Optimum Design of Directive Antenna Arrays Subject to Random Variations

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This paper discusses the optimum design of discrete, directive antenna arrays of arbitrary geometrical configuration in space, when the excitations and spatial positions of the elements vary in a random fashion about their nominal values. Under certain assumptions the expected power pattern of an array turns out to be the power pattern of the nominal array, plus a "background" power level which has the same dependence on direction as the pattern of a single element. A set of excitations which maximizes the theoretical directivity of an array may correspond to a superdirective design, in which the background power level will completely swamp the desired pattern unless the excitations and positions of the elements are controlled with extraordinary precision. A method is given for maximizing the gain of the array while holding the expected background power level constant, when the precision with which the excitations and positions can be controlled is known. The method is illustrated with numerical examples.

1. INTRODUCTION

The effects of random variations on antenna patterns have recently been discussed in a number of papers which treat more or less special cases. Several authors[1, 2, 3] are concerned with linear Chebyshev-designed broadside arrays, and in particular with the effects of manufacturing variations on the patterns of slotted waveguide arrays. Ruze[4] has derived the expected pattern of a plane, rectangular array in which the positions of the radiators are rigidly fixed, while only their amplitudes and phases are variable. Less attention seems to have been given to the possibility that the positions of the elements may also vary in a random fashion, even though this latter situation can arise in any electromagnetic or acoustic array whose elements are not rigidly supported.

[1] L. L. Bailin and M. J. Ehrlich, I.R.E. Trans., PGAP-1, pp. 85–106, Feb., 1952.
[2] D. Ashmead, I.R.E. Trans., PGAP-4, pp. 81–92, Dec., 1952.
[3] H. F. O'Neill and L. L. Bailin, I.R.E. Trans., PGAP-4, pp. 93–102, Dec., 1952.
[4] J. Ruze, Nuovo Cimento, **9**, Supp. 3, pp. 364–380, 1952.

This paper derives a relatively simple statistical expression for the expected power pattern of an array when the excitations and positions of its elements are subject to independent random variations. The result is not restricted to linear or even to plane arrays, but is valid for arrays of arbitrary geometrical configuration in space. In the special case when the excitations are variable and the positions of the elements are either fixed or subject to random displacements with a spherically symmetric distribution, the expected power pattern is the pattern of the nominal array, plus a "background" power level which has the same dependence on direction as the pattern of a single element. The expected background power level is proportional, for small errors, to the sum of the mean-square errors in excitation and position.

We consider in particular the application of our results to the problem of designing superdirective arrays. A superdirective array is one having a beamwidth in radians much less than the reciprocal of the largest dimension of the array in wavelengths. It is well known that such narrow beams can be designed on paper, but only by employing heavy cancellation between adjacent elements. If the excitations and positions of the elements of a superdirective array are not controlled with great accuracy, the background power level due to random errors will completely swamp the desired pattern. This corresponds to the familiar fact that a really superdirective array has to be constructed with extraordinarily high precision in order to give anything like its calculated performance.

A method is given for computing the excitations which maximize the gain of an array of any specified geometrical configuration, while holding the expected background power level due to random variations constant. It is assumed that the excitation coefficients can all be controlled to the same per cent accuracy, and that the element displacements, if any, are distributed with spherical symmetry. If the random variations are taken to be zero, or if no restrictions are placed on the background power level, the present procedure becomes equivalent to the methods which have been described in recent papers[5, 6] on the maximum gain of an arbitrary array.

An interesting result of the analysis relates to arrays in which the elements are all excited with equal amplitudes, and with phases such that their fields add in phase in a specified direction in space. This arrangement may justifiably be called the normal excitation, since it is

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[5] A. I. Uzkov, Dokl. Akad. Nauk SSSR, **53,** pp. 35–38, 1946.

[6] A. Bloch, R. G. Medhurst, and S. D. Pool, Proc. Inst. Elect. Engrs., **100,** Pt. III, pp. 303–314, 1953.

often adopted in practice as a means of obtaining a beam in the desired direction. It is proved that, of all possible excitations of the array which produce a main lobe in the given direction, the normal excitation leads to a pattern which is most insensitive to the effects of random errors. Furthermore, if all the elements have the same ohmic resistance, the normal excitation produces the highest power flow in the direction of the main beam for a given rate of heating the elements.

It is not invariably true that maximum gain is incompatible with minimum sensitivity of the pattern to random errors. The normal excitation maximizes the gain of any array of isotropic elements in which the distance from every element to every other element is an integral number of half wavelengths. However, for most arrays the gain can be increased above that obtainable with the normal excitation, at the expense of a (possibly enormous) increase in the sensitivity of the pattern to random errors; and some sort of compromise will have to be struck.

Several types of symmetry are commonly found in antenna arrays, and some of these symmetries force very simple relationships to hold among certain of the optimum excitation coefficients. We discuss the use of symmetry to reduce the amount of computation necessary in designing an optimum array. As an example, these considerations are applied to the design of an array of four elements located at the corners of a tetrahedron, and also to a four-element end-fire array. Curves are obtained which illustrate the relationship of gain to pattern sensitivity for several such arrays of different dimensions in wavelengths.

It is shown that if an arbitrary array of isotropic elements is excited successively to have maximum directivity in different directions, the average value of the maximum directivity over all directions in space is equal to the number of elements in the array. The excitation required to produce maximum directivity will naturally depend on direction, and considerations of pattern sensitivity are ignored. The significance of this result is that if an array configuration permits an abnormally high gain in a certain direction, as is theoretically possible, for example, with a very short end-fire array, then there exist other directions in which the maximum gain of the same array is abnormally low.

2. STATISTICAL FORMULATION

Consider an antenna array of n elements. We shall call the elements radiators, though they may equally well be thought of as receivers. We assume that each element has the same directivity pattern $\mathbf{s}(\mathbf{u})$ with respect to a fixed set of axes,* where $\mathbf{u}$ is a unit vector representing a direction in space, and $\mathbf{s}(\mathbf{u})$ is a complex-valued vector function giving the amplitude, phase, and polarization of the radiation field over a large sphere centered at the element. For acoustic fields, $\mathbf{s}(\mathbf{u})$ is a scalar function. The average density of power flow in the direction $\mathbf{u}$ is proportional to

$$s^2(\mathbf{u}) = \mathbf{s}(\mathbf{u}) \cdot \mathbf{s}^*(\mathbf{u}) \tag{1}$$

where an asterisk denotes the complex conjugate quantity.

Let the excitation of the kth element be A_k. The complex numbers $A_1, A_2, \cdots, A_n$ will be called the excitation coefficients of the array. Let the position vector of the kth element relative to an arbitrary origin be $\mathbf{R}_k$. The field strength produced by the array at the point at the end of the vector $R\mathbf{u}$ from the origin will, for large R, be proportional to

$$\mathbf{s}(\mathbf{u}) f(\mathbf{u})/R \tag{2}$$

and the average density of power flow at the same point will be proportional to

$$\Phi(\mathbf{u})/R^2 \tag{3}$$

where the power directivity pattern $\Phi(\mathbf{u})$ is

$$\Phi(\mathbf{u}) = s^2(\mathbf{u}) \mid f(\mathbf{u}) \mid^2 \tag{4}$$

and the array factor $f(\mathbf{u})$ is

$$f(\mathbf{u}) = \sum_{k=1}^{n} A_k \exp (i\beta \mathbf{R}_k \cdot \mathbf{u}) \tag{5}$$

As usual, β denotes 2π divided by the wavelength λ.

Let us assume now that the excitation coefficients and the positions of the elements actually have some random scatter about their mean or expected values. We shall calculate the expected values of the field and power patterns. These expected values may be regarded as averages taken over a large number of different arrays, or they may be thought of as long-term time averages for a single array whose parameters vary with time in a random fashion. We can adopt the latter point of view when dealing with an array whose elements are not rigidly intercon-

* No difficulties would result in the statistical analysis from assuming a different pattern for each element, but the added generality would complicate some of the following work, and it is unnecessary for the great majority of practical arrays. We also ignore the possibility that the orientations of the elements might be subject to random variations.

nected, but are subject to displacements relative to one another in the course of time.

To make matters precise, assume that the excitation coefficients are given by

$$A_k = a_k + \alpha_k \tag{6}$$

where a_k is the expected value of A_k and the α_k's are independent random complex variables with mean zero. Also let

$$\mathbf{R}_k = \mathbf{r}_k + \boldsymbol{\varrho}_k \tag{7}$$

where $\mathbf{r}_k$ is the expected value of the position vector $\mathbf{R}_k$ and the $\boldsymbol{\varrho}_k$'s are independent random vectors with mean (0, 0, 0), all having the same statistical distribution.

We can now write down the expected values of the field and power patterns. Denoting expected values by angular brackets, we find for the field strength,

$$\begin{aligned} \langle \mathbf{s}(\mathbf{u})f(\mathbf{u})\rangle &= \mathbf{s}(\mathbf{u}) \sum_{k=1}^{n} \langle A_k \rangle \langle \exp\,(i\beta\mathbf{R}_k\cdot\mathbf{u})\rangle \\ &= \mathbf{s}(\mathbf{u}) \sum_{k=1}^{n} a_k \exp\,(i\beta\mathbf{r}_k\cdot\mathbf{u}) \langle \exp\,(i\beta\boldsymbol{\varrho}_k\cdot\mathbf{u})\rangle \\ &= \langle \exp\,(i\beta\boldsymbol{\varrho}\cdot\mathbf{u})\rangle\, \mathbf{s}(\mathbf{u})f_0(\mathbf{u}) \end{aligned} \tag{8}$$

where $\boldsymbol{\varrho}$ is a random vector having the same distribution as the $\boldsymbol{\varrho}_k$'s, and $f_0(\mathbf{u})$ is the nominal array factor

$$f_0(\mathbf{u}) = \sum_{k=1}^{n} a_k \exp\,(i\beta\mathbf{r}_k\cdot\mathbf{u}) \tag{9}$$

which results when the excitation coefficients and positions all have their expected values.

The norm of the general array factor may be written

$$\begin{aligned} | f(\mathbf{u}) |^2 &= \sum_{k=1}^{n} A_k \exp\,(i\beta\mathbf{R}_k\cdot\mathbf{u}) \sum_{j=1}^{n} A_j^* \exp\,(-\,i\beta\mathbf{R}_j\cdot\mathbf{u}) \\ &= \sum_{k=1}^{n} {\sum_{j=1}^{n}}' (a_k+\alpha_k)(a_j^*+\alpha_j^*) \exp[i\beta(\mathbf{r}_k-\mathbf{r}_j)\cdot\mathbf{u}] \exp[i\beta(\boldsymbol{\varrho}_k-\boldsymbol{\varrho}_j)\cdot\mathbf{u}] \\ &\qquad + \sum_{k=1}^{n} (a_k+\alpha_k)(a_k^*+\alpha_k^*) \end{aligned} \tag{10}$$

where the primed summation sign indicates that the term for which $j = k$ is to be omitted. Taking expected values and recalling that the random variables are independent, we obtain

$$\langle | f(\mathbf{u}) |^2 \rangle$$

$$= \sum_{k=1}^{n} \sum_{j=1}^{n}{}' a_k a_j^* \exp [i\beta(\mathbf{r}_k - \mathbf{r}_j)\cdot\mathbf{u}] \langle \exp (i\beta\boldsymbol{\varrho}_k\cdot\mathbf{u}) \rangle \langle \exp (-i\beta\boldsymbol{\varrho}_j\cdot\mathbf{u}) \rangle$$

$$+ \sum_{k=1}^{n} | a_k |^2 + \sum_{k=1}^{n} \langle | \alpha_k |^2 \rangle \qquad (11)$$

$$= | \langle \exp (i\beta\boldsymbol{\varrho}\cdot\mathbf{u}) \rangle |^2 | f_0(\mathbf{u}) |^2 + \sum_{k=1}^{n} \langle | \alpha_k |^2 \rangle$$

$$+ [1 - | \langle \exp (i\beta\boldsymbol{\varrho}\cdot\mathbf{u}) \rangle |^2] \sum_{k=1}^{n} | a_k |^2$$

where the last step follows by adding and subtracting the terms with $j = k$ which were omitted from the double sum.

Multiplying through by the power pattern $s^2(\mathbf{u})$ of a single element gives the expression for the expected power pattern of the array, namely

$$\langle \Phi(\mathbf{u}) \rangle = | \langle \exp (i\beta\boldsymbol{\varrho}\cdot\mathbf{u}) \rangle |^2 \Phi_0(\mathbf{u}) + s^2(\mathbf{u}) \sum_{k=1}^{n} \langle | \alpha_k |^2 \rangle$$

$$+ [1 - | \langle \exp (i\beta\boldsymbol{\varrho}\cdot\mathbf{u}) \rangle |^2] s^2(\mathbf{u}) \sum_{k=1}^{n} | a_k |^2 \qquad (12)$$

where the power pattern of the nominal array is

$$\Phi_0(\mathbf{u}) = s^2(\mathbf{u}) | f_0(\mathbf{u}) |^2 \qquad (13)$$

3. SPECIAL CASES

If the positions of the elements of the array are supposed to be exactly known and rigidly fixed, then the displacement vectors $\boldsymbol{\varrho}_k$ are identically zero, and the general result derived above reduces to

$$\langle \Phi(\mathbf{u}) \rangle = \Phi_0(\mathbf{u}) + s^2(\mathbf{u}) \sum_{k=1}^{n} \langle | \alpha_k |^2 \rangle \qquad (14)$$

Equation (14) has a simple physical interpretation. It asserts that the expected power pattern is the power pattern of the nominal array, plus a "background" power level which has the same dependence on direction as the pattern of an individual radiator, and is proportional to the sum of the mean-square errors of the excitation coefficients. Of course for any particular array the background power will not have exactly the directional dependence $s^2(\mathbf{u})$, but will exhibit fluctuations depending on the particular set of errors in the excitations.[7] However, in order to have the over-all pattern be a good approximation to the nominal pattern $\Phi_0(\mathbf{u})$, it is necessary to hold the expected value of the background power well below the maximum value of $\Phi_0(\mathbf{u})$. We shall discuss the implications of this requirement presently.

If the displacements of the elements are not identically zero, then we denote the cartesian components of the vector $\boldsymbol{\rho}$ by ξ, η, ζ, and their joint probability distribution by $P(\xi, \eta, \zeta)$. $P(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta$ represents the probability that the end of the vector $\boldsymbol{\rho}$, drawn from the origin, will lie in the volume element $d\xi\, d\eta\, d\zeta$ centered at (ξ, η, ζ). Then the expected value of $\exp(i\beta\boldsymbol{\rho}\cdot\mathbf{u})$ is given by

$$\langle \exp(i\beta\boldsymbol{\rho}\cdot\mathbf{u})\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[i\beta(\xi u_x + \eta u_y + \zeta u_z)]\, P(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta \tag{15}$$

where u_x, u_y, u_z are the components of the unit vector $\mathbf{u}$.

In order to evaluate the integral (15) for any particular array, one should choose the most plausible joint distribution function $P(\xi, \eta, \zeta)$ that his physical insight permits. In general $\langle \exp(i\beta\boldsymbol{\rho}\cdot\mathbf{u})\rangle$ will depend upon the direction of $\mathbf{u}$; but if the distribution of $\boldsymbol{\rho}$ is spherically symmetric,* the expected value will be independent of direction. In this case $P(\xi, \eta, \zeta)$ is a function only of the magnitude ρ of the displacement vector. To evaluate the integral (15), take the ζ-axis parallel to $\mathbf{u}$, and let $p(\rho)\, d\rho$ be the probability that the length of the displacement vector lies between ρ and $\rho + d\rho$. Then

$$\begin{aligned}\langle \exp(i\beta\boldsymbol{\rho}\cdot\mathbf{u})\rangle &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \exp(i\beta\rho\cos\theta) p(\rho)\sin\theta\, d\rho\, d\theta\, d\varphi \\ &= \int_0^{\infty} [(\sin\beta\rho)/\beta\rho]\, p(\rho)\, d\rho = \langle(\sin\beta\rho)/\beta\rho\rangle\end{aligned} \tag{16}$$

If the cartesian components of $\boldsymbol{\rho}$ are assumed to be normally and independently distributed with mean zero and equal variances $\sigma^2/3$, so that the root-mean-square value of ρ is σ, one has

$$p(\rho) = 3(6/\pi)^{1/2}(\rho^2/\sigma^3)\exp(-3\rho^2/2\sigma^2) \tag{17}$$

and

$$\langle \exp(i\beta\boldsymbol{\rho}\cdot\mathbf{u})\rangle = \exp(-\beta^2\sigma^2/6) \tag{18}$$

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[7] The statistical distribution of these fluctuations has been discussed by Ruze in Reference 4.

* Under some circumstances the displacement vectors may be constrained to lie in a plane. If their distribution is circularly symmetric, and if we confine our attention to directions $\mathbf{u}$ lying in the plane of the displacements, the results will be formally similar to those obtained for the spherically symmetric case.

For *any* spherically symmetric distribution of $\boldsymbol{\rho}$ we may define a parameter δ^2 by

$$\delta^2 = | \langle \exp (i\beta\boldsymbol{\rho}\cdot\mathbf{u}) \rangle |^{-2} - 1 \tag{19}$$

If σ, the root-mean-square value of ρ, is small compared to the wave-length, then

$$\delta^2 \approx \beta^2\sigma^2/3 \tag{20}$$

for the normal distribution (17), and also for other distributions which taper off for large ρ with comparable or greater rapidity.

From equations (12) and (19) we obtain the normalized expected power pattern for a spherically symmetric distribution of displacements, namely

$$(1 + \delta^2) \langle \Phi(\mathbf{u}) \rangle = \Phi_0(\mathbf{u}) + s^2(\mathbf{u}) \left[(1 + \delta^2) \sum_{k=1}^{n} \langle | \alpha_k |^2 \rangle + \delta^2 \sum_{k=1}^{n} | a_k |^2 \right] \tag{21}$$

Again the expected pattern turns out to be the nominal pattern plus a background level with the same distribution as the pattern of a single element.

In what follows we shall idealize the problem somewhat by assuming that the excitation coefficients A_k can all be controlled to the same relative accuracy.[8] Precisely we suppose there is a small number ϵ such that

$$\langle |\alpha_k|^2 \rangle = \epsilon^2 | a_k |^2, \qquad k = 1, 2, \cdots, n \tag{22}$$

Thus (21) becomes

$$(1 + \delta^2) \langle \Phi(\mathbf{u}) \rangle = \Phi_0(\mathbf{u}) + \Delta^2 s^2(\mathbf{u}) \sum_{k=1}^{n} | a_k |^2 \tag{23}$$

where

$$\Delta^2 = (1 + \delta^2)\epsilon^2 + \delta^2 \approx \delta^2 + \epsilon^2 \tag{24}$$

and the last approximation is valid if δ^2 is small compared to unity.

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[8] Ruze's assumptions in Reference 4 amount to taking $| A_k |$ and $\arg A_k$ as independent random variables with means I_k, ϑ_k, where the coefficients of the desired pattern are $I_k \exp (i\vartheta_k)$. All the phases are assumed to be normally distributed with the same variance, and the variance of the amplitude $| A_k |$ is taken proportional to $| I_k |^2$. From these assumptions equation (22) follows. Ruze's result looks more complicated than equation (14) mainly because the expected value of A_k differs from $I_k \exp (i\vartheta_k)$ by a constant factor.

4. CRITERIA FOR GOOD PATTERNS

The preceding statistical analysis is equally valid whether the nominal pattern $\Phi_0(\mathbf{u})$ is to have some specified beam shape for a particular application, or merely to provide as narrow a beam as possible. However the latter case is of greater practical interest, and so we shall consider henceforth only the design of highly directive or pencil beam arrays.

We suppose that the number and configuration of the radiators are fixed in advance, and we have to choose the excitation coefficients to obtain the desired pattern of the nominal array. We shall denote the set of n complex numbers $(a_1, a_2, \cdots, a_n)$ by the single symbol a. Our problem is to find a way of exciting the array so as to produce a narrow beam in some direction $\mathbf{u}_0$, i.e., we must choose a in such a way as to make $\Phi_0(\mathbf{u}_0)$ small for all directions $\mathbf{u}$ not near $\mathbf{u}_0$. When we try to formulate the problem more precisely than this we find a large number of alternatives.

As a mathematically tractable criterion for a good pattern, we shall stipulate that a is to be chosen so as to minimize the generalized gain function

$$G(a) = \frac{\Phi_0(\mathbf{u}_0)}{\dfrac{1}{4\pi}\displaystyle\int_S \Phi_0(\mathbf{u})w(\mathbf{u})\, d\Omega} \tag{25}$$

where $w(\mathbf{u})$ is a non-negative weight function which may be chosen at pleasure, $d\Omega$ is an element of solid angle, and S is the surface of the unit sphere. Since $\Phi_0(\mathbf{u})$ is proportional to the density of power flow in the direction $\mathbf{u}$, $G(a)$ represents the ratio of power flow in the direction $\mathbf{u}_0$ to a weighted average of power flow in all directions over a sphere. If one were interested in the radiation pattern only in some plane containing $\mathbf{u}_0$, the weighted average could be taken in all directions around a circle with only minor changes in the formal analysis.

A few comments may clarify the significance of the function $w(\mathbf{u})$. If $w(\mathbf{u})$ is taken to be identically unity, then $G(a)$ is just the conventional gain of an array with the set of excitations a. However, it is well known that merely maximizing the gain does not always produce a pattern with low side lobes. If it is important to prevent large fields from being radiated in specified directions, one may choose $w(\mathbf{u})$ to be unity over the set of unwanted directions and zero elsewhere; and in principle even more complicated choices of $w(\mathbf{u})$ could be made to discourage side lobes. For receiving arrays, $w(\mathbf{u})$ may similarly be chosen to discriminate against the reception of spurious signals from particular directions.

In terms of the excitation coefficients, the function $G(a)$ has a simple formal expression, namely

$$G(a) = s^2(\mathbf{u}_0) \frac{\left| \sum_{k=1}^{n} a_k \exp(i\beta\mathbf{r}_k \cdot \mathbf{u}_0) \right|^2}{\sum_{k=1}^{n} \sum_{j=1}^{n} h_{jk} a_k a_j^*} \tag{26}$$

where

$$h_{jk} = \frac{1}{4\pi} \int_S \exp\,[i\beta(\mathbf{r}_k - \mathbf{r}_j) \cdot \mathbf{u}] s^2(\mathbf{u}) w(\mathbf{u})\, d\Omega \tag{27}$$

and obviously

$$h_{kj} = h_{jk}^* \tag{28}$$

The coefficients h_{jk} depend only on the weight function, the pattern of a single element, and the positions of the elements (in terms of wavelength). When the elements are isotropic and the weight function is identically unity, we have

$$h_{jk} = \frac{\sin \beta r_{jk}}{\beta r_{jk}} \tag{29}$$

where r_{jk} is the distance from radiator j to radiator k. In any case $G(a)$ is the quotient of two Hermitian forms in the excitation coefficients, and can therefore be maximized by standard mathematical techniques.

It may be noted that the use of an integrated criterion such as (25) for keeping the fields small away from the main beam does not absolutely guarantee the absence of undesirably high side lobes in particular directions for any given array. To be sure of keeping all side lobes below a certain level, we should choose a to maximize some such expression as

$$T(a) = \frac{\Phi_0(\mathbf{u}_0)}{\max\limits_{\Omega} \Phi_0(\mathbf{u})} \tag{30}$$

where the denominator is the maximum value of $\Phi_0(\mathbf{u})$ over a chosen set of directions Ω not containing $\mathbf{u}_0$. For equispaced linear arrays the criterion (30) leads to the Chebyshev design procedure first described by Dolph. However in general it is a much harder mathematical problem to maximize $T(a)$ than to maximize $G(a)$, and for that reason we shall not employ $T(a)$ here.

5. RESTRICTIONS ON SUPERDIRECTIVE ARRAYS

Even when a set of excitation coefficients maximizing the gain function $G(a)$ has been found, there is no assurance that this set will be a satisfactory one on which to base the construction of a physical array. There is a further restriction on the solution: the array must be excited in such a way that when it is constructed it is likely to have a pattern $\Phi(\mathbf{u})$ which differs from the nominal pattern $\Phi_0(\mathbf{u})$ by an acceptably small amount.

From Section 3 the expected power pattern is of the form

$$\Phi_0(\mathbf{u}) + \Delta^2 s^2(\mathbf{u}) \sum_{k=1}^{n} | a_k |^2 \tag{31}$$

where Δ^2 includes the effects of both excitation and position errors. The background power level relative to the main lobe of the nominal pattern is then just

$$\frac{\Delta^2 s^2(\mathbf{u}) \sum_{k=1}^{n} | a_k |^2}{\Phi_0(\mathbf{u}_0)} = \frac{\Delta^2 s^2(\mathbf{u})}{s^2(\mathbf{u}_0)} K(a) \tag{32}$$

where

$$K(a) = \frac{\sum_{k=1}^{n} | a_k |^2}{\left| \sum_{k=1}^{n} a_k \exp (i\beta \mathbf{r}_k \cdot \mathbf{u}_0) \right|^2} \tag{33}$$

$K(a)$ is a function measuring the susceptibility of the pattern to random errors in the excitations and positions of the elements. Since in practice Δ^2 is never zero, an array with too large a value of $K(a)$ will be unacceptable.

Although the function $K(a)$ has been introduced as a result of statistical considerations, it can also be interpreted in terms of the efficiency of the array as an energy radiator. If we imagine the elements to have a certain ohmic resistance, and the excitation coefficients to correspond to the element currents, then $\Sigma \, | a_k |^2$ is a measure of the power which is lost in the form of heat, and $K(a)$ is proportional to the ratio of dissipated power to power density in the direction $\mathbf{u}_0$. Thus a large value of $K(a)$ corresponds to large circulating currents in the array, and to high ohmic losses for a given rate of radiation of power in the direction of the main beam.

As a simple example of arrays which can have arbitrarily high gain at the expense of large values of $K(a)$, let us consider an end-fire array of length L pointing in the direction $\mathbf{u}_0$ (say the direction of the z-axis). The array will have $n + 1$ elements situated at $z = 0, L/n, 2L/n, \cdots, L$. Let the expected excitation coefficient of the radiator at $z = kL/n$ be

$$a_k = (-)^k C_{n,k} \exp (ik\beta L/n) \tag{34}$$

where the $C_{n,k}$'s are binomial coefficients. Then from (9) the norm of the nominal array factor is

$$\begin{aligned} | f_0(\mathbf{u}) |^2 &= | 1 - \exp [i(\mathbf{u}_0 \cdot \mathbf{u} + 1)\beta L/n] |^{2n} \\ &= 2^{2n} \sin^{2n} [(1 + \cos \theta)\beta L/2n] \\ &= 2^{2n} \sin^{2n} [(\beta L \cos^2 \tfrac{1}{2}\theta)/n] \end{aligned} \tag{35}$$

where θ is the angle between the direction $\mathbf{u}$ and the direction of the array. If L is fixed and n is large, then approximately

$$| f_0(\mathbf{u}) |^2 \approx (2\beta L/n)^{2n} \cos^{4n} \tfrac{1}{2}\theta. \tag{36}$$

Taking n large enough, one obtains an arbitrarily sharp beam and an arbitrarily high gain. On the other hand,

$$\sum_{k=1}^{n} | a_k |^2 = \sum_{k=1}^{n} C_{n,k}^{\ 2} = \frac{(2n)!}{(n!)^2} \approx \frac{2^{2n}}{(n\pi)^{1/2}} \tag{37}$$

the last approximation being valid for large n. Hence for L fixed and n large, we have from (33),

$$K(a) \approx (n/\beta L)^{2n}(n\pi)^{-1/2} \tag{38}$$

If Δ is a typical figure like 0.01, it is clear that this end-fire array is totally useless, in spite of its high theoretical gain, if the spacing L/n is much less than $1/\beta = \lambda/2\pi$.

The array just considered exhibits a characteristic feature of superdirective arrays. All such arrays depend for their narrow beamwidths on heavy cancellation between the fields of closely spaced radiators. In practice the radiators can be adjusted with only finite precision, and the random errors which contribute to the background power level will not cancel out, on the average, as do the fields of the nominal elements in all directions except that of the main beam. Hence the background power level will completely swamp the main beam unless the array is designed with extraordinarily high precision.

It turns out that merely maximizing the gain function $G(a)$ of an array with interelement spacings of less than about a quarter wavelength is likely to lead to a superdirective design with an unacceptably large value of $K(a)$. To get useful results, therefore, we should maximize $G(a)$ subject to the auxiliary condition that $K(a)$ is not to exceed a preassigned value. A method for doing this is given in the next section.

We note that it would appear possible on paper to design an array in a limited volume with high $G(a)$ and low $K(a)$ by using a large number of closely packed elements. For example, suppose we have an array with a high value of $G(a)$. Now suppose each element to be divided into m very nearby elements, each with $1/m$ times the original excitation. The nominal field pattern, which determines the theoretical gain, is essentially unchanged, while $K(a)$ is divided by m. The catch is that a really superdirective array with a reasonable value of Δ^2 would require a colossal number of elements to reduce $K(a)$ to an acceptable value. Furthermore our statistical arguments have been based on the assumption that the excitation coefficients may be regarded as *independent* random variables when the antenna is built. If the elements are packed close together it seems unlikely that the excitations remain independent; then the use of $K(a)$ to determine the precision which must be maintained is no longer justified.

In dealing with superdirective arrays it should always be remembered that a superdirective antenna is a high-Q, small-bandwidth device, inasmuch as the amount of reactive energy in the near field of the antenna is very large compared to the energy radiated per cycle. The resulting stringent physical limitations on superdirectivity have been exhibited by Chu.[9] We shall not, however, discuss questions of bandwidth here.

6. OPTIMUM DESIGNS TAKING ACCOUNT OF FINITE PRECISION

The procedure for maximizing the gain function $G(a)$ of a definite array, while requiring $K(a)$ not to exceed a specified value, depends upon certain theorems which will now be stated. The proofs are given in the appendix.

Theorem I. Let $\mu \geqq 0$ and let $a[\mu]$ denote the set of excitation coefficients satisfying the system of linear equations

$$\sum_{j=1}^{n} h_{kj}a_j[\mu] + \mu a_k[\mu] = \exp\,(-i\beta\mathbf{r}_k\cdot\mathbf{u}_0) \tag{39}$$

$k = 1, 2, \cdots, n$. *Of all possible choices of a satisfying*

$$K(a) \leqq K(a[\mu]) \tag{40}$$

the maximum value of $G(a)$ is obtained when $a = a[\mu]$.

The parameter μ is essentially a Lagrangian undetermined multiplier, such as is commonly used in the calculus of variations and in the determination of maxima or minima subject to a constraint. When designing a directive array we may select a reasonable value of $K(a)$, say K_0, with an eye to the precision with which we expect to construct the array.

[9] L. J. Chu, J. App. Phys., **19**, pp. 1163–1175, 1948.

What we should then like to do is choose the value of μ for which

$$K(a[\mu]) = K_0 \tag{41}$$

and solve (39) with this value of μ. Unfortunately we cannot determine μ simply or directly from the condition (41), so we may have to make several trials with different values of μ. If our first guess does not yield a value of K sufficiently close to K_0, the direction to proceed is indicated by

Theorem II. Both $K(a[\mu])$ and $G(a[\mu])$ are monotone nonincreasing functions of μ; that is, $dK(a[\mu])/d\mu \leqq 0$ and $dG(a[\mu])/d\mu \leqq 0$.

Fairly simple expressions may be obtained (see equations (A4), (A5), and (A20) through (A24) of the appendix) for $a[\mu]$, $K(a[\mu])$, and $G(a[\mu])$ in terms of the eigenvalues and eigenvectors of the Hermitian matrix (h_{jk}), and the parameter μ. To make a thorough study of any particular array configuration, one might well start by computing the eigenvalues and eigenvectors. Then it would be easy to plot K and G against μ over any desired range.

A restriction on the possible values of $K(a)$ is given by

Theorem III. For any array with n elements,

$$K(a) \geqq 1/n \tag{42}$$

and the coefficients which yield the value $1/n$ are

$$a_k = \exp(-i\beta\mathbf{r}_k \cdot \mathbf{u}_0), \quad k = 1, 2, \cdots, n \tag{43}$$

up to a constant proportionality factor.

The choice of coefficients (43) means that all elements have equal amplitudes and the phases are chosen to make their contributions add in phase in the direction $\mathbf{u}_0$. This may be called the normal excitation; it is often adopted in practice as a means of obtaining a beam in the direction $\mathbf{u}_0$. The patterns so obtained are most insensitive to random errors in the excitation coefficients, although frequently at the expense of rather disagreeable side lobes. Since, as pointed out in Section 5, $K(a)$ also measures the efficiency of the array as an energy radiator, Theorem III shows that the normal excitation produces the highest power flow in the direction $\mathbf{u}_0$ for a given rate of heating the elements.

If no auxiliary condition is imposed on the value of $K(a)$, we have

Theorem IV. The coefficients which maximize the gain function $G(a)$ absolutely are obtained by putting $\mu = 0$ in equations (39), so that $a = a[0]$, where

$$\sum_{j=1}^{n} h_{kj}a_j[0] = \exp(-i\beta\mathbf{r}_k \cdot \mathbf{u}_0) \tag{44}$$

$k = 1, 2, \cdots, n$.

Equations (44) are equivalent to those given by Bloch, Medhurst, and Pool[10] as a result of a differently phrased argument. They could also be obtained by following up Uzkov's remark[11] that the problem of maximizing the directivity of an arbitrary array amounts to the problem of choosing an orthogonal basis for a complex linear vector space generated by the patterns of the individual radiators. In principle both these papers show how to determine the maximum gain in the general case when each element of the array has a different directivity pattern; but they say nothing about precision requirements.

It is not invariably true that maximum gain is incompatible with minimum sensitivity to random errors. For example, the normal excitation of Theorem III, which minimizes $K(a)$, simultaneously maximizes the conventional gain of any array of isotropic elements in which the distance from every element to every other element is an integral number of half wavelengths. Under this condition (29) shows that $h_{jk} = \delta_{jk}$; and the values of G and K obtained from the solution of (39) are independent of μ. An example is a linear array with half-wavelength spacing between adjacent elements; another is the three-dimensional configuration of four elements at the vertices of a tetrahedron of edge $\lambda/2$. If the elements of these arrays are not isotropic, then the normal excitation does not give quite the maximum gain; but the difference will be small if the beamwidth of the array factor is narrow compared to the beamwidth of the element pattern. For a general array with interelement spacings much less than a half wavelength, however, there may be a great deal of difference between the excitation and pattern for minimum sensitivity ($\mu = \infty$) and the excitation and pattern for maximum gain ($\mu = 0$). Illustrative examples are worked out in Section 8.

7. SYMMETRIC ARRAYS

To find optimum excitation coefficients for an n-element array by the method of the preceding section requires the solution of a system of n simultaneous linear equations, and this can be laborious if n is large. Fortunately several types of symmetry are commonly found in antenna arrays. Some of these symmetries force very simple relationships to hold among certain of the optimum excitation coefficients, and can therefore be used to obtain an immediate reduction of the order of the system of equations (39).

By a *symmetry* S of an antenna configuration we shall mean any combination of translations, rotations, and reflections which leaves the con-

[10] Reference 6, p. 304, equation (6b).
[11] Reference 5, p. 37.

figuration invariant. Let Sk denote the number of the element of the array into which the kth element is carried by the symmetry S. Some simple symmetries and the corresponding relationships which can exist between a_k and a_{Sk} are listed in Table I. These relationships hold for the solutions $a_k[\mu]$ of equations (39) if the origin of the coordinate system is symmetrically placed,* i.e., if the origin is invariant under the symmetry operation S. In the last four entries of the table we assume that the pattern $s^2(\mathbf{u})$ of a single element and the weight function $w(\mathbf{u})$ appearing in the definition of the generalized gain depend only on the angle between $\mathbf{u}$ and $\mathbf{u}_0$.

To illustrate the method of proving these relationships consider the second one. Reflection in a plane normal to $\mathbf{u}_0$ carries a point $\mathbf{r}$ into

$$S\mathbf{r} = \mathbf{r} - 2(\mathbf{r}\cdot\mathbf{u}_0)\mathbf{u}_0 \tag{45}$$

and for any points $\mathbf{x}$ and $\mathbf{y}$

$$\mathbf{u}_0\cdot S\mathbf{x} = -\mathbf{u}_0\cdot\mathbf{x} \tag{46}$$

$$S\mathbf{x}\cdot S\mathbf{y} = \mathbf{x}\cdot\mathbf{y} \tag{47}$$

From (46) it follows that

$$s^2(-S\mathbf{u})w(-S\mathbf{u}) = s^2(\mathbf{u})w(\mathbf{u}) \tag{48}$$

since by assumption $s^2(\mathbf{u})$ and $w(\mathbf{u})$ depend only on the angle between $\mathbf{u}$ and $\mathbf{u}_0$, that is, on $\cos^{-1}(\mathbf{u}\cdot\mathbf{u}_0)$. From the last three equations and (27) we conclude that

$$\begin{aligned} h_{Sk,Sj} &= \frac{1}{4\pi}\int_S \exp\left[-i\beta(\mathbf{r}_{Sj} - \mathbf{r}_{Sk})\cdot S\mathbf{u}\right]s^2(-S\mathbf{u})w(-S\mathbf{u})\,d\Omega \\ &= \frac{1}{4\pi}\int_S \exp\left[-i\beta(\mathbf{r}_j - \mathbf{r}_k)\cdot\mathbf{u}\right]s^2(\mathbf{u})w(\mathbf{u})\,d\Omega \\ &= h_{kj}^* \end{aligned} \tag{49}$$

We also have

$$\exp(-i\beta\mathbf{r}_{Sk}\cdot\mathbf{u}_0) = [\exp(-i\beta\mathbf{r}_k\cdot\mathbf{u}_0)]^* \tag{50}$$

The system of equations (39) can be written in the form

$$\sum_{j=1}^{n} h_{Sk,Sj}a_{Sj} + \mu a_{Sk} = \exp(-i\beta\mathbf{r}_{Sk}\cdot\mathbf{u}_0) \tag{51}$$

* If the origin is not symmetrically placed it may be necessary to multiply $a[\mu]$ by a suitably chosen complex constant in order to get a set of excitation coefficients which satisfy the relationships of Table I.

or, using (49) and (50),

$$\sum_{j=1}^{n} h_{kj}^* a_{Sj} + \mu a_{Sk} = [\exp(-i\beta \mathbf{r}_k \cdot \mathbf{u}_0)]^* \tag{52}$$

or finally, taking complex conjugates,

$$\sum_{j=1}^{n} h_{kj} a_{Sj}^* + \mu a_{Sk}^* = \exp(-i\beta \mathbf{r}_k \cdot \mathbf{u}_0) \tag{53}$$

Comparing (39) and (53) we see that a_k and a_{Sk}^* are solutions of the same system of linear equations. The determinant of the system is not zero, by equation (A7) of the appendix, and hence

$$a_k = a_{Sk}^* \tag{54}$$

Proofs for the other cases in Table I are similar.

In cases 3 and 5 of Table I the relationship $a_{Sk} = a_k$ reduces the system of equations (39) to a system of order equal to the number of different classes of symmetric points in the array. In cases 1, 2, and 4 the relationship $a_{Sk} = a_k^*$ is more difficult to exploit. If h_{jk} is a real matrix, which will be true whenever

$$s^2(-\mathbf{u})w(-\mathbf{u}) = s^2(\mathbf{u})w(\mathbf{u}) \tag{55}$$

the system (39) may be rewritten as the following pair of real systems:

$$\begin{aligned} \sum_{j=1}^{n} h_{kj} \operatorname{Re} a_j + \mu \operatorname{Re} a_k &= \cos(\beta \mathbf{r}_k \cdot \mathbf{u}_0) \\ \sum_{j=1}^{n} h_{kj} \operatorname{Im} a_j + \mu \operatorname{Im} a_k &= -\sin(\beta \mathbf{r}_k \cdot \mathbf{u}_0) \end{aligned} \tag{56}$$

Now using

$$\operatorname{Re} a_{Sj} = \operatorname{Re} a_j, \qquad \operatorname{Im} a_{Sj} = -\operatorname{Im} a_j \tag{57}$$

the order of these systems reduces to about $n/2$ each.

TABLE I — RELATIONSHIPS AMONG OPTIMUM EXCITATION COEFFICIENTS IN SYMMETRIC ANTENNA ARRAYS.

Case	Symmetry S	Relationship
1	Reflection in a point	$a_{Sk} = a_k^*$
2	Reflection in a plane normal to $\mathbf{u}_0$	$a_{Sk} = a_k^*$
3	Reflection in a plane parallel to $\mathbf{u}_0$	$a_{Sk} = a_k$
4	180° rotation about an axis perpendicular to $\mathbf{u}_0$	$a_{Sk} = a_k^*$
5	Rotation about an axis parallel to $\mathbf{u}_0$	$a_{Sk} = a_k$

It may be noted that the relationships listed in Table I still apply if instead of $G(a)$ the function to be maximized is the main-beam to side-lobe ratio mentioned at the end of Section 4, namely

$$T(a) = \frac{\Phi_0(\mathbf{u}_0)}{\max_{\Omega} \Phi_0(\mathbf{u})} \tag{58}$$

subject to the auxiliary condition on $K(a)$. In this case we assume that the set Ω of directions in which radiation is to be kept small has rotational symmetry about the direction $\mathbf{u}_0$, that is, if $\mathbf{u}_1$ is in Ω any other $\mathbf{u}$ for which $\mathbf{u} \cdot \mathbf{u}_0 = \mathbf{u}_1 \cdot \mathbf{u}_0$ is also in Ω.

8. NUMERICAL EXAMPLES

We shall apply the preceding theory to two array configurations which illustrate some points of interest without requiring unnecessarily heavy computation. For simplicity throughout this section we consider only isotropic elements and adopt the conventional definition of gain, so that the matrix elements h_{jk} are of the form given by equation (29), namely $(\sin \beta r_{jk})/\beta r_{jk}$.

The first example consists of four elements situated at the vertices of a tetrahedron of edge length l (Fig. 1). The unit vector $\mathbf{u}_0$ is taken to be along the altitude of the tetrahedron which passes through the element a_4 and is perpendicular to the plane containing a_1, a_2, and a_3. This array is invariant under rotations of 120° about an axis containing $\mathbf{u}_0$, and so by Case 5 of Table I we have

$$a_1 = a_2 = a_3 \tag{59}$$

It is easily shown that the system of equations (39) for the optimum coefficients reduces to the two equations:

$$\begin{gathered} \left[1 + \mu + \frac{2 \sin \beta l}{\beta l}\right] a_1 + \frac{\sin \beta l}{\beta l} a_4 = 1 \\ \frac{3 \sin \beta l}{\beta l} a_1 + (1 + \mu) a_4 = \exp\left[-i(2/3)^{1/2} \beta l\right] \end{gathered} \tag{60}$$

After solving (60), the functions $G(a)$ and $K(a)$ may be computed from their definitions (26) and (33).

The optimum gain G is plotted against the sensitivity function K in Fig. 1 for $l = \lambda/2$, $\lambda/4$, $\lambda/8$, and $\lambda/16$. As a matter of interest a few values of the parameter μ are shown on the curves. The curve for $l = \lambda/2$ consists of the single point $K = 1/4$, $G = 4$. This is an array with half-

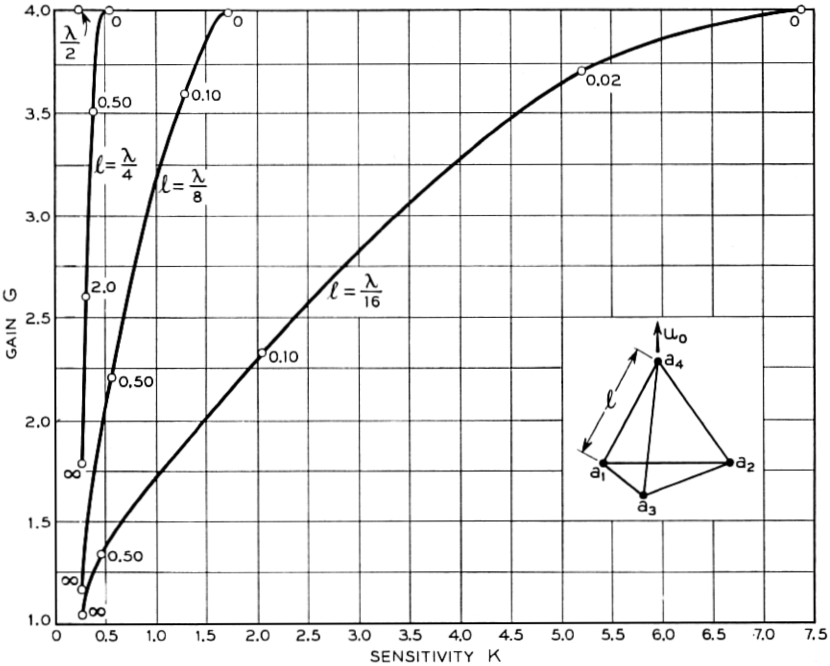

Fig. 1 — Gain G versus sensitivity K for optimum tetrahedral arrays of various edge lengths l. Some values of the parameter μ are shown on the curves.

wavelength interelement spacing, for which, as noted at the end of Section 6, minimum sensitivity and maximum gain are obtained simultaneously. The lower ends of the curves ($\mu = \infty$) all represent normal excitation and minimum sensitivity; in accord with Theorem III the limiting value of K is $\frac{1}{4}$. The corresponding gain decreases as the size of the array decreases. As the condition on K is relaxed by decreasing the parameter μ, the gain increases to a value very near 4 in all cases, and the sensitivity increases to larger and larger values for the smaller arrays. The maximum values of G, which occur for $\mu = 0$, and the corresponding values of K are listed in Table II.

As a second example we consider a four-element equispaced linear end-fire array with interelement spacing l (Fig. 2). This array is symmetric with respect to its center point, and so by Case 1 of Table I,

$$\begin{aligned} a_4 &= a_1^* \\ a_3 &= a_2^* \end{aligned} \tag{61}$$

Equations (39) break up into two pairs of real equations of the form

TABLE II — MAXIMUM GAIN G AND CORRESPONDING SENSITIVITY K FOR FOUR-ELEMENT TETRAHEDRAL AND LINEAR ARRAYS WITH INTERELEMENT SPACINGS l/λ WAVELENGTHS.

l/λ	Tetrahedral		Linear	
	G	K	G	K
½	4.000	0.2500	4.000	0.2500
¼	3.960	0.5405	12.77	2.065
⅛	3.990	1.901	15.21	1.07×10^2
1/16	3.997	7.371	15.80	6.6×10^3
1/32	—	—	15.95	4.2×10^5
0	4.000	∞	16.00	∞

(56), namely:

$$\left[1 + \mu + \frac{\sin 3\beta l}{3\beta l}\right] \operatorname{Re} a_1 + \left[\frac{\sin \beta l}{\beta l} + \frac{\sin 2\beta l}{2\beta l}\right] \operatorname{Re} a_2 = \cos 3\beta l/2$$
$$\left[\frac{\sin \beta l}{\beta l} + \frac{\sin 2\beta l}{2\beta l}\right] \operatorname{Re} a_1 + \left[1 + \mu + \frac{\sin \beta l}{\beta l}\right] \operatorname{Re} a_2 = \cos \beta l/2 \tag{62}$$

and

$$\left[1 + \mu - \frac{\sin 3\beta l}{3\beta l}\right] \operatorname{Im} a_1 + \left[\frac{\sin \beta l}{\beta l} - \frac{\sin 2\beta l}{2\beta l}\right] \operatorname{Im} a_2 = -\sin 3\beta l/2$$
$$\left[\frac{\sin \beta l}{\beta l} - \frac{\sin 2\beta l}{2\beta l}\right] \operatorname{Im} a_1 + \left[1 + \mu - \frac{\sin \beta l}{\beta l}\right] \operatorname{Im} a_2 = -\sin \beta l/2 \tag{63}$$

Plots of G against K, as computed from the above equations, are shown in Fig. 2 for arrays with $l = \lambda/4$, $\lambda/8$, $\lambda/16$, and $\lambda/32$. Note that the scale of Fig. 2 is different from the scale of Fig. 1. The point for $l = \lambda/2$, which again falls at $K = ¼$, $G = 4$, is not shown because it happens to coincide with the lower end of the curve for $l = \lambda/4$. For spacings less than $\lambda/2$ the curves rise to values of gain greater than 4, the limiting value of gain for infinitesimal spacing being 16. However, for spacings less than $\lambda/4$ the sensitivity becomes very large for the higher gains, so that the upper ends of the curves are far off the horizontal scale in Fig. 2. The theoretical maximum gains and corresponding sensitivities are listed in Table II.

A considerable difference is evident in the behavior of small tetrahedral and small linear arrays, where by "small" we mean interelement spacings appreciably less than a half wavelength.* Sensitivity considerations

* It is probably not meaningful to make a direct comparison between tetrahedral and linear arrays of the same spacing l, on account of their very different proportions. For example, the tetrahedral array will fit into a sphere of diameter $1.225l$, while the linear array requires a sphere of diameter $3l$.

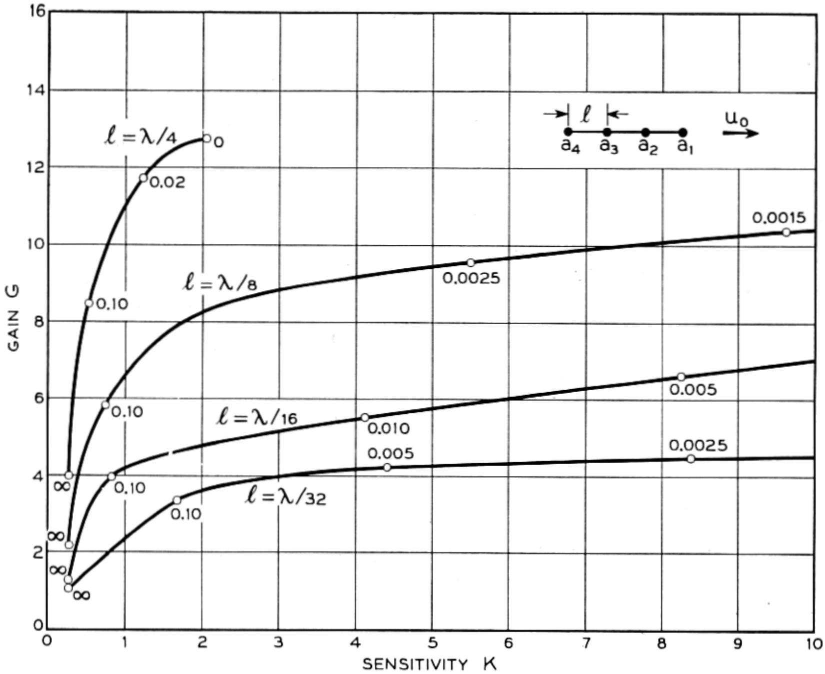

Fig. 2 — Gain G versus sensitivity K for optimum four-element linear end-fire arrays with various interelement spacings l. Some values of the parameter μ are shown on the curves.

being ignored for the moment, the small linear array can have much higher gains *in the end-fire direction* than the tetrahedral array can have in any direction. On the other hand, by a suitable choice of excitation the tetrahedral array can exhibit very nearly the same gain in an arbitrary direction, while in directions different from end-fire the small linear array cannot produce nearly so high a gain. These observations serve to introduce a more general result, as follows:

Let $\Gamma(\mathbf{u})$ be the maximum (conventional) gain which can be achieved in the direction $\mathbf{u}$ with an array of n isotropic elements, located at arbitrary preassigned positions in space. The value of $\Gamma(\mathbf{u})$ and the excitation necessary to obtain it will of course depend upon $\mathbf{u}$. The following theorem is proved in the appendix:

Theorem V. The average value over all directions in space of the maximum gain $\Gamma(\mathbf{u})$ of an array of isotropic elements is equal to the number of elements in the array; that is,

$$\frac{1}{4\pi}\int_S \Gamma(\mathbf{u})\, d\Omega = n \qquad (64)$$

From this theorem it is obvious that the tetrahedral array, which comes as close to spherical symmetry as is possible with four elements, will have a maximum gain of approximately 4 in every direction, while the small linear array, which has a maximum gain of nearly 16 in the end-fire direction, must have a maximum gain much smaller than 4 in most other directions. In the limiting case of an infinitesimally short linear array of n elements, it may be shown by a simple extension of Uzkov's derivation[12] that the maximum gain which can be obtained in the direction $\mathbf{u}$ is

$$\Gamma(\mathbf{u}) = \sum_{k=0}^{n-1} (2k + 1)P_k^2 (\cos \theta) \tag{65}$$

where $P_k(\cos \theta)$ is a Legendre polynomial and θ is the angle between $\mathbf{u}$ and the direction of the array. $\Gamma(\mathbf{u})$ is equal to n^2 when $\theta = 0$, but its value decreases rapidly as θ varies away from zero. It is easy to verify Theorem V in this special case, since we have

$$\begin{aligned} \frac{1}{4\pi} \int_S \Gamma(\mathbf{u})\, d\Omega &= \frac{1}{2} \sum_{k=0}^{n-1} (2k + 1) \int_0^\pi P_k^2 (\cos \theta) \sin \theta \, d\theta \\ &= \frac{1}{2} \sum_{k=0}^{n-1} (2k + 1)[2/(2k + 1)] = n \end{aligned} \tag{66}$$

An infinitesimal array is not in itself of practical importance, since for gains greater than unity the excitation coefficients and the sensitivity are infinite; but the above example does illustrate that if an array can have an abnormally high gain in certain directions, there must be other directions in which its maximum gain is abnormally low. This fact may have applications to the design of arrays whose physical orientation is fixed, while the main beam is caused to scan by electrically varying the excitations of the various elements.

In conclusion we should like to point out that the design procedure given in this paper applies to any array in which the number of elements and their configuration are specified in advance. In many practical cases the array to be used will be so specified, and our method then yields the optimum design. One could, of course, ask a more general question: Given n elements and a sphere of fixed diameter, how should the elements be placed inside the sphere and how should they be excited to maximize the gain in a specified direction, while holding the function $K(a)$, which measures the sensitivity of the pattern to random variations, below a preassigned value? The restriction to a definite number of

[12] Reference 5, pp. 37–38.

elements is essential, since as pointed out in Section V we could produce on paper an arbitrarily high gain with an arbitrarily small $K(a)$ by packing colossal numbers of elements into the sphere. It may well be true (though a proof seems to be lacking) that if there is no restriction on pattern sensitivity, the maximum gain possible with n elements is obtained by arranging them in an end-fire array of infinitesimal length; but the excitation coefficients and $K(a)$ are then infinite. It is not self-evident that the end-fire arrangement will be optimum if $K(a)$ is required to have a reasonable, finite value. The very interesting problem of determining optimum array configurations for finite values of $K(a)$ appears, however, to be considerably more difficult than the problem of determining optimum excitations for arrays of fixed configuration.

Appendix

PROOFS OF OPTIMIZATION THEOREMS

Let H denote the Hermitian matrix (h_{jk}), a the n-vector whose components are the excitation coefficients, and e the n-vector whose components are

$$e_k = \exp(-i\beta \mathbf{r}_k \cdot \mathbf{u}_0), \qquad k = 1, 2, \cdots, n \tag{A1}$$

The product Ha of the matrix H and the vector a is the vector whose jth component is

$$(Ha)_j = \sum_{k=1}^{n} h_{jk} a_k, \quad j = 1, 2, \cdots, n \tag{A2}$$

and the inner product (x, y) of two vectors is defined as

$$(x, y) = \sum_{k=1}^{n} x_k y_k^* \tag{A3}$$

The functions $G(a)$ and $K(a)$ defined by (26) and (33) may accordingly be written

$$G(a) = \frac{|(a, e)|^2}{(Ha, a)} \tag{A4}$$

$$K(a) = \frac{(a, a)}{|(a, e)|^2} \tag{A5}$$

Theorem I. Consider the system of equations

$$Ha + \mu a = e, \tag{A6}$$

where μ is a scalar parameter. If $\mu \geqq 0$, then for any nonzero a whatever,

$$(Ha, a) + \mu(a, a) = \frac{1}{4\pi} \int_S | f_0(\mathbf{u}) |^2 s^2(\mathbf{u}) w(\mathbf{u}) \, d\Omega + \mu(a, a) > 0 \tag{A7}$$

so that the Hermitian form (A7) is positive definite and

$$\det (h_{kj} + \mu\delta_{kj}) \neq 0$$

Then the system of equations (A6) does have a solution $a[\mu]$.

From the complex conjugates of (A6) and (A7) we find that

$$(a[\mu], e) \neq 0 \tag{A8}$$

so that $a[\mu]$ is not orthogonal to e. Hence an arbitrary vector a may be written in the form

$$a = \alpha a[\mu] + b \tag{A9}$$

where α is a complex scalar and b is a vector orthogonal to e, so that

$$(b, e) = 0 \tag{A10}$$

Note that if α were zero, we should have

$$f_0(\mathbf{u}_0) = (b, e) = 0 \tag{A11}$$

corresponding to no radiation in the direction $\mathbf{u}_0$; so we may assume $\alpha \neq 0$. Since $G(a)$ and $K(a)$ are quotients of Hermitian forms in the components of a, their values do not change when a is multiplied by a constant. Hence it suffices to consider excitations of the form

$$a = a[\mu] + b \tag{A12}$$

where b satisfies (A10). Such excitations leave (a, e) constant, so we need only study the behavior of (a, a) and (Ha, a) as functions of b.

To satisfy the condition

$$K(a) \leqq K(a[\mu]) \tag{A13}$$

it is necessary that

$$(a[\mu], b) + (b, a[\mu]) + (b, b) \leqq 0 \tag{A14}$$

Then we have

$$\begin{aligned}(Ha, a) &= (Ha[\mu], a[\mu]) + (Ha[\mu], b) + (Hb, a[\mu]) + (Hb, b) \\ &= (Ha[\mu], a[\mu]) + (Ha[\mu], b) + (b, Ha[\mu]) + (Hb, b) \\ &= (Ha[\mu], a[\mu]) + (e, b) - \mu(a[\mu], b) \\ &\qquad + (b, e) - \mu(b, a[\mu]) + (Hb, b) \\ &\geqq (Ha[\mu], a[\mu]) + \mu(b, b) + (Hb, b) \\ &\geqq (Ha[\mu], a[\mu])\end{aligned} \tag{A15}$$

where the second step follows because H is Hermitian, the third step on account of (A6), and the fourth step on account of (A10) and (A14). The equality signs in the last two steps hold if and only if $b = 0$. Hence the only possible solution to the problem of maximizing $G(a)$ under the condition $K(a) \leqq K(a[\mu])$ is given by $a = \alpha a[\mu]$, where α is a complex scalar. This proves Theorem I.

Theorem II. Since H is a positive definite Hermitian matrix, it has n positive real eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, and n linearly independent eigenvectors $v^{(1)}, v^{(2)}, \cdots, v^{(n)}$, which satisfy

$$Hv^{(r)} = \lambda_r v^{(r)}, \qquad r = 1, 2, \cdots, n \tag{A16}$$

The eigenvectors may be taken as orthonormal, so that

$$(v^{(r)}, v^{(s)}) = \delta_{rs} \tag{A17}$$

We may expand the right-hand side of equations (A6) in terms of the vectors $v^{(r)}$; thus

$$e = \sum_{r=1}^{n} E_r v^{(r)} \tag{A18}$$

where

$$E_r = (e, v^{(r)}), \qquad r = 1, 2, \cdots, n \tag{A19}$$

Assuming a similar expansion of the solution of (A6), write

$$a[\mu] = \sum_{r=1}^{n} c_r v^{(r)} \tag{A20}$$

Substituting (A18) and (A20) into (A6) and using (A16) and (A17), we find without difficulty,

$$c_r = E_r/(\lambda_r + \mu) \tag{A21}$$

We can now write down expressions for the Hermitian forms occurring in $G(a)$ and $K(a)$. Using (A18), (A20), and (A21), and the orthogonality condition (A17), we obtain

$$(a[\mu], e) = \sum_{r=1}^{n} | E_r |^2/(\lambda_r + \mu) \tag{A22}$$

$$(a[\mu], a[\mu]) = \sum_{r=1}^{n} | E_r |^2/(\lambda_r + \mu)^2 \tag{A23}$$

$$(Ha[\mu], a[\mu]) = \sum_{r=1}^{n} \lambda_r | E_r |^2/(\lambda_r + \mu)^2 \tag{A24}$$

Note that all three expressions are positive real-valued, and that

$$d(a[\mu], e)/d\mu = -(a[\mu], a[\mu]) \tag{A25}$$

From the preceding expressions and the definition (A5) of $K(a)$, we have

$$\frac{dK(a[\mu])}{d\mu} = \frac{1}{(a[\mu], e)^3}\left[(a[\mu], e)\,\frac{d(a[\mu], a[\mu])}{d\mu} + 2(a[\mu], a[\mu])^2\right] \tag{A26}$$

The first factor on the right is positive, and the terms in square brackets become

$$-2\sum_{r=1}^{n}\frac{|E_r|^2}{(\lambda_r+\mu)}\sum_{s=1}^{n}\frac{|E_s|^2}{(\lambda_s+\mu)^3} + 2\left[\sum_{r=1}^{n}\frac{|E_r|^2}{(\lambda_r+\mu)^2}\right]^2 \leqq 0 \tag{A27}$$

by Schwarz's inequality. Hence $K(a[\mu])$ is a nonincreasing function of μ.

Now let $\mu_2 > \mu_1$ and let K_1, K_2, G_1, G_2 denote $K(a[\mu_1])$, $K(a[\mu_2])$, $G(a[\mu_1])$, $G(a[\mu_2])$. By Theorem I

$$G_2 = \max_{K(a)\leqq K_2} G(a) \leqq \max_{K(a)\leqq K_1} G(a) = G_1 \tag{A28}$$

since we have just proved $K_2 \leqq K_1$. This completes the proof of Theorem II.

Theorem III. To find the minimum possible value of $K(a)$, let

$$a = e + b \tag{A29}$$

where

$$(b, e) = 0 \tag{A30}$$

The coefficient of e may be taken as unity, since if it were zero, $K(a)$ would be infinite. When a is given by (A29), (a, e) has the constant value

$$(a, e) = n \tag{A31}$$

and making use of (A30),

$$(a, a) = n + (b, b) \geqq n \tag{A32}$$

Referring back to (A5), we see that $K(a)$ achieves the minimum value $1/n$ when $a = e$; this is Theorem III.

Theorem IV. To maximize $G(a)$ when there is no restriction on $K(a)$, set $\mu = 0$ in the system of equations (A6). We do not require (A13) or (A14), but from (A15) we still get

$$\begin{aligned}(Ha, a) &= (Ha[0], a[0]) + (e, b) + (b, e) + (Hb, b)\\ &= (Ha[0], a[0]) + (Hb, b)\\ &\geqq (Ha[0], a[0])\end{aligned} \tag{A33}$$

which shows that $G(a)$ is maximum when $a = a[0]$, and thus proves Theorem IV.

Theorem V. Setting $\mu = 0$ in (A23) and (A24) and using (A19), we obtain from (A4) an expression for the maximum gain in any direction **u**, namely

$$\Gamma(\mathbf{u}) = \sum_{r=1}^{n} \lambda_r^{-1} \mid E_r \mid^2 = \sum_{r=1}^{n} \sum_{s=1}^{n} \sum_{t=1}^{n} \lambda_r^{-1} e_s v_s^{(r)*} e_t^* v_t^{(r)} \qquad \text{(A34)}$$

The dependence on **u** appears in the components of e, where

$$e_k = \exp(-i\beta \mathbf{r}_k \cdot \mathbf{u}), \qquad k = 1, 2, \cdots, n \qquad \text{(A35)}$$

Averaging over all directions in space gives

$$\frac{1}{4\pi} \int_S e_s e_t^* \, d\Omega = \frac{\sin \beta r_{st}}{\beta r_{st}} = h_{st} \qquad \text{(A36)}$$

the identification with the matrix element h_{st} being made from equation (29) on the assumption that the elements of the array are isotropic and the conventional definition of gain is used. It follows now from (A16) and (A17) that

$$\begin{aligned} \frac{1}{4\pi} \int_S \Gamma(\mathbf{u}) \, d\Omega &= \sum_{r=1}^{n} \sum_{s=1}^{n} \sum_{t=1}^{n} \lambda_r^{-1} v_s^{(r)*} h_{st} v_t^{(r)} \\ &= \sum_{r=1}^{n} \sum_{s=1}^{n} \lambda_r^{-1} \delta_{rs} \lambda_s = n \end{aligned} \qquad \text{(A37)}$$

which proves Theorem V.