Boolean Matrices and the Design of Combinational Relay Switching Circuits

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(Manuscript received August 30, 1954)

The p-terminal generalization of a two-terminal switching function is a matrix of switching functions representing the conditions under which the terminals are interconnected. The properties of these "switching matrices" are studied, and examples are given to show how they may be employed effectively in the design of switching circuits. Some basic problems are outlined and a bibliography is attached.

CONTENTS

1. INTRODUCTION

Matrices over a Boolean algebra, or simply *Boolean matrices*, are rectangular arrays of elements from a Boolean algebra. These arrays are subject to appropriate rules of operation, some of which are analogous to the rules of operation for ordinary matrices, whereas others reflect the

Boolean character of the elements. The purpose of this paper is to present those properties of Boolean matrices which have application to the design of combination relay logic circuits, and to develop fundamental aspects of this application.

In this paper, we shall assume a knowledge of the elementary aspects of Boolean algebra and of its application to the design of switching circuits.[1,2] We shall use the following notations for the Boolean operations:

$+$ denotes the Boolean sum or union,

$\cdot$ denotes the Boolean product or intersection,

$\leqq$ denotes inclusion.

In order to be able to operate most naturally with Boolean matrices, we use the system in which the parallel connection of contacts x and y is represented by $x + y$ while their series connection is represented by $x \cdot y$ or just xy. We use 0 to denote an open circuit or contact and 1 to denote a closed circuit or contact. The Boolean algebra from which the elements of our Boolean matrices are selected is the set "S" of 2^{2^n} Boolean or switching functions of n variables $x_1, x_2, \cdots, x_n$.

2. MATRICES ASSOCIATED WITH COMBINATIONAL CIRCUITS

We shall be concerned with the analysis and synthesis of combinational relay circuits, that is, circuits which may be represented symbolically as in Fig. 1. Here there are indicated n coils $x_1, x_2, \cdots, x_n$, which determine respectively the conditions $x_1, x_2, \cdots, x_n$ of contacts in the box. We call $x_1, x_2, \cdots, x_n$ the *inputs* or *input variables* of the circuit. The outputs of the circuit are the interconnections between the terminals $1, 2 \cdots, p$, which are established as a result of energization of certain of the coils. It is assumed that the contacts of the circuit are all in the box and that these contacts are operated solely by the coils

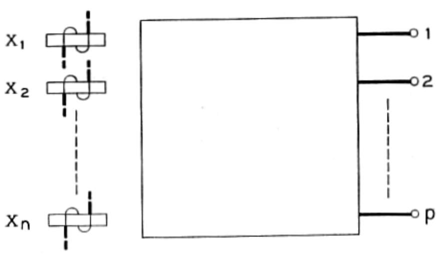


Fig. 1

$X_1, X_2, \cdots, X_n$, which are controlled entirely from outside the box. The design of such circuits is discussed in Reference 3, Chapter 6.

In a combinational circuit, after a brief operate-time, the state f_{ij} of the connection between the terminals i and j depends only on the combination of values assumed by the input variables $x_1, x_2, \cdots, x_n$, and hence may be represented as a Boolean function of these variables:

$$f_{ij} = f_{ij}(x_1, x_2, \cdots, x_n).$$

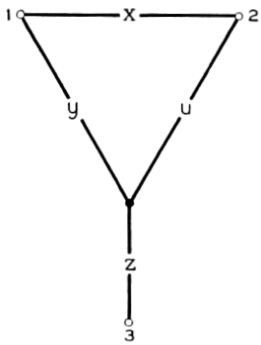
(Since a terminal i is always connected to itself, we define $f_{ii} = 1$ for each i.) The p^2 functions so obtained may be used as the elements of a $p \times p$ symmetric Boolean matrix which we call the output matrix "F" of the circuit:

$$F = [f_{ij}].$$

Thus, for example, the output matrix of a simple, three-terminal circuit is illustrated in Fig. 2. (In Fig. 2 and in succeeding figures, small rings are used to denote the p terminals of a network, and black dots are used to denote non-terminal nodes which simply serve as connecting points.)

Two p-terminal combinational circuits with the same output matrix are called *equivalent*. Equivalence is denoted by the symbol "$\sim$" in this paper.

With a given circuit, we can associate a second type of Boolean matrix in the following manner. First we select and number certain nodes in the circuit, using the numbers $p + 1, p + 2, \cdots, p + k$. These we call *non-terminal nodes* to distinguish them from the *terminal nodes* $1, 2, \cdots, p$ of the circuit. The non-terminal nodes are so chosen that between any two of the $p + k$ nodes of the circuit there appears at most a single contact or a group of single contacts in parallel. Moreover, we assume that every



$$\begin{bmatrix} 1 & x+yu & xuz+yz \\ x+yu & 1 & xyz+uz \\ xuz+yz & xyz+uz & 1 \end{bmatrix}$$

Fig. 2

contact of the network is included in the connection between some pair of nodes.

Let p_{ij} represent the "connection" between nodes i and j. This symbol has the value 0 if there is no connection at all and 1 if there is a short circuit, but otherwise it is the symbol denoting a single contact or is a sum of such symbols. The matrix "P" of order $p + k$:

$$P = [p_{ij}]$$

is then called a *primitive connection matrix* of the circuit. For example, if in Fig. 2 we select a single non-terminal node in the obvious way and number it "4", we obtain the primitive connection matrix

$$P = \begin{bmatrix} 1 & x & 0 & y \\ x & 1 & 0 & u \\ 0 & 0 & 1 & z \\ y & u & z & 1 \end{bmatrix}.$$

A third type of matrix which we often associate with a circuit falls, in a sense, between the primitive connection matrix and the output matrix. We call it just a *connection matrix*, "C". (This term was originated by Warren Semon at the Harvard Computation Laboratory.) In such a matrix, the entries are switching functions of two-terminal circuits connecting the nodes, both terminal and non-terminal, of the circuit, but the number of non-terminal nodes selected need not be large enough to lead to a primitive connection matrix. However, it is assumed that enough non-terminal nodes are selected so that all the contacts of the network are accounted for in the resulting two-terminal circuits. The situation is illustrated in Fig. 3. For certain purposes, a connection matrix such as this is as useful as a primitive one, or more so.

The output matrix of a circuit may also be regarded as a connection matrix if that appears desirable, but its primary importance lies in the fact that it is *the generalization of the switching function of a two-terminal circuit.* (Note that the output matrix of a two-terminal circuit is simply

$$F = \begin{bmatrix} 1 & f \\ f & 1 \end{bmatrix}$$

where f is the switching function of the circuit, and thus tells us nothing more nor less than the switching function f itself.)

The fundamental problem of the *analysis* of a combinational circuit involves writing a connection matrix corresponding to the circuit, thereafter deducing the corresponding output matrix or other relevant information. This presents relatively few difficulties. The problem of *synthesis*, on the other hand, is much more difficult, for it involves translating given operate-requirements into the form of an output or connection matrix and deducing therefrom a primitive or a near-primitive connection matrix corresponding to an optimal or at least to an economical realization of the requirements. We shall discuss these problems in the order stated, but first we need to indicate some properties of the type of Boolean matrix we use in studying switching circuits.

3. THE ALGEBRA OF SWITCHING MATRICES

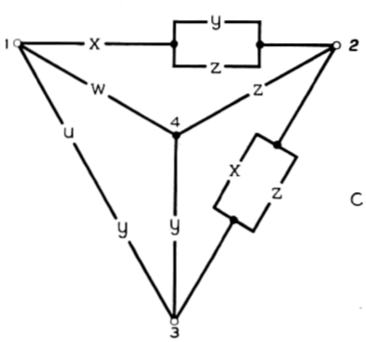
3.1. *Basic Definitions and Properties*

The Boolean matrices which are useful in switching theory have all 1's on the main diagonal. Any Boolean matrix of this kind, with its remaining elements chosen from a switching algebra S, will be called a *switching matrix*. When, as in later sections of this paper, only relay contacts are used as switching elements, the resulting switching matrices are all *symmetric*. Throughout this section, we discuss the more general case, however, by way of laying the groundwork for a later extension of these methods to electronic circuits.

Consider now the set "M" of all switching matrices of order m with elements from S. We make the following definitions, where $A = [a_{ij}]$, $B = [b_{ij}]$, $\cdots$, are matrices of M:

(1) *Equality:* $A = B$ if and only if $a_{ij} = b_{ij}$ for all i and j.

(2) *Sum:* $A + B = [a_{ij} + b_{ij}]$, that is, the sum is formed by adding corresponding elements. The sum of two switching matrices is again a



$$C = \begin{bmatrix} 1 & x(y+z) & uy & w \\ x(y+z) & 1 & x+z & z \\ uy & x+z & 1 & y \\ w & z & y & 1 \end{bmatrix}$$

Fig. 3

switching matrix. It corresponds to connecting the elements a_{ij} and b_{ij} in parallel between nodes i and j throughout the circuit.

(3) *Logical Product:* $A * B = [a_{ij} \cdot b_{ij}]$, that is, the logical product is found by multiplying corresponding elements throughout. The logical product of two switching matrices is again a switching matrix. It corresponds to connecting the elements a_{ij} and b_{ij} in series between i and j.

(4) *Complement:* $A' = [\alpha_{ij}]$ where $\alpha_{ij} = a_{ij}'$ if $i \neq j$, but $\alpha_{ii} = 1$ for all i. This operation corresponds to replacing all the two-terminal circuits corresponding to the a_{ij} $(i \neq j)$ by their complements, recognizing the fact that the connection of a terminal to itself is invariable.

(5) *Inclusion:* $A \leqq B$ ("A is included in B" or "A is contained in B") if and only if $a_{ij} \leqq b_{ij}$ for all i and j. Also, $B \geqq A$ is equivalent to $A \leqq B$. If $A \leqq B$, then any combination of values of the input variables which results in a path from i to j in the circuit corresponding to A, also results in such a path in the circuit corresponding to B.

(6) *Zero Matrix:* The zero matrix Z has $a_{ij} = 0$ for $i \neq j$ but $a_{ii} = 1$ for all i. This corresponds to open circuits between all pairs of terminals.

(7) *Universal Matrix:* The universal matrix U has $a_{ij} = 1$ for all i and j. It corresponds to short circuits between all pairs of terminals.

(8) *Matrix Product:*

$$AB = \left[\left(\sum_{k=1}^{m} a_{ik} b_{kj}\right)\right].$$

The rule here is the same as for ordinary matrices. A^p means $AA \cdots A$ to p factors. The matrix product of two switching matrices is again a switching matrix, but since the product of symmetric matrices is not necessarily symmetric, this product does not always have meaning in the case of relay switching circuits.

(9) *Multiplication by a Scalar:* $\alpha A = A\alpha = [\beta_{ij}]$ where α belongs to S and $\beta_{ij} = \alpha a_{ij}$ if $i \neq j$, but $\beta_{ii} = 1$ for all i. Thus αA is again a switching matrix.

(10) *Transpose:* $A^T = [\alpha_{ij}]$ where $\alpha_{ij} = a_{ji}$.

Using these definitions, it is not difficult to prove the following fact:

Theorem 3.1.1: With respect to the meanings of $=$ *and* $\leqq$, *and the operations* $+$, $*$, *and* $'$ *as here defined, the switching matrices of order m over a switching algebra S constitute a Boolean algebra.*

This result is due to Lunts.[5] As a consequence of this theorem, *we may employ all the rules of Boolean algebra in operating with switching matrices.*

When the matrix product, the transpose, and multiplication by scalars are also taken into account, many of the familiar rules of ordinary matrix algebra are seen to persist. Some additional rules result from the combination of these operations with the Boolean ones, but most of them will not be used in this paper. A list of the more fundamental properties is given in the appendix. Properties of the class of *all* Boolean matrices are discussed in Reference 6.

3.2. *A Useful Theorem*

The following result is useful in establishing several basic theorems concerning the analysis of circuits. The theorem was first stated by Lunts.[5]

Theorem 3.2.1: If A is any switching matrix of order m, then there exists a positive integer $q \leqq m - 1$ such that

$$A \leqq A^2 \leqq \cdots \leqq A^q = A^{q+1} = \cdots .$$

First we note that if $A^h = [\alpha_{ik}]$ then, since $a_{jj} = 1$,

$$A^{h+1} = \left[\sum_k \alpha_{ik} a_{kj}\right] = \left[\alpha_{ij} + \sum_{k \neq j} \alpha_{ik} a_{kj}\right].$$

Thus the ij-entry of A^{h+1} contains α_{ij} so that $A^h \leqq A^{h+1}$ for all positive integers h.

To complete the proof, it will suffice to show that $A^{m-1} = A^m$. Hence consider any off-diagonal entry of A^m. (The diagonal entries are all 1.) It may be written in the form

$$\sum_{k_1 \cdots k_{m-1}} a_{ik_1} a_{k_1 k_2} \cdots a_{k_{m-2} k_{m-1}} a_{k_{m-1} j} .$$

There are $m + 1$ subscripts here, so that not all can be distinct. Consider now any term of this sum. If $j = k_s$ for some s, the term takes the form

$$a_{ik_1} \cdots a_{k_{s-1} j} a_{j k_{s+1}} \cdots a_{k_{m-1} j}$$

which is contained in the term

$$a_{ik_1} \cdots a_{k_{s-1} j}$$

of the ij-entry of A^s and hence in the ij-entry of A^{m-1}, by what has already been proved. A similar conclusion holds if $i = k_s$. If neither i nor j is equal to any k, then $k_s = k_r$ for some s and r and the term takes the form

$$a_{ik_1} \cdots a_{k_{s-1} k_r} a_{k_r k_{s+1}} \cdots a_{k_{r-1} k_r} a_{k_r k_{r+1}} \cdots a_{k_{m-1} j}$$

which is contained in the term

$$a_{ik_1} \cdots a_{k_{s-1} k_r} a_{k_r k_{r+1}} \cdots a_{k_{m-1} j} .$$

But this too is contained in the ij-entry of A^{m-1}. Thus we may conclude that $A^{m-1} \geqq A^m$. But we have already shown that $A^{m-1} \leqq A^m$. Hence $A^{m-1} = A^m$, and the theorem follows.

In the generic case, $q = m - 1$. However, because of special behavior of the elements of A, we may have $q < m - 1$, as is frequently the case when switching circuits are under consideration.

4. THE ANALYSIS OF COMBINATIONAL CIRCUITS

4.1 *The Star-Mesh Transformation and the Reduced Connection Matrix*

The basic problem of analysis is, as stated earlier, the determination of the relation between any given connection matrix and the corresponding output matrix. To accomplish this, we show first how to obtain from a given circuit an equivalent circuit using one less non-terminal node in the formation of the connection matrix. This operation may then be repeated until there are no non-terminal nodes in the accounting. The method is to formalize the Y-Δ or star-mesh transformation (Reference 3, page 94).

Consider a non-terminal node r in a combinational network with p terminal nodes and k non-terminal nodes, and with a corresponding connection matrix C, not necessarily primitive. Connections c_{ir} and c_{rj} provide a path from node i to node j if and only if $c_{ir}c_{rj} = 1$. Let us now replace the connections of the given circuit by others such that between each pair of nodes i and j (neither of which is r) there appears circuitry corresponding to the function $c_{ij} + c_{ir}c_{rj}$ and remove all connections c_{ir} between node r and other nodes of the circuit. Thus node r is effectively removed from the circuit, but the output of the circuit on the remaining nodes will be the same as before.

Matrixwise, this operation proceeds as follows. To remove a non-terminal node r, we add to each entry c_{ij} of C the product of the entry

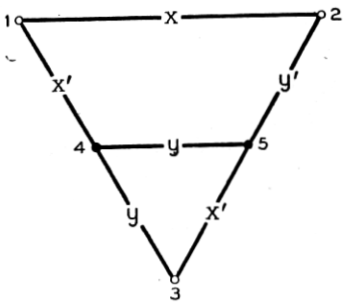


Fig. 4

c_{ir} in row i and column r of C by the entry c_{rj} in column j and row r, thereafter deleting row r and column r from C.

This operation of removing a node may be repeated until, after k steps, no non-terminal nodes remain. The resulting $p \times p$ connection matrix will be called a *reduced connection matrix* of the original circuit and will be denoted by the symbol C_0. The process is illustrated for the circuit of Fig. 4. The matrix obtained from C by the removal of node j is denoted by $C_{(j}$, etc.

$$P = \begin{bmatrix} 1 & x & 0 & x' & 0 \\ x & 1 & 0 & 0 & y' \\ 0 & 0 & 1 & y & x' \\ x' & 0 & y & 1 & y \\ 0 & y' & x' & y & 1 \end{bmatrix}, \qquad C_{(5} = \begin{bmatrix} 1 & x & 0 & x' \\ x & 1 & x'y' & 0 \\ 0 & x'y' & 1 & y \\ x' & 0 & y & 1 \end{bmatrix},$$

$$C_0 = C_{(4(5} = \begin{bmatrix} 1 & x & x'y \\ x & 1 & x'y' \\ x'y & x'y' & 1 \end{bmatrix}.$$

The given circuit is needlessly complicated, considering its output. It will be simplified presently.

The process of removing a node may of course be reversed in certain cases. This reversal is a simple matter when the entries of the output or connection matrix contain the proper terms. Thus, for example, we have

$$\begin{bmatrix} 1 & a + \alpha\beta & b + \alpha\delta \\ a + \alpha\beta & 1 & c + \beta\delta \\ b + \alpha\delta & c + \beta\delta & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & a & b & \alpha \\ a & 1 & c & \beta \\ b & c & 1 & \delta \\ \alpha & \beta & \delta & 1 \end{bmatrix}.$$

Again, starting with the matrix C_0 associated with Fig. 4, we observe that because of the common factor x', the entries $x'y$ and $x'y'$ could have arisen from the removal of a node. In fact, it is readily checked that

$$\begin{bmatrix} 1 & x & x'y \\ x & 1 & x'y' \\ x'y & x'y' & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & x & 0 & y \\ x & 1 & 0 & y' \\ 0 & 0 & 1 & x' \\ y & y' & x' & 1 \end{bmatrix}.$$

The circuit corresponding to the second of these matrices is shown in Fig. 5. It has one less contact than the circuit with which we began, the node insertion having replaced the two contacts x' by one. As this example suggests, the "insertion of nodes" will appear to be an important tool in the synthesis of circuits.

4.2. *The Fundamental Theorem*

The relationship between connection and output matrices may now be established. (A less general form of the following theorem was first stated by Lunts.[5])

Theorem 4.2.1: If C is any connection matrix of a p-terminal circuit, if C_0 is the corresponding reduced connection matrix, and if F is the output matrix of the circuit, then there exists an integer k, $1 \leqq k < p$, such that $C_0^{p-k} = F$.

In the case of the matrix C_0 obtained in the preceding section, we have, for example, $C_0^2 = C_0$ so that C_0 is itself the output matrix of the circuit.

From Theorem 3.2.1 it follows that there exists an integer k, $1 \leqq k < p$, such that $C_0^{p-k} = C_0^{p-k+1} = \cdots$. It only remains to show that $C_0^{p-k} = F$. For this purpose it is sufficient to show that $C_0^{p-1} = F$.

Let us denote the elements of C_0 by c_{ij}. Then the ij-entry of C_0^2 is the function

$$c_{i1}c_{1j} + c_{i2}c_{2j} + \cdots + c_{ip}c_{pj} .$$

Since multiplication means "and" and addition means "or", this function is 1 for $i \neq j$ when and only when the input variables are such that there is a path from i to j, either directly (because of the term $c_{ij}c_{jj} = c_{ij}$) or via some intermediate node r. Similarly, the ij-entry of

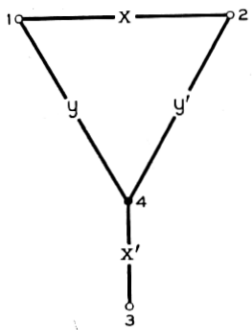


Fig. 5

C_0^3 is a function which is 1 when and only when the input variables are such that there is a path from i to j, proceeding directly, or via one intermediate node, or via two intermediate nodes. Continuing thus, since no path requires more than $p - 2$ intermediate nodes, we see that the ij-entry of C_0^{p-1} is a function which is 1 when and only when the circuit variables are such as to interconnect i and j. That is, $C_0^{p-1} = F$.

The following two corollaries are immediate:

Corollary 4.2.2: The reduced circuit matrix of a two-terminal circuit is the output matrix of the circuit.

Corollary 4.2.3: The ij-entry of F may be found by considering the circuit as a two-terminal circuit connecting i and j and removing all the other nodes.

4.3. *Characterization of an Output Matrix*

Certainly any symmetric switching matrix may be interpreted as a connection matrix of a combinational relay circuit. However, a natural question to ask at this point is, "When is a given symmetric switching matrix also an output matrix?" The answer is given in

Theorem 4.3.1: The necessary and sufficient condition that a symmetric switching matrix C be an output matrix is that $C^2 = C$.

Suppose first $C^2 = C$. Multiplying both sides repeatedly by C, we conclude $C^{p-1} = C$, where p is the order of C. That is, C is its own output matrix.

Conversely, suppose C is an output matrix. Denote any reduced connection matrix of the circuit by C_0. Then, using Theorems 4.2.1 and 3.2.1, we have $C = C_0^{p-1} = C_0^{2p-2} = C^2$, and the theorem is proved.

4.4 *Redundant Elements*

In the synthesis of a circuit it is often helpful to detect and remove, or to insert, what we call *redundant elements*. These are elements whose replacement by open circuits (in the parallel case) or by short circuits (in the series case) will not alter the output of the circuit.

To illustrate these notions, we consider first the following connection matrix in which redundant terms are bracketed:

$$\begin{bmatrix} 1 & x + [y] & y & z + [xu] \\ x + [y] & 1 & y & u \\ y & y & 1 & [yuz] \\ z + [xu] & u & [yuz] & 1 \end{bmatrix}.$$

The term y in the 1,2-entry is redundant since there is a path from 1 to 3 and another from 3 to 2, hence one from 1 to 2, if $y = 1$. Thus the term y in the 1,2-position (and of course also the term y in the 2,1-position) may be dropped, that is, may be replaced by zero, and the output matrix of the circuit will not be altered. We could have reasoned in the same way that the y in the 1,3-entry or the 2,3-entry is redundant, but only *one* of these three y's may be removed. (The reader should note that the key to this situation is the appearance of two identical elements in the same row, along with a third identical element in the same column as one of them.)

Similarly, after the y in the 1,2-position has been removed, the term xu in the 1,4-entry may be seen to be redundant since there is a path from 1 to 2 if $x = 1$ and from 2 to 4 if $u = 1$. (The key here is the fact that the factors x and u of xu appear as terms of other entries in row 1 and column 4.) Finally, the entry yuz in the 3,4-position is redundant. In fact, there is a path from 3 to 2 if $y = 1$ and from 2 to 4 if $u = 1$, so that there is a path from 3 to 4 when $yu = 1$, regardless of the value of z. The successive deletion of these redundant terms, in brackets, evidently yields a primitive connection matrix which is equivalent to the original matrix in the sense that both lead to the same output matrix.

To illustrate the removal of redundant factors, we consider the connection matrix

$$\begin{bmatrix} 1 & x & z + [x']y \\ x & 1 & x \\ z + [x']y & x & 1 \end{bmatrix}.$$

Here the factor x' in the 1,3-entry is redundant since there is a path from 1 to 3 when y is 1, regardless of the condition of x, as a figure will readily show. This factor may therefore be dropped (replaced by 1) without altering the output matrix. This amounts to adding a redundant term x to the 1,3-entry because of the x's in the 1,2 and 2,3 positions. Then the rule $x + x'y = x + y$ accounts for the removal of x'.

The importance of these ideas is that we can reverse both processes whenever this is of avail in the synthesis of a circuit, as will appear later.

It should not be overlooked that the removal (or insertion) of redundant elements must either proceed in successive steps or be capable of being so arranged, since such an operation alters the corresponding circuit, and hence may alter the conditions for redundancy of other terms. Sometimes the insertion of redundant terms renders other terms redundant so that they in turn may be removed. Later, we shall employ this fact to good advantage.

It may of course be possible to replace elements of a connection matrix by other functions than 0 and 1, but we do not employ such substitutions in this paper. These substitutions have been characterized completely by Semon.[4]

5. THE SYNTHESIS OF COMBINATIONAL CIRCUITS

5.1. *The Truth-Table Method of Synthesis*

The "truth-table" approach may be employed in synthesis in the same way that it is in the two-terminal case. For example, the output of a four-terminal circuit is specified by Table 1, where x and y are input variables.

The necessary and sufficient condition that given output requirements be consistent is clearly that whenever $f_{ij} = f_{jk} = 1$, then $f_{ik} = 1$ also. It is readily checked that this condition is satisfied in the case of this example.

We have, from the f_{12}-column, $f_{12} = x'y' + x'y = x'$. Similarly, $f_{13} = x'y + xy = y$, $f_{14} = 0$, $f_{23} = x'y$, $f_{24} = xy'$, $f_{34} = x'y'$. Thus we have

$$F = \begin{bmatrix} 1 & x' & y & 0 \\ x' & 1 & x'y & xy' \\ y & x'y & 1 & x'y' \\ 0 & xy' & x'y' & 1 \end{bmatrix}.$$

Let us consider this output matrix as a connection matrix of the desired circuit. Then, since we have a path from 2 to 1 if $x' = 1$ and from 1 to 3 if $y = 1$, the 23-entry $x'y$ is redundant and may be replaced by zero. (The resulting matrix is no longer an output matrix, of course.) Now we insert a node "5" to separate the products xy' and $x'y'$ in the

TABLE I

x	y	f_{12}	f_{13}	f_{14}	f_{23}	f_{24}	f_{34}	Non-Vanishing Product
0	0	1	0	0	0	0	1	$x'y'$
0	1	1	1	0	1	0	0	$x'y$
1	0	0	0	0	0	1	0	xy'
1	1	0	1	0	0	0	0	xy

last column:

$$\begin{bmatrix} 1 & x' & y & 0 & 0 \\ x' & 1 & 0 & 0 & x \\ y & 0 & 1 & 0 & x' \\ 0 & 0 & 0 & 1 & y' \\ 0 & x & x' & y' & 1 \end{bmatrix}.$$

The reader should check this by removing the additional node, and should *observe how this insertion replaced two* y' *contacts by one.* The circuit corresponding to this primitive connection matrix is shown in Fig. 6.

In many cases, the output matrix or a suitable connection matrix may be written by inspection. Any techniques useful in simplifying switching functions may of course be applied in computing the f_{ij}. However, the "simplest" forms of these functions are not necessarily the most useful, when it comes to matrix methods of synthesis. These points will be illustrated by later examples.

5.2. *Matrix Synthesis of Two-Terminal Circuits*

In the previous section, we gave an example of how one may deduce from an output matrix a primitive connection matrix of an economical realization of the circuit. In this section we illustrate the procedure further by applying it to some simple two-terminal circuits.

Geometrically, synthesis of a two-terminal circuit amounts to the selection of an appropriate set of nodes and connecting links joining the

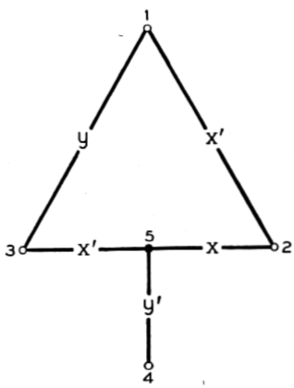


Fig. 6

two terminals. Algebraically, this is accomplished by beginning with the output matrix

$$\begin{bmatrix} 1 & f \\ f & 1 \end{bmatrix}$$

of the two-terminal circuit and reducing the complexity of its entries by successive node insertions until a primitive connection matrix is finally obtained. Operations with redundant elements are ordinarily an essential part of the process, as the following examples show.

Example A. $f = ABC + AD + BD + CD$.

First we factor f in some convenient way, say into the form $f = A(BC + D) + BD + CD$, and by the insertion of a node remove the first term, that is, render it redundant so that it may be replaced by zero:

$$\begin{bmatrix} 1 & A(BC + D) + BD + CD \\ A(BC + D) + BD + CD & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & BD + CD & A \\ BD + CD & 1 & BC + D \\ A & BC + D & 1 \end{bmatrix}.$$

Suppose now that we decide to remove the term CD from the 1,2-entry. We note first that there is a path from 1 to 2 if $BD = 1$ and from 2 to 3 if $D = 1$, i.e., from 1 to 3 if $BD = 1$. Hence we may add the redundant term BD to the 1,3-entry:

$$\begin{bmatrix} 1 & BD + CD & A + [BD] \\ BD + CD & 1 & BC + D \\ A + [BD] & BC + D & 1 \end{bmatrix}.$$

(Brackets around a term denote that it is redundant.) Now we can insert a fourth node which removes the terms CD, BD, BC from the 1,2-, 1,3-, and 2,3-entries respectively. This yields the matrix

$$\begin{bmatrix} 1 & [BD] & A & D \\ [BD] & 1 & D & C \\ A & D & 1 & B \\ D & C & B & 1 \end{bmatrix}.$$

Here the entry BD is redundant, for we have a path from 1 to 4 if $D = 1$, from 4 to 3 if $B = 1$, from 3 to 2 if $D = 1$, hence from 1 to 2 if $BD = 1$. Dropping the two BD's we obtain the primitive connection matrix,

$$\begin{bmatrix} 1 & 0 & A & D \\ 0 & 1 & D & C \\ A & D & 1 & B \\ D & C & B & 1 \end{bmatrix}$$

which is a wiring diagram for the bridge circuit shown in Fig. 7.

This work can all be performed without recopying. Thus, the matrix

$$\begin{bmatrix} 1 & [A(BC+D)]+[BD]+[CD] & A+[BD] & D \\ [A(BC+D)]+[BD]+[CD] & 1 & [BC]+D & C \\ A+[BD] & [BC]+D & 1 & B \\ D & C & B & 1 \end{bmatrix}$$

gives the desired result. Brackets are drawn around all terms, whether originally present or inserted, which are ultimately removed because of redundancy.

Example B. $f = A'B + AB' + AC$.

One possible procedure is indicated by the following matrix:

$$\begin{bmatrix} 1 & [A'B]+[AB']+[AC] & A & A & A' \\ [A'B]+[AB']+[AC] & 1 & C & B' & B \\ A & C & 1 & 0 & 0 \\ A & B' & 0 & 1 & 0 \\ A' & B & 0 & 0 & 1 \end{bmatrix}.$$

Here the three terms of f were removed one at a time, proceeding from right to left, by inserting three additional nodes. The corresponding circuit, shown in Fig. 8, contains an unnecessary A-contact, even though we have arrived at a primitive connection matrix.

An alternative procedure is indicated in the matrix:

$$\begin{bmatrix} 1 & [A'B]+[A(B'+C)] & A' & A \\ [A'B]+[A(B'+C)] & 1 & B & B'+C \\ A' & B & 1 & 0 \\ A & B'+C & 0 & 1 \end{bmatrix}$$

in which the $A'B$ term was removed first. The corresponding, more economical circuit is given in Fig. 9.

These examples illustrate the important facts that (1) *the matrix representation is not prejudiced in favor of series-parallel circuitry* and (2) *the circuit finally obtained depends on the steps used in obtaining a primitive connection matrix.*

5.3. *Further Examples With More Than Two Terminals*

The examples of the preceding section were introduced primarily for illustrative purposes. We now introduce two examples designed to indicate the power of the method.

First we construct a circuit simultaneously realizing all sixteen switching functions of two variables. (This circuit was first obtained by a

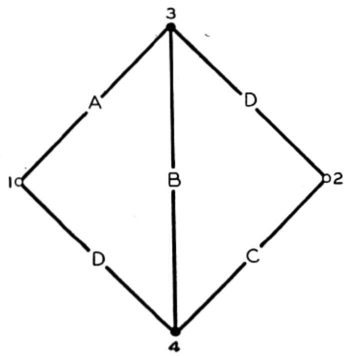


Fig. 7

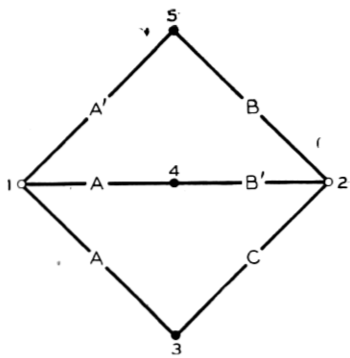


Fig. 8

student in the switching course at M.I.T. and was later proved by Shannon, in an unpublished memorandum, to be minimal.)

A connection matrix of order 17 (sixteen terminals and ground) may be written at once. It has the 16 functions in the G row and column, and all other off-diagonal entries are zero. For convenience, we omit

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	G	
1																0	1
	1										y'	y				x	2
		1														y	3
			1						y'	y						x'	4
				1												y'	5
					1	x'								$x + y$		$[x'] + y'$	6
						1							$x + y'$			$[x'] + y$	7
							1	x								$[x] + y'$	8
								1								$[x] + y$	9
									1							$[x'y']$	10
										1						$[x'y]$	11
											1					$[xy']$	12
												1				$[xy]$	13
													1			$[(x' + y)(x + y')]$	14
														1		$[(x' + y')(x + y)]$	15
															1	1	16
																1	G

all entries below the diagonal. Entries which are zero throughout are omitted entirely. (Off-diagonal entries which are not in the G-column are inserted redundant terms whose presence will be explained.)

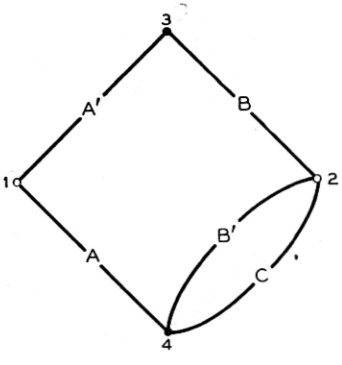


Fig. 9

First we note that since terminals 4, 10, and 11 are grounded when and only when $x' = 1$, $x'y' = 1$, and $x'y = 1$ respectively, redundant entries y' and y may be inserted in the 4,10- and 4,11-positions respectively. Then, however, the 10,G- and 11,G-entries become redundant and may be replaced by zeros. (The terms y and y' which we inserted then lose their redundancy, of course.) In the same way, we may put redundant terms y' and y in the 2,12- and 2,13-entries respectively, thereafter replacing the now redundant 12,G- and 13,G-entries by zeros.

With the entries $x'y' + xy$ and $x'y + xy'$ in the 14,G- and 15,G-positions factored as indicated in the matrix, we see next that the same type of operation permits insertion of redundant entires $x + y'$ and $x + y$ in the 7,14- and 6,15-positions respectively, after which the 14,G- and 15,G-entries become redundant and may be removed.

Finally, we note that the common term x' in the 6,G- and 7,G-entries permits the insertion of a redundant x' in the 6,7-position. This in turn renders the x''s in the 6,G- and 7,G-entries redundant, so that they may be replaced by zeros. Finally, in the same way, we insert a redundant x in the 8,9-position, after which the x's in the 8,G- and 9,G-entries may be replaced by zeros.

The resulting primitive connection matrix corresponds to the circuit shown in Fig. 10.

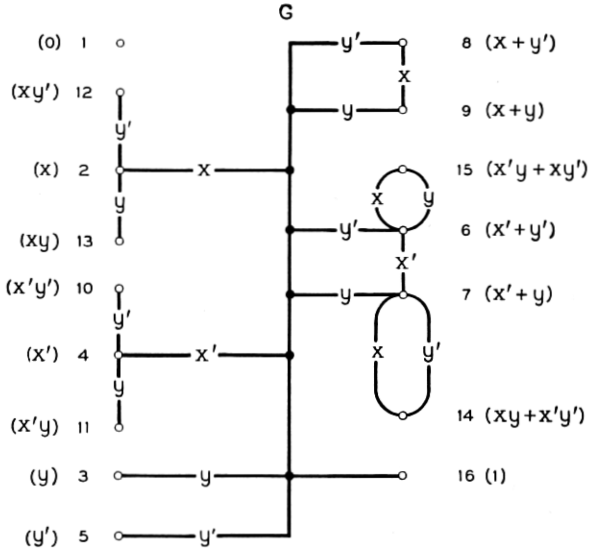


Fig. 10

TABLE II

Relays Operated	Leads Grounded
v, w	1
v, x	2
w, x	1, 2
v, y	3
w, y	1, 3
x, y	2, 3
v, z	1, 2, 3
w, z	4
x, z	1, 4
y, z	2, 4
None	None

As a second example, consider a circuit (Reference 3, page 124) with five relays, v, w, x, y, z and leads 1, 2, 3, 4, whose operate conditions are given in Table II.

No other combinations of relays operated occur, so that we "don't care" what happens in the circuit for such combinations. Nor do we care if ungrounded terminals are interconnected. Taking account of these assumptions, it may now be seen that switching functions expressing the conditions under which the various leads are grounded are as shown in Table III.

These functions allow us to write a connection matrix for the desired circuit:

$$\begin{array}{ccccc} (1) & (2) & (3) & (4) & (G) \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & wz' + w'zy' \\ 0 & 1 & 0 & 0 & xz' + x'zw' \\ 0 & 0 & 1 & 0 & vz + yz' \\ 0 & 0 & 0 & 1 & zv' \\ wz' + w'zy' & xz' + x'zw' & vz + yz' & zv' & 1 \end{bmatrix}.$$

The insertion of a node "5" to remove the terms in column "G" which contain the factor z immediately suggests itself since 4 z-contacts might thus be replaced by a single z-contact. This would require, however, preliminary insertion of suitable redundant terms in place of certain zero-entries of this matrix. The additional column and row for node "5" and the requisite terms whose redundancy must be investigated in the light of the don't-care conditions are shown in the next matrix. (Proposed redundant terms are listed above the diagonal only.)

TABLE III

Lead	Function
(1)	$wz' + w'zy'$
(2)	$xz' + x'zw'$
(3)	$vz + yz'$
(4)	zv'

In the 1,2-position, the product $x'w'y'$ would have to be inserted, as shown. But when $x'w'y' = 1$, vz may be 1 also. May 1 and 2 be con-

$$\begin{bmatrix} \begin{bmatrix} 1 & (x'w'y') & (vw'y') & (v'w'y') & wz' + w'zy' \\ 0 & 1 & (vw'x') & (v'w'x') & xz' + x'zw' \\ 0 & 0 & 1 & 0 & vz + yz' \\ 0 & 0 & 0 & 1 & zv' \\ wz' + w'zy' & xz' + x'zw' & vz + yz' & zv' & 1 \end{bmatrix} & \begin{matrix} w'y' \\ w'x' \\ v \\ v' \\ z \end{matrix} \\ \begin{matrix} w'y' & w'x' & v & v' & z \end{matrix} & 1 \end{bmatrix}$$

nected when $vz = 1$? A check of the table of operate-conditions shows that 1 and 2 are both to be grounded when $vz = 1$, so that the insertion of this term is harmless.

Next, in the 1,3-position, the entry $vw'y'$ would have to be inserted. When this factor is 1, we may also have $xz = 1$, but v and x and z are never all simultaneously operated, so this causes no trouble. However, we may alternatively have $xz' = 1$ or $x'z = 1$. May 1 and 3 be connected when v and x or v and z are both operated? The table shows that, in either case, 1 and 3 may be connected since neither is grounded in the first case, but both are grounded in the second. Thus the term $vw'y'$ may be safely inserted.

The term $vw'x'$ in the 2,3-position brings trouble, however, for when $vy = 1$, only 3 is to be grounded, whereas 2 and 3 could be connected in this case. If we abandon the attempt to remove the term vz from the 3,G-entry at this step, the difficulty is eliminated, for the required redundant terms then violate none of the operate conditions of the circuit.

The terms containing z' may also be removed from column G by the insertion of a node "6". This replaces three z-contacts by just one. The insertion of both nodes 5 and 6 is indicated in the following matrix. It is left to the reader to complete the checking of the redundant terms.

The redundant terms we inserted were of course absorbed again by the node-insertion process. We now have a connection matrix which

$$
\begin{array}{ccccccc}
(1) & (2) & (3) & (4) & (G) & (5) & (6 \\
1 & 0 & 0 & 0 & [wz'] + [w'zy'] & w'y' & w \\
0 & 1 & 0 & 0 & [xz'] + [x'zw'] & w'x' & x \\
0 & 0 & 1 & 0 & vz + [yz'] & 0 & y \\
0 & 0 & 0 & 1 & [zv'] & v' & 0 \\
[wz'] + [w'zy'] & [xz'] + [x'zw'] & vz + [yz'] & [zv'] & 1 & z & z \\
w'y' & w'x' & 0 & v' & z & 1 & 0 \\
w & x & y & 0 & z' & 0 & 1
\end{array}
$$

is not primitive but which cannot be further simplified by node insertion because of the absence of appropriate common factors. The corresponding circuit is shown in Fig. 11.

The reader may check that the requirements are satisfied and that no leads are improperly grounded. Although there are 12 contacts in this realization, only 21 springs are required because of the three possible transfers. Leads 1, 2, and 4 are connected when none of the relays are operated, but otherwise no ungrounded leads are connected.

5.4. *Other Transformations of a Connection Matrix*

We have seen how the removal and insertion of nodes by the Y-Δ transformation may be used in the analysis and synthesis of networks.

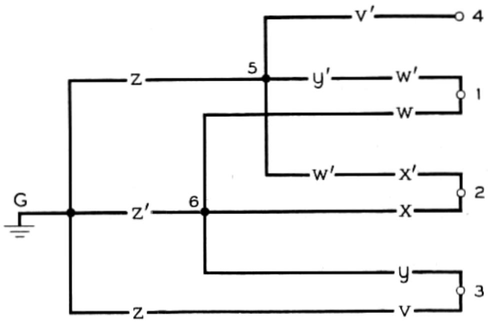


Fig. 11

There are, of course, other methods of transforming a connection matrix without altering the output. One of these is the Δ-Y transformation, which is simply the dual of the one just mentioned, and which we now explain.

Consider three nodes i, j, k of a network, as indicated in Fig. 12(a). This part of the network may be replaced by the network of Figure 12(b). This replacement is what is known as the "Δ-Y transformation" (Reference 3, page 93).

Matrixwise, this transformation is simple to execute. We first mark the ij and ji, jk and kj, ki and ik entries of the connection matrix, say by bracketing them. Then, in a new column, we enter in rows i, j, k the sums of the marked entries in those rows. The rest of the column is filled out with zeros except for the diagonal entry, which — as always — is 1. The bracketed elements are then replaced by zeros and the matrix is completed in symmetric fashion. The reduced connection matrix appearing in Section 4.1 is used to provide an example:

$$\begin{bmatrix} 1 & [x] & [x'y] \\ [x] & 1 & [x'y'] \\ [x'y] & [x'y'] & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & x+y \\ 0 & 1 & 0 & x+y' \\ 0 & 0 & 1 & x' \\ x+y & x+y' & x' & 1 \end{bmatrix}.$$

The result is a primitive connection matrix, but not as simple a one as we had before. However, a redundant x may be inserted in the 1,2-

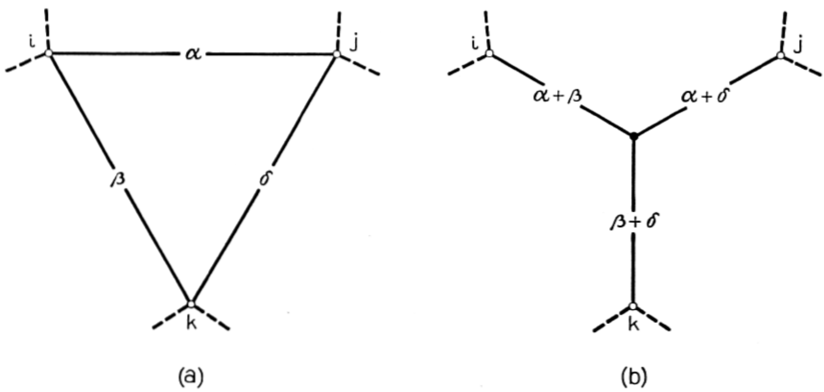


Fig. 12

position after which the x's in the 1,4- and 2,4-positions become redundant. When these are removed, the matrix is again that of the circuit in Fig. 5.

This transformation is clearly indicated for consideration when the addition of two entries in the same row results in a considerable simplification. It is probable that still other transformations — useful, like this one, under special conditions — could be developed. However, because of its simple geometric and algebraic significance, the node-insertion operation seems likely to remain most useful of all.

6. CONCLUSION

In this paper we have outlined the basic properties of switching matrices and have applied them to both the analysis and the synthesis of combinational relay circuits. It is clear that we have not reduced circuit design to a "cook-book" procedure, but our experience with a variety of design problems (not all of which are reported here) leads us to believe that the method shows considerable promise of becoming a "practical" tool, and that further study is justified along the following lines:

(a) The work done in References 4 and 7 on circuits with unilateral elements should be extended in the hope of devising a tool comparable to node-insertion for synthesis.

(b) The class of all transformations of a connection matrix which leave the associated output matrix invariant should be characterized and their application in synthesis should be studied. (From an algebraic point of view, the interesting thing here is that allowable transformations of a connection matrix need not leave its order invariant.)

(c) The manner in which "don't-care" conditions enter into synthesis should be more extensively studied. In the presence of such conditions, the class of connection matrices giving rise to an acceptable output matrix is of course considerably more extensive than it would be otherwise.

(d) The possibility of characterizing a primitive connection matrix of a minimal network should be investigated. This may be related to synthesis by a minimum number of nodes and complete absence of redundant elements.

Other problems have, of course, suggested themselves. We have listed what seem to be the more important ones; the reader will undoubtedly formulate others for himself.

7. APPENDIX

The Basic Properties of Switching Matrices

If A, B, C are any switching matrices of order m over a switching algebra with n variables, then with respect to the definitions given in Section 3, we have the following Boolean rules of operation:

$$A + A = A \qquad A * A' = Z$$
$$A * A = A \qquad U + A = U$$
$$A + B = B + A \qquad U * A = A$$
$$A * B = B * A \qquad A + A' = U$$
$$A + (B + C) = (A + B) + C \qquad (A * B)' = A' + B'$$
$$A * (B * C) = (A * B) * C \qquad (A + B)' = A' * B'$$
$$A + B * C = (A + B) * (A + C) \qquad (A')' = A$$
$$A * (B + C) = A * B + A * C \qquad A + A * B = A$$
$$Z + A = A \qquad A + A' * B = A + B$$
$$Z * A = Z \qquad A \leqq A$$
$$A \leqq B \text{ and } B \leqq A \text{ if and only if } A = B$$
$$A \leqq B \text{ and } B \leqq C \text{ imply } A \leqq C$$
$$A \leqq B \text{ if and only if } A * B = A$$
$$A \leqq B \text{ if and only if } A + B = B$$
$$Z \leqq A \leqq U \text{ for all } A.$$

Every Boolean matrix C has a canonical expansion

$$C = \sum_{k=0}^{2^n-1} C_k p_k$$

and a dual canonical expansion

$$C = \prod_{\substack{* \\ k=0}}^{2^n-1} (\Gamma_k + s_k U)$$

where p_k are the fundamental products formed from $x_1, x_2, \cdots, x_n$ and the s_k are the fundamental sums. The ij-entries of C_k and Γ_k are the values associated with p_k and s_k respectively in the ordinary canonical expansions of the entry c_{ij} of C.

When the not-characteristically-Boolean operations of forming the transpose and the matrix product are introduced, we find that the following properties hold, among others. Many are familiar, others are not.

$$AB \neq BA \text{ ordinarily} \qquad U^p = U$$
$$A(B + C) = AB + AC \qquad Z^p = Z$$
$$(A + B)C = AC + BC \qquad (A^p)^q = A^{pq}$$
$$AZ = ZA = A \qquad A^pA^q = A^{p+q}$$
$$(A^T)^T = A \qquad AU = UA = U$$
$$(A^T)' = (A')^T \qquad (AB)C = A(BC)$$
$$(A + B)^T = A^T + B^T \qquad A(B * C) \leqq AB * AC$$
$$(A * B)^T = A^T * B^T \qquad (A * B)C \leqq AC * BC$$
$$(AB)^T = B^TA^T$$

$A \leqq B$ implies $AC \leqq BC$ and $CA \leqq CB$, but not conversely.

8. ACKNOWLEDGMENT

The authors wish to thank their colleagues E. P. Stabler and S. H. Washburn of the Technical Staff of Bell Telephone Laboratories for their very substantial help and encouragement in the preparation of this article.

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