Stability of Negative Impedance Elements in Short Transmission Lines

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Until recently, voice frequency repeaters of the two-way type have been applied almost exclusively to electrically long transmission lines. Now, negative impedance repeaters are used in quantity in the exchange telephone plant, and applications to electrically short lines arise more frequently. Because lower over-all transmission losses can be obtained by utilizing the lower phase shift in short lines, a different engineering approach to the application of E-type negative impedance repeaters is desirable.

This paper outlines a general method whereby transmission performance and stability can be related to the characteristics of a symmetrical repeater located in the center of a short transmission line. The theory is particularly applicable to negative impedance repeaters. A tandem arrangement of short sections of transmission line, where each section has a centrally located repeater, can be classed as a line loaded with negative impedance.

1. INTRODUCTION

For a period of about 40 years voice frequency repeaters have been engineered to provide amplification for both directions of conversation in two-wire telephone lines. Some of these repeaters have been operated in lines over 50 miles long; others have been operated in lines shorter than 10 miles. Yet practically all, including negative impedance repeaters of the E-type[1], have been associated with transmission lines which can be classed as electrically long in that they have exceeded one half wavelength at the highest frequency in the pass-band.

Within the past few years the need for two-way amplification in electrically short lines in the exchange area plant has become increasingly evident. A short section of line has limited phase shift at voice frequencies, and advantage can be taken of this fact in the repeater design, to reduce the over-all attenuation below that obtainable with design methods applicable to electrically long lines. Furthermore, the use of negative impedance devices such as E-type repeaters has made it possible to consider engineering the repeater as an integral part of an electrically short line. This method of design is a logical one because in addition to the reduction in over-all attenuation, the image impedances seen looking into the line terminals are modified by the addition of the repeater. In effect, the philosophy of the hybrid coil and the 22-type repeater is discarded along with the idea that the image impedance of the repeater must match the characteristic impedance of the line. Where the repeater is located a distance less than one quarter wavelength (at a frequency of 4,000 cps) from either line terminal, better transmission performance generally can be obtained by a mismatch between the image impedance of the repeater and the characteristic impedance of the line.

Once a change in philosophy in matching the repeater to the line impedance is made, it becomes easier to forget the repeater as a separate device and to treat it as an integral part of the line in the way a loading coil would be treated. Hence, interest is centered upon the propagation constant and image impedances of the repeatered or loaded line and the transmission characteristics of the device itself are subordinated to this end.

When a two-wire repeater, or its equivalent in the form of a network of active elements, is inserted in a transmission line, stability (freedom from oscillation) becomes a prime consideration. In electrically short lines, the image impedance as well as the loss of the over-all line is a function of the degree of stability desired, which in turn will depend upon the requirements of the system in which the repeatered line must operate.

It is the purpose of this paper to relate transmission characteristics with stability, for a repeater in the form of a symmetrical active network located in the center of an electrically short transmission line. The equations shown and the method of solution are particularly applicable to the fundamental design of E-type negative impedance repeaters in transmission lines wherein the repeater is located less than a quarter wavelength from the line terminals. The frequency at which this wavelength is determined is the highest desired in the pass-band.

The problem is attacked by taking the general case of the symmetrical two-wire repeater located in the center of a transmission line as shown in Fig. 1(a) and substituting for the repeater the equivalent lattice shown in Fig. 1(b). This lattice consists of series arms, $Z_A/2$, and shunt arms, $2Z_B$. The method described herein is general and can be applied to any type of impedance in either the series or the shunt arm. However, be cause the specific application considered here is for the E-type repeater, Z_A is specified as an open circuit stable negative impedance and Z_B is specified as a short circuit stable negative impedance. This is designated on Fig. 1(b) where these impedances are defined as the ratio of two polynominals which are functions of the complex frequency variable p. It is understood that these impedances will have negative resistance components at some real frequencies because the term negative impedance is used herein to describe an impedance whose resistive component is negative within some band of frequencies.

Three conditions are considered:

(a) An E1 or E2 repeater of negative impedance Z_A in series with the line (special case where Z_B is infinite).

(b) An E3 repeater of negative impedance Z_B shunted across the line (special case where Z_A is zero).

(c) The E23 repeater in the line (case of Fig. 1(b), which is the lattice equivalent of the bridged T arrangement of the E23 repeater).

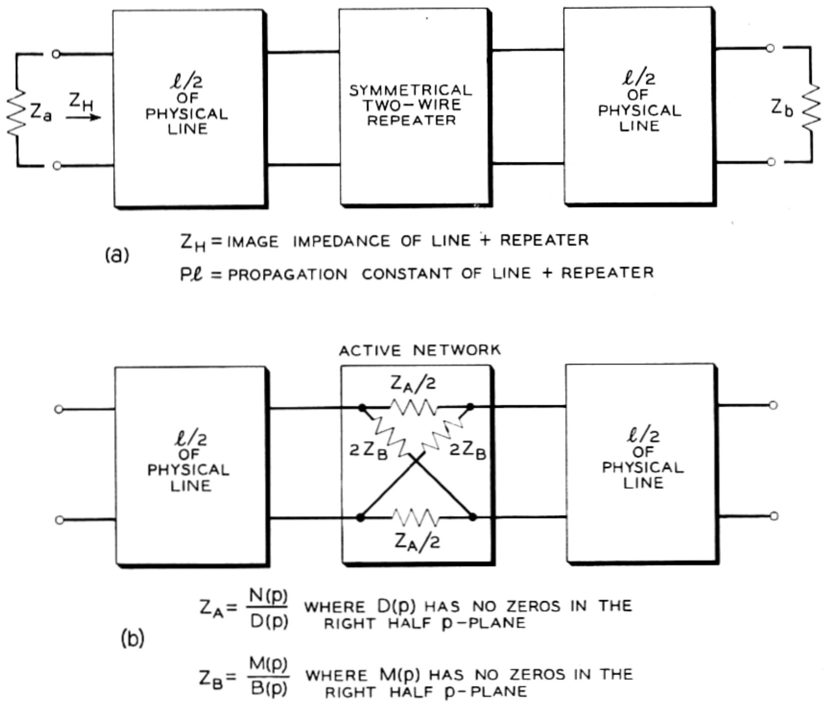


Fig. 1 — Symmetrical two-wire repeater in transmission line. (a) Schematic. (b) Equivalent circuit.

2. GENERAL STABILITY CRITERIA

Before the specific objective is considered, stability criteria will be reviewed briefly and a stability theorem applicable to symmetrical linear four-pole networks will be described.

2.1. *Basic Stability Equation*

The stability of the network of Fig. 1(a) can be determined from an examination of the roots in the complex frequency plane of the equation:

$$1 - \left[\frac{Z_H - Z_a}{Z_H + Z_a}\right] \cdot \left[\frac{Z_H - Z_b}{Z_H + Z_b}\right] e^{-2P\ell} = 0 \tag{1}$$

where:

Z_H = Image impedance of the over-all line

$P\ell$ = Propagation constant of the over-all line

Z_a = Impedance of one line termination

Z_b = Impedance of the other termination

The quantity on the left hand side of the equation is the reciprocal of the interaction factor[2] and its use as a measure of stability has been discussed by F. B. Llewellyn.[3]

As pointed out by Llewellyn Eq. (1) bears a striking similarity to the famous Nyquist equation for stability of feedback amplifiers, usually written

$$(1 - \mu\beta) = 0$$

In both cases the fundamental requirement for stability is that the equations should have no roots in the right half complex frequency plane.

In specific cases, the Nyquist criterion for stability can be applied by plotting on the complex plane as a function of real frequency the factors in (1) which correspond to $\mu\beta$, and seeing whether the plot encircles the point $(1, j0)$.

In general, however, the fact that the point $(1, j0)$ is outside such a plot is not in itself proof of stability. This ambiguity in the interpretation of the diagram can be resolved if the factors involved in (1) are evaluated at complex frequencies.

It should be noted that a separate plot at real frequencies would be required for each combination of terminating impedance Z_a and Z_b. The assumption of particular values for Z_a and Z_b would naturally lead to specialized stability criteria and it was to avoid this that Llewellyn made the alternative assumption that the system should be stable with any combination of passive terminating impedances. Since in a practical telephone system the network of Fig. 1(a) has to be stable when Z_a and Z_b are arbitrary passive impedances, Llewellyn's results are applicable to the cases considered herein. However, since his criterion is stated in terms of the image impedances and loss of the network, it is not in a form which can be readily applied in a design problem involving negative impedance loading.

For design purposes, what is required is a relationship between stability, the properties of the physical line and the negative impedance repeaters. This relationship can be found directly by means of the bisection theorem given in the following section.

2.2. *A Stability Theorem For Active Four Poles*

The symmetrical network of Fig. 1(a) is a particular case of the somewhat more general type of symmetrical structure shown on Fig. 2(a), to which the theorem to be discussed in this section applies.

Referring to Fig. 2(a), N is a symmetrical network in the sense that its external characteristics are such as to make the terminal pairs (1, 1′) and (2, 2′) electrically indistinguishable. For example, N may be a symmetrical T network with fairly obvious symmetry or it may be a two-way repeater with somewhat less apparent structural symmetry.

For simplicity, the theorem will be stated in terms of the network in Fig. 2(b) which has complete structural symmetry in the sense that the networks N_1 and N_2 are the mirror images of each other in the plane of symmetry AB. In this case, the open and short circuit impedances of the bisected network are the impedances looking into the terminals (1, 1′) or (2, 2′) with the terminals in the plane of symmetry AB respectively open and short circuited.

To apply the theorem in the more general case of Fig. 2(a), the open and short circuit impedances of the bisected network should be interpreted as the impedances of the series and shunt arms of the lattice network which is electrically equivalent to N. Methods of determining these impedances by external measurements on the network are discussed by Bode.[4]

In a particular application of the theorem in this paper, the symmetrical network consists of a transmission line of length ℓ with an active network at its center in the form of a lattice structure. This situation is shown in Fig. 1(b) and it is a slightly more complicated form of symmetry than the simple mirror image symmetry of Fig. 2(b). The situation at the middle of this network is indicated in Fig. 2(c) and it can be shown that the arm impedances of the over-all equivalent lattice are given in the manner indicated in the figure. From what has been said above, these impedances will also be the Z_{Short} and Z_{Open} of the theorem.

With regard to the terminating impedances Z_a and Z_b of Fig. 2, the theorem is based on the assumption that the network must be stable when these impedances assume any arbitrary passive values. This is also the requirement in the transmission line problem considered in this paper.

Statement Of The Theorem

A necessary and sufficient condition for a structurally symmetrical linear four-pole to be stable with any combination of passive terminating

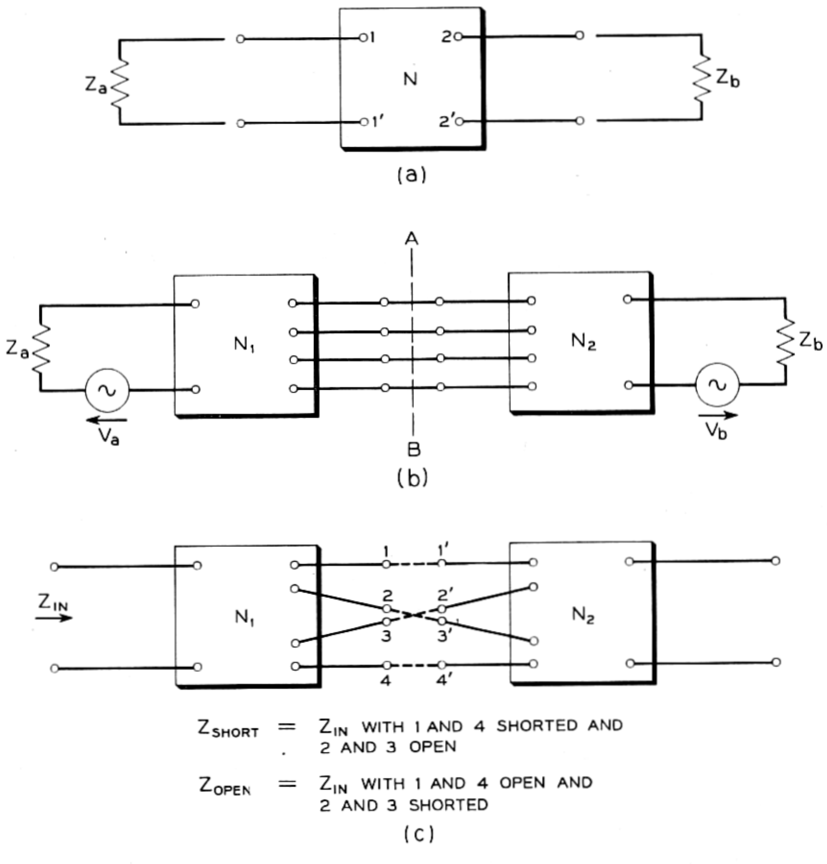


Fig. 2 — Symmetrical linear four-pole with passive terminations. (a) General symmetrical network. (b) Simple bisection. (c) Lattice bisection.

impedances is that the open and short circuit impedances of the bisected network shall be positive real impedance functions. These open and short circuit impedances are the input impedances of either half of the network when the terminals in the plane of bisection are respectively open and short circuited. The term passive impedance as used herein denotes a positive real impedance function.

The theorem is proved in Appendix A and therein also are found the requirements for an impedance function to be positive real.

3. SERIES NEGATIVE IMPEDANCE LOADING

This is the case of Fig. 1(b) where Z_B is infinite and where Z_A is a negative impedance of the open circuit stable type. It also represents the installation of an E1 or E2 repeater in the center of an electrically short transmission line.

3.1. *Stability*

Consider Fig. 3(a) where a negative impedance of the open circuit stable type (Z_A) is shown in the center of a physical transmission line. The problem is to determine the equations which relate transmission characteristics with stability for all passive terminating impedances.

According to the bisection theorem stated in 2.2 the network of Fig. 3(a) will be stable for all passive impedance terminations provided the open and short circuit impedances of the bisected network are positive real. The short circuit impedance of the bisected network is shown in Fig. 3(b) and is represented by Z_H multiplied by Tanh $P\ell/2$. This must be made positive real. The open circuit impedance of the bisected network is shown in Fig. 3(c) and is expressed as Z_H divided by Tanh $P\ell/2$. This open circuit impedance is positive real because it equals the open circuit impedance of one half of the physical line, Z_{OC}. Thus it has no direct bearing on stability but does contribute the relationship:

$$\frac{Z_H}{\operatorname{Tanh} P\ell/2} = \frac{Z_o}{\operatorname{Tanh} \gamma\ell/2} = Z_{OC} \tag{2}$$

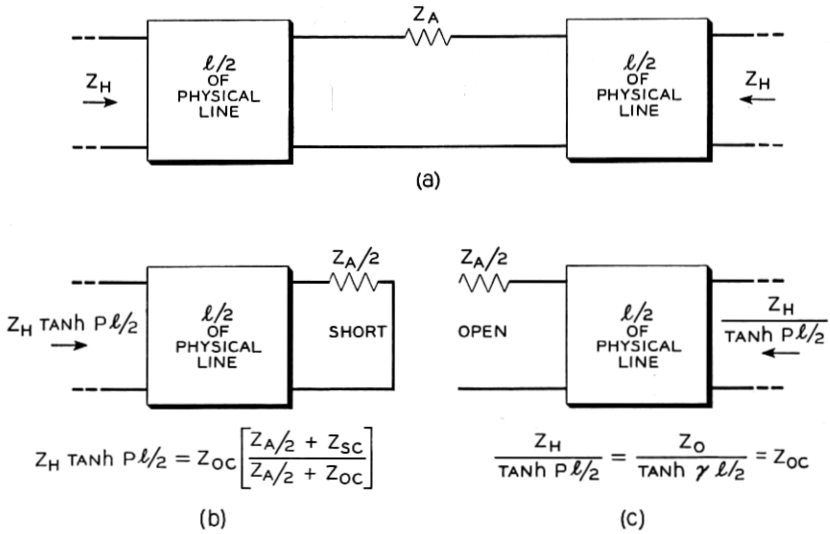
where:

Z_H = Image impedance of the line with Z_A

$P\ell$ = Propagation constant of the line with Z_A

Z_O = Characteristic impedance of the physical line

$\gamma\ell$ = Propagation constant of length ℓ of physical line



where:

Z_H = Image impedance of physical line with Z_A
P = Propagation constant per unit length — line with Z_A
ℓ = Length of physical line
Z_O = Characteristic impedance of physical line
γ = Propagation constant per unit length of physical line
$Z_{OC} = \dfrac{Z_O}{\text{Tanh } \gamma\ell/2}$
$Z_{SC} = Z_O \text{ Tanh } \gamma\ell/2$.

Fig. 3 — Application of bisection theorem to series loading. (a) Schematic. (b) Short circuit impedance of bisected network. (c) Open circuit impedance of bisected network.

Equation (2) demonstrates an important relationship which has been known ever since the discovery of coil loading. It is worthwhile repeating here because the network in Fig. 3(a) is, in fact, a single section of line loaded with a series impedance Z_A. Equation (2) demonstrates that the midsection impedance and propagation constant of the loaded line bear the same relationship to each other as the corresponding parameters of the nonloaded line bear to each other. Thus the general relationship between propagation constant and midsection impedance of a loaded line is to this extent independent of the loading element.

With regard to stability, the application of the bisection theorem to Fig. 3(a) has shown that the basic criterion for stability is that Z_H Tanh $P\ell/2$ must be a positive real impedance function. This impedance can be expressed in terms of physical line parameters and Z_A as follows:

$$Z_H \text{ Tanh } P\ell/2 = Z_O \text{ Tanh } (M + \gamma\ell/2) \tag{3}$$

where

$$M = \text{Tanh}^{-1} \frac{Z_A}{2Z_O}$$

Z_O = Characteristic impedance of the physical line

There are two requirements which must be placed upon Z_O Tanh $(M + \gamma\ell/2)$ for it to be positive real. One of them is that in the following equation, R shall be a positive resistance at all frequencies.

$$Z_O \text{ Tanh } (M + \gamma\ell/2) = R \pm jX \tag{4}$$

The other requirement will be reserved until (4) has been discussed.

The limit of stability will be approached as R approaches zero. If (4) is taken to the limit of stability then:

$$Z_O \text{ Tanh } (M + \gamma\ell/2) = \pm jX \tag{5}$$

where as before

$$M = \text{Tanh}^{-1} \frac{Z_A}{2Z_O}$$

If at a single frequency all values of $Z_A/2$ which satisfy (5) are plotted on the Z-plane, their locus will trace a circle because jX is a straight line and the relationship is a bilinear transformation.[5] Formulas for the centers and radii of these circles are given in Appendix B. If, as is the case with telephone cable, the characteristic impedance Z_O has a negative angle, this trace will appear as shown in Fig. 4. The region inside this

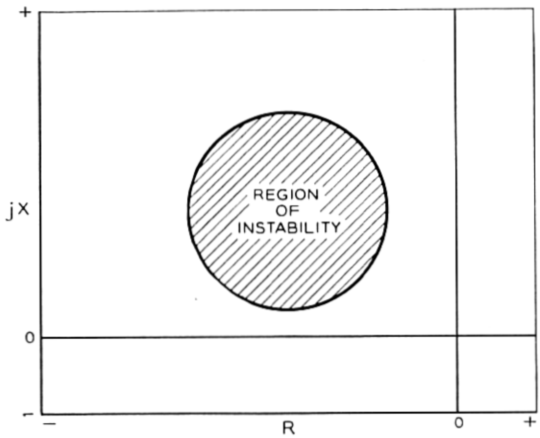


Fig. 4 — Circle for determining the stability of a loading element in a line.

circle corresponds to negative values of R in (4). For stability, the negative impedance $Z_A/2$ must lie outside this circle. However, this is only a part of the stability criterion.

The other requirement on Z_O Tanh $(M + \gamma\ell/2)$ for it to be positive real is discussed in Appendix A3. In order to apply this second requirement it is necessary to expand Z_O Tanh $(M + \gamma\ell/2)$ in terms of the circuit parameters as follows:

$$Z_O \operatorname{Tanh}(M + \gamma\ell/2) = Z_{OC} \left[\frac{\dfrac{Z_A}{2} + Z_{SC}}{\dfrac{Z_A}{2} + Z_{OC}} \right] \tag{6}$$

where

$$Z_{SC} = Z_O \operatorname{Tanh} \gamma\ell/2$$

$$Z_{OC} = \frac{Z_O}{\operatorname{Tanh} \gamma\ell/2}$$

Then from Appendix A3 it can be seen that stability will be obtained providing the real part of $(Z_A/2) + Z_{SC}$ is a positive resistance and providing (4) is satisfied. Thus, the second requirement for stability is that the magnitude of the real part of $Z_A/2$ at real frequencies shall not exceed the real part of Z_{SC}, the short circuit impedance of $\ell/2$ of physical line.

This second requirement while sufficient is not necessary as will be discussed later. However, in many practical applications it is not unduly restrictive.

The graphical meaning of the stability requirements can be seen from Fig. 5. Here a family of stability circles has been drawn for a cable circuit at three different frequencies. As the attenuation increases with frequency the circles decrease in size. At each frequency, $-Z_{SC}$ is shown. It falls on the circumference of the corresponding circle. As the frequency increases, $-Z_{SC}$ will rotate clockwise and at some frequency will be on the left edge of the circle. The first requirement for stability means that at any given frequency $Z_A/2$ cannot lie inside the corresponding circle. The second requirement for stability, means that $Z_A/2$ must lie to the right of $-Z_{SC}$ on Fig. 5 at all frequencies. It also means that the trace of $Z_A/2$ over the frequency range cannot enclose this family of circles. The locus on the Z-plane will be similar to that shown on Fig. 5. This is difficult to show graphically on a single plane because it may

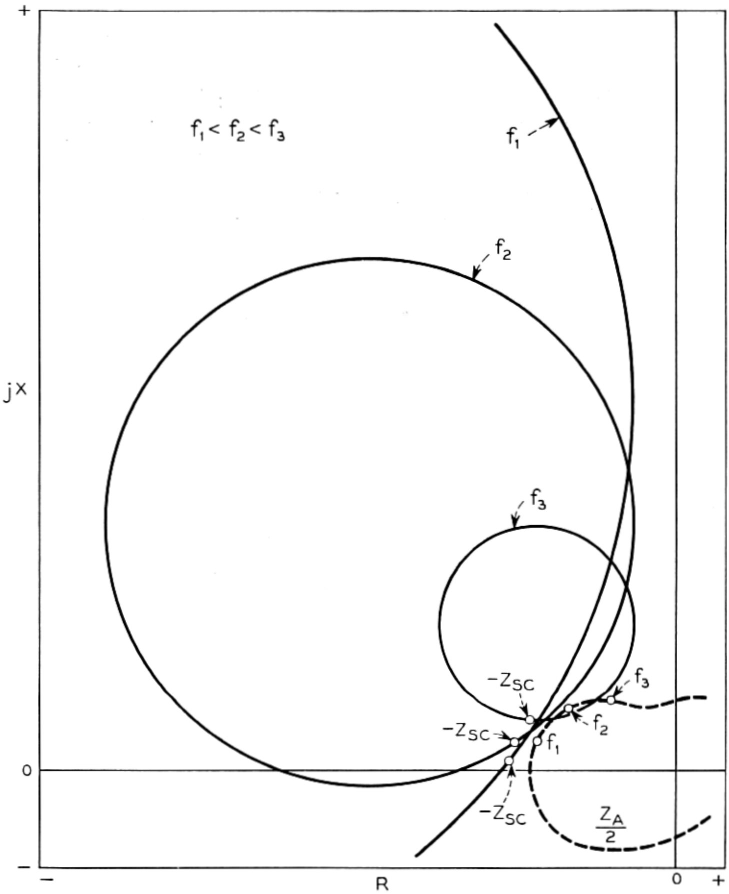


Fig. 5 — Stability circles for $Z_A/2$.

appear that the trace of $Z_A/2$ does go through the family of circles. However, where it apparently passes through the family of circles it does so at a lower frequency. At no given frequency does the trace of $Z_A/2$ lie within its stability circle.

The second requirement for stability, namely that the real part of $(Z_A/2) + Z_{SC}$ should be positive, is a sufficient condition but not a necessary one. A necessary and sufficient condition, when $Z_A/2$ is open circuit stable, is derived in Appendix A3. With reference to Fig. 5 this condition requires that the plot of $Z_A/2$ will not enclose any of the stability circles.

3.2. *Applications*

Two important classes of problems can be solved by applying the stability considerations of the previous section:

(a) The determination of the lowest value of attenuation possible with stability for all passive impedance terminations if the image impedance Z_H is restricted by over-all system requirements;

(b) The determination of Z_A and the allowable variations in it consistent with stability.

The problem of determining the lowest attenuation in a loading section like that of Fig. 3(a) consistent with stability for all passive impedance terminations when the impedance Z_H is given can be solved by the following method.

It has been shown in Section 3.1 that the first requirement for stability is that the real part of the short circuit impedance (shown on Fig. 3 as Z_H Tanh $P\ell/2$) shall have a positive real part at all real frequencies. By substituting for Tanh $P\ell/2$ its value from (2) this requirement may be written as

$$\frac{Z_H^2}{Z_{OC}} = R \pm jX \qquad (7)$$

where R is a positive resistance at all frequencies.

As R approaches zero from positive values, the limit of stability will be approached and in the limit

$$\frac{Z_H^2}{Z_{OC}} = \pm jX \qquad (8)$$

Likewise, by substituting values from (2), (8) can be expressed in the alternate form as

$$Z_{OC} \text{ Tanh}^2 P\ell/2 = \pm jX \qquad (9)$$

If the phase as well as the magnitude of Z_H is to be specified in the problem, it should be noted that Z_H must be specified as a passive impedance. This is necessary because Z_H is the square root of the product of the open and short circuit impedances of the bisected network. Since these separately must be positive real, Z_H must also be positive real. This means that Z_H^2/Z_{OC}, which equals the short circuit impedance of the bisected network, will have no roots in the right half, complex frequency plane. To check for stability in this case all that is required is to insure that (7) is satisfied for the length of the physical line selected. Section length enters into (7) because Z_{OC} is the open circuit impedance of one half this length of line. If the system proved unstable by this check the only recourse would be to solve for a section length which would prove stable.

If the magnitude only of Z_H had been fixed then for each section length there would be a choice of phase angle for Z_H within the limits prescribed by system requirements. For solution in this case (8) could be interpreted to mean that for stability

$$\text{the angle of } \frac{Z_H^2}{Z_{OC}} < 90 \text{ degrees} \tag{10}$$

Eq. (2) can be rearranged as follows:

$$\text{Tanh } P\ell/2 = \frac{Z_H}{Z_{OC}} \tag{11}$$

From (10) and (11) it may be shown that when the magnitude of Z_H is fixed, the closer the loading section is brought to instability by adjusting the angle of Z_H, the lower will be the attenuation of the section. Thus this angle should be adjusted for minimum stability.

When the limit of stability is reached, (8) indicates that Z_H^2/Z_{OC} will be a pure reactance but the question arises as to whether the angle of Z_H should be adjusted to make the sign of the reactance positive or negative. The choice here is based on two observations. First, that if the angle of Z_{OC} is negative, as it generally is for cable circuits, minimum overall attenuation will result when the reactance on the right hand side of (8) is positive. Second, from (6) it may be shown that if the short circuit impedance of the bisected network is a positive reactance, $Z_A/2$ will lie on the circumference of the stability circles of Fig. 5 on the arc to the right of $-Z_{SC}$. In the case of lines such as are under consideration here, where the top frequency in the pass band is less than one quarter wave length, the second stability requirement which is discussed in Section 3.1 thereby will be satisfied within the pass band of frequencies.

Therefore, when the magnitude alone of Z_H has been specified the design procedure is to select an angle for Z_H such that the angle of Z_H^2/Z_{OC} is just less than $+90$ degrees. In (8), Z_H^2/Z_{OC} then will be nearly equal to $+jX$ and the magnitude of X will be given by the ratio of the magnitude of Z_H^2 to the magnitude of Z_{OC}. The magnitude and angle of Z_H now is determined for any chosen frequency in the band and the value of the negative impedance $Z_A/2$ required to give the desired value of Z_H can be obtained from the equation:

$$Z_A = 2Z_{OC} \frac{Z_O^2 - Z_H^2}{Z_H^2 - Z_{OC}^2} \tag{12}$$

where Z_O is the characteristic impedance of the nonloaded line.

The value found for Z_A may turn out to be unrealizable as such. It will have to be synthesized and a compromise made. The practical value can be checked for stability by the graphical method explained in Section 3.1. Here any correction for stability can be made which might be necessary. With the realizable value of Z_A, the final values for attenuation and also for Z_H then will have to be found.

The value of Z_H can be obtained from the following equation.

$$Z_H = Z_{OC} \sqrt{\frac{\frac{Z_A}{2} + Z_{SC}}{\frac{Z_A}{2} + Z_{OC}}} \tag{13}$$

The propagation constant P can be found from

$$\text{Tanh } P\ell/2 = \sqrt{\frac{\frac{Z_A}{2} + Z_{SC}}{\frac{Z_A}{2} + Z_{OC}}} \tag{14}$$

The attenuation per section with $Z_A/2$ included can be determined from the following equation wherein the logarithmic rather than the hyperbolic form has been used.

$$\text{Attenuation (db per section)} = 20 \log_{10} \left| \frac{1 + \frac{Z_H}{Z_{OC}}}{1 - \frac{Z_H}{Z_{OC}}} \right| \tag{15}$$

and the angle of $\left[1 + \frac{Z_H}{Z_{OC}} \Big/ 1 - \frac{Z_H}{Z_{OC}} \right]$ in degrees is the phase shift of the loaded line section.

The class of problem represented by (b) at the head of this section can be solved by the graphical method for determining the stability of $Z_A/2$ as outlined in Section 3.1.

3.3. *Characteristics*

This type of loading has several interesting characteristics. First, zero attenuation over any frequency band, no matter how narrow, when $Z_A/2$ is adjusted to the limit of stability as defined by (5) and (6) is unrealizable. This can be seen from (14) together with the fact that Z_{SC} and Z_{OC} represent passive impedances and that $Z_A/2$ is a negative impedance of the open circuit stable type. All three impedances when shown on the impedance plane with increasing frequency will rotate in the clockwise direction. For stability, the locus of $Z_A/2$ must not enclose either $-Z_{SC}$ or $-Z_{OC}$. At the limit of stability, however, $Z_A/2$ must lie on the circles of stability which are shown on Fig. 5. This means a changing relationship between these three impedances with frequency which is incompatible with zero attenuation over any frequency band. Second, a flat response can be realized over a band of frequencies, in general, only at the expense of increasing the loss at the lower frequencies above that required for stability. Third, as the length of the physical line is increased, it becomes more and more difficult to obtain a low over-all loss and yet avoid the enclosure of the stability circles of Fig. 5 with a realizable design of $Z_A/2$. The practical limit here appears to be one quarter wave length of physical line at the highest frequency it is desired to pass.

4. SHUNT NEGATIVE IMPEDANCE LOADING

This case where Z_B is shunted across the line conductors at the mid-point of a physical line (Fig. 1(b) where Z_A is zero) can be classed as shunt type negative impedance loading. The negative impedance Z_B is of the short circuit stable type such as the impedance produced by the E3 repeater. Hence, this case can represent an E3 repeater bridged across the conductors of an electrically short transmission line.

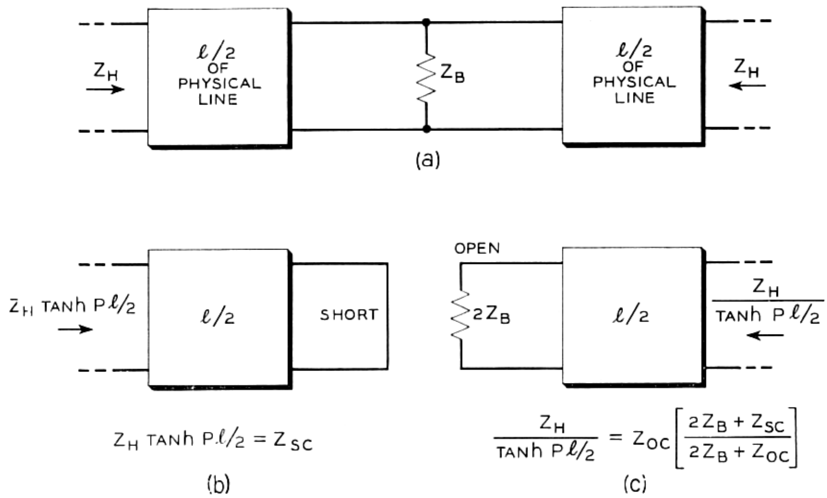


Fig. 6 — Application of bisection theorem to shunt loading. (a) Schematic. (b) Short circuit impedance of bisected network. (c) Open circuit impedance of bisected network.

4.1. *The Stability Equations*

The same method described in detail in Section 3.1 is used here to determine stability. The short and open circuit impedances of the bisected network are obtained as shown in Fig. 6.

As seen from Fig. 6(b) the short circuit impedance of the bisected network is positive real being equal to Z_{SC}, the short circuit impedance of the physical line of length $\ell/2$.

$$Z_H \text{ Tanh } P\ell/2 = Z_O \text{ Tanh } \gamma\ell/2 = Z_{SC} \tag{16}$$

The open circuit impedance of the bisected network [Fig. 6(c)] determines stability. Thus for stability:

$\dfrac{Z_H}{\text{Tanh } P\ell/2}$ must be a positive real impedance function.

A substitution from (16) for Tanh $P\ell/2$ above will yield the following requirement for stability:

$$\frac{Z_H^2}{Z_{SC}} = R \pm jX \tag{17}$$

where R must be a positive resistance and Z_H a positive real impedance function.

The limit of stability is reached as R goes to zero. Therefore, the limit of stability for all passive impedance terminations can be expressed by:

$$\frac{Z_H^2}{Z_{SC}} = \pm jX \tag{18}$$

A similar equation can be obtained in terms of Tanh $P\ell/2$ rather than Z_H.

$$\frac{Z_{SC}}{\text{Tanh}^2 P\ell/2} = \pm jX \tag{19}$$

4.2. *Analysis*

From Eq. (18) the sole criterion for stability is that the angle of

$$\frac{Z_H^2}{Z_{SC}} < 90° \tag{20}$$

provided that Z_H is a positive real impedance function.

The value of the propagation constant can be found from (16) to be:

$$\text{Tanh } P\ell/2 = \frac{Z_{SC}}{Z_H} \tag{21}$$

and from the short circuit and open circuit impedances, Fig. 6(b) and Fig. 6(c), the following are found

$$\text{Tanh } P\ell/2 = \sqrt{\frac{Z_{SC}(2Z_B + Z_{OC})}{Z_{OC}(2Z_B + Z_{SC})}} \tag{22}$$

and

$$Z_H = \sqrt{Z_{SC}Z_{OC}\left[\frac{2Z_B + Z_{SC}}{2Z_B + Z_{OC}}\right]} \tag{23}$$

In order to translate stability into engineering parameters, the open circuit impedance, $Z_H/\text{Tanh } P\ell/2$, can be expressed in parameters of the physical line and Z_B.

$$\frac{Z_H}{\text{Tanh } P\ell/2} = Z_{OC}\left[\frac{2Z_B + Z_{SC}}{2Z_B + Z_{OC}}\right] \tag{24}$$

By reasoning similar to that used in Appendix A and in Section 3.1 it can be shown that the requirement for stability will be met if

$$Re\left(\frac{1}{2Z_B} + \frac{1}{Z_{OC}}\right) \text{ is positive} \tag{25}$$

where Z_B is a negative impedance of the short circuit stable type and providing

$$Z_{OC}\left[\frac{2Z_B + Z_{SC}}{2Z_B + Z_{OC}}\right] = R \pm jX \tag{26}$$

where R is a positive resistance at all frequencies.

The limit of stability will be reached as R approaches zero.

If at any given frequency all values of $2Z_B$, which fulfill (26) when R is zero, are plotted on the Z-plane they will trace out the same circle as found before in Section 3.1 where Z_A was likewise examined (if the frequency and physical line parameters are the same in both cases). Thus $2Z_B$ should not lie inside the stability circle of Fig. 4.

However, in order to meet the restriction imposed by (25) upon Z_B, the locus of $2Z_B$ when plotted over the frequency range from zero to infinity must enclose this family of circles as shown in Fig. 7. This can be established in much the same way as in Section 3.1 where it was proved that the locus of $Z_A/2$ must *not* enclose this family of circles.

Here, in the case of shunt loading, as in the case of series loading, zero attenuation is inconsistent with stability for all passive impedance terminations. Likewise, with shunt loading designed for minimum stability over the pass band the attenuation will vary with frequency.

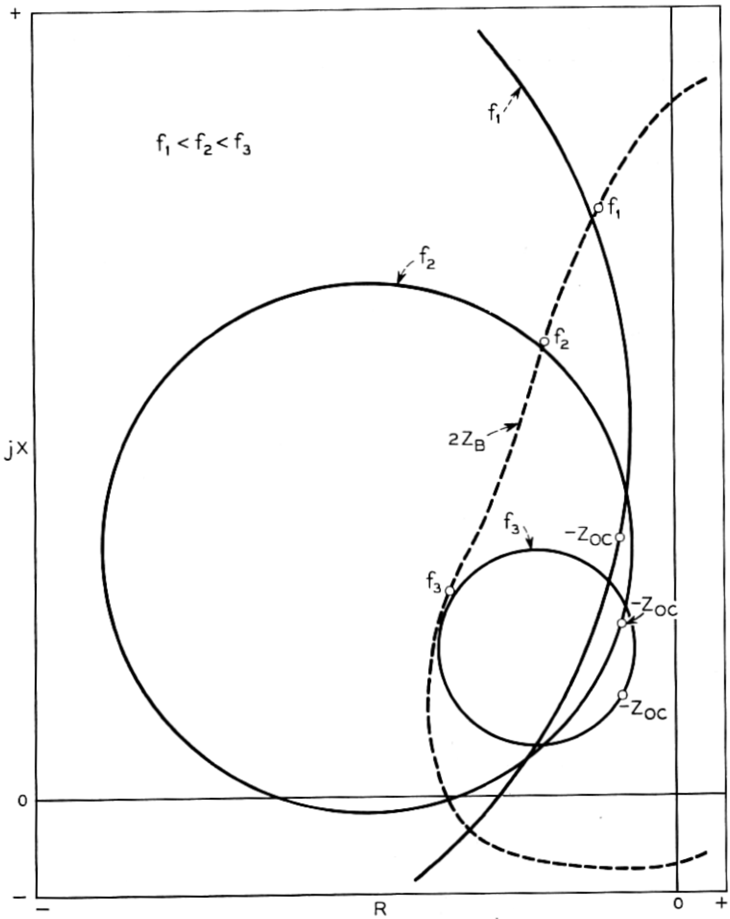


Fig. 7 — Stability circles for $2Z_B$.

5. LATTICE LOADING

The general case of Fig. 1(b) is where the combination of an E2 and an E3 repeater is located in the center of an electrically short line. Although the actual E23 repeater is connected as a bridged T arrangement[1] the equivalent lattice form is used herein for simplicity in explanation. What is said as applied to the lattice structure applies also to the bridged *T*.

Fig. 8(a) shows a lattice network of negative impedances connected at the midpoint of a line of length ℓ. Negative impedance Z_A is open circuit stable; Z_B is short circuit stable.

The short circuit impedance Z_H Tanh $P\ell/2$ is shown in Fig. 8(b). It is the same as that shown in Fig. 3(b) for the case where Z_A is used alone. Thus a requirement for stability in the limit is that

$$Z_{OC}\left[\frac{\frac{Z_A}{2}+Z_{SC}}{\frac{Z_A}{2}+Z_{OC}}\right] = \pm jX \tag{27}$$

where $Re\ [(Z_A/2) + Z_{SC})]$ is positive.

Stability can be determined exactly as explained in Section 3.1.

The open circuit impedance $Z_H/\text{Tanh } P\ell/2$ is shown in Fig. 8(c). It is the same as the case where Z_B is used alone. Thus the limiting requirement for stability is that

$$Z_O\left[\frac{2Z_B+Z_{SC}}{2Z_B+Z_{OC}}\right] = \pm jX \tag{28}$$

where

$$Re\left(\frac{1}{2Z_B}+\frac{1}{Z_{OC}}\right) \text{ is positive.}$$

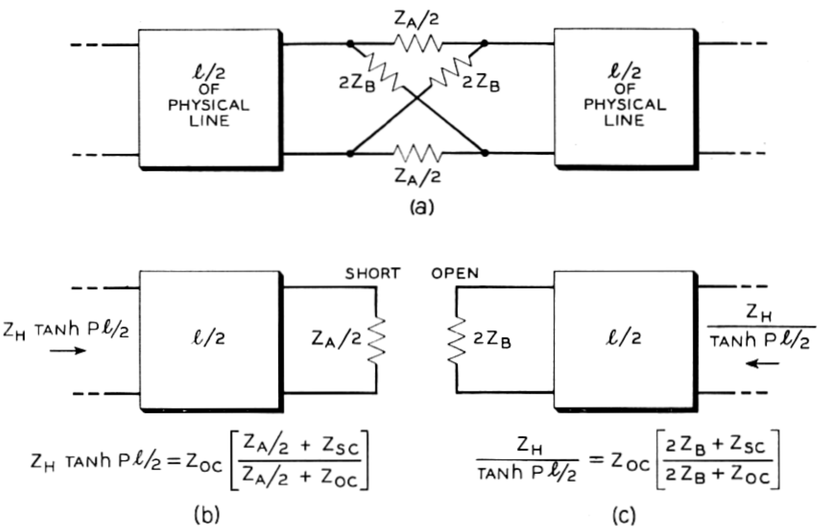


Fig. 8 — Application of bisection theorem to lattice loading. (a) Schematic. (b) Short circuit impedance of bisected network. (c) Open circuit impedance of bisected network.

Stability can be determined as explained in Section 4.2 for (26).

Thus stability with Z_A is determined independently of Z_B and the converse is true also. In regard to stability each negative impedance can be designed without considering the other. The image impedance, Z_H, and propagation constant, $P\ell$, of the resulting line [Fig. 8(a)] will depend upon both Z_A and Z_B, however.

Equations for the image impedance Z_H and the propagation constant of the repeatered line can be expressed as follows:

$$Z_H = Z_{OC} \sqrt{\frac{\left[\frac{Z_A}{2} + Z_{SC}\right][2Z_B + Z_{SC}]}{\left[\frac{Z_A}{2} + Z_{OC}\right][2Z_B + Z_{OC}]}} \tag{29}$$

and

$$P\ell/2 = \text{Tanh}^{-1} \sqrt{\frac{\left[\frac{Z_A}{2} + Z_{SC}\right][2Z_B + Z_{OC}]}{\left[\frac{Z_A}{2} + Z_{OC}\right][2Z_B + Z_{SC}]}} \tag{30}$$

From what has been said in the preceding sections it should be evident that when both Z_A and Z_B are designed to the limit of stability in a telephone cable section the image impedance and propagation constant will be as follows:

$$Z_H = \sqrt{(+jX_1)(-jX_2)} \tag{31}$$

and

$$\text{Tanh}\, P\ell/2 = \sqrt{\frac{+jX_1}{-jX_2}} \tag{32}$$

where

$$+jX_1 = Z_O \text{ Tanh } (M + \gamma\ell/2)$$

$$M = \text{Tanh}^{-1} Z_A/2Z_O \qquad \text{at limit of stability}$$

$$-jX_2 = Z_O \text{ Tanh } (N + \gamma\ell/2)$$

$$N = \text{Tanh}^{-1} 2Z_B/Z_O \qquad \text{at limit of stability}$$

From these last two equations it is apparent that at the limit of stability Z_H is a resistance in the pass band and the attenuation of the repeatered section can be zero and the system be stable for all passive impedance terminations. Furthermore, zero db attenuation can be realized theoretically over the pass band.

6. SUMMARY

The stability and transmission characteristics have been outlined for a single section considering three separate systems of negative impedance loading and a summary is shown in Fig. 9. A general practical restriction on the use of these systems is that the negative impedance or impedances shall be located in the center of the section and shall be less than one quarter wave length from the line terminals at the highest frequency it is desired to pass. The condition for stability has been taken that each section must be stable for all passive impedance terminations.

The important features can be outlined as follows.

6.1. *Series Negative Impedance Loading*

With a series loading element, Z_A, stability is determined solely by the short circuit impedance of the bisected section. The attenuation of the section must be finite for stability. Where the loading section is designed to the limit of stability, the transmission-frequency response will vary with frequency in the pass band; and the magnitude of the image impedance $|Z_H|$ will tend to increase with frequency within this band. A flat transmission-frequency response is possible only at the expense of greater loss at the lower frequencies than is required for stability.

6.2. *Shunt Negative Impedance Loading*

With a shunt loading element, Z_B, stability is determined solely by the open circuit impedance of the bisected section. The attenuation of the section must be finite for stability. Where the loading element is designed to the limit of stability, the attenuation will vary with frequency in the pass band and increase at frequencies outside the band. The magnitude of the image impedance $|Z_H|$ will tend to decrease in the pass band as the frequency increases.

A flat transmission-frequency response with shunt loading is possible only at the expense of greater loss at the higher frequencies than is required for stability.

6.3. *Loading with a Lattice or Equivalent Bridged T Network*

Loading with a lattice network having series arms of $Z_A/2$ and shunt arms of $2Z_B$ both of which are negative impedances, the former open circuit stable, the latter short circuit stable, will have the following characteristics.

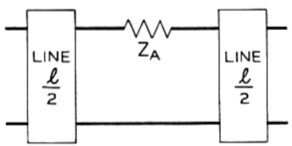
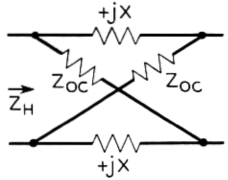
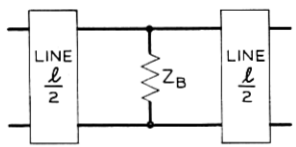
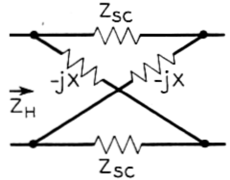
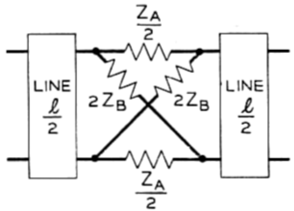
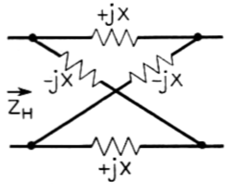
TYPE OF LOADING	SCHEMATIC	AT THE LIMIT OF STABILITY: OVERALL EQUIVALENT CIRCUIT	AT THE LIMIT OF STABILITY: STABILITY REQUIREMENT	TRANSMISSION CHARACTERISTICS (OF LOADED LINE)
SERIES			$Z_{OC}\left[\frac{\frac{Z_A}{2}+Z_{SC}}{\frac{Z_A}{2}+Z_{OC}}\right] = +jX$ WHERE: $Re\left(\frac{Z_A}{2}+Z_{SC}\right)$ IS POSITIVE	$\text{TANh } P\frac{\ell}{2} = \sqrt{\frac{\frac{Z_A}{2}+Z_{SC}}{\frac{Z_A}{2}+Z_{OC}}}$ $Z_H = Z_{OC}\sqrt{\frac{\frac{Z_A}{2}+Z_{SC}}{\frac{Z_A}{2}+Z_{OC}}}$
SHUNT			$Z_{OC}\left[\frac{2Z_B+Z_{SC}}{2Z_B+Z_{OC}}\right] = -jX$ WHERE: $Re\left(\frac{1}{2Z_B}+\frac{1}{Z_{OC}}\right)$ IS POSITIVE	$\text{TANh } P\frac{\ell}{2} = \sqrt{\frac{Z_{SC}}{Z_{OC}}\left[\frac{2Z_B+Z_{OC}}{2Z_B+Z_{SC}}\right]}$ $Z_H = \sqrt{Z_{SC}Z_{OC}\left[\frac{2Z_B+Z_{SC}}{2Z_B+Z_{OC}}\right]}$
LATTICE			REQUIREMENT SAME AS FOR SERIES PLUS THAT FOR SHUNT	$\text{TANh } P\frac{\ell}{2} = \sqrt{\left[\frac{\frac{Z_A}{2}+Z_{SC}}{\frac{Z_A}{2}+Z_{OC}}\right]\left[\frac{2Z_B+Z_{OC}}{2Z_B+Z_{SC}}\right]}$ $Z_H = Z_{OC}\sqrt{\left[\frac{\frac{Z_A}{2}+Z_{SC}}{\frac{Z_A}{2}+Z_{OC}}\right]\left[\frac{2Z_B+Z_{SC}}{2Z_B+Z_{OC}}\right]}$

Fig. 9 — Summary.

The short circuit impedance of the bisected section will determine the stability of the negative impedances in the series arms. The open circuit impedance of the bisected section will determine the stability of the negative impedances in the shunt arms.

Zero attenuation is theoretically possible as a limit in the pass band of frequencies consistent with stability for all passive impedance terminations.

The image impedance for zero attenuation will be a positive resistance

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Appendix A

To prove that: A necessary and sufficient condition for a structurally symmetrical linear four-pole to be stable with any combination of passive terminating impedances is that the open and short circuit impedances of the bisected network shall be positive real. These open and short circuit impedances are the input impedances of either half of the network when the terminals in the plane of bisection are respectively open and short circuited.

A.1. *Proof of Necessity*

Consider the network of Fig. 2(b), in the text, representing a linear four-pole which is structurally symmetrical in the sense that the right half of the network is the mirror image of the left half in the plane of symmetry AB.

Assuming the network is stable, the necessity of the condition in the theorem will be established if the open and short circuit impedances of the bisected network are shown to be positive real. Stability is used here in the sense that the response to an impressed signal will die out upon removal of the excitation.

For an impedance function $Z(p)$ of the complex frequency variable $p = \alpha + i\omega$ to be positive real it is sufficient to show that the following four conditions are satisfied.[6]

1. $Z(p)$ has no zeros in the right half p-plane.

2. Zeros of $Z(p)$ on the boundary of the right half p-plane are simple and at them dZ/dp = a positive real constant.

3. The real part of $Z(i\omega) \geqq 0$ for all values of ω.

4. The imaginary part of $Z(p) = 0$ whenever the imaginary part of $p = 0$.

The fourth condition is always satisfied by physical networks and will be assumed true without proof.

To show that condition three is satisfied, consider the determinant Δ of the entire network in Fig. 2(b) in terms of its open circuit impedance parameters and the arbitrary passive terminations Z_a and Z_b

$$\Delta = \begin{vmatrix} Z_{11} + Z_a & Z_{12} \\ Z_{12} & Z_{11} + Z_b \end{vmatrix} \qquad \text{(A1)}$$

Since the network is stable by hypothesis, Δ as a function of the complex variable p can have no zeros in the right half of the p-plane. Since the definition of stability requires that a response will die out on the removal of the excitation, zeros of Δ on the boundary of the right half p-plane are excluded.

If $Z_a = Z_b = Z$, in Eq. (A1), Δ may be expanded into the product of two factors as follows

$$\Delta = (Z_{11} + Z - Z_{12})(Z_{11} + Z + Z_{12}) \qquad \text{(A2)}$$

From what has been said above, neither of the factors in (A2) can have zeros on the imaginary p axis or in other words at real frequencies.

If $Z_{11} = R_{11} + jX_{11}$ or in general if $Z_{rs} = R_{rs} + jX_{rs}$, equation (A2) may be rewritten in the following form.

$$\Delta = [R_{11} - R_{12} + R + j(X_{11} - X_{12} + X)] \\ [R_{11} + R_{12} + R + j(X_{11} + X_{12} + X)] \qquad \text{(A3)}$$

Remembering that Z is an arbitrary passive terminating impedance, X can always be chosen to nullify either of the imaginary parts in the above two factors. Moreover, since neither factor has a root at real frequencies and since R can be given any positive value, it is obviously necessary that their real parts shall be positive, thus:

$$R_{11} - R_{12} + R > 0 \qquad R_{11} + R_{12} + R > 0 \qquad \text{(A4)}$$

Since R, the real part of the terminating impedance, is not negative, the limiting situation in the above conditions will occur when $R = 0$ and it follows that both $R_{11} - R_{12}$ and $R_{11} + R_{12}$ must be positive. It can also be concluded from this that R_{11} is positive, though this is obvious from the fact R_{11} is the resistive component of the open circuit input impedance of a stable network.

The result just established which may be expressed by stating that $R_{11} > |R_{12}|$, is identical to the Gewertz condition for symmetrical linear networks mentioned by F. B. Llewellyn.[3]

From standard network theory, the open and short circuit input impedances of the bisected network of Fig. 2(b) are given by the following equations

$$Z_{\text{Open}} = Z_{11} + Z_{12} \qquad Z_{\text{Short}} = Z_{11} - Z_{12} \tag{A5}$$

where

Z_{Open} = Open circuit input impedance of the bisected network

Z_{Short} = Short circuit input impedance of the bisected network

By applying the Gewertz condition to (A5) it is clear that the open and short circuit impedances must have positive real parts which establishes requirement 3 for a positive real function. To show that Z_{Open} and Z_{Short} satisfy conditions 1 and 2 for a positive real function, set Z equal to zero in equation (A2). This reduces Δ to the product of Z_{Open} and Z_{Short}. Since Δ has no roots inside or on the boundary of the right half p-plane this must also be true of Z_{Open} and Z_{Short}. Hence, they each meet all the requirements for positive real functions which completes the proof of necessity.

A.2. *Proof of Sufficiency*

In the proof of sufficiency, Z_{Open} and Z_{Short} are assumed to be positive real and it must be shown that the network of Fig. 2(b) is stable when terminated in arbitrary passive impedances.

The proof depends on Bartlett's Bisection Theorem.[4] According to this theorem, a lattice network with arm impedances Z_{Open} and Z_{Short} will have exactly the same external characteristics as the symmetrical network of Fig. 2(b). Since Z_{Open} and Z_{Short} are assumed to be positive real, the lattice arms can be realized with passive impedances. This means that the lattice network will be stable with any combination of passive terminating impedances and it follows that this must likewise be true of the equivalent circuit of Fig. 2(b).

A.3. *Special Applications of the Theorem*

In some special applications of the stability theorem it is only necessary to ensure that the open circuit impedance and the short circuit impedance of the bisected symmetrical network have positive real parts. This is more lenient than that these impedances shall be positive real as stated in the theorem. The other main condition for positive realness which requires that the roots be located in the left half p-plane (complex frequency-plane) will be guaranteed automatically by placing special requirements on some of the network elements.

As an example of such a situation consider the circuit of Fig. 3(a) consisting of a transmission line of length ℓ with a negative impedance Z_A located at its center.

If Z_{OC} and Z_{SC} are respectively the open and short circuit impedances of the nonloaded line of length $\ell/2$ the open and short circuit impedances of the same length of line with loading may be written down as follows.

$$Z_{\text{Open}} = Z_{OC} \tag{A6}$$

$$Z_{\text{Short}} = Z_{OC} \frac{\left[\frac{Z_A}{2} + Z_{SC}\right]}{\left[\frac{Z_A}{2} + Z_{OC}\right]} \tag{A7}$$

In this example, the open circuit impedance of the bisected network Z_{Open}, is obviously positive real since it equals the open circuit impedance of a length $\ell/2$ of nonloaded line, which is passive.

It will now be shown that a sufficient condition for the short circuit impedance Z_{Short} of the bisected network to be positive real is that its resistive component shall be positive at all frequencies, providing that Z_A is a negative impedance of the open-circuit-stable type having a resistive component which is always less in magnitude than the real part of $2Z_{SC}$.

To show this, assume the real part of Z_{Short} is positive and that all the impedances being considered are rational. To satisfy the rationality requirement in the case of transmission line impedances, the line may be considered as the limit of a lumped element network. Since Z_A is open circuit stable by hypotheses, it can have no poles inside or on the boundary of the right half p-plane. Likewise, since Z_{OC} and Z_{SC} are passive impedances they have neither poles nor zeros in the right half p-plane. With these facts in mind, consider the expression on the right hand side of equation (A7). The only zeros which this function can have are those due to $(Z_A/2) + Z_{SC}$. As the complex variable p traces a path around the boundary of the right half complex frequency plane, the impedance function $(Z_A/2) + Z_{SC}$ will trace out a closed curve in the Z-plane. Since it has been assumed that the magnitude of the real part of $Z_A/2$ is always less than the real part of Z_{SC} the closed curve will lie entirely in the right half Z-plane and cannot therefore enclose the origin. It follows from complex variable theory[7] and the rational nature of all the impedance functions involved that $(Z_A/2) + Z_{SC}$ must have an equal number of zeros and poles in the right half p-plane. Since it has no poles in that area it has no zeros either hence, Z_{Short} has no zeros in, or on the boundary of the right half p-plane.

If, in addition, Z_{Short} has a positive real part as assumed, requirements 1, 2 and 3 for a positive real function are met. Taking requirement 4 as being true without proof, it follows that Z_{Short} is a positive real function.

Thus, if Z_{Short} has a positive real part and Z_A has the properties attributed to it above, the network of Fig. 3(a) will be stable with arbitrary passive terminations.

As mentioned at the beginning of this section, the situation just considered relates to a sufficient condition for stability when the impedance Z_A is open circuit stable.

A necessary condition for stability when Z_A is open circuit stable, may be obtained from an examination of (A7). As pointed out in considering the sufficient condition for stability, the only roots which Z_{Short} can have in the right half p-plane are due to the factor $(Z_A/2) + Z_{SC}$. Since this factor has no poles in the right half p-plane it follows from complex variable theory[7] that if $Z_A/2 + Z_{SC}$ is plotted on the Z plane as a function of real frequency, the number of times the plot encircles the origin gives the number of zeros which the function has in the right half p-plane. Since Z_{Short}, for stability, can have no zeros in this area, it follows that the plot of $Z_A/2 + Z_{SC}$ must not encircle the origin. This last statement is equivalent to saying that $Z_A/2$ must not encircle $-Z_{SC}$.

If the necessary condition just established, is combined with the stability requirement that $Z_A/2$ cannot enter the stability circles discussed in connection with Fig. 5, it will be seen readily that a necessary and sufficient condition for stability may be laid down as follows.

If in the circuit of Fig. 3(a), Z_A is open circut stable then a necessary and sufficient condition for the transmission line with series loading to be stable is that the plot of $Z_A/2$ as a function of real frequency shall not encircle any of the stability circles associated with the line of length $\ell/2$. These stability circles are shown in Fig. 5.

Appendix B

EQUATIONS FOR THE STABILITY CIRCLE

At any given frequency the equation for the stability circle can be expressed in the following formulas in terms of the rectangular coordinates (R, X) of the center and the radius (r).

$$R = -\frac{(R_I^2 - X_I^2) + (R_{OC}^2 + X_{OC}^2)}{2\,R_{OC}} \tag{B1}$$

$$X = \frac{X_I R_I}{R_{OC}} \tag{B2}$$

$$r = \sqrt{R^2 + X^2 - \left[R_I^2 - X_I^2 + \frac{2X_{OC}X_IR_I}{R_{OC}}\right]} \tag{B3}$$

where:

R_I = The resistance component of the characteristic impedance, Z_O, of the physical line

X_I = The reactance component of the characteristic impedance, Z_O, of the physical line

R_{OC} = The resistance component of the open circuit impedance of $\ell/2$ of physical line

X_{OC} = The reactance component of the open circuit impedance of $\ell/2$ of physical line

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