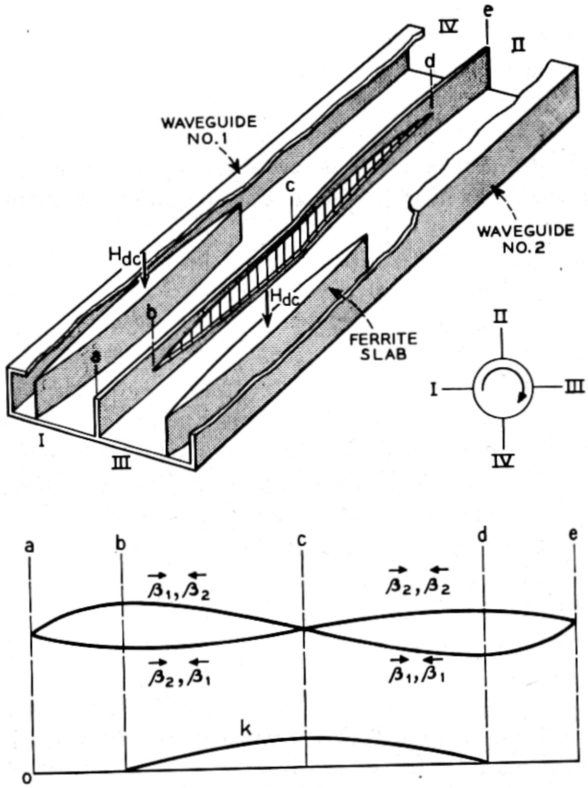

Fig. 9 — Rectangular waveguide circulator employing mode warping.

* See page 86 of Reference 3.

Two rectangular waveguides are coupled by a long divided aperture. At the left hand end, both waveguides are occupied by thin tapered slabs of ferrite located off center in the waveguide cross section. These terminate in knife edges at the center where the coupling aperture is largest. From here on the dividing partition is deflected so as to alter the phase constants of the two waveguides. We may identify four distinct regions in this coupler. The regions between *a* and *b*, and between *d* and *e* are for the purpose of matching to the waveguide terminals. In the regions between *b* and *c*, and between *c* and *d* mode warping occurs. Both of these regions are equivalent to 45-degree twists of the birefringent prototype, and power in either of their input terminals will be equally divided between the two waveguides at the center cross section *c*. Thus they operate as broadband hybrids. Section *cd* is like one half of the structure of Fig. 6, and it operates in the same manner. Section *bc*, on the other hand, operates like a non-reciprocal hybrid, and by virtue of its transversely magnetized ferrite slabs, it has non-reciprocal phase constants as shown at the bottom of the illustration. For propagation from left to right, the ferrite in waveguide 1 exhibits a permeability greater than one, while the ferrite in waveguide 2 exhibits a permeability less than one. For propagation from right to left, the situation is reversed. We may therefore analyze the behavior as follows:

A wave entering at terminal I will, by the time it arrives at cross section b, be travelling in the normal mode having the higher phase constant. As it passes through section *bc* the mode will be warped so that at *c* the power will be equally divided between the two waveguides in the even symmetric mode, which has the higher phase constant. Mode warping will continue through section *cd*, and all of the power will be transferred to waveguide 2 which has the higher phase constant at section *d*. Thus, power entering at I will be delivered to II.

A wave entering at II will return to cross section c with power equally divided in the even symmetric mode. From this point on, however, the situation is changed because of the non-reciprocal behavior of the ferrite. Now, waveguide 2 will have a phase constant which is higher than waveguide 1, and the wave which is travelling in the higher phase constant mode will be delivered to terminal III. The circulation order of the terminals is therefore as shown by the circulator symbol at the right of the waveguide.

In this structure, as in the one which preceded it, we note that the average phase constant is not necessarily constant. Nevertheless, the coupling coefficient and phase constant difference are varied sinusoidally as shown in Fig. 4.

NORMAL MODES ON A RAPID TWIST

So far we have shown that a slowly twisting birefringent medium can be used as a prototype for the design of broadband directional couplers having any desired power division ratio. We have not yet said anything about the length such a coupler must have, nor about how the twist may vary with distance. While a constant rate of twist was assumed, this is not necessary. The rate of twist can be varied arbitrarily along the coupler provided only that the total twist angle is such as to give the desired power division. However, the twist rate must not exceed some maximum value or the assumption that the polarization of the wave will twist with the medium will no longer be satisfied. Practically speaking, it would appear that the rate of twist should be constant and equal to the maximum permissible value, since to use a smaller rate in some portions of the coupler would only make the coupler longer than necessary.

It is evident at this point that we need to know more about the maximum twist rate and what happens if it is exceeded in the interest of making the coupler short. Our previous assumption of a gradual twist amounted to the assumption that the normal modes at any cross-section of the medium were linearly polarized along the principal axes of birefringence for the cross section, i.e., were the same as the local normal modes. We will now show that if the twist is rapid the normal modes are perturbed and are no longer linearly polarized.

Figs. 10(a) and 10(b) show symbolically two successive cross sectional views of a rapid twist having principal axes of birefringence A and B.

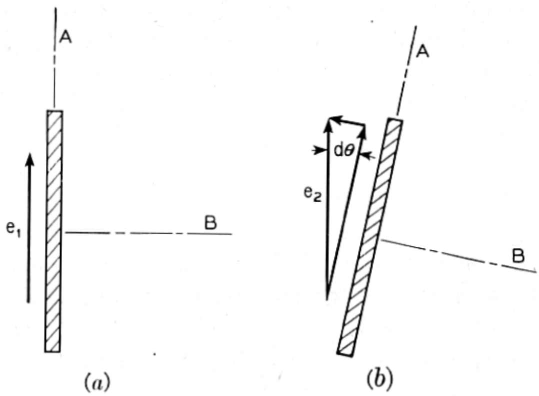

Fig. 10 — Coupling produced between A and B polarizations in a rapidly twisting birefringent medium.

We assume that the two cross sections are an infinitesimal distance apart, and hence that the angle $d\theta$ is infinitesimal. Between these cross sections we assume the axes of birefringence are fixed as shown in Fig. 10(a). Thus, a wave with polarization e_1 along the A axis at the input will emerge from the first infinitesimal step with unchanged polarization as shown by e_2. On entering the second step we see that both the A and B polarizations will be excited. Now if the fin twists at a constant rate $d\theta/dz = \rho$, then an electric field polarized along A is coupled to an electric field polarized along B as given by

$$de_b = -e_a\, d\theta; \qquad de_a = +e_b\, d\theta. \tag{22}$$

As before, we have assumed that there is no scattering or attenuation and that backward traveling components may be neglected. We can now write the basic equations for waves of polarization A and B.

$$\frac{de_a}{dz} = -j\beta_a e_a + e_b \frac{d\theta}{dz} \tag{23}$$

$$\frac{de_b}{dz} = -\, j\beta_b e_b - e_a \frac{d\theta}{dz} \tag{24}$$

These equations when solved simultaneously give

$$e_a = \varepsilon^{-j\beta_0 z}\, [C_a\, \varepsilon^{-j(\sqrt{\Delta^2+\rho^2})z} + D_a\, \varepsilon^{+j(\sqrt{\Delta^2+\rho^2})z}] \tag{25}$$

$$e_b = \varepsilon^{-j\beta_0 z}\, [C_b\, \varepsilon^{-j(\sqrt{\Delta^2+\rho^2})z} + D_b\, \varepsilon^{+j(\sqrt{\Delta^2+\rho^2})z}] \tag{26}$$

where β_0 and Δ have the meanings given in (3) and (4), and the coefficients C and D are related by

$$\frac{C_b}{C_a} = +\, j\left[\frac{\Delta - \sqrt{\Delta^2 + \rho^2}}{\rho}\right] = \frac{D_a}{D_b} \tag{27}$$

These equations say that in such a twist medium there are two normal modes of propagation. The mode with the larger phase constant is obtained when $D_b = 0$ and $D_a = 0$. For this mode

$$\frac{e_b}{e_a} = +\, j\left[\frac{\Delta - \sqrt{\Delta^2 + \rho^2}}{\rho}\right] \tag{28}$$

If the twist rate is low ($\rho \to 0$), e_b is very small relative to e_a. Therefore, this mode is characterized by having most of its energy polarized along the A axis. It is elliptically polarized because e_b lags e_a by 90°.

If C_a and C_b equal zero, we have the other normal mode which is also elliptically polarized, but has most of its energy polarized along the B axis and therefore has the lower phase constant.

If we launch all of the wave power into such a twist section in one of the normal modes, (25) and (26) tell us that the wave will continue in this mode until the twist is terminated.

These conclusions may now be translated back into our fixed frame of reference (vertical and horizontal axes) and we can determine how power will be transferred from one polarization to the other. If ρ is small enough, the conclusion will be the same as given in Fig. 5. But if ρ is large, a vertically polarized input wave (which is a local normal mode) no longer corresponds to one of the normal modes for the medium. Consequently, some of both modes will be excited, and since they travel at different velocities, the output polarization will depend upon the electrical length of the coupler and hence upon the frequency. The power transfer curves will have ripples as indicated in Fig. 11. This picture is similar to that of the pendulums shown in Fig. 1, and we can now see that the reason is much the same, namely, that an initial condition where all of the energy is in one pendulum does not correspond to a normal mode of the system.

Equation (27) makes it possible to determine the size of these ripples, or to determine the twist rate ρ which will make them smaller than a specified value. If the medium is excited as shown in Fig. 10(a), the worst axial ratio (e_b/e_a) will be

$$\upsilon \cong 2\left|\frac{C_b}{C_a}\right| = 2\left(\frac{\sqrt{\Delta^2 + \rho^2} - \Delta}{\rho}\right) \tag{29}$$

for $| C_b/C_a |^2 \ll 1$. Solving for ρ we obtain

$$\rho \cong \upsilon\Delta \tag{30}$$

Now $\Delta = \pi/\lambda_\Delta$ where λ_Δ is the birefringent wavelength equal to $2\pi/(\beta_a - \beta_b)$. Therefore

$$\rho = \frac{\pi\upsilon}{\lambda_\Delta} \tag{31}$$

For a complete power transfer

$$\rho z = \frac{\pi}{2}$$

Finally,

$$z = \frac{\lambda_\Delta}{2\upsilon} \tag{32}$$

This gives the length of the coupler in birefringent wavelengths which

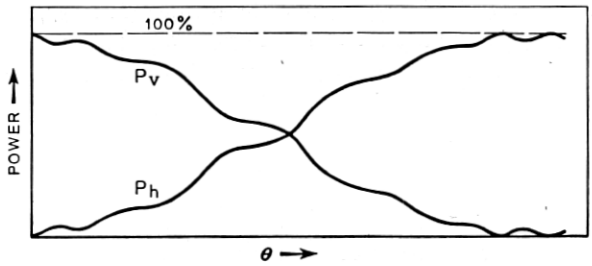

Fig. 11 — Power transfer in a rapid twist.

will insure that the axial ratio does not exceed v. (A birefringent wavelength is comparable to a beat wavelength λ_b defined earlier, and is the distance for which a 360° difference in phase shift exists for the two principal polarizations.)

Thus, to take a specific example, suppose we wish to employ a constant pitch twist which will completely transfer the power from vertical to horizontal polarization, and we wish to keep the power in the unwanted output polarization (vertical) more than 20 db down from the wanted polarization. Then $v = .1$, and $z = 5\lambda_\Delta$.

Since a birefringent wavelength is equal to several wavelengths in the medium, it is clear that the bandwidth of such a coupler has been purchased at the expense of considerable length. As a matter of fact we may note that a conventional coupler with matched phase constants must be made one half a beat wavelength long to provide complete power transfer. Thus, if the twist had the same beat wavelength as the conventional coupler, it would be ten times as long in order to keep the crosstalk more than 20 db down over a very broad band.

Even though the field will in general be elliptically polarized, there will be points along the coupler where the two normal modes are in phase and the polarization will be linear. Thus, by making the coupler the right length, the power transfer can be made correct even though $|C_b/C_a|$, and hence v, is quite appreciable. This makes it possible to make the coupler shorter and still couple power as desired; but the price paid is increased frequency dependence. In other words, by making the coupler shorter for the same total twist angle, the ripples of Fig. 11 will become more pronounced; but by choosing exactly the right length relative to the ripple position, all of the power can still be transferred to the desired polarization. It can be shown that the shortest twist section which can be made in this way is $\sqrt{3}$ times as long as a non-twisted birefringent medium having the same birefringence $(\beta_a - \beta_b)$. It will be where λ_0 is the average wavelength in the medium corresponding to β_0. Thus, the whole coupler will be twenty wavelengths long. From (39) we can determine that

$$\rho_{max} = 0.025\, \beta_0$$

and this means that the phase constant difference between modes I and II which we assumed was constant actually varies by less than one per cent. At the ends of the coupler, the fact that $\rho = 0$ means that modes I and II are linearly polarized along the axes of birefringence. However, the normal modes require some of both I and II, and from (36) we find that for the almost I mode

$$\left| \frac{e_{II}}{e_I} \right| = 0.0025 \tag{43}$$

This ratio is a measure of the excitation of the undesired II mode which will be produced if the input polarization is the local normal mode polarized along the A axis. Since a similar mismatch occurs at the opposite end and also in the middle where the rate of twist begins to decrease, the total crosstalk between the output polarizations may be four times the above figure. Thus, at the output end

$$\left| \frac{e_a}{e_b} \right|_{max} = 0.01 \tag{44}$$

and the worst crosstalk into the undesired polarization is thus 40 db down from the desired output.

Let us compare this with a uniform twist which produces the same crosstalk. From (32) we obtain,

$$z = 250\, \lambda_0$$

Therefore, the uniform twist would have to be 12.5 times as long as the non-uniform twist to do the same job. While this analysis for the variable twist is only approximate, it indicates a marked superiority for the variable twist coupler.

This conclusion is also borne out by a study of the possibility of producing wave coupling by a series of step twists in a birefringent medium, where each step is one half a birefringent wavelength in length. This analysis is given in the appendix. It is shown there that if there are a large enough number of steps so that the total angle between steps is small, then the angles between steps starting at one end and passing through to the other end of the coupler should have a binomial distribution. This will result in the slowest departure from the desired power division as frequency changes. If we regard such a long multi-step twist as an approximation to a smooth twist of the same length, we again conclude that the smooth twist should start off and finish with a zero twist rate, and should have the maximum twist rate at the center.

CONCLUSION

It has been shown that Cook's scheme for producing broadband directional couplers by variation of the phase constants, may be generalized by simultaneously varying the coupling coefficient. For the simplest case, the difference between the phase constants should vary cosinusoidally and the coupling coefficient should vary sinusoidally with distance along the coupler. Such a programming of the coupling parameters corresponds directly to a twisted birefringent medium (such as a metal waveguide having a flattened cross section) where the rate of twist is constant. Since this medium is easy to analyze and to visualize physically, it has been used as a prototype for the design of a number of different types of coupler, all of which are based on the same principle. This principle, which in retrospect sounds rather obvious, is simply that in order to avoid interference effects between two modes of propagation in a multimode waveguide system we should avoid exciting more than one of the normal modes. Also by gently warping the waveguide structure, it is possible to warp the field configuration of the desired normal mode so that it will produce the required power division at the terminals of the system without appreciably scattering power into other unwanted modes.

By avoiding wave interference, such couplers should in principle be independent of frequency. However, the requirement that warping be smooth and gradual also dictates that these couplers must be many wavelengths long. It may turn out that they will be most useful in the millimeter wavelength range where such electrical lengths are physically short. Expressions have been given for the uniform twist which allow one to compute how long the coupler must be in order to meet specified requirements on power division at the terminals. These may of course be applied equally well to any of the other types of coupler by making use of the equivalence equations (17), (18), and (19).

The importance of varying coupling coefficient as well as the phase constant difference for a pair of coupled waves is particularly evident at the ends of a complete power transfer coupler. If we let $k = 0$ in (13) and (14), we see that the local normal modes correspond to all of the power in one or the other of the two waveguide terminals. If k is not zero, it is necessary that the phase constant difference be infinite to achieve the same objective, and this is not possible practically.

Finally, it has been noted that a constant rate of twist for the twist prototype coupler is not necessarily the best. In fact a coupler may be made shorter for specified performance, or will give better performance for a specified length if the twist rate is maximum in the middle and approaches zero at the ends.

Appendix

Multi-Step Mode Transformation

We will show the way in which power can be transferred from one to another of two coupled modes by means of a series of step-mode transformations. The analysis will be made for the special case of a birefringent medium and the results can then be translated for any other type of medium.

Specifically, we assume that the modes to be coupled are vertically and horixontally polarized dominant waves in a round waveguide. Initial excitation e_0 is vertically polarized. This is launched into a first Δ 180° section* set at an angle $\theta_1/2$ from the vertical. As a result, the output of this section is a rotated polarization at angle θ_1 from the vertical. This is introduced into a second Δ 180° section set at an angle $\theta_1 + \theta_2/2$ from the vertical. The output of this section is then at angle $\theta_1 + \theta_2$, and so forth. The polarizations between sections will then appear as

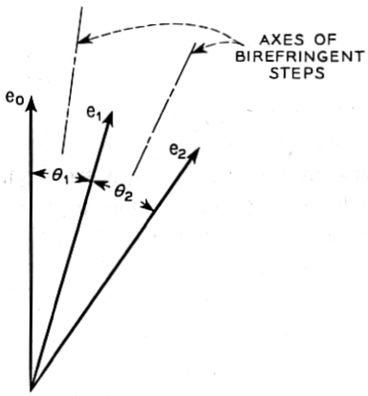

Fig. 12 — Step-twist birefringent medium.

* A section of linearly birefringent medium for which the difference in phase shift for the two principal polarization is 180°.

shown in Fig. 12 by e_1, e_2, $\cdots$, and the dashed lines represent the orientations of the principal axes of the Δ 180° sections. If the Δ 180° sections work perfectly, the output power has pure linear polarization at some arbitrary angle $\theta_1 + \theta_2 + \cdots + \theta_n$ which gives the desired final distribution of power between vertical and horizontal polarizations.

Now if the frequency varies from its design value causing a departure of the differential phase shift from Δ 180° by a small angle δ, it can be shown that a portion of the input polarization e_0 will not be rotated to the angle θ_1. Thus, at the output of the first section e_1 will be split into two components

$$e_1 = e_0 \cos \delta \, \underline{|\theta_1} + j e_0 \sin \delta \, \underline{|0}$$

where $\underline{|\theta}$ means, "polarized at angle θ". This splitting recurs at each successive section. At the end of the second section, we will have

$$e_2 = e_0 \cos^2 \delta \, \underline{|\theta_1 + \theta_2} + j e_0 \sin \delta \cos \delta \, \underline{|\theta_1} + j e_0 \sin \delta \cos \delta \, \underline{|2\theta_1 + \theta_2} - e_0 \sin^2 \delta \, \underline{|0}$$

etc.

If now we take the total field polarized at right angles to the desired output polarization, we will have:

1 Section:

$$j e_0 \sin \delta \sin \theta_1$$

2 Section:

$$- e_0 \sin^2 \delta \sin (\theta_1 + \theta_2)$$
$$+ j e_0 \sin \delta \cos \delta (\sin \theta_2 - \sin \theta_1)$$

3 Section:

$$- j e_0 \sin^3 \delta \sin (\theta_1 + \theta_2 + \theta_3)$$
$$- \sin^2 \delta \cos \delta \begin{bmatrix} \sin (-\theta_1 - \theta_2) \\ \sin (-\theta_1 + \theta_3) \\ \sin (\theta_2 + \theta_3) \end{bmatrix}$$
$$+ j \sin \delta \cos^2 \delta \begin{bmatrix} \sin (\theta_1) \\ \sin (-\theta_2) \\ \sin (\theta_3) \end{bmatrix}$$

4 Section:

$$+\, e_0 \sin^4 \delta \sin (\theta_1 + \theta_2 + \theta_3 + \theta_4)$$

$$-je_0 \sin^3 \delta \cos \delta \begin{cases} \sin (-\theta_1 - \theta_2 - \theta_3) \\ \sin (-\theta_1 - \theta_2 + \theta_4) \\ \sin (-\theta_1 + \theta_3 + \theta_4) \\ \sin (\theta_2 + \theta_3 + \theta_4) \end{cases}$$

$$-e_0 \sin^2 \delta \cos^2 \delta \begin{cases} \sin (\theta_1 + \theta_2) \\ \sin (\theta_1 - \theta_3) \\ \sin (\theta_1 + \theta_4) \\ \sin (-\theta_2 - \theta_3) \\ \sin (-\theta_2 + \theta_4) \\ \sin (\theta_3 + \theta_4) \end{cases}$$

$$je_0 \sin \delta \cos^3 \delta \begin{cases} \sin (-\theta_1) \\ \sin (\theta_2) \\ \sin (-\theta_3) \\ \sin (\theta_4) \end{cases}$$

5 Section:

$$je_0 \sin^5 \delta \sin (\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5)$$

$$e_0 \sin^4 \delta \cos \delta \begin{cases} \sin (-\theta_1 - \theta_2 - \theta_3 - \theta_4) \\ \sin (-\theta_1 - \theta_2 - \theta_3 + \theta_5) \\ \sin (-\theta_1 - \theta_2 + \theta_4 + \theta_5) \\ \sin (-\theta_1 + \theta_3 + \theta_4 + \theta_5) \\ \sin (\theta_2 + \theta_3 + \theta_4 + \theta_5) \end{cases}$$

$$-je_0 \sin^3 \delta \cos^2 \delta \begin{cases} \sin (\theta_1 + \theta_2 + \theta_3) \\ \sin (\theta_1 + \theta_2 - \theta_4) \\ \sin (\theta_1 + \theta_2 + \theta_5) \\ \sin (\theta_1 - \theta_3 - \theta_4) \\ \sin (\theta_1 - \theta_3 + \theta_5) \\ \sin (\theta_1 + \theta_4 + \theta_5) \\ \sin (-\theta_2 - \theta_3 - \theta_4) \\ \sin (-\theta_2 - \theta_3 + \theta_5) \\ \sin (-\theta_2 + \theta_4 + \theta_5) \\ \sin (+\theta_3 + \theta_4 + \theta_5) \end{cases}$$

$$-e_0 \sin^2 \delta \cos^3 \delta \begin{cases} \sin(-\theta_1 - \theta_2) \\ \sin(-\theta_1 + \theta_3) \\ \sin(-\theta_1 - \theta_4) \\ \sin(-\theta_1 + \theta_5) \\ \sin(+\theta_2 + \theta_3) \\ \sin(+\theta_2 - \theta_4) \\ \sin(+\theta_2 + \theta_5) \\ \sin(-\theta_3 - \theta_4) \\ \sin(-\theta_3 + \theta_5) \\ \sin(+\theta_4 + \theta_5) \end{cases}$$

$$+je_0 \sin \delta \cos^4 \delta \begin{cases} \sin(\theta_1) \\ \sin(-\theta_2) \\ \sin(\theta_3) \\ \sin(-\theta_4) \\ \sin(\theta_5) \end{cases}$$

Now upon examining these voltage components representing power in the undesired polarization, we find that certain of the bracketed sums will vanish merely by making the structure symmetrical. Thus, in the 5 section case, if $\theta_1 = \theta_5$, $\theta_2 = \theta_4$, the $\sin^4 \cos$ and $\sin^2 \cos^3$ brackets vanish. By setting each of the remaining brackets equal to zero, we obtain several simultaneous equations which allow us to solve for the remaining θ ratios. In general, this may be an algebraically complicated procedure. However, the reader can verify by inspection that if all of the angles are assumed very small so that

$$\sin \theta \to \theta,$$

a binomial distribution will cause all remaining brackets to vanish. The only unwanted voltage term to remain will be

$$e_0 \sin^n \delta \sin(\theta_1 + \theta_2 + \cdots + \theta_n).$$

Since the sum of the θ's is simply the total twist angle which we assume is held constant, then clearly the voltage departure from linear output polarization as a function of δ (or frequency) varies as $\sin^n \delta$ where n is the number of sections used. This demonstrates the bandwidth advantage to be gained from using a large number of steps.

We may conclude that if a total twist angle is achieved by a very large number of very small step twists between Δ 180° sections, this may be considered an approximation of a long continuous twist. Since the optimum design for the step twists calls for a binomial distribution, it is evident that the continuous twist should also approximate this proportioning by having a very small $d\theta/dz$ at the ends, and a maximum $d\theta/dz$ in the center. This conclusion bears out what was said earlier about starting the continuous twist with $d\theta/dz = 0$.

As a special case where four sections are used to obtain a total twist angle of 90°, an exact solution gives

$$\begin{aligned} \theta_1 &= 10.2° \\ \theta_2 &= 34.8° \\ \theta_3 &= 34.8° \\ \theta_4 &= 10.2° \end{aligned}$$

whereas the binomial solution would have been

$$\begin{aligned} \theta_1 &= 11.25° \\ \theta_2 &= 33.75° \\ \theta_3 &= 33.75° \\ \theta_4 &= 11.25° \end{aligned}$$

If a step twist design of a certain number of steps is chosen as a model, the design for some other type of coupler may be determined by using equations 17, 18, and 19 to solve for the phase constants and coupling coefficient in each step. The β_a and β_b are the phase constants for the parallel and perpendicular polarizations in the Δ 180° steps. The values of θ for the several steps will be $\theta_1/2$, $\theta_1 + \theta_2/2$, $\theta_1 + \theta_2 + \theta_3/2$, etc., since these are the angles at which the Δ 180° plates were oriented. Each section of the coupler will of course be of the same length as the equivalent Δ 180 section in the twist coupler.