Interchannel Interference in FM and PM Systems Under Noise Loading Conditions

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One of the principal sources of interchannel interference in multichannel FM and PM systems is the dependence of the attenuation and phase shift of the transmission path on frequency. Here we study the interference produced by a simple kind of such a dependence, namely that due to a single echo. Besides being important in themselves, the results are of interest because they may be used to estimate the interference in other, more complicated cases.

In this paper expressions are derived for the interchannel interference produced by echoes of relatively small amplitude. Several important special cases are studied in detail and curves that simplify the computation of the interference power are presented.

CONTENTS

INTRODUCTION

The systems we shall consider are of the FDM-FM and FDM-PM types; that is, systems in which the composite signal wave (the "base-band signal") from a group of carrier telephone channels in frequency division multiplex (FDM) is transmitted by frequency modulation (FM) or phase modulation (PM). Such methods are currently being used in the Bell System to send large groups of telephone channels by microwave radio. If the FM signal is accompanied by echoes, which may be due to reflections in the equipment or in the transmission medium, the wave of instantaneous frequency versus time is distorted in a nonlinear manner and interchannel interference occurs. Here we shall be concerned with this interference.

The distortion has been analyzed in a number of previous publications[1-4] for the case in which the base-band signal may be represented by one or more sine waves. However, when the number of telephone channels is not small, the sine wave representation becomes unwieldy because a large number of both low and high order modulation products must be considered. Here we avoid this difficulty, at the cost of somewhat more complex analysis, by using a band of random noise to represent the multiplex signal.

It has been found in practice that such a random noise signal of appropriate bandwidth and power adequately simulates a composite speech signal. For studies involving interchannel interference the energy corresponding to some particular telephone channel is removed. When such a wave is impressed on the frequency modulator and the resulting FM wave is transmitted, detected, and finally demodulated, the received output in the originally clear channel represents interchannel interference. Measurements of this type have been discussed by Albersheim and Schafer.[3]

One of the principal sources of interchannel interference is the variation of the attenuation and phase shift of the transmission path with frequency. An echo produces a simple form of such a variation. In fact, it is often possible to estimate the effect produced by a more complex type of variation by comparing it with a roughly equivalent echo.

It is the purpose of this paper to develop formulas for the interchannel interference produced by echoes of relatively small amplitude in FM and PM systems. General expressions are given in the form of definite integrals which may be evaluated by numerical integration. Approximations are obtained in a number of cases of importance and numerical tables and curves are furnished to facilitate applications to specific problems.

We wish to express our thanks to Miss Mary Corr and Miss Barbara Fischer who have performed the rather lengthy computations required for this investigation.

1. DEVELOPMENT OF GENERAL FORMULAS

Since the tolerable amount of interchannel interference in multiplex telephony is small, the practically important situations are characterized by echoes which are relatively small relative to the main transmitted wave. It is necessary to consider in detail only the case of one small echo. Effects of small multiple echoes may then be calculated by superposition of the effects of single echoes. Our distortion problem then reduces to the following: An original signal

$$E_0(t) = E \sin [pt + \varphi(t)] \tag{1.1}$$

is received along with an echo

$$E_1(t) = rE \sin [p(t - T) + \varphi(t - T)] \tag{1.2}$$

Here r represents the ratio of echo amplitude to the amplitude of the principal received component, p is the unmodulated carrier frequency, T is the delay difference of the paths and $\varphi(t)$ is the phase variation caused by the multichannel signal.

Calculation of the phase of the original signal plus its echo is equivalent to calculating the value of θ in the representation

$$\sin x + r \sin (x + y) = V \sin (x + \theta) \tag{1.3}$$

with x, y, r, V, and θ real numbers. V and θ are determined as functions of r and y by the two equations obtained from the $\sin x$ and $\cos x$ portions of (1.3). When $r \ll 1$, V approaches unity, and θ is proportional to r; direct expansion in powers of θ shows that to a first order approximation

$$\theta = r \sin y \tag{1.4}$$

In our case,

$$x = pt + \varphi(t) \tag{1.5}$$

$$y = \varphi(t - T) - \varphi(t) - pT \tag{1.6}$$

Hence the phase error produced by the echo is

$$\theta(t) = r \sin [v(t) - pT], \tag{1.7}$$

$$v(t) = \varphi(t - T) - \varphi(t). \tag{1.8}$$

Our problem is to study the power spectrum of $\theta(t)$ when $\varphi(t)$ is a random noise wave.

The treatment will be restricted to the practically important case in which the noise source is free from dc and discrete sinusoidal components. A sufficient mathematical condition, which we shall adopt to insure this, is that the power spectrum of the noise have limited total fluctuation. We shall assume also that the noise wave is a member of a Gaussian ensemble and hence that all the statistical properties are derivable from either its power spectrum or autocorrelation function. Which of the two quantities we use at any stage of the computations is a matter of relative expediency. We shall adopt a uniform notation in that an ensemble of time functions $x(t)$ will be said to have a power spectrum $w_x(f)$ and an autocorrelation function $R_x(\tau)$, i.e.,

$$R_x(\tau) = \text{ave}\,[x(t)x(t + \tau)] = \int_0^\infty w_x(f) \cos 2\pi\tau f \, df \tag{1.9}$$

$$w_x(f) = 4 \int_0^\infty R_x(\tau) \cos 2\pi f\tau \, d\tau \tag{1.10}$$

The power spectrum is proportional to the mean square of the response of a filter of bandwidth df centered at f when $x(t)$ is applied as input.

In the echo distortion problem, the power spectrum $w_\varphi(f)$ is the quantity given and the power spectrum $w_\theta(f)$ is the one desired. The corresponding autocorrelation functions $R_\varphi(\tau)$ and $R_\theta(\tau)$ are calculable from the corresponding power spectra and vice versa. The power spectrum is the more convenient choice as a function to compute when we deal with the noise response of a linear system with known transfer admittance $Y(if)$, since the power spectrum of the response is merely $|\, Y(if) \,|^2$ times that of the input. For example, differentiation is equivalent to transmission through a transfer admittance $i2\pi f$ and hence multiplies the power spectrum by $4\pi^2 f^2$. The inverse process of integration divides the power spectrum by $4\pi^2 f^2$. When a highly nonlinear operation is performed on the noise, or when the operation is a linear one more simply described in the time domain, the autocorrelation function may be simpler to compute.

The first step in the solution of the problem is to evaluate the statistics of the noise wave ensemble,

$$\{v(t)\} = \{\varphi(t - T) - \varphi(t)\} \tag{1.8}$$

in terms of the statistics of $\varphi(t)$. This can readily be done in terms of power spectra by calculating the transfer admittance of a two-path transmission system such as the right-hand member of (1.8) defines, or in terms of autocorrelation functions by averaging the appropriate time functions. The latter procedure gives:

$$\begin{aligned} R_v(\tau) &= \text{ave}\,[v(t)\, v(t + \tau)] \\ &= \text{ave}\,\{[\varphi(t - T) - \varphi(t)][\varphi(t - T + \tau) - \varphi(t + \tau)]\} \\ &= \text{ave}\,[\varphi(t - T)\, \varphi(t - T + \tau)] + \text{ave}\,[\varphi(t)\, \varphi(t + \tau)] \\ &\qquad - \text{ave}\,[\varphi(t - T)\, \varphi(t - T + \tau + T)] \\ &\qquad - \text{ave}\,[\varphi(t)\, \varphi(t + \tau - T)] \\ &= 2R_\varphi(\tau) - R_\varphi(\tau + T) - R_\varphi(\tau - T) \end{aligned} \tag{1.11}$$

In the averaging process we have made use of the facts that the average of the sum is equal to the sum of the averages and that $\varphi(t)$ is a stationary ensemble, allowing us to replace $t - T$ by t when averaging. Since the two-path transmission system is linear and invariable, the $v(t)$ ensemble remains Gaussian. The corresponding power spectra relations are, from (1.10),

$$w_v(f) = 4\, w_\varphi(f) \sin^2 \pi f T \tag{1.12}$$

Our next step called for by (1.7) requires the evaluation of the auto-correlation function of a sine function of a band of noise. The solution may be obtained* from (3.2 — 7) of Reference 5, which gives a general theorem for a Gaussian noise ensemble expressible in our notation as

$$\begin{aligned} &\text{ave}\,\{\exp\,[iax(t) + ibx(t+\tau)]\} \\ &\qquad = \exp\,[-\frac{a^2+b^2}{2} R_x(0) - abR_x(\tau)] \end{aligned} \tag{1.13}$$

In this equation a and b are real constants. Multiplying both sides by the constant exp $(i\beta)$, β real, and equating real parts gives the more directly applicable result:

$$\begin{aligned} &\text{ave}\,\{\cos\,[ax(t) + bx(t+\tau) + \beta]\} \\ &\qquad = \exp\left[-\frac{a^2+b^2}{2} R_x(0) - abR_x(\tau)\right]\cos\beta \end{aligned} \tag{1.14}$$

Referring now to (1.7) we write

$$\begin{aligned} R_\theta(\tau) &= \text{ave}\,\{r \sin\,[v(t) - pT]\, r \sin\,[v(t+\tau) - pT]\} \\ &= \frac{r^2}{2}\,\text{ave}\,\{\cos\,[v(t) - v(t+\tau)]\} \\ &\qquad - \frac{r^2}{2}\,\text{ave}\,\{\cos\,[v(t) + v(t+\tau) - 2pT]\} \\ &= \frac{r^2}{2}\exp\,[-R_v(0) + R_v(\tau)] \\ &\qquad - \frac{r^2}{2}\exp\,[-R_v(0) - R_v(\tau)]\cos 2pT \\ &= \frac{r^2}{2}\, e^{-R_v(0)}[e^{R_v(\tau)} - e^{-R_v(\tau)}\cos 2pT] \end{aligned} \tag{1.15}$$

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* This application of the characteristic functions method of attack has been employed in similar situations by D. Middleton in Reference 6 and M. K. Zinn in unpublished memoranda.

Note that since the operation of taking the sine is nonlinear, the $\theta(t)$ ensemble is not Gaussian and neither the power spectrum nor the autocorrelation is sufficient to give a complete statistical description. Either will be sufficient for our purposes however.

The power spectrum of the distortion ensemble $\theta(t)$ is now determinable from the Fourier cosine transform of 4 $R_\theta(\tau)$ as indicated by (1.10). The remaining steps are simplified, however, if we perform two preliminary operations on $R_\theta(\tau)$ before the final Fourier transform is calculated.

We first observe that if $R_\theta(\tau)$ contains a component C which does not vary with τ, the ensemble $\theta(t)$ contains a dc component $\bar{\theta} = \sqrt{C}$. The presence of C complicates the integration and hence we subtract out such a term before taking the Fourier transform. The value of C is given by

$$\begin{aligned} C &= \lim_{\tau \to \infty} R_\theta(\tau) \\ &= \frac{r^2}{2} e^{-R_v(0)} \lim_{\tau \to \infty} [e^{R_v(\tau)} - e^{-R_v(\tau)} \cos 2pT] \\ &= \frac{r^2}{2} e^{-R_v(0)} (1 - \cos 2pT) \end{aligned} \tag{1.16}$$

In calculating the limit we have made use of the fact that, for our assumed noise wave, $R_v(\tau)$ must approach zero as τ becomes infinite. Our autocorrelation function of interest is thereby reduced to

$$R_{\theta-\bar{\theta}}(\tau) = \frac{r^2}{2} e^{-R_v(0)} \{e^{R_v(\tau)} - 1 - [e^{-R_v(\tau)} - 1] \cos 2pT\} \tag{1.17}$$

which is the autocorrelation function of the ensemble $\theta(t) - \bar{\theta}$.

Our second preliminary operation on $R_\theta(\tau)$ is suggested by our ultimate goal, namely the calculation of the interchannel interference spectrum $w_c(f)$ (i.e., $w_c(f)\, df$ is the average power received in an idle channel of width df when the other channels are carrying signals). We note that the spreading of the original spectrum into initially vacant frequency ranges occurs solely because of the nonlinear dependence of $R_{\theta-\bar{\theta}}(\tau)$ on $R_v(\tau)$. The disturbance received in an idle channel is therefore produced by this nonlinearity. The part of $R_{\theta-\bar{\theta}}(\tau)$ which varies linearly with $R_v(\tau)$ may be expected to represent the linear transmission, and the difference between $R_{\theta-\bar{\theta}}(\tau)$ and its linear portion to represent the nonlinear transmission. In other words, subtracting the linear portion from $R_{\theta-\bar{\theta}}(\tau)$ is equivalent to removing the linear transmission from any channel without disturbing the nonlinear contributions from the remaining channels. These considerations lead us to set the autocorrelation function $R_c(\tau)$ corresponding to $w_c(f)$ equal to $R_{\theta-\bar{\theta}}(\tau)$ minus its linear portion. The work of Appendix I shows that this equality is rigorously true.

In order to perform the subtraction it is convenient to write (1.17) in the form

$$R_{\theta-\bar{\theta}}(\tau) = F[R_v(\tau)] \tag{1.18}$$

where in our case for a general variable z,

$$F(z) = \frac{r^2}{2} e^{-R_v(0)}[e^z - 1 - (e^{-z} - 1) \cos 2pT] \tag{1.19}$$

The portion of $R_{\theta-\bar{\theta}}(\tau)$ which varies linearly with $R_v(\tau)$ is $F'(0)\, R_v(\tau)$ where

$$F'(z) = \frac{r^2}{2} e^{-R_v(0)}(e^z + e^{-z} \cos 2pT) \tag{1.20}$$

$$F'(0) = \frac{r^2}{2} e^{-R_v(0)}(1 + \cos 2pT) \tag{1.21}$$

According to the foregoing discussion and Appendix I, the autocorrelation function and the power spectrum of the interchannel interference are given by

$$\begin{aligned} R_c(\tau) &= R_{\theta-\bar{\theta}}(\tau) - F'(0)R_v(\tau) \\ &= \frac{r^2}{2} e^{-R_v(0)} \{e^{R_v(\tau)} - 1 - R_v(\tau) \\ &\qquad - [e^{-R_v(\tau)} - 1 + R_v(\tau)] \cos 2pT\} \end{aligned} \tag{1.22}$$

$$w_c(f) = 4 \int_0^\infty R_c(\tau) \cos 2\pi f\tau \, d\tau \tag{1.23}$$

The power spectrum of $\theta(t) - \bar{\theta}$ may be simply expressed in terms of $w_c(f)$ without further integration. We find by applying the Fourier transform relationship of (1.9) and (1.10) to (1.22):

$$\begin{aligned} w_{\theta-\bar{\theta}}(f) &= 4 \int_0^\infty [R_c(\tau) + F'(0)R_v(\tau)] \cos 2\pi f\tau \, d\tau \\ &= w_c(f) + F'(0)w_v(f) \\ &= w_c(f) + 4F'(0)w_\varphi(f) \sin^2 \pi fT \\ &= w_c(f) + 2r^2e^{-R_v(0)}(1 + \cos 2pT)w_\varphi(f) \sin^2 \pi fT \end{aligned} \tag{1.24}$$

where we have used (1.12) and (1.21). The power spectrum $w_\theta(f)$ of θ is obtained by adding to (1.24) a spire at $f = 0$ to represent the power in $\bar{\theta}$, the dc portion of θ.

The results obtained up to this point may be summarized as follows: Let $\varphi(t)$ be the phase variation produced by the impressed multichannel signal. The echo (1.2) produces a phase error $\theta(t)$ (1.7) in the received signal. $\theta(t)$ has a dc component $\bar{\theta}$ given by the square root of (1.16). $\theta(t) - \bar{\theta}$ is a function of time which fluctuates about zero and has the power spectrum $w_{\theta-\bar{\theta}}(f)$ given by (1.24). Now consider the problem of computing the interchannel interference. One procedure would be to consider the case in which all but one of the channels are loaded. Then the power spectrum $w_\varphi(f)$ of $\varphi(t)$ would have a narrow slot in it corresponding to the zero power in the unloaded channel. The values of $w_{\theta-\bar{\theta}}(f)$ computed from this $w_\varphi(f)$ for values of f within the slot would give the channel interference spectrum (for phase modulation). However, it turns out that as the slot width approaches zero, these values of $w_{\theta-\bar{\theta}}(f)$ approach those of $w_c(f)$ where $w_c(f)$ is computed from (1.23) on the assumption that all channels are loaded. In other words the $w_\varphi(f)$ used in the calculation of (1.23) has no slot. (Actually the same value of $w_c(f)$ is obtained whether $w_\varphi(f)$ has a slot or not. Of course, the slot must be vanishingly narrow).

Almost all of the preceding work pertains to the power spectrum of the phase error $\theta(t)$. For the sake of completeness we shall give the power spectrum of the complete phase angle, $Q(t) = \varphi(t) + \theta(t)$, of the output. In order to obtain all of the $0(r^2)$ term it is necessary to add another term to the approximation (1.7):

$$\theta(t) = r \sin [v(t) - pT] - \frac{r^2}{2} \sin [2v(t) - 2pT] + 0(r^3) \quad (1.25)$$

The autocorrelation function for $Q(t)$ may be shown to be

$$\begin{aligned} R_Q(\tau) &= R_\varphi(\tau) + R_\theta(\tau) + \text{ave } [\varphi(t)\, \theta(t + \tau) + \varphi(t + \tau)\, \theta(t)] \\ &= R_\varphi(\tau) + R_\theta(\tau) - rR_v(\tau)\, e^{-R_v(0)/2} \cos pT \\ &\quad + r^2 R_v(\tau)\, e^{-2R_v(0)} \cos 2pT + 0(r^3) \end{aligned} \quad (1.26)$$

The average has been evaluated by using

$$\begin{aligned} &\text{ave } \{\varphi(t \pm \tau) \sin [\varphi(t - T) - \varphi(t) - pT]\} \\ &\quad = [R_\varphi(T \pm \tau) - R_\varphi(\tau)] \cos pT \exp [R_\varphi(T) - R_\varphi(0)] \end{aligned} \quad (1.27)$$

and the corresponding result for 2φ and $2pT$ ($R_{2\varphi}(\tau) = 4R_\varphi(\tau)$). Equation (1.27) may be obtained from the result similar to (1.13) which contains the three variables $\varphi(t)$, $\varphi(t \pm \tau)$, $\varphi(t - T)$ instead of $x(t)$ and $x(t + \tau)$.

The power spectrum $w_Q(f)$ of $Q(t)$ may be obtained from (1.26) by replacing the autocorrelation functions in the coefficients by the corresponding power spectra. By using (1.24) we may obtain the following expression for the power spectrum of the fluctuating portion $Q(t) - \bar{Q}$ of $Q(t)$, $\bar{Q} = \bar{\theta}$ being the dc portion of $Q(t)$:

$$w_{Q-\bar{Q}}(f) = w_\varphi(f) + w_c(f) + w_v(f)[-r\, e^{-R_v(0)/2} \cos pT + r^2\, e^{-R_v(0)} \cos^2 pT + r^2\, e^{-2R_v(0)} \cos 2pT] \qquad (1.28)$$

The expression (1.12) for $w_v(f)$ shows that it is proportional to $w_\varphi(f)$. Hence, assuming $w_\varphi(f)$ to have a narrow slot corresponding to an idle channel, it is seen that (1.28) is in agreement with the fact that only $w_c(f)$ contributes to the interchannel interference spectrum.

We remark that for echoes produced by reflections in wave guides the angle $2pT$ in (1.22) and (1.24) usually contains many multiples of 2π. In most cases it would in fact be reasonable to average the effect of the term $\cos 2pT$ by assuming the angle to be uniformly distributed throughout 2π radians. This gives the result zero for the term since plus and minus values are symmetrically distributed. We shall, however, carry the term along in our formulas since it is of importance when the delays are small, as in multipath radio propagation, and when our results are used to estimate interchannel interference in general.

2. APPLICATIONS TO FLAT NOISE SIGNAL

In general $w_\varphi(f)$ depends on the type of preemphasis used in the channel multiplex signal. Two representative conditions will be studied: (1) phase modulation (PM) in which channels at equal level are impressed on a phase modulator, or the more usual equivalent situation of equal level channels which are differentiated before being impressed on a frequency modulator, and (2) frequency modulation (FM) in which channels at equal level are impressed on a frequency modulator. The appropriate power spectra are:

$$\text{PM:} \quad w_\varphi(f) = P_0, \qquad f_a < f < f_b \qquad (2.1)$$

$$\text{FM:} \quad w_\varphi(f) = P_0/4\pi^2 f^2, \qquad f_a < f < f_b \qquad (2.2)$$

The latter form results because the phase is the time integral of the instantaneous frequency, which has a flat spectrum. In PM, P_0 is expressed in (radians)^2/cps, but in FM, P_0 is expressed in (radians/sec.)^2/cps. In the case of most common practical interest, the band of frequencies between 0 and f_a is relatively narrow compared to the range $f_b - f_a$. We have accordingly assumed that f_a approaches zero. In the FM case we cannot set $f_a = 0$ immediately because the power spectrum would then become unbounded at the origin. It is found, however, that finite limits are approached for the actual quantities of interest which we compute.

When the spectrum $w_\varphi(f)$ of the signal has the form (2.1) or (2.2), expression (1.23) for the interchannel interference spectrum may be written as

$$w_c(f) = r^2(2\pi f_b)^{-1}(G - H \cos 2pT) \tag{2.3}$$

where $w_c(f)\, df$ is measured in (radians)^2 and

$$G = 2e^{-R_v(0)} \int_0^\infty [e^{R_v(\tau)} - R_v(\tau) - 1] \cos au\, du \tag{2.4}$$

$$H = 2e^{-R_v(0)} \int_0^\infty [e^{-R_v(\tau)} + R_v(\tau) - 1] \cos au\, du \tag{2.5}$$

$$u = 2\pi f_b \tau \qquad U = 2\pi f_b T \qquad a = f/f_b \tag{2.6}$$

For PM:

$$R_v(\tau) = P_0 f_b \left[\frac{2 \sin u}{u} - \frac{\sin (u + U)}{u + U} - \frac{\sin (u - U)}{u - U}\right] \tag{2.7}$$

where $P_0 f_b$ is the mean square value, in (radians)^2, of the PM signal $\varphi(t)$.

For FM:

$$R_v(\tau) = A[-2(1 - \cos U)\cos u - 2uSi(u) + (u + U)Si(u + U) + (u - U)Si(u - U)] \tag{2.8}$$

$$A = (P_0 f_b)/(2\pi f_b)^2$$

In (2.8) $P_0 f_b$ is the mean square value (measured in (radians per second)^2) of the FM signal $\varphi'(t)$. Consequently, $A = (\sigma/f_b)^2$ where σ is the rms frequency deviation, in cycles per second, of the signal. Equations (2.7) and (2.8) follow from (2.1), (2.2) and (1.11).

When $R_v(0)$ is small compared to unity, the same is true of $R_v(\tau)$, and (2.4) and (2.5) lead to the approximations

$$G \approx H \approx \int_0^\infty R_v^2(\tau) \cos au\, du \tag{2.9}$$

The interchannel interference calculated from (2.9) is that produced by the "second order modulation products" and is studied, together with other approximations, in Sections 3 and 4.

The quantity of interest in practice is the ratio of average interference power in the idle channel to average signal power in an adjacent channel. For PM the average interference and signal powers (in a narrow channel centered on frequency f) are the channel bandwidth times $w_c(f)$ and $w_\varphi(f)$, respectively. When the number of channels is large, the power spectrum does not change appreciably in going from one channel to the next and we may write

$$\frac{P_I}{P_S} = \frac{w_c(f)}{w_\varphi(f)} = \frac{r^2}{2\pi P_0 f_b}(G - H \cos 2pT) \tag{2.10}$$

as the desired interference-to-signal power ratio.

For FM the average interference and signal powers in a narrow channel are $(2\pi f)^2 \times$ (channel bandwidth) times $w_c(f)$ and $w_\varphi(f)$, respectively. These powers are measured in (radians per second)^2. When we take the ratio P_I/P_S, the $(2\pi f)^2 \times$ (channel bandwidth) cancels out and we have from (2.2) and (2.3)

$$\frac{P_I}{P_S} = \frac{r^2 a^2}{2\pi A}(G - H \cos 2pT) \tag{2.11}$$

3. APPROXIMATIONS FOR G AND H — PHASE MODULATION

Table 3.1 contains various approximate expressions for the quantities G and H which enter expression (2.3) for the channel interference spectrum $w_c(f)$. The notation is explained in Section 2, and the results apply to the case in which $w_\varphi(f)$ has the flat power spectrum (2.1).

Case 1 gives the exact expressions developed in Section 2. Case 2 is the "second order modulation approximation", valid when $R_v(0) \ll 1$. Evaluation of the integral (2.9) for G with the help of

$$\int_{-\infty}^{\infty} \frac{\sin u \sin (u - U)}{u(u - U)} \cos au \, du = \begin{cases} \pi U^{-1} \cos\left(\frac{aU}{2}\right) \sin\left(U - \frac{aU}{2}\right), & 0 \leqq a \leqq 2 \\ 0, & 2 \leqq a \end{cases}$$

leads to the following expression for the quantity J appearing in Table 3.1:

TABLE 3.1 — PHASE MODULATION

Case No.	Restrictions on Parameters U and $P_0 f_b$	G and H	Notes
1	No restrictions, $w_\varphi(f)$ defined by (2.1)	G defined by (2.4) H defined by (2.5) $R_v(\tau)$ defined by (2.7)	$P_0 f_b = \text{ave } [\varphi^2(t)]$ $\frac{P_I}{P_s} = \frac{r^2}{2\pi P_0 f_b}(G - H \cos 2pT)$, $U = 2\pi f_b T$, $a = f/f_b$
2	$2P_0 f_b \left(1 - \frac{\sin U}{U}\right) \ll 1$	$G \approx H \approx 2\pi(P_0 f_b)^2 J$, J defined by (3.1)	"2nd Order Modulation" approx., J tabulated in Table 3.2, 10 $\log_{10} J$ plotted in Fig. 5.1
3	$P_0 f_b U^2/3 \ll 1$, $U \ll 1$	$G \approx H \approx 2\pi(P_0 f_b)^2 \frac{U^4}{240} \cdot [12 - 30a + 20a^2 - a^5]$	Special case of Case 2
4	$2P_0 f_b \ll 1$, $U \gg 1$	$G \approx H \approx 2\pi(P_0 f_b)^2 \cdot \left(1 - \frac{a}{2}\right)\left(1 + \frac{\cos aU}{2}\right)$	Special case of Case 2 G is a rapidly oscillating function of a
5	$P_0 f_b U^2/3 \gg 1$, $U \ll 1$	$G \gg H$, $G \approx (10\pi/P_0 f_b U^2)^{1/2} \exp(-10a^2/4P_0 f_b U^2)$	When $U \ll 1$ Case 3 applies when $P_0 f_b$ is small Case 5 applies when $P_0 f_b$ is large
6	$U \gg 1$	$G \approx e^{-b_0}[I(b_0, a) + 2I(b_1, a) \cos aU]$, $H \approx e^{-b_0}[I(-b_0, a) + 2I(-b_1, a) \cos aU]$, $b_0 = 2P_0 f_b$, $b_1 = -P_0 f_b$ $I(b, a)$ studied in Appendix III	When $P_0 f_b \ll 1$, Case 6 reduces to Case 4. When $P_0 f_b \gg 1$, $G \gg H$. As a increases, G oscillates between G^+ and G^- given in Table 3.3. See Table 5.1

$$J = \left(1 - \frac{a}{2}\right)\left(1 + \frac{\cos aU}{2}\right) + \frac{\sin(2U - aU)}{4U} - \frac{2}{U} \cos\left(\frac{aU}{2}\right) \sin\left(U - \frac{aU}{2}\right) \tag{3.1}$$

Values of J are tabulated in Table 3.2 for various values of a and k where $U = k\pi/4$ (or $T = k/(8f_b)$). J is zero when a exceeds 2.

Cases 3 and 4 in Table 3.1 follow directly from Case 2. In order to

TABLE 3.2 — VALUES OF J

k	$a = 0$	0.25	0.50	0.75	1.00	1.25
0	0	0	0	0	0	0
1	0.018	0.009	0.003	0.001	0.002	0.004
2	0.227	0.115	0.041	0.011	0.023	0.057
3	0.794	0.434	0.161	0.049	0.098	0.231
4	1.500	0.903	0.352	0.123	0.250	0.524
5	1.924	1.284	0.527	0.224	0.458	0.816
6	1.924	1.385	0.585	0.332	0.659	0.936
7	1.712	1.231	0.505	0.427	0.773	0.800
8	1.500	0.994	0.375	0.506	0.750	0.494
9	1.335	0.800	0.325	0.579	0.602	0.228
10	1.245	0.658	0.425	0.657	0.405	0.180
11	1.307	0.544	0.643	0.725	0.262	0.358
12	1.500	0.472	0.883	0.752	0.250	0.601

obtain Case 5 we note that when $U \ll 1$, expression (2.7) gives

$$R_v(\tau) \approx -U^2 P_0 f_b \frac{d^2}{du^2} \frac{\sin u}{u}. \tag{3.2}$$

The corresponding integrals for G and H could, if required, be investigated by the methods used to study Lewin's integral in Appendix III. However here we consider only the case where $U^2P_0f_b$ is so large that (1) $\exp R_v(\tau)$ is the dominant term in the integral (2.4) for G, and (2) most of the contribution to the value of the integral comes from the region around $u = 0$. The results for Case 5 then follow from (2.4) and the fact that (3.2) becomes

$$R_v(\tau) \approx U^2 P_0 f_b \left(\frac{1}{3} - \frac{u^2}{10}\right)$$

When $U \gg 1$, expression (2.7) shows that $R_v(\tau)$ is small except when u is near 0 or near U:

$$\begin{aligned} R_v(\tau) &\approx 2\, P_0 f_b\, u^{-1} \sin u, && u \text{ near } 0 \\ R_v(\tau) &\approx -P_0 f_b (u - U)^{-1} \sin (u - U), && u \text{ near } U \end{aligned} \tag{3.3}$$

These approximations, expression (2.4) for G, and the definition (A3–1) of $I(b, a)$ give the results stated in Case 6. Just as in Case 4, G is a rapidly oscillating function of a when U is large. It oscillates between G^+ and G^- where

$$G^{\pm} = e^{-b_0}[I(b_0, a) \pm 2I(b_1, a)] \tag{3.4}$$

Values of G^+ and G^- are given in Table 3.3 for various values of $a = j/f_b$ and $P_0 f_b$ [in (radians)^2].

TABLE 3.3 — VALUES OF G^+ AND G^-. THE UPPER NUMBER OF AN ENTRY IS G^+ AND THE LOWER NUMBER IS G^-.

$P_0 f_b$ (radians)^2	$a = 0$	0.25	0.50	0.75	1.00	1.25
0.25	0.384 0.160	0.338 0.144	0.292 0.126	0.245 0.107	0.197 0.087	0.148 0.066
0.50	1.018 0.504	0.906 0.464	0.789 0.415	0.665 0.357	0.537 0.291	0.406 0.222
0.75	1.55 0.89	1.39 0.83	1.22 0.75	1.04 0.65	0.845 0.533	0.645 0.411
1.00	1.90 1.22	1.73 1.15	1.53 1.05	1.32 0.92	1.07 0.76	0.830 0.596
2.00	2.16 1.86	2.04 1.80	1.88 1.68	1.68 1.52	1.43 1.31	1.17 1.07
4.00	1.59 1.57	1.56 1.54	1.50 1.48	1.40 1.40	1.28 1.28	1.14 1.14

4. APPROXIMATIONS FOR G AND H — FREQUENCY MODULATION

The various cases which we shall consider for FM are roughly similar to those considered in Section 3 for PM, and are listed in Table 4.1. The power spectrum $w_\varphi(f)$ is assumed to be that given by (2.2). As pointed out in Section 2, the average FM signal power in a narrow channel of width Δf centered on frequency f is assumed to be $P_0 \Delta f$ (radians/second)^2 if $0 < f < f_b$ and zero if $f_b < f$. The average interchannel interference power is $(2\pi f)^2\, w_c(f) \Delta f$ (radians/second)^2. When $f > f_b$ this gives all of the power present in the frequency interval Δf.

Case 1, Table 4.1, gives the exact expressions for G and H, and Case 2 corresponds to the "second order modulation" approximation which holds when $R_v(0) \ll 1$, $R_v(0)$ being computed from (2.8). K is a function of $a = f/f_b$ and $U = 2\pi f_b T$ which may be obtained by writing the integral (2.9) as

$$G \approx \pi f_b \int_{-\infty}^{\infty} R_v^{\,2}(\tau) e^{2\pi i f \tau}\, d\tau = 4^{-1} \pi f_b \int_{-\infty}^{\infty} w_v(x) w_v(f - x)\, dx \quad (4.1)$$

where $R_v(\tau)$ and $w_v(f)$ are taken to be even functions. From the definitions (1.12) and (2.2) for $w_v(f)$ and $w_\varphi(f)$ we have

$$w_v(f) = \begin{cases} P_0 (\pi f)^{-2} \sin^2 \pi f T, & |f| < f_b \\ 0, & |f| > f_b \end{cases} \quad (4.2)$$

TABLE 4.1 — FREQUENCY MODULATION

Case No.	Restrictions on Parameters U and A	G and H	Notes
1	No restriction, $w_\varphi(f)$ defined by (2.2)	G defined by (2.4) H defined by (2.5) $R_v(\tau)$ defined by (2.8)	$A = (\sigma/f_b)^2$, σ = rms $[d\varphi(t)/dt]/2\pi$, σ = rms frequency deviation of signal measured in cycles/sec. $P_I/P_S = [r^2a^2/(2\pi A)] \cdot [G - H \cos 2pT]$
2	$2A[USi(U) - 1 + \cos U] \ll 1$	$G \approx H \approx 2\pi A^2 a^{-2} UK$ K defined by (4.5) and (4.6)	"2nd Order Modulation" approx. K tabulated in Table 4.2. See Fig. 5.2
3	$AU^2 \ll 1$, $U \ll 1$.	$G \approx H \approx \pi A^2 U^4 (2 - a)/4$, $0 \leqq a \leqq 2$	Special case of Case 2 $UK \approx a^2 U^4 (2 - a)/8$
4	$AU\pi \ll 1$, $U \gg 1$	$G \approx H \approx \begin{cases} 2\pi^2 A^2 U a^{-2}, & 0 < a < 1 \\ \pi^2 A^2 U, & a = 1 \\ 0, & a > 1 \end{cases}$	Special case of Case 2
5	$U \ll 1$, Lewin's case	$G \approx e^{-b} I(b, a)$ $H \approx e^{-b} I(-b, a)$ $b = AU^2$	Case 5 agrees with Case 3 when $b \ll 1$. $G \gg H$ for $b \gg 1$. $I(b, a)$ defined and tabulated in Appendix III. See Fig. 5.4.
6	$4A \gg 1$, $U \gg 1$	$G \approx [\pi y_1/(A \sinh y_1)]^{1/2} \exp[-2A(\cosh y_1 - 1)]$ y_1 defined by $\frac{a}{2A} = \int_0^{y_1} \frac{\sinh v}{v} dv$	$G \gg H$ Value of G is independent of U. See Fig. 5.3.
7	$B \gg 1$, $U \approx 1$, $B = A[1 - U^{-1} \sin U]$	$G \approx [\pi/B]^{1/2} \exp[-a^2/(4B)]$	$G \gg H$

where the lower limit f_a of the frequency band is taken to be very close to zero. When $f > 2f_b$ the value of (4.1) is zero. When we take $0 < f < 2f_b$ the limits of integration in (4.1) are $x = f - f_b$ and $x = f_b$. Changing the variable of integration in (4.1) from x to $y = 2\pi xT$ converts (4.1) into

$$4\pi A^2U^3 \int_{\alpha-U}^{U} y^{-2}(\alpha - y)^{-2} \sin^2 \frac{y}{2} \sin^2 \left(\frac{\alpha - y}{2}\right) dy \tag{4.3}$$

where $\alpha = 2\pi fT = aU$. By partial fractions

$$\alpha^2 y^{-2}(\alpha - y)^{-2} = y^{-2} + 2\alpha^{-1}y^{-1} + (\alpha - y)^{-2} + 2\alpha^{-1}(\alpha - y)^{-1} \tag{4.4}$$

Considerations of symmetry show that the $\alpha - y$ terms on the right contribute the same amount to (4.3) as do the y terms. When the y^{-2} term is converted into y^{-1} by an integration by parts, (4.3) may be expressed in terms of $Si(x)$ and $Ci(x)$ functions. In this way it may be shown that the approximation (4.3) for G and H has the value

$$2\pi A^2U^3K/\alpha^2 = 2\pi\, A^2U\, K/a^2$$

where

$$\begin{aligned} K = {} & (-U^{-1} + \beta^{-1})(1 - \cos U)(1 - \cos \beta) \\ & + (Si\, U - Si\, \beta)(1 + \cos \alpha - 2\alpha^{-1} \sin \alpha) \\ & + (Si\, 2U - Si\, 2\beta)(-\cos \alpha + \alpha^{-1} \sin \alpha) \\ & + (Ci\, 2U - Ci\, 2\,|\,\beta\,| - Ci\, U + Ci\,|\,\beta\,|)(\sin \alpha + \alpha^{-1} \cos \alpha) \\ & + \alpha^{-1}(2 + \cos \alpha)[\log_e (U/|\,\beta\,|) - Ci\, U + Ci\,|\,\beta\,|\,] \end{aligned} \tag{4.5}$$

Here $\alpha = aU$ and $\beta = \alpha - U = (a - 1)U$. When $f = 2f_b$, a has the value 2 and K is zero, as it should be. When $f = f_b$, i.e., when $a = 1$ and $\alpha = U$,

$$\begin{aligned} K = {} & (1 + \cos U - 2U^{-1} \sin U)\, Si\, U \\ & + (-\cos U + U^{-1} \sin U)\, Si\, 2U \\ & + (Ci\, 2U - Ci\, U - \log_e 2)(\sin U + U^{-1} \cos U) \\ & + U^{-1}\, (2 + \cos U)(\log_e U + .577.. - Ci\, U) \end{aligned} \tag{4.6}$$

Values of K are tabulated in Table 4.2 for various frequencies and delays ($a = f/f_b$ and $k = 4U/\pi = 8f_bT$). Cases 3 and 4 in Table 4.1 show that when U is very small $K \approx a^2U^3(2 - a)/8$ and when U is very large $K \approx \pi$, $\pi/2$, or 0 according to whether $a < 1$, $a = 1$, or $a > 1$.

Case 3 in Table 4.1 is a special case of Case 2 which may be obtained

TABLE 4.2 — VALUES OF K

k	$a = 0$	0.25	0.50	0.75	1.00	1.25
0	0	0.0	0.0	0.0	0.0	0.0
1	0	0.006	0.022	0.041	0.058	0.068
2	0	0.048	0.164	0.305	0.422	0.473
3	0	0.141	0.490	0.890	1.20	1.26
4	0	0.286	0.961	1.74	2.19	1.45
5	0	0.453	1.57	2.64	3.05	2.62
6	0	0.673	2.16	3.36	3.41	2.45
7	0	0.897	2.69	3.68	3.13	1.73
8	0	1.14	3.13	3.65	2.40	0.883
9	0	1.39	3.45	3.34	1.62	0.370
10	0	1.66	3.68	2.95	1.20	0.378
11	0	1.93	3.79	2.64	1.32	0.741
12	0	2.20	3.79	2.51	1.89	1.10

by letting U become very small in (4.3). The value of P_I/P_S corresponding to Case 3 has been given by Albersheim and Schafer.[3] Case 4 may be obtained by letting U become large in (4.5) and (4.6).

When U is very small, expression (2.8) for $R_v(\tau)$ becomes

$$R_v(\tau) \approx AU^2u^{-1} \sin u \tag{4.7}$$

Assuming $AU^2 \ll 1$ and substituting (4.7) in the "second order modulation" approximation (2.9) for G and H gives us another derivation of Case 3 (see (A3–2)). However, if the rms frequency deviation of the signal is so large that AU^2 is not small, even though $U \ll 1$, we have Case 5, the case investigated by Lewin.[2] The formulas given in Table 4.1 are obtained when (4.7) is set in the integrals (2.4) and (2.5) for G and H, and the results compared with the definition (A3–1) of $I(b, a)$.

When U is very large, expression (2.8) for $R_v(\tau)$ becomes

$$R_v(\tau) \approx \begin{cases} A[\pi U - 2\cos u - 2uSi(u)], & 0 \le u < U \\ 0, & U < u \end{cases} \tag{4.8}$$

Substituting (4.8) in (2.9) and integrating by parts twice leads to another derivation of Case 4. The expression for G given in Case 6 is obtained from (4.8) and (2.4) by the method outlined in Appendix II. $A\pi U$ is assumed to be so large that most of the contribution to the value of the integral (2.4) for G comes from the region around $u = 0$. It is also assumed that $R_v(\tau) + 1$ is negligible in comparison with $\exp R_v(\tau)$ in this region. This leads to the approximation

$$G \approx 2\int_0^\infty e^{2A[1-\cos u - uSi(u)]} \cos au \, du \tag{4.9}$$

which holds when $U \gg 1$ and $A\pi U \gg 1$.

The expression for G in Case 6 is merely the leading term in the asymptotic expansion arising from the saddle point at $u = iy_1$. When further terms in this expansion are obtained (using, for example, equation (10.4) of Reference 7) it is found that the expression for Case 6 should be multiplied by

$$1 + \frac{[4cs - 5y_1 + s^2(y_1^{-1} - 2y_1)]}{48As^3} + \cdots. \tag{4.10}$$

where c and s denote $\cosh y_1$ and $\sinh y_1$, respectively. The next term consists of $1/A^2$ times a function of y_1, and so on. When y_1 becomes small, as it does when A becomes large or a becomes small, (4.10) becomes

$$1 + \frac{1}{48A} + \frac{103}{13824A^2} + \cdots.$$

However, comparison of Tables 4.3 and 4.4 shows that the formula of Case 6 gives fairly reliable values of G when A is as small as 0.5 (U must be large, of course).

When $(1 - a)$ is small and $A \ll 1$, but U still large enough to make $A\pi U \gg 1$, (4.9) gives

$$G \approx Aa^{-2}\left[\pi + 2 \text{ arc tan } \frac{(1 - a)}{A\pi}\right] + 0(A^2/a^2) \tag{4.11}$$

This may be obtained by letting A become small in

$$G \approx 4(A/a)^2 \int_0^\infty e^{2Ay} Si(u) F \, du \tag{4.12}$$

$$y = 1 - \cos u - uSi(u)$$

$$F = -2 \cos auSi(u) + Si[(1 - a)u] + Si[(1 + a)u]$$

which may be obtained from (4.9) by integrating by parts twice.

The formula for G given in Case 7, Table 4.1, is obtained when U is taken to be of order unity and A is assumed to be so large that only the exponential term in the integrand of (2.4) is of importance. Most of the contribution comes from around $u = 0$ where

$$R_v(\tau) = R_v(0) - u^2A(1 - U^{-1} \sin U) + \cdots .$$

When $U \gg 1$, and $A \gg 1$, Cases 6 and 7 both give

$$G \approx (\pi/A)^{1/2} \exp [-a^2/(4A)]$$

which leads to an expression for P_I/P_S similar to one given by Alber-

TABLE 4.3 — VALUES OF G AND H OBTAINED BY NUMERICAL INTEGRATION

	$a = 0$	0.25	0.50	0.75	1.00	1.25
G for $U = 3$						
$A = 0.125$	0.690	0.651	0.599	0.466	0.326	0.224
0.25	1.53	1.46	1.29	1.06	0.808	0.561
0.50	2.17	2.09	1.97	1.63	1.38	0.993
H for $U = 3$						
$A = 0.125$	0.424	0.393	0.335	0.260	0.182	0.111
0.25	0.574	0.528	0.447	0.339	0.233	0.136
0.50	0.283	0.255	0.211	0.156	0.104	0.039
G for $U = 6$						
$A = 0.125$	2.62	2.42	1.85	1.18	0.631	0.257
0.25	3.26	3.05	2.50	1.81	1.17	0.695
0.50	2.55	2.46	2.21	1.86	1.47	1.10
H for $U = 6$						
$A = 0.125$	0.802	0.705	0.465	0.227	0.067	0.0031
0.25	0.266	0.231	0.148	0.064	0.0126	0.0035
0.50	0.0092	0.0079	0.0048	0.0018	0.00017	0.00019
	$A = 0.0625$	0.125	0.25	0.50		
G for $a = 1$						
$U = 1.5$	0.0126	0.045	0.155	0.446		
3	0.111	0.326	0.808	1.38		
6	0.245	0.631	1.17	1.47		
12	0.300	0.657	1.16	1.47		

sheim and Schafer[3] for long delay. When $U \ll 1$ and $AU^2 \gg 1$, Cases 5 and 7 both give

$$G \approx (6\pi A^{-1} U^{-2})^{1/2} \exp [-3a^2 A^{-1} U^{-2}/2]$$

Before the approximations listed in Table 4.1 were developed, a number of values of G and H were obtained from (2.4) and (2.5) by numerical integration. These values, given in Table 4.3, are the best we have and may be used to check the various approximations.

As an example of the values given by our approximations we take the case $U = 6$. In Table 4.4, "G — Case 6" has been computed from the formula given in Case 6, Table 4.1 (which assumes $U \to \infty$). When these values are compared with the corresponding ones in Table 4.3, it is seen that the agreement is not good for $A = 0.125$. Better agreement is shown by "G — Improved Case 6" in which the values are computed from (2.4) and (4.8) by the method of Appendix II. It differs from "Case 6" in that $F(u)$ of (A2–1) is $\exp [R_v(\tau)] - R_v(\tau) - 1$ instead of merely $\exp [R_v(\tau)]$.

TABLE 4.4 — APPROXIMATE VALUES OF G FOR $U = 6$

	$a = 0$	0.25	0.50	0.75	1.00	1.25
G—Case 6						
$A = 0.125$	5.01	4.13	2.54	1.32	0.63	0.28
0.25	3.54	3.26	2.58	1.80	1.14	0.66
0.50	2.51	2.42	2.17	1.82	1.43	1.06
G—Improved Case 6						
$A = 0.125$	2.62	2.37	1.74	1.06	0.56	0.26
0.25	3.17	2.94	2.39	1.72	1.10	0.66
0.50	2.50	2.41	2.17	1.82	1.42	1.06

5. INTERCHANNEL INTERFERENCE POWER

The values of P_I/P_S, the ratio of the interchannel interference power to the signal power, may be computed from the formulas (2.10) and (2.11) when G and H are known. One would like to have curves giving P_I/P_S for representative combinations of echo delay, signal power, and channel position which are likely to occur in practice. However, the large number of such combinations coupled with the difficulty of computing G and H leads us to restrict ourselves mostly to curves for Cases 2 and 6 in Tables 3.1 and 4.1. In all cases the signal power P_S (per cps) is taken to be equal to the constant value P_0 (measured in $(\text{radians})^2/\text{cps}$ for PM and in $(\text{radians/sec})^2/\text{cps}$ for FM) over the signal band $(0, f_b)$, and is zero outside this band.

Case 2 is the "second order modulation" approximation which, roughly speaking, applies when the echo delay is very short or when the rms deviation of the phase angle (for PM) or of the frequency (for FM) is small. Case 6 applies when the echo delay is very long.

(*a*) *"Second order modulation" approximation for PM — Table 3.1, Case 2.* Since $G \approx H$, equation (2.10) may be written as

$$\begin{aligned} 10 \log_{10} (P_I/P_S) &\approx \rho + D_1 + D_2 + D_3 \\ \rho &= 10 \log_{10} r^2 \\ D_1 &= 10 \log_{10} (1 - \cos 2pT) \\ D_2 &= 10 \log_{10} (P_0 f_b) \\ D_3 &= 10 \log_{10} J \end{aligned} \tag{5.1}$$

where ρ is the reflection coefficient expressed in decibels, D_1 is a quantity which varies rapidly with T and whose representative value is zero. J is the function of a and U defined by equation (3.1). The quantities P_0,

f_b, a, U are defined by equation (2.1) and Case 1 of Table 3.1. P_0f_b is the average signal power in (radians)^2, $a = f/f_b$ gives the channel position and $U = 2\pi f_b T$ measures the echo delay.

The approximation (5.1) holds when $2P_0f_b(1 - U^{-1} \sin U)$ is small in comparison with unity.

Fig. 5.1 shows D_3 plotted as a function of f_bT for various values of a. The values of J which were used were taken from Table 3.2. Values of J for $U \ll 1$ may be obtained from the expression given for G in Case 3, Table 3.1. Case 4 gives another special case.

(b) *Large delay, $U \gg 1$ for PM — Table 3.1, Case 6*. When U is very large, G is a rapidly oscillating function of a. When in addition P_0f_b is small, Case 4 shows that J fluctuates between $(1 - a/2)/2$ and $3(1 - a/2)/2$. The corresponding fluctuations in D_3 are noticeable in Fig. 5.1 for the larger values of f_bT. If $2P_0f_b$ is large compared to unity (and the delay is large), equation (5.1) no longer holds. In this case $G \gg H$ and we may write (2.10) as

$$\begin{aligned} 10 \log_{10} (P_I/P_S) &\approx \rho + D_4 \\ \rho &= 10 \log_{10} r^2 \\ D_4 &= 10 \log_{10} (2\pi P_0 f_b)^{-1} G \\ G &\approx e^{-b_0}[I(b_0, a) + 2I(b_1, a) \cos aU] \end{aligned} \tag{5.2}$$

where b_0 and b_1 are defined in Case 6, Table 3.1. It is seen that as a increases from 0 to 1, D_4 oscillates rapidly between limits D_4^+ and D_4^- corresponding to G^+ and G^- which are defined by equation (3.4) and tabulated in Table 3.3. Table 5.1 gives values of D_4^+ and D_4^- computed from Table 3.3.

The entries corresponding to the values 0.25 and 0.50 for P_0f_b must be used with caution in equation (5.2) since they do not satisfy $2P_0f_b \gg 1$ and H is not negligible in comparison with G.

(c) *"Second order modulation" approximation for FM — Table 4.1, Case 2*. By making use of the expressions for G and H given in Case 2 we may write equation (2.11) as

$$\begin{aligned} 10 \log_{10}(P_I/P_S) &\approx \rho + D_1 + D_2' + D_3' \\ D_2' &= 10 \log_{10} A \\ D_3' &= 10 \log_{10} UK \end{aligned} \tag{5.3}$$

where ρ and D_1 are defined by (5.1) and A by (2.8). K is the function of a and U defined by (4.5). A is proportional to the signal power: $A =$

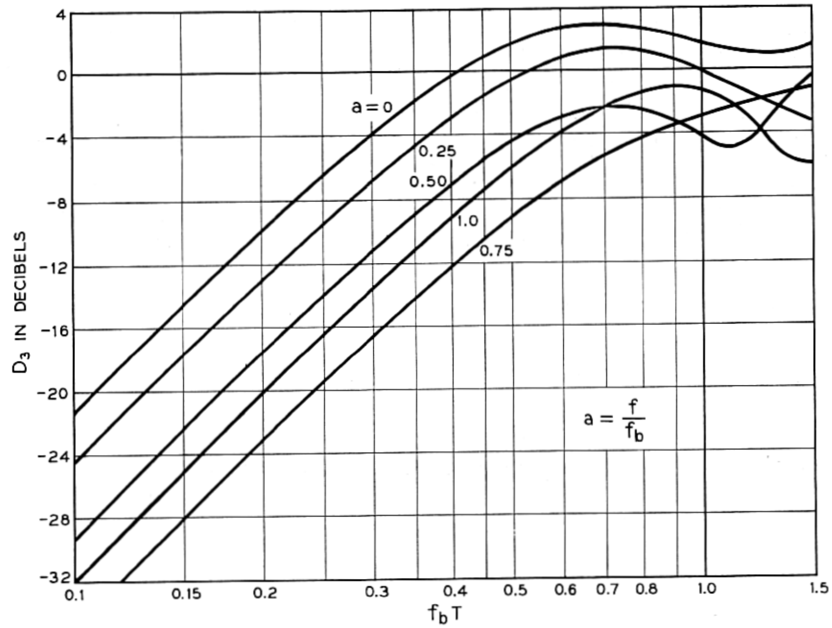

Fig. 5.1 — For "Second Order Modulation" the ratio P_I/P_S for PM depends upon D_3 as shown by equation (5.1).

Table 5.1 — Values of D_4^+ and D_4^-. The Upper Number of an Entry is D_4^+ and the Lower One is D_4^-

	$a = 0$	0.25	0.50	0.75	1.00—
$P_0 f_b$ (radians)[2]					
0.25	−6.12	−6.68	−7.31	−8.07	−9.01
	−9.92	−10.37	−10.96	−11.67	−12.56
0.50	−4.89	−5.40	−6.00	−6.75	−7.67
	−7.94	−8.31	−8.79	−9.45	−10.34
0.75	−4.83	−5.31	−5.87	−6.57	−7.47
	−7.24	−7.56	−8.01	−8.63	−9.46
1.00	−5.20	−5.61	−6.13	−6.78	−7.67
	−7.12	−7.38	−7.77	−8.33	−9.17
2.00	−7.65	−7.89	−8.25	−8.74	−9.44
	−8.30	−8.44	−8.74	−9.17	−9.82
4.00	−11.99	−12.08	−12.25	−12.55	−12.94
	−12.05	−12.13	−12.31	−12.55	−12.94

$(\sigma/f_b)^2$ where σ is the rms frequency deviation of the signal in cycles per second. D_2' and D_2 play similar roles in (5.3) and (5.1).

The approximation (5.3) holds when $2A[USi(U) - 1 + \cos U]$ is small in comparison with unity.

The values of K given in Table 4.2 lead to the curves for D_3' shown in Fig. 5.2. When $U \ll 1$, Case 3 shows that $D_3' \approx 10 \log_{10} [a^2U^4(2 - a)/8]$, and Case 4 shows that when $U \gg 1$ (provided $a < 1$ and $A\pi U \ll 1$) $D_3' \approx 10 \log_{10} (\pi U) = 12.95 + 10 \log_{10} f_bT$.

(*d*) *Large delay, $U \gg 1$ for FM — Table 4.1, Case 6.* It has just been pointed out that when U becomes very large, P_I/P_S depends upon the delay only through the term $D_3' \approx 10 \log_{10} \pi U$ (neglecting the rapidly varying term D_1) if $AU\pi \ll 1$. If $AU\pi \gg 1$, P_I/P_S becomes independent of U as $U \to \infty$. This follows from the fact that the formulas of Case 6 allow us to write (2.11) as

$$\begin{aligned} 10 \log_{10} (P_I/P_S) &= \rho + D_4' \\ D_4' &\approx 10 \log_{10} [Ga^2(2\pi A)^{-1}] \end{aligned} \tag{5.4}$$

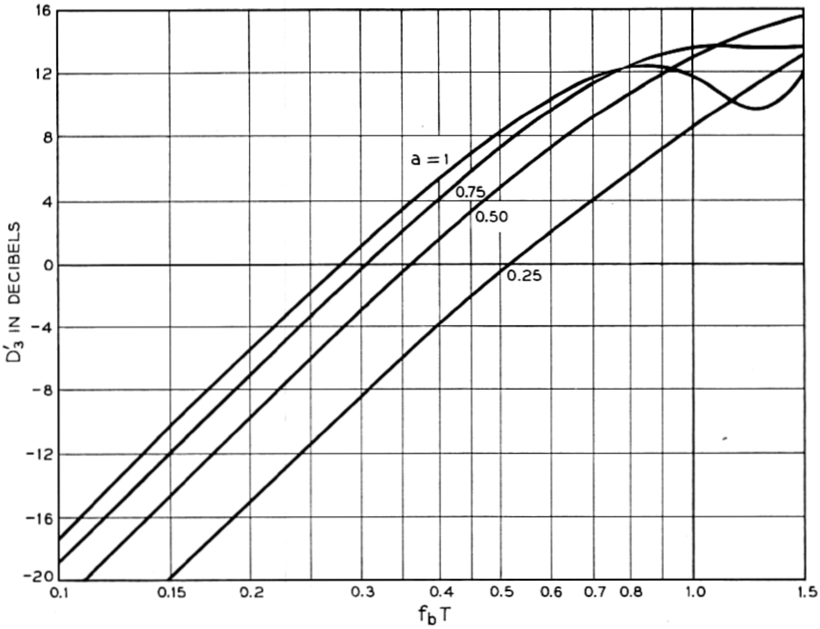

Fig. 5.2 — For "Second Order Modulation" the ratio P_I/P_S for FM depends on D_3' as shown by equation (5.3).

where ρ is defined by (5.1) and G depends only on a and A through

$$G \approx [\pi y_1/A \sinh y_1]^{1/2} \exp [-2A (\cosh y_1 - 1)]$$
$$\frac{a}{2A} = \int_0^{y_1} \frac{\sinh v}{v} dv \tag{5.5}$$

Fig. 5.3 shows D_4' plotted as a function of A for various values of a. It is assumed that $A\pi U \gg 1$.

(e) *Small delay and large rms frequency deviation for FM — Table 4.1, Cases 5 and 7*. It turns out that Case 5 (Lewin's, case, $U \ll 1$) and Case 7 ($A \gg 1$ and U of order unity) may be combined into a single case by taking the quantity b in the formulas of Case 5 to be $6A(1 - U^{-1} \sin U)$ instead of AU^2. When $U \ll 1$, Case 5 is obtained. When $A \gg 1$ the asymptotic expansion for $I(b, a)$ leads to Case 7 if U is $0(1)$.

In order to put this combined case in a form suited to calculation we write (2.11) as

$$\frac{P_I}{P_S} = \frac{r^2 a^2 G}{2\pi A}\left(1 - \frac{H}{G} \cos 2pT\right)$$
$$\approx \frac{r^2 6(1 - U^{-1} \sin U) a^2 e^{-b} I(b, a)}{2\pi b}\left(1 - \frac{H}{G} \cos 2pT\right) \tag{5.6}$$
$$10 \log_{10} (P_I/P_S) \approx \rho + D_1' + D_5' + D_6'$$

where ρ is given by (5.1) and

$$D_1' = 10 \log_{10} (1 - (H/G) \cos 2pT)$$
$$D_5' = 10 \log_{10} (1 - U^{-1} \sin U)$$
$$D_6' = 10 \log_{10} 6a^2 e^{-b} I(b, a)/(2\pi b) \tag{5.7}$$
$$b = 6A(1 - U^{-1} \sin U)$$

Fig. 5.4 shows values of D_6', computed from the values of $I(b, a)$ given in Appendix III, plotted as a function of b for various values of a. The maximum value of 3 db for D_1' occurs when $AU^2 \ll 1$ and $\cos 2pT = -1$. When AU^2 is large D_1' is approximately zero.

Fig. 5.5 and 5.6 show, in a rough way, the regions in which the various approximations apply. For PM, the delay and the rms phase deviation (measured by $U = 2\pi f_b T$ and $(P_0 f_b)^{1/2}$, respectively) are the parameters which determine the type of approximation to be used. The regions in the $[(P_0 f_b),^{1/2} U]$ plane shown in Fig. 5.5 are marked with the numbers 2a, 3, 4, 5, 6b where the integer indicates the case number in Table 3.1 and the letters a and b refer to Cases (a) and (b) in this section. Fig. 5.6 is

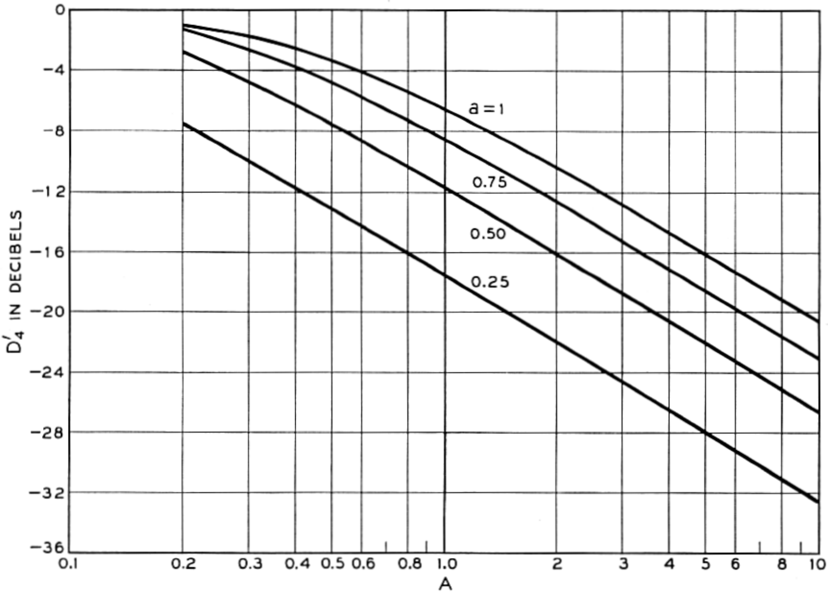

Fig. 5.3 — For long delayed echoes the ratio P_I/P_S for FM depends upon D_4' as shown by equation (5.4).

the corresponding figure for FM. The coordinates are U and $A^{1/2}$, where $A^{1/2}(=\sigma/f_b)$ measures the rms frequency deviation. The region numbers are 2c, 3, 4, 5, 6d, 7e where the integers indicate the case number in Table 4.1. It will be noted that there are regions where no approximation is available. However, an answer may always be obtained by numerical integration of equations (2.4) and (2.5) for G and H.

Fig. 5.7 shows values of P_I/P_S for the top channel ($a = 1$) where the interference is often at a maximum in an FM system. The coordinates ($A^{1/2}$, f_bT) are essentially the same as those of Fig. 5.6. In order to simplify the plotting, the phase angle $2pT$ is assumed to be such that $\cos 2pT$ is zero so that the contours are given by

$$\text{Constant} = 10 \log_{10} \left(\frac{P_I}{r^2 P_S} \right) = 10 \log_{10} \left(\frac{G}{2\pi A} \right)$$

The contours have been obtained in part from the various approximations where applicable and in part from values obtained by numerical computation from the exact expression. While there are, of necessity, some areas of uncertainty in Fig. 5.7, it should be adequate for most engineering purposes. No corresponding curves have been computed for the case of phase modulation.

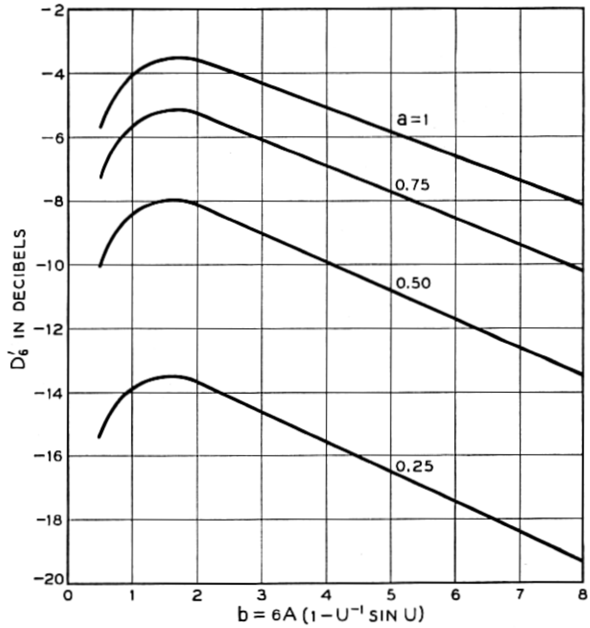

Fig. 5.4 — Under the condition of large rms frequency deviation and small delay the ratio P_I/P_S for FM depends upon D_6' as shown by equation (5.6).

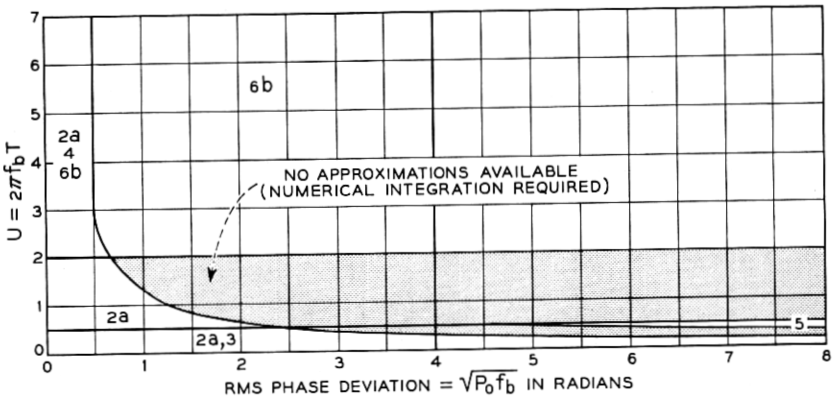

Fig. 5.5 — Regions of validity for the various approximations for PM. The integers refer to case numbers in Table 3.1 and the letters to cases discussed in Section 5. This figure and Fig. 5.6 are intended to give only an idea of the relative positions of the regions.

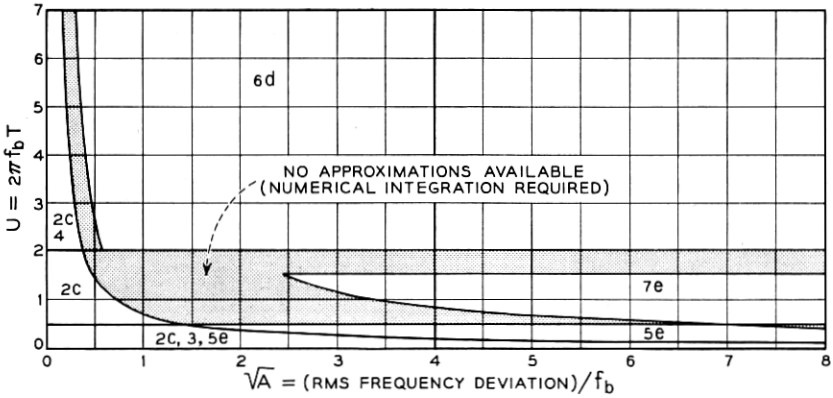

Fig. 5.6 — Regions of validity for the various approximations for FM. The integers refer to case numbers in Table 4.1 and the letters to cases discussed in Section 5.

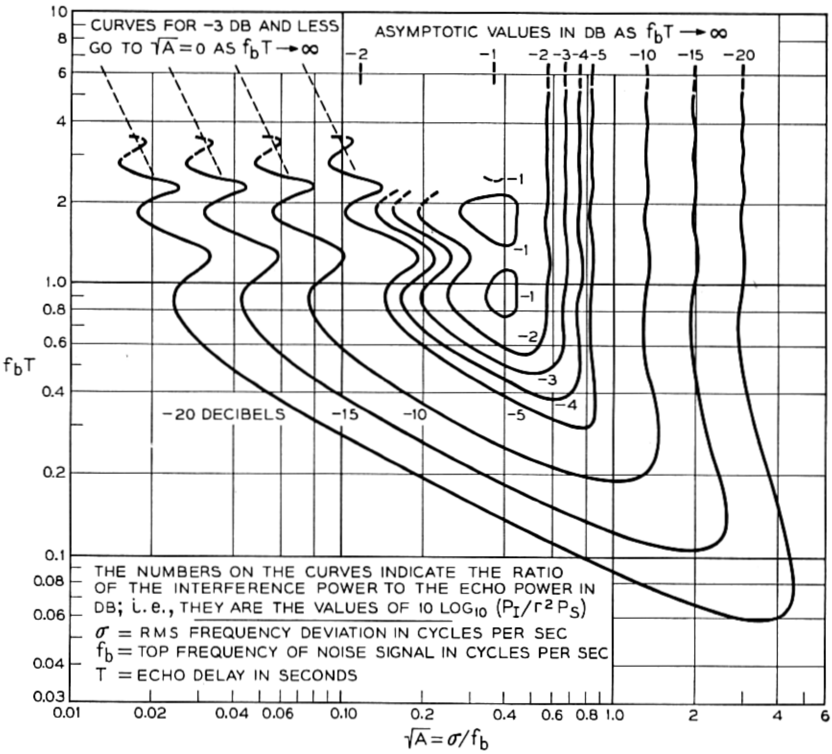

Fig. 5.7 — Contours of constant interference in the top channel of a multichannel FM system.

It should be noticed that Fig. 5.7 is plotted for the case $a = f/f_b = 1$. If Fig. 5.7 were plotted for values of a slightly less than unity there would not be much change except in the upper left hand corner (A small and f_bT large), where the interference would tend to be 3 db stronger. This discontinuous behavior as a passes through unity is shown by Case 4, Table 4.1 where $U \gg 1$ and $A\pi U \ll 1$. When $U \gg 1$ and $A\pi U \gg 1$, as occurs when $f_bT \to \infty$ (with A held fixed), equation (4.11) gives

$$10 \log_{10} \frac{P_I}{r^2 P_S} \approx 10 \log_{10} \left[\frac{1}{2} + \frac{1}{\pi} \arctan \left(\frac{1-a}{A\pi} \right) \right]$$

when $A \ll 1$. This shows that at $f_bT = \infty$, the discontinuity arises for values of A near $(1 - a)/\pi$. When $A \gg 1$ and $U \gg 1$, we have

$$10 \log_{10} \frac{P_I}{r^2 P_S} \approx -15 \log_{10} A - 10 \log_{10} \frac{2\pi^{1/2}}{a^2} - \frac{4.34a^2}{4A}$$

which changes only slowly as $a \to 1$.

6. USE OF EQUIVALENT ECHO TO ESTIMATE INTERCHANNEL INTERFERENCE

When a steady sinusoid $\exp(i\omega t)$ is applied to a transmission medium of the sort we have under consideration, the output is $\exp(i\omega t - \alpha - i\beta)$ where α and β are the attenuation and phase shift, respectively. Distortionless transmission occurs when α is constant and β has a constant slope over the essential range of frequencies. Departures from these ideal cónditions cause interchannel interference in multichannel FM and PM systems. Our evaluation of the interference caused by a small echo may equally well be regarded as an evaluation of interference for a particular kind of amplitude and phase distortion, namely that given by

$$\alpha \approx -r \cos \omega T, \qquad \beta \approx r \sin \omega T. \tag{6.1}$$

These expressions are obtained by writing $\exp(i\omega t) + r \exp[i\omega(t - T)]$ in the form $\exp(i\omega t - \alpha - i\beta)$ when $|r| \ll 1$.

The analysis given by E. D. Sunde in Section 1 of Reference 8 shows that a minimum phase system in which $\beta = r \sin \omega T$ also has $\alpha = -r \cos \omega T$ as in (6.1). This suggests a procedure for calculating interchannel interference from phase data alone when (1) the distortion is known to be of the minimum phase type and (2) the variation of phase with frequency can be approximated by a sine function. In such cases we can apply our echo analysis directly by identifying r as the amplitude of the phase oscillation and T as the reciprocal of the period.

In carrier multiplex systems the sinusoidal approximation need hold only in the region around the carrier frequency f_0 where $2\pi f_0 = p$ and p is the radian frequency appearing in equation (1.1). For FM we shall, in this section, arbitrarily take the region to extend from $f_0 - 4\sigma$ to $f_0 + 4\sigma$ where σ is the rms frequency deviation of the signal. For PM and the signal power spectrum given by equation (2.1) we may take the region to be $f_0 \pm 4f_b(P_0f_b/3)^{1/2}$.

A special case occurs when the nonlinear portion of β may be represented as $a_2(f - f_0)^2/2$ in the region of interest. We can think of $r \sin \omega T$ as going through several oscillations between $f = 0$ and $f = f_0$, and that a maximum, if $a_2 < 0$, (or a minimum, if $a_2 > 0$) of $r \sin \omega T$ occurs at $f = f_0$, i.e. at $\omega = p$. The band of interest is taken to be narrow enough to lie in the immediate vicinity of the maximum. This sort of curve fitting is permissible since equation (2.3) shows that the interchannel interference depends on the carrier frequency only through the term $\cos 2pT$. Furthermore, constant terms and terms linear in frequency in the expression for β do not affect the amount of interchannel interference.

At the maximum mentioned in the preceding paragraph $\omega = p$, $\sin pT = 1$ and $\cos 2pT = -1$. Near this maximum $r \sin \omega T$ is

$$r \cos 2\pi(f - f_0)T \approx r - r4\pi^2(f - f_0)^2T^2/2 \tag{6.2}$$

In order that this approximation may hold over the region $f_0 \pm 4\sigma$ (for FM), we require

$$2\pi(4\sigma)T \leqq 1$$

We take T to be as large as possible, namely

$$T = 1/8\pi\sigma \tag{6.3}$$

in order to make r as small as possible, since our work assumes $r \ll 1$. Comparison of $a_2(f - f_0)^2/2$ with (6.2) gives

$$r = -a_2(2\pi T)^{-2} = -16a_2\sigma^2 \tag{6.4}$$

which must be small compared to unity if our results are to be used. This result holds for $a_2 < 0$. When $a_2 > 0$ expression (6.3) still holds for T but now $r = 16a_2\sigma^2$. Similar expressions hold for PM. These values of r and T may now be inserted in our formulas to determine the interchannel interference.

From the definitions of A and U it may be shown that (6.3) is equivalent to $AU^2 = 1/16$. Therefore the "second order modulation" approximation given by Case 2 in Table 4.1 may be used. Also $G - H \cos 2pT$ is approximately equal to $2G$ because $\cos 2pT = -1$. It turns out that the second order modulation approximation may also be used in the PM case.

When a_2 is sufficiently small, considerations such as those above show that the ratio of the interference power for $\beta = a_2(f - f_0)^2/2$ (radians) to the received signal power at the frequency f is

$$\begin{aligned} P_I/P_S &= (a_2\sigma f_b/2)^2\, a^2(2 - a) \text{ for FM} \\ P_I/P_S &= (a_2 f_b^2/2)^2(P_0 f_b)(12 - 30a + 20a^2 - a^5)/30 \text{ for PM} \end{aligned} \tag{6.5}$$

Here $a = f/f_b$ where $(0, f_b)$ is the frequency band of the signal. The rms frequency deviation of the signal for FM is σ cps, and the rms phase deviation of the signal for PM is $(P_0 f_b)^{1/2}$ radians. The first equation in (6.5) comes from a special case of Case 2, namely, Case 3 of Table 4.1, and requires the additional assumption $f_b/4\sigma \ll 1$ (corresponding to $U \ll 1$). The second equation requires a similar additional assumption.

Appendix I

DERIVATION OF A GENERAL THEOREM ON THE INTERCHANNEL INTERFERENCE SPECTRUM

Let $w_a(f)$ be a finite power spectrum having limited total fluctuation for $0 \leqq f \leqq \infty$. Let the total power be finite so that the integral of $w_a(f)$ from $f = 0$ to $f = \infty$ converges absolutely. We define two auxiliary spectra

$$w_\epsilon(f) = \begin{cases} w_a(f), & |f - f_0| < \epsilon \\ 0, & |f - f_0| > \epsilon \end{cases}$$

$$w_b = w_a(f) - w_\epsilon(f) \tag{A1-1}$$

and note that the autocorrelations corresponding to these spectra must satisfy

$$R_b(\tau) = R_a(\tau) - R_\epsilon(\tau) \tag{A1-2}$$

We consider the problem of transmitting the ensemble having the spectrum $w_b(f)$ through a system in which the input and output autocorrelation functions, $\Psi_1(\tau)$ and $\Psi_2(\tau)$, respectively, are related by

$$\Psi_2(\tau) = F[\Psi_1(\tau)]$$

Here $F(z)$ and its derivatives $F'(z)$ and $F''(z)$ are assumed to be finite continuous functions which exist over the range of z of interest.

Let the output spectrum corresponding to the input spectrum $w_b(f)$ be $w_B(f)$. We shall show that as $\epsilon \to 0$ the value of $w_B(f)$ in the range

$|f - f_0| < \epsilon$ approaches

$$4 \int_0^\infty \{F[R_a(\tau)] - F'(0)R_a(\tau)\} \cos 2\pi f\tau \, d\tau \qquad \text{(A1-3)}$$

When we multiply this expression by 2ϵ and set $f = f_0$, we obtain the power appearing at the output of the system in an unloaded channel of width 2ϵ centered on f_0. Here we are not interested in values of $w_B(f)$ outside the range $|f - f_0| < \epsilon$.

First we note several properties of autocorrelation functions. From

$$R(\tau) = \int_0^\infty w(f) \cos 2\pi f\tau \, df$$

it follows that $|R(\tau)| \leqq R(0)$. Also, if $w(f)$ has limited total fluctuation in the interval $(0, \infty)$, the Riemann-Lesbesgue lemmas* and the absolute convergence of the integral for $R(0)$ show that $R(\tau) = 0(1/\tau)$ as $\tau \rightarrow \infty$. Thus we may find positive numbers A, B, C such that for $0 < \tau$ and any ϵ less than some fixed value

$$\begin{aligned} &|R_a(\tau)| < A/\tau \\ &|R_\epsilon(\tau)| \leqq R_\epsilon(0) = \int_{f_0-\epsilon}^{f_0+\epsilon} w_a(f)\, df < B\epsilon \qquad \text{(A1-4)} \\ &|R_\epsilon(\tau)| < C/\tau \end{aligned}$$

By the extended theorem of the mean the autocorrelation function corresponding to $w_B(f)$ is

$$\begin{aligned} R_B(\tau) &= F[R_b(\tau)] = F[R_a(\tau) - R_\epsilon(\tau)] \\ &= F[R_a(\tau)] - F'[R_a(\tau)]R_\epsilon(\tau) + r \qquad \text{(A1–5)} \\ |r| &= 2^{-1}R_\epsilon^2(\tau)\, |F''[R_a(\tau) - \theta R_\epsilon(\tau)]| < R_\epsilon^2(\tau)D \end{aligned}$$

where $0 \leqq \theta \leqq 1$ and D is a positive number such that $|F''(z)| < D$. Then

$$w_B(f) = 4 \int_0^\infty R_B(\tau) \cos 2\pi f\tau \, d\tau = I_1 - I_2 + I_3$$

$$\begin{aligned} I_1 &= 4 \int_0^\infty F[R_a(\tau)] \cos 2\pi f\tau \, d\tau \\ I_2 &= 4 \int_0^\infty F'[R_a(\tau)]R_\epsilon(\tau) \cos 2\pi f\tau \, d\tau \qquad \text{(A1-6)} \\ I_3 &= 4 \int_0^\infty r \cos 2\pi f\tau \, d\tau \end{aligned}$$

* See, for example, Whittaker and Watson, Modern Analysis, 4th edition, p. 172.

Since $F'[R_a(\tau)] = F'(0) + s$, where by the mean value theorem

$$| s | = | R_a(\tau) F''[\theta R_a(\tau)] | < | R_a(\tau) | D$$
$$0 \leqq \theta \leqq 1$$

I_2 may be written as the sum of two integrals, the second of which has an absolute value not greater than

$$4 \int_0^\infty | R_a(\tau) D R_\epsilon(\tau) | \, d\tau = 4D \int_0^T | R_a(\tau) R_\epsilon(\tau) | \, d\tau + 4D \int_T^\infty | R_a(\tau) R_\epsilon(\tau) | \, d\tau < 4D R_a(0) B \epsilon T + 4D \int_T^\infty AC\tau^{-2} \, d\tau = 4D[R_a(0) B\epsilon T + AC/T]$$

where T is an arbitrary number and we have used the inequalities (A1-4). Choosing $T = \epsilon^{-1/2}$ shows that the last expression is $0(\epsilon^{1/2})$. Hence

$$I_2 = 4F'(0) \int_0^\infty R_\epsilon(\tau) \cos 2\pi f\tau \, d\tau + 0(\epsilon^{1/2})$$
$$= 0(\epsilon^{1/2}) + F'(0) \begin{cases} w_a(f), & | f - f_0 | < \epsilon \\ 0, & | f - f_0 | > \epsilon \end{cases}$$

Therefore in the range $| f - f_0 | < \epsilon$, which comprises the only frequencies of interest in the channel interference spectrum,

$$I_2 = 4F'(0) \int_0^\infty R_a(\tau) \cos 2\pi f\tau \, d\tau + 0(\epsilon^{1/2})$$

By use of the inequalities for $| r |$ and $| R_\epsilon(\tau) |$ we see that

$$| I_3 | < 4 \int_0^\infty R_\epsilon^2(\tau) D d\tau < 4D \int_0^T B^2 \epsilon^2 d\tau + 4D \int_T^\infty C^2 \tau^{-2} d\tau = 4D(B^2\epsilon^2 T + C^2/T)$$

If we choose $T = 1/\epsilon$ this expression is $0(\epsilon)$.

When we collect our results and let ϵ become vanishingly small, we see that expression (A1-6) for $w_B(f)$ approaches (A1-3) for frequencies in the range $| f - f_0 | < \epsilon$. Thus in the limit as $\epsilon \to 0$ we may use the autocorrelation function

$$F[R_a(\tau)] - F'(0) R_a(\tau)$$

to compute the interchannel interference spectrum. This is the result used in (1.22).

Appendix II

APPROXIMATE EVALUATION OF INTEGRALS OF A CERTAIN TYPE

The problem of evaluating the integral G defined by (2.4) is quite a difficult one. Here we shall outline a method which often may be used to obtain an idea of the order of the magnitude of such an integral.

Let $F(u)$ be an even analytic function of u such that the major contribution to the value of

$$I(a) = 2\int_0^\infty F(u)\cos au\, du = \int_{-\infty}^{\infty} F(u)e^{iau}\, du \tag{A2-1}$$

comes from a saddle point on the positive imaginary u axis. Then the "method of steepest descents" suggests that an approximate value of $I(a)$ may be obtained by the following procedure.

1. Set $f(y) = F(iy)$ and plot $z = d[\log f(y)]/dy = f'(y)/f(y)$ as a function of y.

2. Draw the horizontal line $z = a$. Suppose its first intersection with the curve obtained in step 1 is at $y = y_1$, and let the slope of the curve, determined either graphically or by differentiation, be $(dz/dy)_{y_1}$ at y_1.

3. Then $I(a) \approx [2\pi/(dz/dy)_{y_1}]^{1/2} f(y_1)e^{-ay_1}$ (A2-2)

It should be noted that (A2-2) cannot be used indiscriminately. Thus, it does not work well for $F(u) = 1/(1 + u^4)$ because there is no saddle point on the imaginary u-axis. However, when it is applied to integrals of the type encountered in our study it appears to do fairly well, as Table 4.4 shows.

Appendix III

LEWIN'S INTEGRAL

Here we study the integral

$$I(b, a) = \int_{-\infty}^{\infty} [e^{bu^{-1}\sin u} - 1 - bu^{-1}\sin u]\cos au\, du \tag{A3-1}$$

which occurs in several limiting cases in our work and which has been studied by Lewin[2] for $a = 0$ and $a = 1$.

When the exponential term is expressed as a power series in $(b \sin u)/u$ and the result integrated termwise, we obtain

$$I(b, a) = \sum_{n=2}^{\infty} A_n b^n / n! \tag{A3-2}$$

$$A_n = \int_{-\infty}^{\infty} \left(\frac{\sin u}{u}\right)^n \cos au \, du$$

$$= \frac{2\pi}{2^n(n-1)!} \sum_{m=0} (-)^m C_m{}^n (n - 2m + a)^{n-1}$$

$$A_n \sim (6\pi/n)^{1/2} \exp\left[-3a^2/(2n)\right]$$

where $C_m{}^n = n!/m!(n-m)!$ and the last term in the summation for A_n is the last one for which $n - 2m + a$ is positive (assuming $a \neq$ integer) and for which $m \leqq n$. When $n \geqq 2$, A_n is a continuous function of a. Tables III A and III B were computed from (A3-2).

When b is small, the first term in (A3-2) gives, for $0 \leqq a \leqq 2$,

$$I(b, a) \approx b^2 \pi (2 - a)/4 \tag{A3-3}$$

and when b is a large positive number the contribution of the exponential term in the region around $u = 0$ gives

$$I(b, a) \approx (6\pi/b)^{1/2} \exp\left(b - \frac{3a^2}{2b}\right) \tag{A3-4}$$

Lewin has given more careful approximations for the $a = 0$ and $a = 1$ cases.

When b is large and negative most of the contribution comes from around $u = \pm 3\pi/2$ where $\exp[(b \sin u)/u]$ attains its largest values. It is found that

$$I(-\beta, a) = 10.76\, \beta^{-1/2} \cos(4.49a) e^{0.217\beta - 2.30a^2\beta^{-1}} + R \tag{A3-5}$$

where $\beta = -b$ is a large positive number and R is a remainder term. The numbers in (A3-5) are related to the value $u_0 = 4.493 \ldots$ where $(\sin u)/u$ has a minimum.

Computation shows that the value $\beta = 8$ is not large enough to make the leading term in (A3-5) a good approximation for $I(-\beta, a)$. In order to obtain a better approximation we write

$$\begin{aligned} I(-\beta, a) \approx\ & 2 \int_0^y \exp[-\beta u^{-1} \sin u] \cos au \, du \\ & + 2 \int_0^y [-1 + \beta u^{-1} \sin u] \cos au \, du \\ & + \int_y^{\infty} (\beta u^{-1} \sin u)^2 \cos au \, du \end{aligned} \tag{A3-6}$$

TABLE III A — $e^{-b}I(b, a)$ FOR $b > 0$

b	e^b	$a = 0$	0.25	0.50	0.75	1.00	1.25
0	1	0.0	0.0	0.0	0.0	0.0	0.0
0.5	1.649	0.272	0.241	0.209	0.176	0.142	0.107
1.0	2.718	0.761	0.685	0.602	0.511	0.414	0.314
2.0	7.389	1.560	1.440	1.291	1.117	0.919	0.713
3.0	20.08	1.913	1.801	1.645	1.448	1.215	0.968
4.0	54.60	1.974	1.888	1.751	1.566	1.341	1.098
5.0	148.4	1.905	1.844	1.731	1.571	1.372	1.153
6.0	403.4	1.794	1.751	1.660	1.525	1.356	1.166
7.0	1097.	1.680	1.649	1.575	1.463	1.320	1.157
8.0	2981.	1.576	1.552	1.492	1.398	1.277	1.138

TABLE III B — $I(b, a)$ FOR $b < 0$

b	$a = 0$	.25	.50	.75	1.0	1.25
0	0.0	0.0	0.0	0.0	0.0	0.0
−0.5	0.349	0.300	0.254	0.210	0.167	0.125
−1.0	1.25	1.06	0.885	0.723	0.576	0.432
−2.0	4.16	3.41	2.76	2.20	1.76	1.34
−3.0	8.03	6.37	4.97	3.88	3.14	2.46
−4.0	12.6	9.66	7.23	5.49	4.55	3.74
−5.0	17.8	13.2	9.40	6.89	5.93	5.19
−6.0	23.6	16.8	11.4	8.00	7.25	6.85
−7.0	30.0	20.7	13.1	8.71	8.48	8.78
−8.0	37.2	24.8	14.5	8.93	9.59	11.0

where y is such that the quantity within the brackets in (A3-1), with $b = -\beta$, is approximately $(\beta u^{-1} \sin u)^2/2$ when $u > y$. For rough work we may take $y = \beta$. The leading term in (A3-5) arises from the contribution of the region around $u = 3\pi/2$ to the value of the first integral in (A3-6). The contributions from the regions around $u = 7\pi/2, 11\pi/2, \ldots$ (if y is large enough) add to the value of R but they are generally small in comparison with the leading term in (A3-5).

Thus we are led to approximate R in (A3-5) by the sum of the second and third integrals (expressed in terms of integral sines and cosines) in (A3-6) with $y = \beta$. When R is replaced by this sum, expression (A3-5) gives values for $b = -8$ which agree fairly well with those in the table.

REFERENCES

1. S. T. Meyers, Nonlinearity in Frequency-Modulation Radio Systems Due to Multipath Propagation, Proc. I.R.E., **34,** pp. 256–265, 1946.
2. L. Lewin, Interference in Multi-Channel Circuits, Wireless Engineer, **27,** pp. 294–304, 1950.
3. W. J. Albersheim and J. P. Schafer, Echo Distortion in the FM Transmission of Frequency-Division Multiplex, Proc. I.R.E., **40,** pp. 316–328, 1952.

4. J. P. Vasseur, Calcul de la distortion d'une onde modulée in fréquence, Annales de Radioélectricité, **8**, pp. 20–35, 1953.
5. S. O. Rice, Mathematical Analysis of Random Noise, B.S.T.J., **23**, pp. 282–332, 1944 and **24**, pp. 46–156, 1945.
6. D. Middleton, The Distribution of Energy in Randomly Modulated Waves, Phil. Mag., **42**, pp. 689–707. July, 1951.
7. S. O. Rice, Diffraction of Plane Radio Waves by a Parabolic Cylinder, B.S.T.J., **33**, pp. 417–504, 1954.
8. E. D. Sunde, Theoretical Fundamentals of Pulse Transmission, B. S. T. J., **33**, pp. 721–788, and **33**, pp. 987–1010, May and July, 1954.