The Potentials of Infinite Systems of Sources and Numerical Solutions of Problems in Semiconductor Engineering

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(Manuscript received August 30, 1954)

Tables and charts are given of mathematical functions related to the potential of a line of point charges. The use of these functions is illustrated by applications to semiconductor resistivity measurements and to calculations of the base resistance of point-contact transistors.

INTRODUCTION

The "method of images" is a simple but elegant technique for the solution of problems in potential theory. Formal solutions to problems involving plane boundaries and point sources can usually be written down immediately from obvious symmetry considerations. The solutions thus obtained, however, are often infinite series whose convergence is unsatisfactory for numerical calculations.

Many techniques have been devised for improving upon the convergence of these infinite sums. It has generally been assumed that the mathematician or engineer must select one of these techniques and apply it to his own special problem. The object of this article is to make such special treatment unnecessary in most cases. To this end, the potentials of certain simple image systems are tabulated.

The use of the tabulations is illustrated by numerical solutions of some problems in semiconductor engineering. Transistors and varistors have become famous because of their non-ohmic properties. But solutions to ohmic flow problems can be useful as first approximations in the design of semiconductor devices and can be very accurate approximations for suitably arranged measurements of the equilibrium resistivity of semiconductors.[1,2] Plane boundaries occur as the physical boundaries of the semiconductor and as junctions between regions of different conductivities and conductivity types. Point sources are the simplest idealizations of point contacts.

The four-point method to be described for the measurement of semiconductor resistivity is used in geophysics for measuring the resistivity of the earth. The mathematical content of this article is applicable to problems other than ohmic flow. The differential equations for ohmic flow are identical with those of electrostatics, heat conduction, and the hydrodynamics of an ideal fluid. In one of the examples given below, the Coulomb energy of an ionic crystal is calculated.

SUPERPOSITION OF AN INFINITE NUMBER OF POINT SOURCES

The strength q of a point source will be defined so that the potential due to the source is q/r, where r is the distance from the source. Sources of negative strength are often called "sinks", but this special designation will not be used in the discussion to follow. All the cases of ohmic flow to be considered here involve point sources of current at a plane surface and it will be shown that

$$q = \frac{I\rho}{2\pi} \tag{1}$$

where I is the current into the plane surface of material of resistivity ρ.

According to the principle of superposition, the potential at a given point due to some configuration of fixed sources is the sum of the potentials that would be established at that point by the various sources making up the configuration. The computations for many problems could be accomplished easily if one could have a table of the potential of a line of equally-spaced point sources of equal value q. But a divergent expression is obtained when one attempts to calculate the potential of such an arrangement of sources by superposing the q/r potentials of the sources. One way to avoid this divergence is to abandon the attempt (implicit in the choice of q/r as the potential of a source) to make the potential at infinity equal to zero. Potentials with respect to some arbitrary point would be finite and could be tabulated.

However, having the potential at infinity equal to zero simplifies the superposition of source systems. All of the problems that could be solved with the potential of a line of point sources can be solved with either of two related arrangements that permit setting the potential at infinity equal to zero. Tabulations of the potentials of these arrangements will be given.

The first scheme that will be considered consists entirely of point sources and is illustrated in Fig. 1. It consists of a line of equally spaced point sources of equal strength q, and a parallel line of opposite sources, except that one of the positive sources and the corresponding negative source have been omitted. The potential at the point P (where the omitted positive source would have been placed) is to be considered and may be written

$$V(P) = \frac{q}{a} M(\lambda) \tag{1}$$

where M is a dimensionless function of the ratio $\lambda = d/a$ of the distance d between the lines of sources to the spacing a of the charges in the lines. It is obvious that $M(\lambda)$ must vanish at $\lambda = 0$.

The second scheme is illustrated in Fig. 2. It consists of a line of equally spaced point sources embedded in a line source of equal and opposite strength per unit length. The potential at the point Q, lying on a perpendicular to the line of charges, erected at one of the point sources

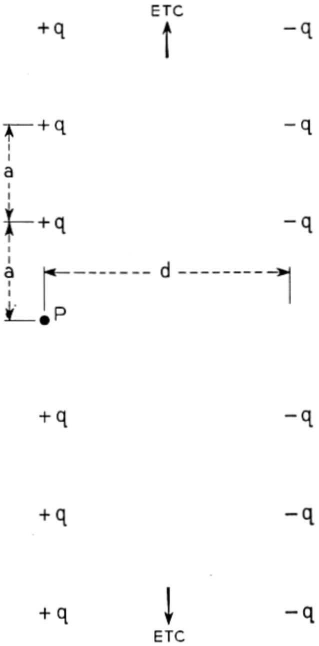


Fig. 1 — Arrangement of sources corresponding to the function M. The potential at point P is $(q/a)\ M(d/a)$.

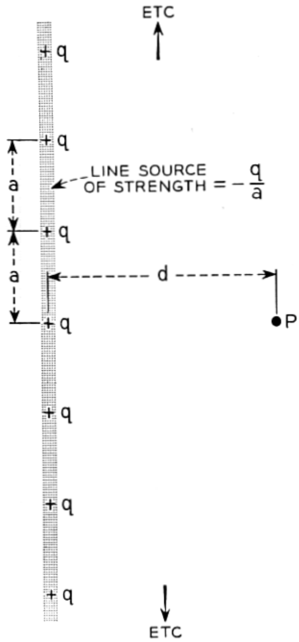


Fig. 2 — Source distribution corresponding to the function N. The potential at point P is $(q/a)\ N(d/a)$.

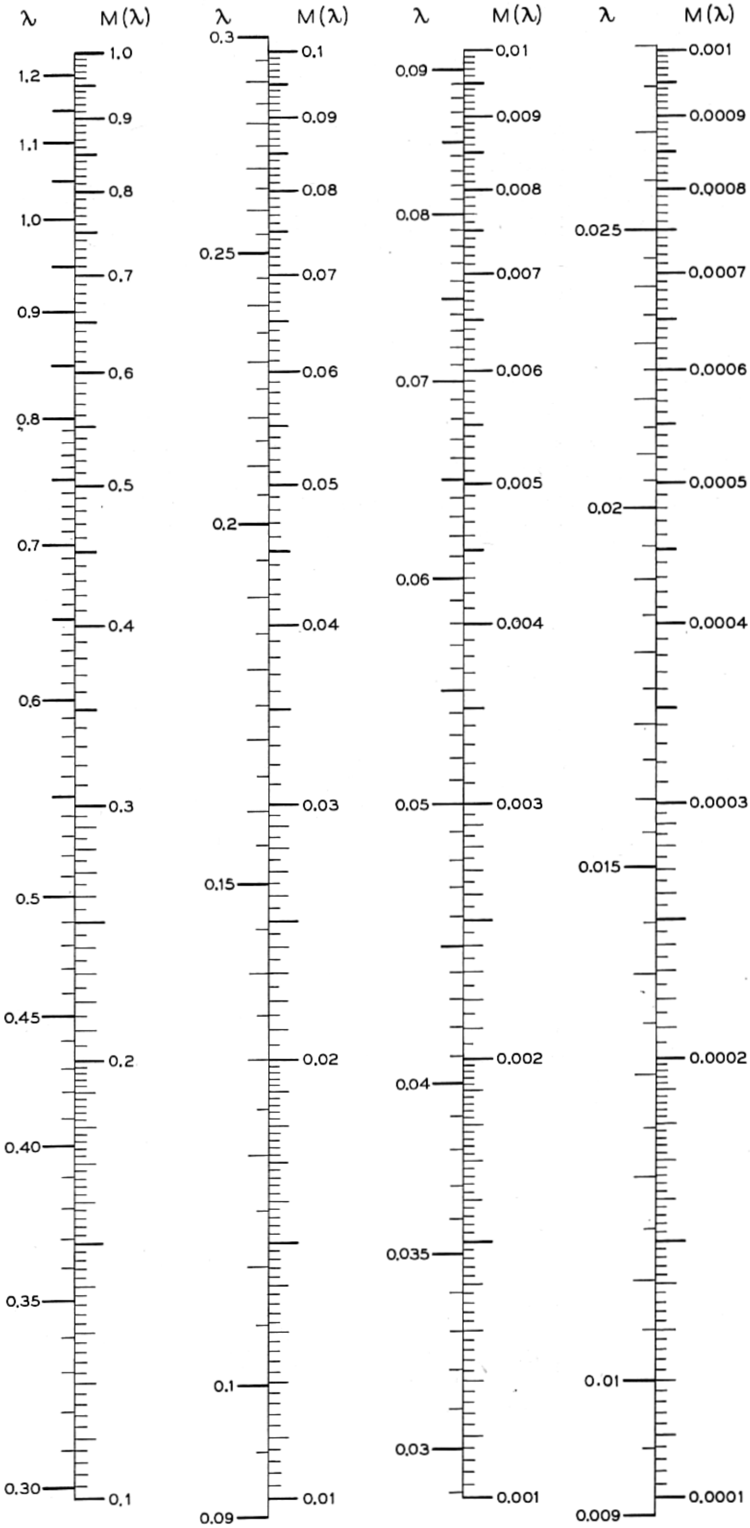


Fig. 3 — The function $M(\lambda) = 2 \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{\sqrt{n^2 + \lambda^2}} \right)$.

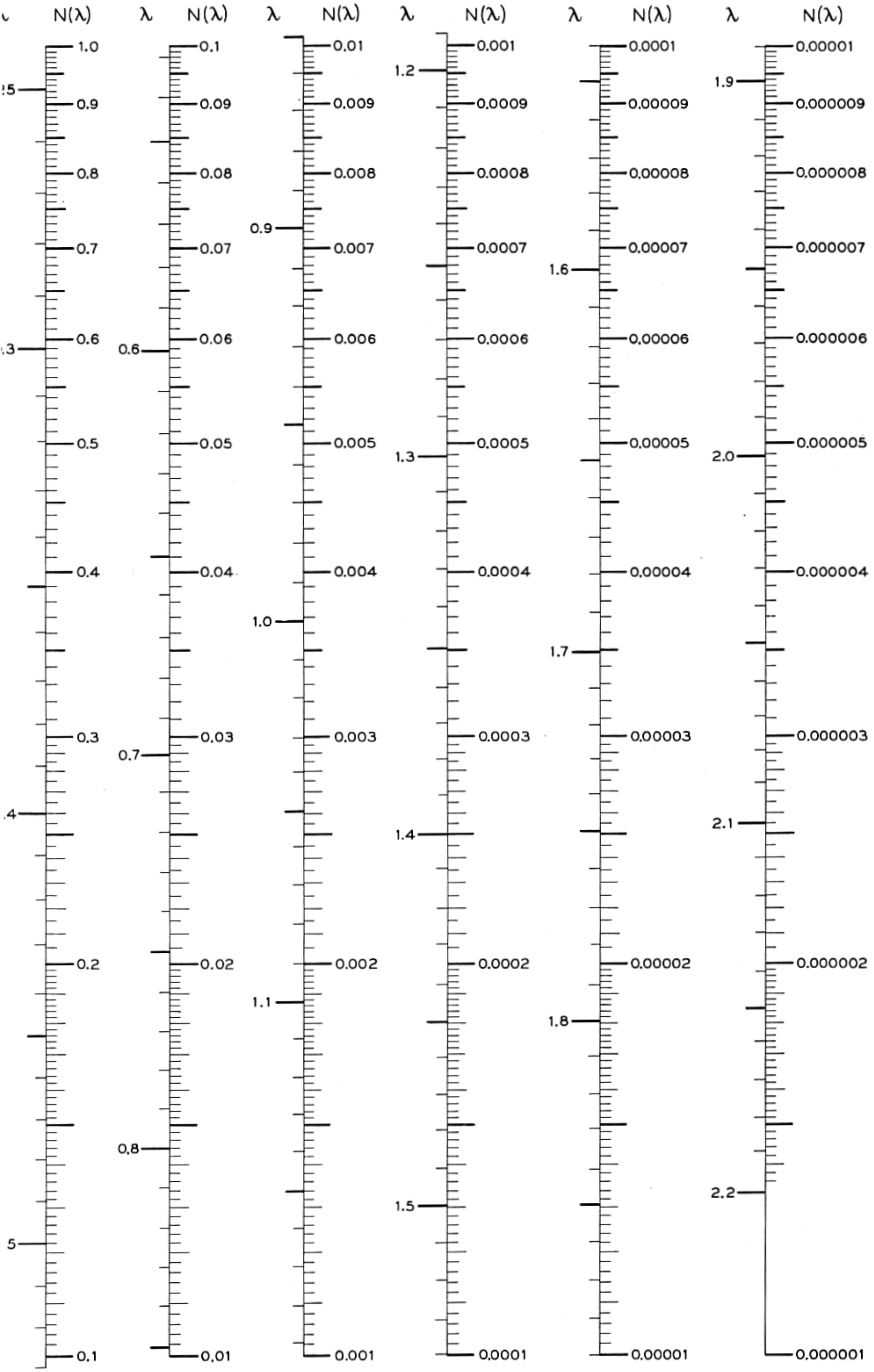


Fig. 4 — The function $N(\lambda) = 2\pi \sum_{n=1}^{\infty} iH_0^{(1)}(i2\pi n\lambda)$.

Table I — Values of the Functions M(λ) and N(λ)

λ	$M(\lambda)$	$N(\lambda)$
0.01	0.000120	90.557677
0.02	0.000481	41.943612
0.03	0.001081	26.087274
0.04	0.001921	18.328465
0.05	0.003000	13.773673
0.06	0.004317	10.803665
0.07	0.005872	8.729459
0.08	0.007662	7.209018
0.09	0.009686	6.053671
0.10	0.011943	5.151023
0.11	0.014432	4.430064
0.12	0.017150	3.843793
0.13	0.020096	3.359908
0.14	0.023266	2.955502
0.15	0.026660	2.613904
0.16	0.030273	2.322701
0.17	0.034105	2.072472
0.18	0.038151	1.855945
0.19	0.042410	1.667423
0.20	0.046877	1.502384
0.21	0.051550	1.357196
0.22	0.056426	1.228910
0.23	0.061501	1.115110
0.24	0.066773	1.013798
0.25	0.072237	0.923312
0.26	0.077889	0.842254
0.27	0.083726	0.769448
0.28	0.089746	0.703889
0.29	0.095942	0.644722
0.30	0.102312	0.591213
0.31	0.108852	0.542726
0.32	0.115557	0.498711
0.33	0.122425	0.458690
0.34	0.129450	0.422244
0.35	0.136629	0.389006
0.36	0.143958	0.358654
0.37	0.151432	0.330903
0.38	0.159048	0.305500
0.39	0.166801	0.282222
0.40	0.174687	0.260868
0.41	0.182703	0.241262
0.42	0.190844	0.223244
0.43	0.199107	0.206671
0.44	0.207486	0.191417
0.45	0.215979	0.177364
0.46	0.224582	0.164411
0.47	0.233289	0.152462
0.48	0.242098	0.141434
0.49	0.251005	0.131248
0.50	0.260006	0.121836
0.51	0.269098	0.113135
0.52	0.278275	0.105086
0.53	0.287536	0.097637
0.54	0.296876	0.090740
0.55	0.306292	0.084353
0.56	0.315780	0.078434
0.57	0.325338	0.072947
0.58	0.334961	0.067860
0.59	0.344647	0.063140
0.60	0.354392	0.058761
0.61	0.364193	0.054696
0.62	0.374047	0.050921
0.63	0.383952	0.047416
0.64	0.393903	0.044159
0.65	0.403900	0.041133
0.66	0.413937	0.038320
0.67	0.424014	0.035705
0.68	0.434127	0.033273
0.69	0.444273	0.031012
0.70	0.454451	0.028907
0.71	0.464658	0.026949
0.72	0.474890	0.025127
0.73	0.485148	0.023431
0.74	0.495426	0.021852
0.75	0.505725	0.020381
0.76	0.516041	0.019012
0.77	0.526373	0.017736
0.78	0.536718	0.016548
0.79	0.547074	0.015440
0.80	0.557441	0.014409
0.81	0.567816	0.013447
0.82	0.578196	0.012551
0.83	0.588582	0.011715
0.84	0.598970	0.010936
0.85	0.609360	0.010210
0.86	0.619750	0.009532
0.87	0.630138	0.008900
0.88	0.640523	0.008311
0.89	0.650904	0.007761
0.90	0.661279	0.007248
0.91	0.671647	0.006770
0.92	0.682007	0.006323
0.93	0.692358	0.005906
0.94	0.702698	0.005518
0.95	0.713027	0.005155
0.96	0.723344	0.004816
0.97	0.733647	0.004499
0.98	0.743935	0.004204
0.99	0.754209	0.003928
1.00	0.764466	0.003671
1.01	0.774706	0.003431
1.02	0.784928	0.003206

TABLE I — VALUES OF THE FUNCTIONS M(λ) AND N(λ) — (*Continued*)

λ	$M(\lambda)$	$N(\lambda)$
1.03	0.795132	0.002996
1.04	0.805316	0.002801
1.05	0.815480	0.002618
1.06	0.825624	0.002447
1.07	0.835746	0.002287
1.08	0.845847	0.002138
1.09	0.855924	0.001999
1.10	0.865979	0.001869
1.11	0.876010	0.001748
1.12	0.886018	0.001634
1.13	0.896000	0.001528
1.14	0.905958	0.001428
1.15	0.915890	0.001336
1.16	0.925797	0.001249
1.17	0.935677	0.001168
1.18	0.945531	0.001092
1.19	0.955358	0.001022
1.20	0.965158	0.000956
1.21	0.974930	0.000894
1.22	0.984675	0.000836
1.23	0.994392	0.000782
1.24	1.004080	0.000731
1.25	1.013740	0.000684
1.26	1.023371	0.000640
1.27	1.032974	0.000599
1.28	1.042547	0.000560
1.29	1.052091	0.000524
1.30	1.061606	0.000490
1.31	1.071091	0.000459
1.32	1.080547	0.000429
1.33	1.089973	0.000401
1.34	1.099369	0.000376
1.35	1.108735	0.000351
1.36	1.118072	0.000329
1.37	1.127378	0.000308
1.38	1.136654	0.000288
1.39	1.145900	0.000270
1.40	1.155115	0.000252
1.41	1.164300	0.000236
1.42	1.173455	0.000221
1.43	1.182580	0.000207
1.44	1.191674	0.000193
1.45	1.200738	0.000181
1.46	1.209772	0.000169
1.47	1.218775	0.000159
1.48	1.227749	0.000148
1.49	1.236691	0.000139
1.50	1.245604	0.000130
1.51	1.254486	0.000122
1.52	1.263338	0.000114
1.53	1.272161	0.000106
1.54	1.280952	0.000100
1.55	1.289715	0.000093
1.56	1.298447	0.000087
1.57	1.307149	0.000082
1.58	1.315821	0.000077
1.59	1.324464	0.000072
1.60	1.333077	0.000067
1.61	1.341660	0.000063
1.62	1.350214	0.000059
1.63	1.358739	0.000055
1.64	1.367234	0.000051
1.65	1.375700	0.000048
1.66	1.384137	0.000045
1.67	1.392544	0.000042
1.68	1.400923	0.000040
1.69	1.409273	0.000037
1.70	1.417594	0.000034
1.71	1.425886	0.000032
1.72	1.434150	0.000030
1.73	1.442385	0.000028
1.74	1.450593	0.000026
1.75	1.458772	0.000025
1.76	1.466923	0.000024
1.77	1.475046	0.000022
1.78	1.483141	0.000020
1.79	1.491208	0.000019
1.80	1.499248	0.000018
1.81	1.507260	0.000017
1.82	1.515244	0.000016
1.83	1.523202	0.000015
1.84	1.531132	0.000014
1.85	1.539036	0.000013
1.86	1.546912	0.000012
1.87	1.554762	0.000011
1.88	1.562585	0.000010
1.89	1.570381	0.000010
1.90	1.578151	0.000009
1.91	1.585895	0.000009
1.92	1.593612	0.000008
1.93	1.601304	0.000008
1.94	1.608970	0.000007
1.95	1.616609	0.000007
1.96	1.624224	0.000006
1.97	1.631812	0.000006
1.98	1.639376	0.000006
1.99	1.646913	0.000005
2.00	1.654426	0.000005
2.10	1.728200	0.000003
2.20	1.799596	0.000001
2.30	1.868737	0.000001
2.40	1.935741	0.000000
2.50	2 000718	0.000000

may be written

$$V(Q) = \frac{q}{a} N(\lambda) \tag{2}$$

if we continue to denote by a the spacing between the point sources and now take d to be the distance between the point Q and the line. As λ becomes infinite, N goes to zero.

The potential of a line of point sources along a perpendicular to the line erected at one of the point sources can be taken at

$$V = \frac{q}{a}\left[\frac{1}{\lambda} - M(\lambda)\right] \tag{3}$$

where $\lambda = d/a$ and d is the distance from the line of sources.

Numerical values of M and N can be obtained with sufficient accuracy for most purposes from the charts in Figs. 3 and 4. More accurate values of M and N are given in Table I for various values of λ.

A useful relation exists between the functions M and N. It is

$$M(\lambda) + N(\lambda) = \frac{1}{\lambda} + 2 \ln \lambda - 2 \ln 2 + 2C, \tag{4}$$

where C is Euler's constant, 0.577215665. That is

$$M(\lambda) + N(\lambda) = \frac{1}{\lambda} + 2 \ln \lambda - 0.231863031$$

The methods of computing M and N are described in the appendix.

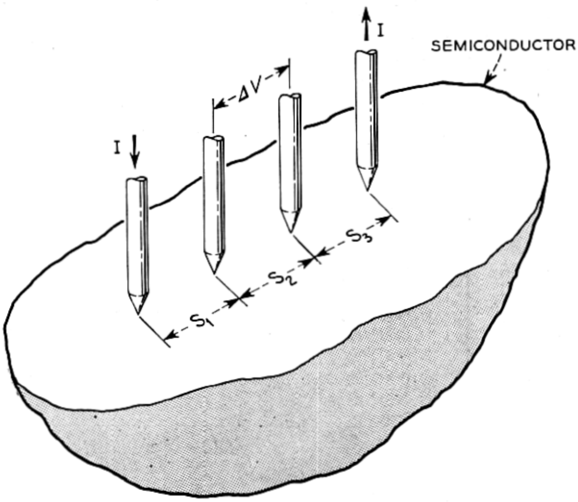


Fig. 5 — Linear four-point probe.

FOUR-POINT RESISTIVITY MEASUREMENTS

Resistivity measurements are regularly being made on germanium with a four-point probe. The basic assumptions are that the resistivity is uniform and that the potential distribution can be accurately approximated by that calculated for ohmic flow. The results of calculations for some geometries have been presented by L. B. Valdes, along with a discussion of the conditions for the validity of the assumption of ohmic flow.[1] In brief, it is required that modulation of the conductivity by injected minority carriers be kept negligible. It is also necessary that the probes used to measure potential difference be sufficiently conducting to pass the current drawn by the voltage measuring instrument. Both of these requirements can usually be met, in the case of germanium, by abrading the surface or by subjecting the contact between probe and germanium to an electric discharge, *i.e.*, "forming" the contact.

The prototype geometry for four-point resistivity measurements is given in Fig. 5, which is taken from Reference 1. The potentials at probes 2 and 3 are

$$\begin{aligned} V_2 &= \frac{q}{S_1} - \frac{q}{S_2 + S_3} \\ V_3 &= \frac{q}{S_1 + S_2} - \frac{q}{S_3} \end{aligned} \tag{5}$$

where q is the strength of a source corresponding to the current I. By By considering a hemispherical surface of infinitesimal radius r centered on one of the current probes, one sees that

$$I = \frac{1}{\rho} \times \text{area} \times E = \frac{1}{\rho}\, 2\pi r^2 \frac{q}{r^2}$$

or

$$q = I\rho/2\pi \tag{6}$$

where ρ is the resistivity and $E = q/r^2$ is the magnitude of the electric field.

The potential difference ΔV between probes 2 and 3 is

$$\Delta V = q\left(\frac{1}{S_1} + \frac{1}{S_3} - \frac{1}{S_2 + S_3} - \frac{1}{S_1 + S_2}\right) \tag{7}$$

It will be assumed in the following examples that $S_1 = S_2 = S_3 = s$ in which case

$$\Delta V = q/s \tag{8}$$

so that

$$\rho = 2\pi s(\Delta V/I). \tag{9}$$

When the four-point probe is applied to a solid that is not approximately semi-infinite, a potential difference will still be observed when a current flows, so we will define an "apparent resistivity" ρ_0 by

$$\rho_0 = 2\pi s(\Delta V/I) \tag{10}$$

The true resistivity will be given by

$$\rho = \rho_0/\text{C.D.} \tag{11}$$

where C.D. is the correction divisor for the particular case.

One of the simplest and most useful departures from the semi-infinite geometry is the slab or slice of finite thickness, free of any conducting coating on its faces (and resting on a non-conducting support during the measurement). Fig. 6 shows the position of the probes and the configuration of sources required for calculating the correction divisor. The required configuration can be obtained by superimposing two of the arrangements shown in Fig. 1 (one with opposite sign) i.e., upon the source and sink used for the semi-infinite geometry. The additional potential at one of the probes due to the additional sources is

$$\frac{q}{2W}\,[M(2s/2w) - M(s/2w)]$$

and the additional potential at the other probe is equal and opposite, so that the additional potential difference is

$$\frac{q}{w}\,[M(2s/2w) - M(s/2w)]$$

In comparison to the potential difference for the semi-infinite case, we find that ΔV is "too high" by a factor

$$\begin{aligned} \text{C.D.} &= \frac{s}{q}\left(\frac{q}{s} + \frac{q}{w}\left[M\left(\frac{2s}{2w}\right) - M\left(\frac{s}{2w}\right)\right]\right) \\ &= 1 + \frac{s}{w}\left[M\left(\frac{s}{w}\right) - M\left(\frac{s}{2w}\right)\right] \end{aligned} \tag{12}$$

This expression is true in general, but is most convenient for slices that are thick in comparison to the spacing of the probes, since the correction terms are then small and errors in reading M from Fig. 3 do not matter much. The correction divisor may be given in terms of the function N

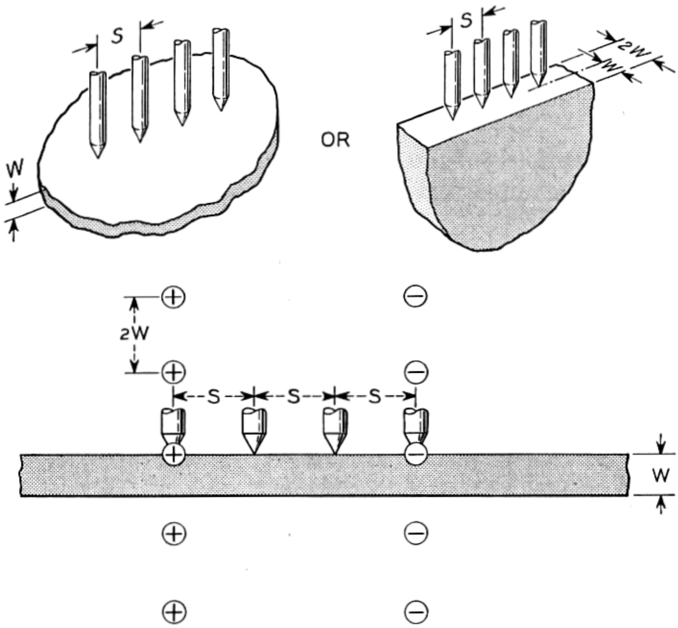
through use of equation (4). The result,

$$\text{C.D.} = \frac{s}{w}\left[2 \ln 2 + N\left(\frac{s}{2w}\right) - N\left(\frac{s}{w}\right)\right] \tag{13}$$

is most useful for thin slices. In fact, if the slice thickness w is less than half the probe spacing s, C.D. $\approx (2 \ln 2)\, s/w$ and equations (10) and (11) give

$$\rho \approx \pi w \Delta V/I \ln 2 \approx 4.53 w \Delta V/I \qquad (\text{for } w \ll \tfrac{1}{2}s) \tag{14}$$

The correction divisor is given in a table in Fig. 6 for a number of ratios of probe spacing to slice thickness. For larger values of this ratio than are



s/w	C.D.	s/w	C.D.	s/w	C.D.
0.1	1.0009	0.6	1.1512	1.2	1.7329
0.2	1.0070	0.7	1.2225	1.4	1.9809
0.3	1.0227	0.8	1.3062	1.6	2.2410
0.4	1.0511	0.9	1.4008	1.8	2.5083
0.5	1.0939	1.0	1.5045	2.0	2.7799
				2.5	3.4674

Fig. 6 — Linear probe on infinite slice with nonconducting faces (or on edge of semi-infinite slice).

found in the table, equation (14) may be used with sufficient accuracy for all practical resistivity measurements.

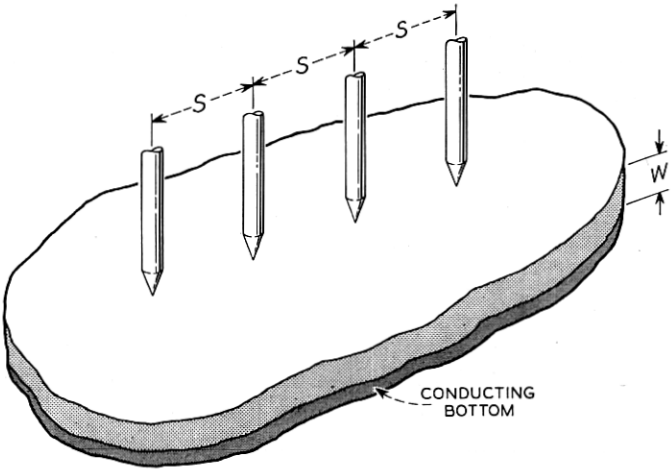
In general, a large correction divisor is desirable because it indicates that a relatively large potential difference is to be measured. For this reason, the use of the four-point probe on thin slices should be and, in fact, is found in practice to be quite satisfactory. However, if the slice is provided with a conducting coating on one side, as in intermediate stages of point-contact transistor fabrication, the measurement is apt to be inaccurate. The correction divisor for this case is

$$\text{C.D.} = 1 + \frac{s}{w}\left[2M\left(\frac{s}{2w}\right) - M\left(\frac{s}{w}\right) - M\left(\frac{s}{4w}\right)\right] \tag{15}$$

or

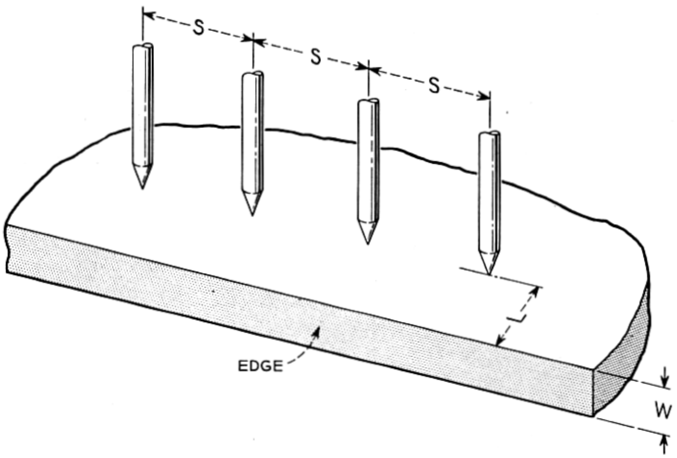
$$\text{C.D.} = \frac{s}{w}\left[N\left(\frac{s}{4w}\right) + N\left(\frac{s}{w}\right) - 2N\left(\frac{s}{2w}\right)\right] \tag{16}$$

Fig. 7 illustrates this configuration and contains a table of the correction divisor. It will be noted that the correction divisor is quite small if the



s/w	C.D.	s/w	C.D.	s/w	C.D.	s/w	C.D.
0.1	.9993	0.5	.9329	1.0	.6833	2.0	.2283
0.2	.9948	0.8	.7960	1.5	.4159	5.0	.0034

Fig. 7 — Linear probe on infinite slice with conducting bottom face.



Correction Divisor for Conducting Case

L/s

s/w	0.1	0.2	0.5	1.0	2.0	5.0	10.0
0.0	0.034	0.124	0.481	0.811	0.962	0.9971	0.9996
0.1	0.03	0.124	0.48	0.81	0.96	0.997	1.0001
0.2	0.03	0.125	0.48	0.81	0.96	1.002	1.0064
0.5	0.04	0.125	0.49	0.83	1.01	1.08	1.090
1.0	0.04	0.142	0.56	1.03	1.34	1.48	1.497
2.0	0.066	0.22	0.95	1.84	2.46	2.72	2.765
5.0	0.146	0.55	2.35	4.58	6.12	6.78	6.894

Correction Divisor for Non Conducting Case

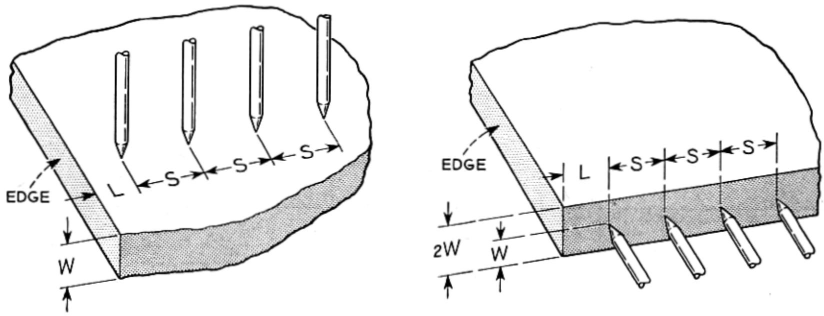
L/s

s/w	0	0.1	0.2	0.5	1.0	2.0	5.0	10.0
0.0	2.000	1.9661	1.8764	1.5198	1.1890	1.0379	1.0029	1.0004
0.1	2.002	1.97	1.88	1.52	1.19	1.040	1.004	1.0017
0.2	2.016	1.98	1.89	1.53	1.20	1.052	1.014	1.0094
0.5	2.188	2.15	2.06	1.70	1.35	1.176	1.109	1.0977
1.0	3.009	2.97	2.87	2.45	1.98	1.667	1.534	1.512
2.0	5.560	5.49	5.34	4.61	3.72	3.104	2.838	2.795
5.0	13.863	13.72	13.32	11.51	9.28	7.744	7.078	6.969
10.0	27.726	27.43	26.71	23.03	18.56	15.49	14.156	13.938

Fig. 8 — Semi-infinite slice — linear probe parallel to edge.

spacing between probe points is much larger than the slice thickness. The two-point measurement to be described below seems preferable to the four-point measurement in the case of thin slices with a conducting coating, because less effort is required to maintain an accurate small spacing between two points than between four points.

The two cases that have been treated thus far were also given by Valdes, but have been mentioned here as simple examples of the use of the M and N functions. An example of a more complicated problem is that of the linear four-point probe on a semi-infinite slice, parallel to the edge of the slice, as shown in Fig. 8. It is useful to note that the



Correction Divisor for Nonconducting Edge

s/w	L/s: 0	0.1	0.2	0.5	1.0	2.0	5.0	10.0	∞
0.0	1.4500	1.3330	1.2555	1.1333	1.0595	1.0194	1.0028	1.0005	1.0000
0.1	1.4501	1.3331	1.2556	1.1335	1.0597	1.0198	1.0035	1.0015	1.0009
0.2	1.4519	1.3352	1.2579	1.1364	1.0637	1.0255	1.0107	1.0084	1.0070
0.5	1.5285	1.4163	1.3476	1.2307	1.1648	1.1263	1.1029	1.0967	1.0939
1.0	2.0335	1.9255	1.8526	1.7294	1.6380	1.5690	1.5225	1.5102	1.5045
2.0	3.7236	3.5660	3.4486	3.2262	3.0470	2.9090	2.8160	2.7913	2.7799
5.0	9.2815	8.8943	8.6025	8.0472	7.5991	7.2542	7.0216	6.9600	6.9315
10.0	18.5630	17.7886	17.2050	16.0944	15.1983	14.5083	14.0431	13.9199	13.8629

Correction Divisor for Conducting Edge

s/w	L/s: 0	0.1	0.2	0.5	1.0	2.0	5.0	10.0	∞
0.0	0.5500	0.6670	0.7445	0.8667	0.9405	0.9806	0.9972	0.9995	1.0000
0.1	0.5517	0.6687	0.7462	0.8683	0.9421	0.9820	0.9982	1.0003	1.0009
0.2	0.5620	0.6788	0.7560	0.8775	0.9502	0.9885	1.0033	1.0056	1.0070
0.5	0.6593	0.7714	0.8402	0.9571	1.0230	1.0615	1.0849	1.0910	1.0939
1.0	0.9754	1.0835	1.1563	1.2796	1.3709	1.4399	1.4864	1.4988	1.5045
2.0	1.8362	1.9938	2.1113	2.3336	2.5129	2.6508	2.7439	2.7685	2.7799
5.0	4.5815	4.9687	5.2605	5.8158	6.2638	6.6088	6.8413	6.9030	6.9315
10.0	9.1629	9.9373	10.5209	11.6315	12.5276	13.2176	13.8060	13.8060	13.8629

Fig. 9 — Semi-infinite slice with probe perpendicular to edge (or quarter-infinite slice with probe on a nonconducting edge).

results from Fig. 6 can be incorporated into the calculations, for we may let the correction divisor

$$\text{C.D.} = \text{C.D. (for nonconducting infinite slices)} \pm \Delta\text{C.D.}, \qquad (17)$$

where the plus sign is used if the edge is nonconducting and the minus sign if the edge is conducting (the faces of the slice are assumed to be nonconducting in either case). Then

$$\Delta\text{C.D.} = \frac{1}{\sqrt{\eta^2 + 1/4}} - \frac{1}{\sqrt{\eta^2 + 1}} + \mu\left[M\left(\mu\sqrt{\eta^2+1}\right) - M\left(\mu\sqrt{\eta^2 + 1/4}\right)\right] \qquad (18)$$

or

$$\Delta\text{C.D.} = \mu\left[\ln\frac{\eta^2+1}{\eta^2+1/4} + N\left(\mu\sqrt{\eta^2+1/4}\right) - N\left(\mu\sqrt{\eta^2+1}\right)\right] \qquad (19)$$

where

$$\mu = \frac{s}{w} \qquad \text{and} \qquad \eta = \frac{L}{s} \qquad (20)$$

As shown in Fig. 8, L is the distance from the edge to the probe, s is the spacing of the probes, and w is the slice thickness. A table of values of the correction divisor is given in Fig. 8.

The situation shown in Fig. 9 is similar to that of Fig. 8 in that the linear four-point probe is near the edge of a semi-infinite slice. But in this case the probe is perpendicular to the edge and L is the distance from the edge to the nearest point of the probe. Here again the correction divisor can be written in terms of the result for infinite slices, equation (17). It is found that

$$\Delta\text{C.D.} = \frac{s}{2w}\left(\frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{\gamma} - \frac{1}{\delta} + M(\gamma) + M(\delta) - M(\alpha) - M(\beta)\right) \qquad (21)$$

or

$$\Delta\text{C.D.} = \frac{s}{2w}\left(2\ln\frac{\gamma\delta}{\alpha\beta} + N(\alpha) + N(\beta) - N(\gamma) - N(\delta)\right) \qquad (22)$$

where

$$\begin{aligned} \alpha &= (L + 1/2 s)/w \\ \beta &= (L + 5/2 s)/w \\ \gamma &= (L + s)/w \\ \delta &= (L + 2s)/w \end{aligned} \qquad (23)$$

Correction divisors can be compounded still further. For example, the correction divisor for the quarter-infinite slice with probe on the diagonal, shown in Fig. 10, may be written to make use of the results for Fig. 9. That is,

$$\text{C.D.} = \text{C.D.} \text{ (Semi-infinite slice with probe perpendicular to non-conducting edge, see Fig. 9)} \pm \Delta\text{C.D.} \tag{24}$$

Again, the plus sign is taken if both edges are nonconducting, the minus sign if both edges are conducting. If α, β, γ, and δ are redefined by

$$\begin{aligned} \alpha &= \frac{1}{2w}\sqrt{L^2 + (S + L)^2} \\ \beta &= \frac{1}{2w}\sqrt{(L + 2s)^2 + (L + 3s)^2} \\ \gamma &= \frac{1}{2w}\sqrt{L^2 + (L + 2s)^2} \\ \delta &= \frac{1}{2w}\sqrt{(L + 3s)^2 + (L + S)^2} \end{aligned} \tag{25}$$

then ΔC.D. can be calculated by inserting equation (25) into equation (21) or (22) and doubling the result. One numerical example will be

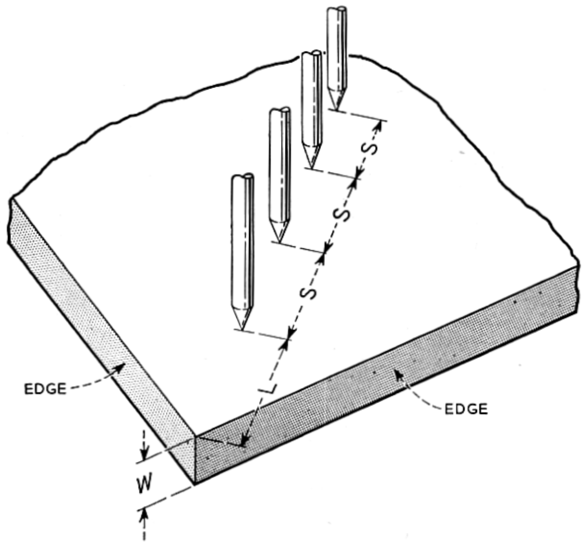


Fig. 10 — Linear probe on quarter-infinite slice.

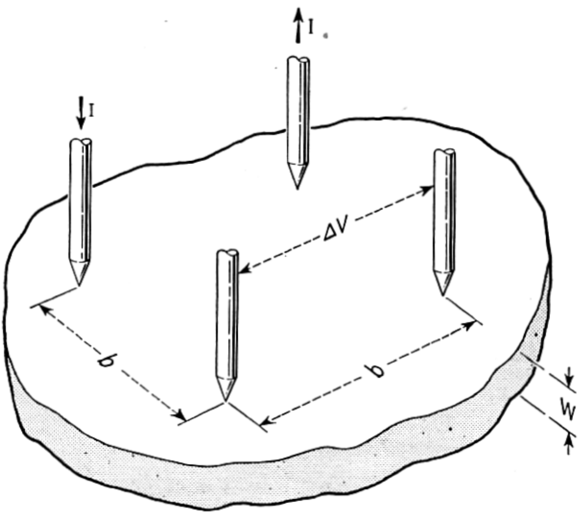
given. If $s = L = w$, C.D. = 2.1097 for nonconducting edges and 1.1665 for conducting edges.

It is evident that tabulation of correction divisors for situations with two or more degrees of freedom is rather tedious. Yet the computation of a correction divisor for a given set of dimensions may be very simple, requiring nothing more than the reading of several values of M or N from Fig. 3 or Fig. 4.

In a four-point resistivity measurement it is not necessary that the points be equally spaced nor that they be colinear. Whatever the chosen arrangement, the correction divisor can be calculated from the M and N functions for any position of the probe on infinite, semi-infinite, or quarter-infinite slices. Numerous less frequently encountered situations may also be treated.

The square arrangement of probe points illustrated in Fig. 11 is useful. When used on the surface of a semi-infinite solid, the resistivity is given by

$$\rho = \rho_0 = \frac{2\pi b}{2 - \sqrt{2}} \frac{\Delta V}{I} \tag{26}$$



b/w	0.1	0.2	0.5	1.0	2.0	5.0	10.0
C.D.	1.0005	1.004	1.057	1.344	2.378	5.916	11.832

Fig. 11 — Square arrangement of points — infinite slice.

where b is the length of the side of the square. When the square probe is used on infinite slices with nonconducting faces, the resistivity calculated from equation (26) must be corrected by dividing by

$$\text{C.D.} = 1 + \frac{b}{(2 - \sqrt{2})w}\left[M\left(\frac{b}{\sqrt{2}w}\right) - M\left(\frac{b}{2w}\right)\right] \tag{27}$$

or

$$\text{C.D.} = \frac{b}{(2 - \sqrt{2})w}\left[\ln 2 + N\left(\frac{b}{2w}\right) - N\left(\frac{b}{\sqrt{2}w}\right)\right] \tag{28}$$

Some values of this correction divisor are given in Fig. 11. For thin slices,

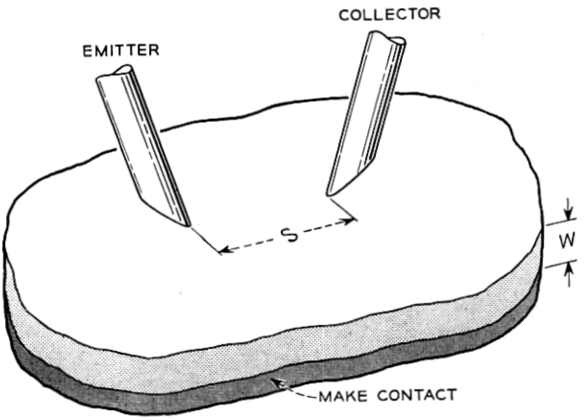
$$\rho \approx \frac{2\pi w \Delta V}{I \ln 2} = 9.06w\,\frac{\Delta V}{I} \qquad \left(w \ll \frac{1}{2}\,b\right) \tag{29}$$

BASE RESISTANCE OF A POINT CONTACT TRANSISTOR AND TWO-POINT RESISTIVITY MEASUREMENTS

The base resistance r_b of a transistor is defined by

$$r_b = (\partial V_\epsilon / \partial I_c)_{I\epsilon} \tag{30}$$

where V_ϵ is the emitter voltage, I_c is the collector current, and I_ϵ is the emitter current. The main contribution to r_b for a wide-spaced point-contact transistor seems to be the ohmic resistivity of the semiconductor. In close-spaced point-contact transistors, the widening of



s/w	0.1	0.2	0.5	1.0	2.0
C.F.	0.931	0.862	0.667	0.401	0.118

Fig. 12 — Base resistance of a point-contact transistor.

the collector space-charge region and the modulation of conductivity by carrier injection have important effects on r_b; it is necessary to calculate the ohmic base resistance to evaluate these effects. The ohmic base resistance for a transistor made on an infinite slice with a base contact on the underside, as shown in Fig. 12, can be calculated from the M or N functions. It is found that

$$r_b = \frac{\rho}{2\pi s}\left(1 - \frac{s}{w}\ln 2 + \frac{s}{2w}\left[M\left(\frac{s}{2w}\right) - M\left(\frac{s}{4w}\right)\right]\right) \tag{30}$$

or

$$r_b = \frac{\rho}{4\pi w}\left[N\left(\frac{s}{4w}\right) - N\left(\frac{s}{2w}\right)\right] \tag{31}$$

These results can be expressed in terms of a correction factor:

$$r_b = \frac{\rho}{2\pi s} \times \text{C.F.} \tag{32}$$

where $\rho/2\pi s$ is the ohmic base resistance for a transistor constructed on a semi-infinite solid with base contact at infinity. Some values of the correction factor are given in Fig. 12.

In measuring the resistivity of a slice which has been provided with a conducting coating on one face, it is better to use two points in the geometry shown in Fig. 12 than to use a four-point probe, for the mechanical reason mentioned previously. The two-point probe also gives better localization of the measurement and reversal of the current affords a valuable check on the assumption of ohmic flow. Obviously, the correction *divisors* for correcting such a resistivity measurement for slice thickness are equal to the base resistance correction *factors* given in Fig. 12.

FURTHER APPLICATIONS OF THE TABULATED FUNCTIONS

In the previous examples, the potential was required only along a line perpendicular to a line of point sources *and passing through one of the sources*. The tabulations are of more general usefulness. For example, $[M(\lambda) - 2M(2\lambda)]q/a$ gives the potential of a line of point sources along a perpendicular bisector of the line between adjacent sources (this potential is zero when $\lambda = 0$; the potential at infinity is $-\infty$.).

The potential of a plane grid of equal point sources is easily obtained from the tabulations. Suppose that the sources have a regular spacing a in one direction and b in the perpendicular direction. There is no loss of generality in assuming that b is greater than or equal to a. The potential at a point a distance d from the plane of sources and on a perpendicular to the plane erected at one of the sources is

$$V = \frac{q}{a} R_k(\lambda) \tag{34}$$

where $\lambda = d/a$, $k = b/a$, and

$$\begin{aligned} R_k(\lambda) &= \frac{1}{\lambda} - M(\lambda) - \frac{1}{k} M\left(\frac{\lambda}{k}\right) \\ &\quad - 2 \sum_{n=1}^{\infty} [M(\sqrt{\lambda^2 + n^2k^2}) - M(nk)] \end{aligned} \tag{35}$$

Equation (4) enables one to transform equation (35) into the following rapidly converging expressions:

$$\begin{aligned} R_k(\lambda) &= \frac{1}{\lambda} - M(\lambda) - 2 \ln \frac{\sinh \pi\lambda/k}{\pi\lambda/k} \\ &\quad - 2 \sum_{n=1}^{\infty} [N(nk) - N(\sqrt{\lambda^2 + n^2k^2})] \end{aligned} \tag{36}$$

$$\begin{aligned} R_k(\lambda) &= N(\lambda) - 2 \ln \frac{\sinh \pi\lambda/k}{\pi\lambda/k} \\ &\quad - 2C - 2 \sum_{n=1}^{\infty} [N(nk) - N(\sqrt{\lambda^2 + n^2k^2})] \end{aligned} \tag{37}$$

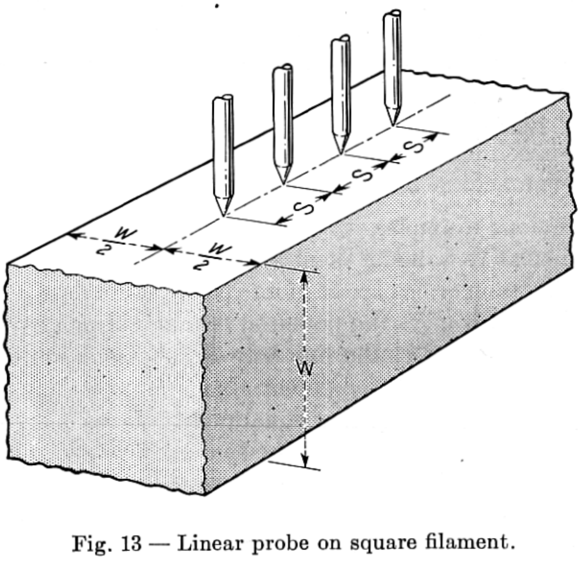


Fig. 13 — Linear probe on square filament.

Equation (36) is best for small values of λ, equation (37) for large values. In either case, one need consider only $n = 1$ to get five-place accuracy.

The potential of a plane grid of sources may be used to calculate the correction divisor for four-point resistivity measurements on filaments. The case of square filaments is illustrated in Fig. 13; the correction divisor for this case is

$$C.D. = \frac{2s}{w}\left[R_2\left(\frac{s}{w}\right) - R_2\left(\frac{2s}{w}\right)\right] \tag{38}$$

For example, when $s/w = 2/3$, C.D. $= 3.10$. This value agrees very well with some experiments in which the reading of a four-point probe on a long germanium filament was compared with the resistivity determined from the potential gradient in the filament when current flowed through its entire length.

The usefulness of the tabulations for three-dimensional arrays of point sources was tested by calculating the Madelung constant α for the Coulomb energy of an ionic crystal of the sodium chloride type. The calculation involves summing the potentials of a number of lines of sources of alternating sign; these potentials are obtained in the same way as for the calculation of the base resistance of a point-contact transistor. Nine terms must be considered to make full use of the accuracy of the tables. It is found that

$$\begin{aligned}\alpha = 2\ln 2 + 4\Bigg[N\left(\frac{1}{2}\right) - N\left(\frac{\sqrt{2}}{2}\right) - 2N(1) + 2N\left(\frac{\sqrt{5}}{2}\right) \\ + N\left(\frac{3}{2}\right) - 2N\left(\frac{\sqrt{10}}{2}\right) + 2N\left(\frac{\sqrt{13}}{2}\right) \\ - N\left(\frac{3}{2}\sqrt{2}\right) - 4N(\sqrt{5})\Bigg]\end{aligned} \tag{39}$$

and a value 1.74755 is obtained. The correct value is 1.747558.[3]

Plane boundaries between media of different but finite resistivities (or different dielectric constants or different thermal conductivities) can be treated by the method of images.[4] When infinite image systems arise in this sort of problem, the tabulations given in this article can be applied.

CONCLUSION

The potential of a line of point sources can be obtained from either of the functions M and N, which have been defined and tabulated. In addition to the tabulated functions, only elementary functions are required for this purpose.

The functions M and N represent the potentials of definite source systems. This fact enables one to visualize the construction of solutions to potential problems involving lines of point sources.

ACKNOWLEDGEMENTS

I am indebted to Mrs. M. B. Prince and to my wife Ingeborg for assistance in the computations and preparation of the illustrations, and to R. W. Hamming for assistance in programming the calculations incorporated in Table 1.

APPENDIX

From Fig. 1, the superposition principle, and the usual q/r potential of a point source, it may be seen that

$$M(\lambda) = 2 \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{\sqrt{n^2 + \lambda^2}} \right) \tag{40}$$

According to the binomial theorem,

$$\frac{1}{\sqrt{n^2 + \lambda^2}} = n^{-1} + \left(-\frac{1}{2} \right) n^{-3} \lambda^2 + \left(-\frac{1}{2} \right)\left(-\frac{3}{2} \right) \frac{1}{2!} n^{-5} \lambda^4 \cdots \tag{41}$$

so that

$$M(\lambda) = -2 \sum_{1}^{\infty} \left[\left(-\frac{1}{2} \right) n^{-3} \lambda^2 + \left(-\frac{1}{2} \right)\left(-\frac{3}{2} \right) \frac{1}{2!} n^{-5} \lambda^4 \cdots \right] \tag{42}$$

or

$$M(\lambda) = \zeta(3)\lambda^2 - \tfrac{3}{4}\zeta(5)\lambda^4 + \cdots , \tag{43}$$

where

$$\zeta(\sigma) = \sum_{n=1}^{\infty} n^{-\sigma} \tag{44}$$

is the Riemann zeta function. Equation (43) indicates a convenient approximation to $M(\lambda)$ for very small λ:

$$M(\lambda) \approx \zeta(3)\lambda^2 = 1.202\lambda^2 \tag{45}$$

The binomial expansion, equation (41), converges more rapidly, the higher the value of n. It is, therefore, worthwhile to split up the sum in equation (40) into two parts. The first part will be a sum of the terms from $n = 1$ to $n = m$, calculated with the original expression for the summand. The second part will be a summation from $n = m + 1$ to $n = \infty$ of the binomial expansion of the summand. It is expedient to take $m = 1$ for $\lambda < 1$. Thus,

$$M(\lambda) = 2 - 2(1 + \lambda^2)^{-1/2} + A_3\lambda^2 - \tfrac{3}{4}A_5\lambda^4 + \tfrac{5}{8}A_7\lambda^6 - \tfrac{35}{64}A_9\lambda^8 + \tfrac{63}{128}A_{11}\lambda^{10} - \tfrac{231}{512}A_{13}\lambda^{12} + \cdots \qquad (46)$$

where

$$A_\sigma = \zeta(\sigma) - 1 = \sum_{n=2}^{\infty} n^{-\sigma} \qquad (47)$$

Table II gives numerical values of the coefficients of equation (46).[5] Table I is based mainly on calculations with $m = 3$. These were done at the Laboratories' IBM installation.

Table II — Numerical Values of Coefficients in Equation (46)

$$M(\lambda) = 2 - 2(1 + \lambda^2)^{-1/2} + .20205\,69032\lambda^2 - .02769\,58163\lambda^4 + .00521\,82984\lambda^6 - .00109\,83398\lambda^8 + .00024\,32335\lambda^{10} - .00005\,53648\lambda^{12} + \cdots$$

Table III — Some Values of the Derivative $M'(\lambda)$

λ	$M'(\lambda)$	λ	$M'(\lambda)$
0.05	0.11982	1.30	0.95001
0.10	0.23734	1.35	0.93512
0.15	0.35040	1.40	0.92004
0.20	0.45709	1.45	0.90489
0.25	0.55586	1.50	0.88975
0.30	0.64556	1.55	0.87471
0.35	0.72545	1.60	0.85982
0.40	0.79519	1.65	0.84513
0.45	0.86714	1.70	0.83068
0.50	0.90467	1.75	0.81649
0.55	0.94528	1.80	0.80259
0.60	0.97736	1.85	0.78900
0.65	1.00174	1.90	0.77568
0.70	1.01926	1.95	0.76270
0.75	1.03077	2.00	0.75003
0.80	1.03709	2.05	0.73768
0.85	1.03900	2.10	0.72564
0.90	1.03719	2.15	0.71391
0.95	1.03229	2.20	0.70249
1.00	1.02487	2.25	0.69136
1.05	1.01541	2.30	0.68053
1.10	1.00431	2.35	0.66999
1.15	0.99195	2.40	0.65972
1.20	0.97862	2.45	0.64973
1.25	0.96457	2.50	0.64000

Table IV — Some Values of the Derivative $N'(\lambda)$

λ	$-N'(\lambda)$	λ	$-N'(\lambda)$
1.00	0.02487	1.55	0.00063
1.05	0.01766	1.60	0.00046
1.10	0.01257	1.65	0.00033
1.15	0.00896	1.70	0.00024
1.20	0.00640	1.75	0.00019
1.25	0.00457	1.80	0.00012
1.30	0.00326	1.85	0.00009
1.35	0.00233	1.90	0.00006
1.40	0.00167	1.95	0.00004
1.45	0.00120	2.00	0.00003
1.50	0.00086		

Although equation (46) certainly converges for all values of λ less than 2, it is not efficient for $\lambda > 1$. For $\lambda > 1$, it is better to work with the function N. According to Madelung,[6]

$$N(\lambda) = 2\pi \sum_{n=1}^{\infty} iH_0^{(1)}(i2\pi n\lambda) \tag{48}$$

where $iH_0^{(1)}(ix)$ is the Hankel function tabulated in Jahnke & Emde.[7]

The functions $M(\lambda)$ and $N(\lambda)$ are tabulated in Table 1. $M(\lambda)$ and $N(\lambda)$ are connected by equation (4). Because of the overlapping regions of λ-values for which the expressions for M and N converge with reasonable rapidity, it is possible to tabulate both M and N for all values of λ. However, for satisfactory graphical interpolation, it is necessary to use the smaller of the two functions, so that M is given in Fig. 3 for small values of λ and N is given in Fig. 4 for large values of λ. Table III gives some values of the derivative, $M'(\lambda)$. The field E of a line of point charges can be calculated from M' in accordance with the equation obtained by differentiating equation (3) with respect to d. Thus

$$E = \frac{q}{a^2}\left[\frac{1}{\lambda^2} + M'(\lambda)\right] \tag{49}$$

The derivative $N'(\lambda)$ is tabulated in Table IV.

1. L. B. Valdes, Resistivity measurements on germanium for transistors, Proc. I.R.E., **42,** pp. 420–427 (1954).
2. L. B. Valdes, Effect of electrode spacing on the equivalent base resistance of point-contact transistors, Proc. I.R.E., **40,** pp. 1429–1434 (1952).
3. J. Sherman, Chem. Revs., **11,** p. 93, (1932).
4. W. R. Smythe, Static and Dynamic Electricity, p. 113, McGraw-Hill, 1939.
5. Values of A_σ were obtained from the British Association for the Advancement of Science Mathematical Tables, **1,** p. xxv.
6. E. Madelung, Physikalische Zeitschrift, **19,** p. 524 (1918).
7. Jahnke & Emde, Functionentafeln, Fourth Edition, pp. 236–242.