Topics in Guided Wave Propagation Through Gyromagnetic Media

Part III — Perturbation Theory and Miscellaneous Results

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Some problems, complete discussion of which would be extremely difficult, are treated approximately by means of perturbation theory. Among these are the partially filled cylindrical waveguide, and the problem of multiple internal reflections in a sample of finite length filling the cross section of a cylindrical guide. Propagation in a ferrite between parallel planes, magnetized along the propagation direction is discussed by the methods described in Part I. The paper concludes with an addendum to Part I — a numerical study of field patterns of the TE_{11}-limit and TM_{11}-limit mode for various dc magnetic fields.

INTRODUCTION

Parts I and II of this paper were devoted to a number of specific propagation problems, whose solutions, though frequently quite complicated, could be discussed with a reasonably modest investment of effort. Unfortunately, not all of these problems pertain to situations met with in actual gyromagnetic devices. Actual devices frequently employ structures whose performance could be predicted only as the result of lengthy computing programs. For example, the microwave gyrator using Faraday rotation usually employs a ferrite sample whose cross-section only partly fills that of the cylindrical waveguide. Although it is easy to formulate the corresponding equation for the propagation constant, the classification and survey, let alone the computation of solutions, would be very difficult to carry out.

Thus, one must often be content with approximate results, and the bulk of the present paper is devoted to perturbation methods. These take as starting point a situation whose propagation problem is essentially solved. The small change in propagation constant due to a slight change in the original state of the system is then calculated. The small change of state may be the weak magnetization of an originally unmagnetized specimen occupying a substantial part of the structure, or the introduction of a very small specimen with arbitrarily large magnetization into the originally empty structure. Under the heading, "Small Magnetization — Arbitrary Sample-Size," we shall discuss the propagation constant for a pencil of ferrite of any radius, coaxial with a cylindrical waveguide, the space between guide wall and pencil being filled with an isotropic medium whose dielectric constant equals that of the ferrite. This is discussed in preparation for the practically more important case of a ferrite pencil of any radius in an air-filled guide. Here the unperturbed state of the system, when the pencil is unmagnetized and therefore isotropic, is already rather complicated and require some preliminary calculations. Under the heading "Small Sample-Size — Arbitrary Magnetization," we consider the case of a thin pencil of ferrite in an originally empty guide.

Another topic, not easily treated except by perturbation methods, is that concerning end effects in samples of finite length. After a preliminary discussion of internal reflections in an extended slab of ferrite (a problem which can be treated rigorously), two cases are considered: a ferrite slug of arbitrary length, closely fitting a cylindrical guide, and a thin disc normal to the guide axis, of arbitrary size. In these cases interest centers around the effect of sample length on Faraday rotation, though for the ferrite slug a subsidiary effect, that of mode conversion, is also mentioned briefly.

It should be emphasized that the perturbation methods employed here are not in themselves novel. They are standard to most linear eigenvalue problems of physics, and have been used in connection with electromagnetic problems by many authors.[1]

The remainder of the paper is devoted to a discussion of a ferrite-filled "cable" in plane parallel form, using the methods of Part I. The treatment is kept in terms of saturation magnetization and magnetizing field, and is based on Polder's equations. The paper concludes with an addendum to Part I, which reports some calculations and graphs of field patterns in a cylindrical waveguide completely filled with ferrite.

1. PERTURBATION METHODS

1.1 *General Method*

A number of authors have made applications of perturbation theory to the problems of propagation in gyromagnetic media and the exposition which follows is included mainly for completeness. We shall develop the subject in the following fashion: it will be supposed that the unperturbed system is a wave guide containing a medium whose permeability and dielectric tensors are diagonal and isotropic, but may vary over the cross section of the guide, although not in the z-direction along the guide. For this system it will be assumed that a complete set of normal modes exists for which appropriate orthogonality relations are known. The perturbation of the system will then consist of changes in the permeability and dielectric tensors of the medium, including the addition of non-diagonal terms. If these changes are to be genuine perturbations, they must be of one of two kinds. Either, the variation in the properties of the medium is confined to a limited region, small in volume in some appropriate sense, in which case its magnitude may be large, or, we may have a small fractional change in the material properties extending over a considerable volume. The fields in the guide may be expanded in the normal modes and a system of equations is developed for the z-dependence of the amplitudes of these modes. These equations are then solved approximately, making use of the smallness of the perturbing terms. The results may then be specialized to the various situations of interest.

Let us suppose that the unperturbed permeability and dielectric constant are $\mu_1(x, y)$ and $\epsilon_1(x, y)$ respectively and that the system is now altered so that it possesses a permeability tensor

$$\begin{Vmatrix} \mu_2(x, y) & -j\kappa(x, y) & 0 \\ j\kappa(x, y) & \mu_2(x, y) & 0 \\ 0 & 0 & \mu_3(x, y) \end{Vmatrix}$$

and a dielectric tensor

$$\begin{Vmatrix} \epsilon_2(x, y) & -j\eta(x, y) & 0 \\ j\eta(x, y) & \epsilon_2(x, y) & 0 \\ 0 & 0 & \epsilon_3(x, y) \end{Vmatrix}.$$

Maxwell's equations for the perturbed system may be written, using the notation of Parts I and II,† in the form:

$$\begin{aligned} \nabla^* H_z - \frac{\partial H_t^*}{\partial z} - j\omega\epsilon_2 E_t - \omega\eta E_t^* &= 0, \\ \nabla^* E_z - \frac{\partial E_t^*}{\partial z} + j\omega\mu_2 H_t + \omega\kappa H_t^* &= 0, \\ \nabla \cdot H_t^* - j\omega\epsilon_3 E_z &= 0, \\ \nabla \cdot E_t^* + j\omega\mu_3 H_z &= 0. \end{aligned} \tag{1}$$

† We omit the vector signs from all transverse vectors, which are sufficiently labelled by the subscript "t."

These may be rearranged as

$$\nabla^* H_z - \frac{\partial H_t^*}{\partial z} - j\omega\epsilon_1 E_t = j\omega(\epsilon_2 - \epsilon_1)E_t + \omega\eta E_t^* = A_t\,,$$

$$\nabla^* E_z - \frac{\partial E_t^*}{\partial z} + j\omega\mu_1 H_t = -j\omega(\mu_2 - \mu_1)H_t - \omega\kappa H_t^* = B_t\,, \tag{2}$$

$$\nabla \cdot H_t^* - j\omega\epsilon_1 E_z = j\omega(\epsilon_3 - \epsilon_1)E_z = A_z\,,$$

$$\nabla \cdot E_t^* + j\omega\mu_1 H_z = -j\omega(\mu_3 - \mu_1)H_z = B_z\,,$$

where A_t, B_t, A_z and B_z are introduced as abbreviations for the terms on the right hand side of the equations. E_z and H_z may be eliminated by substituting in the first two equations the expressions

$$E_z = \frac{\nabla \cdot H_t^* - A_z}{j\omega\epsilon_1}$$

and

$$H_z = \frac{-\nabla \cdot E_t^* + B_z}{j\omega\mu_1}.$$

The two equations so obtained are

$$\frac{1}{j\omega}\nabla^*\left(\frac{\nabla \cdot E_t^*}{\mu_1}\right) + \frac{\partial H_t^*}{\partial z} + j\omega\epsilon_1 E_t = -A_t + \frac{1}{j\omega}\nabla^*\frac{B_z}{\mu_1}$$

and

$$\frac{1}{j\omega}\nabla^*\left(\frac{\nabla \cdot H_t^*}{\epsilon_1}\right) - \frac{\partial E_t^*}{\partial z} + j\omega\mu_1 H_t = B_t + \frac{1}{j\omega}\nabla^*\frac{A_z}{\epsilon_1}. \tag{3}$$

We now suppose that E_t and H_t can be expanded in the form

$$E_t = \sum_n a_n(z)E_{tn}(x, y)$$

and

$$H_t = \sum_n b_n(z)H_{tn}(x, y),$$

where

$$E_{tn}e^{-j\beta_n z} \qquad \text{and} \qquad H_{tn}e^{-j\beta_n z}$$

satisfy the unperturbed form of equations (3) and the boundary condition that tangential E vanishes at the guide wall. These equations have solutions for certain values of β_n only, but, if E_{tn}, H_{tn} are solutions for $\beta_n = c > 0$, then E_{tn}, $-H_{tn}$ are solutions for $\beta_n = -c$. We shall assume hereafter that E_{tn} and H_{tn} pertain to positive β_n values. For a given unperturbed mode it follows that $\frac{b_n(z)}{a_n(z)}$ reverses sign when the direction of propagation reverses. Substituting these series for E_t and H_t in equations (3), one finds

$$\sum_n \left[\frac{db_n}{dz} + j\beta_n a_n\right] H_{tn}^* = -A_t + \frac{1}{j\omega} \nabla^* \frac{B_z}{\mu_1} \tag{4a}$$

and

$$\sum_n \left[\frac{da_n}{dz} + j\beta_n b_n\right] E_{tn}^* = -B_t - \frac{1}{j\omega} \nabla^* \frac{A_z}{\epsilon_1}. \tag{4b}$$

The orthogonality relation between functions of different n has been given by Adler.[2] It is

$$\int E_{tn}^* \cdot \tilde{H}_{tm}\, dS = 0,$$

where $n \neq m$; the tilde denotes the complex conjugate and the integration goes over the cross section of the guide. We consider the fields to be un-normalized and write

$$\int E_{tn}^* \cdot \tilde{H}_{tn}\, dS = \Delta_n ,$$

Clearly

$$\int H_{tn}^* \cdot \tilde{E}_{tn}\, dS = -\tilde{\Delta}_n . \tag{5}$$

Multiplying equation (4b) by $\tilde{H}_{tn}\cdot$ and equation (4a) by $\tilde{E}_{tn}\cdot$ and integrating each expression over the cross section we have

$$-\tilde{\Delta}_n \left[\frac{db_n}{dz} + j\beta_n a_n\right] = -\int A_t \cdot \tilde{E}_{tn}\, dS + \frac{1}{j\omega} \int \tilde{E}_{tn} \cdot \nabla^* \frac{B_z}{\mu_1}\, dS,$$

$$\Delta_n \left[\frac{da_n}{dz} + j\beta_n b_n\right] = -\int B_t \cdot \tilde{H}_{tn}\, dS - \frac{1}{j\omega} \int \tilde{H}_{tn} \cdot \nabla^* \frac{A_z}{\epsilon_1}\, dS.$$

Now we use the identity

$$\underset{\text{surface}}{\int} G_t \cdot \nabla^* F\, dS = -\underset{\text{boundary}}{\int} F(G_t \cdot ds) + \underset{\text{surface}}{\int} F \nabla \cdot G^*\, dS.$$

This yields

$$\int \tilde{E}_{tn}\cdot\nabla^* \frac{B_z}{\mu_1}\,dS = -\int_{b'dry} \frac{B_z}{\mu_1}(\tilde{E}_{tn}\cdot ds) + \int \frac{B_z}{\mu_1}\nabla\cdot\tilde{E}_{tn}^*\,dS,$$

$$= j\omega \int B_z\tilde{H}_{zn}\,dS,$$

since $E_{tn}\cdot ds = 0$. H_{zn} is the H_z field appropriate to the n^{th} mode. Again, we have

$$\int \tilde{H}_{tn}\cdot\nabla^* \frac{A_z}{\epsilon_1} = -\int_{b'dry} \frac{A_z}{\epsilon_1}(\tilde{H}_{tn}\cdot ds) + \int \frac{A_z}{\epsilon_1}\nabla\cdot\tilde{H}_{tn}^*\,dS,$$

$$= -j\omega \int A_z\tilde{E}_{zn}\,dS.$$

since $A_z = 0$ on the boundary. E_{zn} is now the E_z field of the n^{th} mode. Making use of these results we obtain the equations for the amplitudes in the form:

$$\frac{db_n}{dz} + j\beta_n a_n = \frac{1}{\bar{\Delta}_n}\left[\int A_t\cdot\tilde{E}_{tn}\,dS - \int B_z\tilde{H}_{zn}\,dS\right],$$
$$\frac{da_n}{dz} + j\beta_n b_n = \frac{1}{\Delta_n}\left[-\int B_t\cdot\tilde{H}_{tn}\,dS + \int A_z\tilde{E}_{zn}\,dS\right]. \tag{6}$$

Restoring the full expressions for A_t, B_t, A_z and B_z, we have

$$\frac{db_n}{dz} + j\beta_n a_n = \frac{j\omega}{\bar{\Delta}_n}\left[\int (\epsilon_2 - \epsilon_1)E_t\cdot\tilde{E}_{tn}\,dS - j\int \eta E_t^*\cdot\tilde{E}_{tn}\,dS + \int (\mu_3 - \mu_1)H_z\tilde{H}_{zn}\,dS\right], \tag{7a}$$

$$\frac{da_n}{dz} + j\beta_n b_n = \frac{j\omega}{\Delta_n}\left[\int (\mu_2 - \mu_1)H_t\cdot\tilde{H}_{tn}\,dS - j\int \kappa H_t^*\cdot\tilde{H}_{tn}\,dS + \int (\epsilon_3 - \epsilon_1)E_z\tilde{E}_{zn}\,dS\right]. \tag{7b}$$

Equations (7a) and (7b) are, so far, exact, but they involve, on the right hand side, the functions E_t, H_t, E_z and H_z which are still unknown. We are interested in those cases where the integral terms are small, either as a consequence of the terms $(\epsilon_2 - \epsilon_1)$, η and so forth being small, or of their being finite only over a small region. In the first case the fields E_t, H_t, E_z and H_z may be replaced in the integrals by the values which they would have before the perturbation was made. In the second case this is not possible since a large change in the material constants of a region alters the field substantially within that region. Here, then, we have a preliminary problem to solve, namely that of determining the field in the perturbed region in a zero order approximation.

Perturbation problems may be divided into two classes by another distinction. The changes in material properties may be independent of the z-coordinate, so that the new problem is to consider propagation in a uniform guide differing slightly from the original one. Typical of such problems is that of a waveguide containing a ferrite rod of infinite length parallel to the z-axis; the perturbation consisting here of the change in the properties of the rod when it is magnetized. Clearly, in such cases, solutions for which all field components vary as $\exp - j\beta z$ are still possible and the perturbation equations (7) become equations to determine β. On the other hand, there is a class of problems for which the perturbation is confined to a limited region in the z-direction, and we are interested, perhaps, in the reflection and transmission coefficients for a wave incident upon the obstacle. Here, for example, we might think of the case of a disc of ferrite across the guide. If we remain in the range for which perturbation theory is valid the changes in the amplitude of reflected and transmitted waves will be small, but the changes in phase may not be, if the perturbed region is sufficiently long. In the latter case, it would be possible, if the perturbation were uniform in z over the region in which it exists, to find solutions going as $\exp j\beta z$, as described above, and to use these to fit the boundary conditions at the ends. It is also possible if the perturbed region is long, with slowly varying properties, to obtain suitable approximate solutions by the WKB method. Some of these cases will arise in the examples which we treat below.

1.2. *Perturbations Uniform in z*

We consider first the general case in which the perturbation is uniform in z. In the absence of the perturbation the m^{th} mode is to be present. For the fields E_t and H_z, in the perturbed region we write $a_m(z)E_{tm0}(x, y)$ and $a_m(z)H_{zm0}(x, y)$, respectively, where $a_m(z)$ is the amplitude function for the m^{th} mode. If the perturbed region is one in which, for no magnetization, the material properties differ only slightly from their unperturbed values, we may justifiably identify E_{tm0} with E_{tm} and H_{zm0} with H_{zm}. If the material properties are appreciably changed even in the absence of a magnetic field, E_{tm0} and H_{zm0} have to be calculated by an independent method. For a_m we put $A_m e^{-j\beta z}$, where $\beta = \beta_m + \delta\beta$ and $\delta\beta$ is small. Similarly for H_t and E_z, we write $b_m\, H_{tm0}$ and $b_m\, E_{zm0}$, with $b_m = B_m e^{-j\beta z}$. With such assumptions, the m^{th} set of equations (7) gives an equation for β, while any other set, with $n \neq m$, gives the excitation of the n^{th} mode. Substituting in (7) we have

$$j\beta_m A_m - j\beta B_m = \frac{j\omega}{\bar{\Delta}_m}\left(\int [(\epsilon_2 - \epsilon_1)E_{tm0}\cdot\tilde{E}_{tm} - j\eta E_{tm0}{}^*\cdot\tilde{E}_{tm} + (\mu_3 - \mu_1)H_{zm0}\tilde{H}_{zm}]\, dS\right) A_m \equiv jLA_m \tag{8a}$$

and

$$j\beta_m B_m - j\beta A_m = \frac{j\omega}{\Delta_m}\left(\int (\mu_2 - \mu_1)H_{tm0}\cdot\tilde{H}_{tm} - j\kappa H_{tm0}{}^*\cdot\tilde{H}_{tm} + (\epsilon_3 - \epsilon_1)E_{zm0}\tilde{E}_{zm}]\, dS\right) B_m \equiv jMB_m. \tag{8b}$$

Ignoring squares and products of small quantities, one then has

$$\delta\beta = \tfrac{1}{2}(L + M). \tag{9}$$

The first example to be considered is the effect on the propagation in a circularly cylindrical waveguide, when a coaxial, magnetized pencil of ferrite of very small radius is introduced. The guide radius is r_0 and that of the pencil is r_1. Before the ferrite is introduced, $\mu_1 = \mu_0$ and $\epsilon_1 = \epsilon_0$, where μ_0 and ϵ_0 are the free space values. The unperturbed fields are those of the usual TE and TM modes in round guide. It is necessary to calculate first the zero order electric and magnetic fields within the magnetized pencil; it will be sufficient to work out the magnetic case and deduce the electric one by analogy. Since the cross-section of the pencil is very small and transverse propagation effects consequently negligible, the internal field may be calculated by solving a static problem. The transverse magnetic field before the pencil is inserted is H_{tm} and it is assumed that the pencil is so small that over a circle with a few times the radius of the latter, H_{tm} is essentially uniform. We must now solve Laplace's equation for a cylindrical rod immersed in a magnetic field which is to be uniform at large distances. Within the rod, $B_t = \mu H_t - j\kappa H_t{}^*$, and at its surface the usual boundary conditions prevail. Hereafter we write μ for μ_2.

The fields are derivable from potentials Φ_{out}, Φ_{in}, which are of the form

$$\Phi_{\text{in}} = (H_{tm0}\cdot\vec{r}),$$

$$\Phi_{\text{out}} = (H_{tm}\cdot\vec{r}) + \frac{(\vec{a}\cdot\vec{r})}{r^2},$$

where $\vec{r} \equiv (x, y)$, $\vec{a}$ is a constant vector and the coordinate system has its origin at the centre of the rod. Continuity of tangential H at the surface of the rod requires

$$H_{tm0} = H_{tm} + \frac{\vec{a}}{r_1^2}$$

and then

$$\Phi_{\text{out}} = H_{tm}\cdot\vec{r} + \frac{r_1^2}{r^2}(H_{tm0} - H_{tm})\cdot\vec{r}.$$

The normal derivative at the surface of the rod is

$$\frac{1}{r_1}\left[x\frac{\partial\Phi}{\partial x} + y\frac{\partial\Phi}{\partial y}\right]$$

or, externally,

$$\frac{1}{r_1}[H_{tm}\cdot\vec{r} - (H_{tm0} - H_{tm})\cdot\vec{r}].$$

Matching the normal B's at the surface then gives

$$\mu_0[2\,H_{tm} - H_{tm0}] = \mu H_{tm0} - j\kappa H_{tm0}^*,$$

or

$$H_{tm0} = \frac{2\mu_0[(\mu + \mu_0)H_{tm} + j\kappa H_{tm}^*]}{(\mu + \mu_0)^2 - \kappa^2}. \tag{10a}$$

In a similar manner, one would find if the dielectric constant of the rod were ϵ,

$$E_{tm0} = \frac{2\epsilon_0 E_{tm}}{\epsilon + \epsilon_0}. \tag{10b}$$

The longitudinal fields E_{zm} and H_{zm} are unchanged within the rod.

Turning now to the expression (9) for $\delta\beta$, we have in the present-case,

$$\delta\beta = -\frac{\omega}{2\Delta_m}\int_{\text{pencil}} dS\left[\frac{2\epsilon_0(\epsilon - \epsilon_0)}{\epsilon + \epsilon_0}\,|\,E_{tm}\,|^2 + (\epsilon - \epsilon_0)\,|\,E_{zm}\,|^2\right.$$
$$\left. + 2\mu_0\frac{\mu^2 - \mu_0^2 - \kappa^2}{(\mu + \mu_0)^2 - \kappa^2}\,|\,H_{tm}\,|^2 - j\kappa\frac{4\mu_0^2}{(\mu + \mu_0)^2 - \kappa^2}H_{tm}^*\tilde{H}_{tm}\right],$$

where we have anticipated that Δ_m is real, which we verify below. Since the integrand is constant the integral may be replaced by πr_1^2 times the value of the integrand at $r = 0$. We consider now a TE-mode with variation $e^{jn\varphi}$, $n \neq 0$. We shall have,

$$E_{tm} = -j\omega\mu_0 \nabla^* \Psi_m ,$$

$$H_{tm} = -j\beta_m \nabla \Psi_m ,$$

where

$$\Psi_m = J_n \left(u_{nm} \frac{r}{r_0} \right) e^{jn\varphi}, \qquad J_n'(u_{nm}) = 0$$

and

$$\beta_m^2 = \omega^2 \epsilon_0 \mu_0 - \frac{u_{nm}^2}{r_0^2} .$$

For the fields on the axis, one finds for $n = \pm 1$,

$$nH_r = -jH_\varphi = -jn \frac{\beta_m u_{nm}}{2r_0} e^{jn\varphi}$$

and

$$nE_\Phi = jE_r = j\omega_0 n \frac{u_{nm}}{2r_0} e^{jn\varphi}.$$

If $| n | \neq 1$ there is no first order perturbation. We now have

$$\delta\beta = -\frac{\omega}{2\triangle_m} \pi r_1^2 \left[\epsilon_0 \frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0} \omega^2 \mu_0^2 \frac{u_{nm}^2}{r_0^2} + \frac{\mu_0 \beta_m^2 u_{nm}^2}{r_0^2} \frac{\mu + n\kappa - \mu_0}{\mu + n\kappa + \mu_0} \right].$$

But we have

$$\begin{aligned} \triangle_m &= 2\pi \int_0^{r_0} E_{tm}^* \cdot \tilde{H}_{tm} r \, dr, \\ &= -2\pi \cdot \omega\mu_0\beta_m \int_0^{u_{nm}} \left[J_n'(x)^2 + \frac{J_n(x)^2}{x^2} \right] x \, dx, \\ &= -2\pi\omega\mu_0\beta_m \frac{J_n(u_{nm})^2}{2} (u_{nm}^2 - 1). \end{aligned}$$

and then,

$$\delta\beta = \frac{1}{2\beta_m} \cdot \frac{r_1^2}{r_0^2} \cdot \frac{u_{nm}^2}{J_n(u_{nm})^2 (u_{nm}^2 - 1)} \cdot \left[\beta_m^2 \frac{\mu + n\kappa - \mu_0}{\mu + n\kappa + \mu_0} + \beta_0^2 \frac{\epsilon_1 - \epsilon_0}{\epsilon_1 + \epsilon_0} \right]. \tag{11}$$

For TM modes we have

$$E_{tm} = -j\beta_m \nabla \chi_m \qquad E_z = \frac{j_{nm}^2}{r_0^2}\chi_m$$

$$H_{tm} = j\omega\epsilon_0 \nabla^* \chi_m$$

where

$$\chi_m = J_n\left(j_{nm}\frac{r}{r_0}\right)e^{jn\varphi}, \qquad J_n(j_{nm}) = 0$$

and

$$\beta_m^2 = \omega^2\epsilon_0\mu_0 - \frac{j_{nm}^2}{r_0^2}.$$

Proceeding as before, we find,

$$\delta\beta = \frac{1}{2\beta_m}\cdot\frac{r_1^2}{r_0^2}\cdot\frac{1}{J_n'(j_{nm})^2}\left[\beta_0^2\frac{\mu + n\kappa - \mu_0}{\mu + n\kappa + \mu_0} + \beta_m^2\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\right]. \quad (12)$$

A problem which is of some interest, although not of immediate practical significance, is that of a ferrite pencil of arbitrary radius and infinite length in a round wave guide, with the remainder of the waveguide filled with a non-magnetic dielectric, whose dielectric constant, ϵ_1, is equal to that of the ferrite. The ferrite is supposed to be only weakly magnetized. For such a problem, we have,

$$\delta\beta = -\frac{\omega}{2\Delta_m}\int_{\text{pencil}}[(\mu - \mu_0)H_{tm}\cdot\tilde{H}_{tm} - j\kappa H_{tm}^*\cdot\tilde{H}_{tm}]\,dS.$$

H_{tm} is the field of an unperturbed TE or TM mode in the dielectric-filled guide. $\mu - \mu_0$ and κ are supposed small, but r_1, the radius of the pencil need no longer be small.

For TE modes, we have as before (again excluding the case $n = 0$),

$$E_{tm} = -j\omega\mu_0\nabla^*\Psi_m,$$

$$H_{tm} = -j\beta_m\nabla\Psi_m,$$

$$\Psi_m = J_n\left(u_{nm}\frac{r}{r_0}\right)e^{jn\varphi}$$

and

$$\beta_m^2 = \omega^2\epsilon_0\mu_0 - \frac{u_{nm}^2}{r_0^2} = \beta_1^2 - \frac{u_{nm}^2}{r_0^2}.$$

Hence, assuming the pencil coaxial with the guide.

$$-\frac{2\Delta_m}{\omega}\,\delta\beta = 2\pi \int_0^{r_1} (\mu - \mu_0)\beta_m^2 \left[\frac{u_{nm}^2}{r_0^2} J_n'^2\left(u_{nm}\frac{r}{r_0}\right) + \frac{n^2}{r^2} J_n^2\left(u_{nm}\frac{r}{r_0}\right)\right] r\,dr$$

$$+ 2\pi\kappa\beta_m^2 \cdot 2\,\frac{u_{nm}}{r_0}\, n \int_0^{r_1} J_n\left(u_{nm}\frac{r}{r_0}\right) J_n'\left(u_{nm}\frac{r}{r_0}\right) dr,$$

$$= 2\pi\beta_m^2 \left[(\mu - \mu_0) \int_0^{u_{nm}(r_1/r_0)} \left[x J_n'^2(x) + \frac{J_n^2(x)}{x}\right] dx + 2\kappa n \int_0^{u_{nm}(r_1/r_0)} J_n(x) J_n'(x)\,dx\right],$$

$$= 2\pi\beta_m^2\left[(\mu - \mu_0)\left(x J_n(x) J_n'(x) + \frac{x^2}{2} J_n'^2(x) + \frac{x^2 - 1}{2} J_n^2(x)\right)_{x=(r_1/r_0)u_{nm}} + \kappa n J_n^2\left(u_{nm}\frac{r_1}{r_0}\right)\right].$$

Making use of the value of Δ_m found in the preceeding paragraphs, we have

$$\delta\beta = \frac{\beta_m}{(u_{nm}^2 - 1) J_n(u_{nm})^2}\left[\left(\frac{\mu}{\mu_0} - 1\right)\left(x J_n(x) J_n'(x) + \frac{x^2}{2} J_n'^2 x) + \frac{x^2 - 1}{2} J_n^2(x)\right)_{x=u_{nm}(r_1/r_0)} + n\,\frac{\kappa}{\mu_0} J_n^2\left(u_{nm}\frac{r_1}{r_0}\right)\right]. \quad (13)$$

For TM modes,

$$E_{tm} = -j\beta_m \nabla \chi_m\,,$$

$$H_{tm} = j\omega\epsilon_1 \nabla^* \psi_m\,,$$

$$\chi_m = J_n\left(j_{nm}\frac{r}{r_0}\right) e^{jn\varphi},$$

$$\beta_m^2 = \omega^2 \epsilon_1 \mu_0 - \frac{j_{nm}^2}{r_0^2} = \beta_1^2 - \frac{j_{nm}^2}{r_0^2}.$$

In this case,

$$-\frac{2\Delta_m}{\omega}\,\delta\beta = 2\pi(\omega\epsilon_1)^2\left((\mu - \mu_0) \int_0^{r_1} \left[\frac{n^2}{r^2} J_n^2\left(j_{nm}\frac{r}{r_0}\right) + \frac{j_{nm}^2}{r_0^2} J_n'^2\left(j_{nm}\frac{r}{r_0}\right)\right] r\,dr + 2\kappa n \int_0^{r_1} \frac{j_{nm}}{r_0} J_n\left(j_{nm}\frac{r}{r_0}\right) J_n'\left(j_{nm}\frac{r}{r_0}\right) dr\right).$$

The value of Δ_m for this case is

$$\Delta_m = -2\pi\omega\epsilon_1\beta_m \cdot \frac{j_{nm}^2}{2} J_n'^2(j_{nm}).$$

The value of $\delta\beta$ now becomes

$$\delta\beta = \frac{\beta_1^2}{\beta_m j_{nm}^2 J_n^2(j_{nm})} \left[\left(\frac{\mu}{\mu_0} - 1 \right) \left(x J_n(x) J_n'(x) + \frac{x^2}{2} J_n'^2(x) + \frac{x^2 - 1}{2} J_n^2(x) \right)_{x = j_{nm}(r_1/r_0)} + n \frac{\kappa}{\mu_0} J_n^2 \left(j_{nm} \frac{r_1}{r_0} \right) \right]. \tag{14}$$

We note that for a ferrite filled guide with $r_1 = r_0$, the nonreciprocal part of $\delta\beta$ vanishes which confirms a result found in Part I of this paper for weak magnetization.

The very high dielectric constant of the ferrites (about 10) puts rather severe restrictions on the size of the pencils to which perturbation theory is applicable, even for weak magnetization. This limitation would be substantially relaxed if we possessed exact solutions for rods of high dielectric constant inserted into round guide, which could be used as the basis for magnetic perturbation calculations. Unfortunately, the only extensive published calculations of this kind are for dielectric constants less than 3. However, at the suggestion of M. T. Weiss, a calculation of the propagation constant of the lowest mode varying as $e^{\pm j\varphi}$ in a wave guide containing a coaxial dielectric rod ($\epsilon_1 = 10$) has recently been made in the Mathematics Department, for varying rod diameter, but for a single value of guide radius equal to 0.4 times the free space wavelength. With the aid of this information, which was made available to us, the magnetic perturbation calculation has been carried out.

As before, the radius of the guide is r_0 and that of the rod is r_1. The dielectric constant of the rod is ϵ_1. We consider first the propagation in the unmagnetized case. Since we are considering only one mode, namely, that with an angular variation, $e^{j\varphi}$, and of the lowest order radially, we need not identify the E's and H's by a label. We use a subscript "1" for fields in the dielectric and "0" for fields in the empty part of the guide. In general, we have

$$\alpha^2 E_t = -j[\omega\mu\nabla^* H_z + \beta\nabla E_z],$$
$$\alpha^2 H_t = -j[\beta\nabla H_z - \omega\epsilon\nabla^* E_z],$$

where

$$\alpha^2 = \omega^2\epsilon\mu - \beta^2,$$
$$\nabla^2 H_z = -\alpha^2 H_z,$$
$$\nabla^2 E_z = -\alpha^2 E_z.$$

and ϵ, μ refer, for the time being, to the dielectric constant and permeability of the region considered. It will be convenient to put

$$H_{z,t} = \sqrt{\frac{\epsilon_0}{\mu_0}}\,\bar{H}_{z,t}\,; \qquad \frac{\epsilon}{\epsilon_0} = \bar{\epsilon}; \qquad \frac{\mu}{\mu_0} = \bar{\mu};$$

$$\frac{\beta}{\omega\sqrt{\epsilon_0\mu_0}} = \bar{\beta}; \qquad \frac{\alpha}{\omega\sqrt{\epsilon_0\mu_0}} = \bar{\alpha},$$

and to measure lengths in terms of $\dfrac{1}{\omega\sqrt{\mu_0\epsilon_0}}$. We shall continue to use ∇ and ∇^* with the understanding that they refer to the scaled units.

We now have

$$\begin{aligned}
j\bar{\alpha}^2E_t &= \bar{\mu}\nabla^*\bar{H}_z + \bar{\beta}\nabla E_z\,,\\
j\bar{\alpha}^2\bar{H}_t &= \bar{\beta}\nabla\bar{H}_z - \bar{\epsilon}\nabla^*E_z\,,\\
\bar{\alpha}^2 &= \bar{\epsilon}\bar{\mu} - \bar{\beta}^2,\\
\nabla^2\bar{H}_z &= -\bar{\alpha}^2\bar{H}_z\,,\\
\nabla^2E_z &= -\bar{\alpha}^2E_z\,.
\end{aligned}$$

At the surface of the rod E_z, $\bar{H}_z$, E_φ and $\bar{H}_\varphi$ are continuous. We must have

$$\frac{1}{\bar{\alpha}_0^2}\left[\frac{j\bar{\beta}}{r_0}\,E_z(r_1) - \left(\frac{\partial\bar{H}_z}{\partial r}\right)_0\right] = \frac{1}{\bar{\alpha}_1^2}\left[\frac{j\bar{\beta}}{r_1}\,E_z(r_1) - \left(\frac{\partial\bar{H}_z}{\partial r}\right)_1\right]$$

and

$$\frac{1}{\bar{\alpha}_0^2}\left[\left(\frac{\partial E_z}{\partial r}\right)_0 + \frac{j\bar{\beta}}{r_1}\,\bar{H}_z(r_1)\right] = \frac{1}{\bar{\alpha}_1^2}\left[\bar{\epsilon}_1\left(\frac{\partial E_z}{\partial r}\right)_1 + \frac{j\bar{\beta}}{r_1}\,\bar{H}_z(r_1)\right].$$

where

$$\bar{\epsilon}_1 = \frac{\epsilon_1}{\epsilon_0}, \qquad \bar{\alpha}_0^2 = 1 - \bar{\beta}^2 \qquad \text{and} \qquad \bar{\alpha}_1^2 = \bar{\epsilon}_1 - \bar{\beta}^2.$$

$\bar{\mu} = 1$, everywhere, if the unmagnetized ferrite has the permeability of free space.

These equations may be rearranged in the form:

$$\begin{aligned}
\bar{\beta}\left(\frac{1}{\bar{\alpha}_0^2} - \frac{1}{\bar{\alpha}_1^2}\right)E_z(r_1) &= \left\{-\frac{1}{\bar{\alpha}_0^2}\left[r_1\,\frac{1}{\bar{H}_z(r_1)}\left(\frac{\partial\bar{H}_z}{\partial r}\right)_0\right]\right.\\
&\qquad\left.+\frac{1}{\bar{\alpha}_1^2}\left[r_1\,\frac{1}{\bar{H}_z(r_1)}\left(\frac{\partial\bar{H}_z}{\partial r}\right)_1\right]\right\}j\bar{H}_z(r_1),\\
\bar{\beta}\left(\frac{1}{\bar{\alpha}_0^2} - \frac{1}{\bar{\alpha}_1^2}\right)j\bar{H}_z(r_1) &= \left\{-\frac{1}{\bar{\alpha}_0^2}\left[r_1\,\frac{1}{E_z(r_1)}\left(\frac{\partial E_z}{\partial r}\right)_0\right]\right.\\
&\qquad\left.+\frac{\bar{\epsilon}}{\bar{\alpha}_1^2}\left[r_1\,\frac{1}{E_z(r_1)}\left(\frac{\partial E_z}{\partial r}\right)_1\right]\right\}E_z(r_1).
\end{aligned} \tag{15}$$

Within the dielectric, since all fields are bounded at $r = 0$, both E_z and $\bar{H}_z$ are proportional to $J_1(\bar{\alpha}_1 r)$ and, evidently,

$$-\frac{r_1}{\bar{H}_z(r_1)}\left(\frac{\partial \bar{H}_z}{\partial r}\right)_1 \equiv \frac{r_1}{E_z(r_1)}\left(\frac{\partial E_z}{\partial r_1}\right)_1 = \frac{\alpha_1 r_1 J_1'(\alpha_1 r_1)}{J_1(\alpha_1 r_1)} = F(\alpha_1 r_1),$$

in the notation of Part I. $(E_z)_0$ and $(\bar{H}_z)_0$, similarly, will be those two linear combinations of $J_1(\bar{\alpha}_0 r)$ and $Y_1(\bar{\alpha}_0 r)$, which, respectively, vanish and have zero normal derivative at $r = r_0$, in order to ensure the vanishing of the tangential fields there. The functions

$$\frac{r}{(\bar{H}_z)_0}\frac{\partial(\bar{H}_z)_0}{\partial r} \qquad \text{and} \qquad \frac{r}{(E_z)_0}\frac{\partial(E_z)_0}{\partial r}$$

will be called $H(\bar{\alpha}_0 r)$ and $G(\bar{\alpha}_0 r)$ respectively.

Eliminating $E_z(r_1)$ and $H_z(r_1)$ from equations (15) we obtain the characteristic equation of the problem in the form

$$\bar{\beta}^2\left(\frac{1}{\bar{\alpha}_0^2} - \frac{1}{\bar{\alpha}_1^2}\right)^2 = \left(\frac{F(\alpha_1 r_1)}{\bar{\alpha}_1^2} - \frac{H(\bar{\alpha}_0 r_1)}{\bar{\alpha}_0^2}\right)\left(\bar{\epsilon}_1\frac{F(\bar{\alpha}_1 r_1)}{\bar{\alpha}_1^2} - \frac{G(\bar{\alpha}_0 r_1)}{\bar{\alpha}_0^2}\right).$$

The perturbation in the present problem is that due to a mild magnetization of the rod and referring again to equation (9) we have (in unscaled units)

$$\delta\beta = -\frac{\omega}{2\Delta_n}\int_{\text{rod}}[(\mu - \mu_0)H_t\cdot\tilde{H}_t - j\kappa H_t^*\cdot\tilde{H}_t]\,dS,$$

with

$$\Delta_n = \int_{\text{guide}} E_t^*\cdot\tilde{H}_t\,dS.$$

Thus, in the scaled system,

$$\delta\bar{\beta} = -\frac{1}{2}\cdot\frac{\displaystyle\int_0^{r_1}\left[\left(\frac{\mu}{\mu_0} - 1\right)|\bar{H}_t|^2 - j\frac{\kappa}{\mu_0}(\bar{H}_t^*\cdot{}_t\tilde{\bar{H}})\right]r\,dr}{\displaystyle\int_0^{r_0} E_t^*\cdot\tilde{\bar{H}}_t\,r\,dr}. \tag{16}$$

The evaluation of the normalizing integral in the denominator is an exceedingly tedious business and it seems advisable to avoid it. This may be done in the following manner. The characteristic equation has been solved for numerous values of r_1 and $\bar{\beta}$ may be considered to be a reliably known function of r_1. In particular the slope $\frac{d\bar{\beta}}{dr_1}$ is known. But we may also deduce $\frac{d\bar{\beta}}{dr_1}$, by a perturbation calculation in which we start with a rod of radius r_1 and increase the latter to $r_1 + dr_1$ by changing ϵ_0 to ϵ_1 in the shell $r_1 < r < r_1 + dr_1$. For such a perturbation, since E_z and E_φ are continuous at the boundary, they suffer no change when the boundary moves; E_r however is discontinuous and $(E_r)_o = \frac{\epsilon_1}{\epsilon_0}(E_r)_i$ where $(E_r)_o$ and $(E_r)_i$ are the normal fields just outside and just inside the rod. The perturbation formula (unscaled), thus gives

$$\delta\beta = \frac{-\omega}{2\Delta_n}(\epsilon_1 - \epsilon_0)\left(|E_z|^2 + |E_\varphi|^2 + \frac{\epsilon_1}{\epsilon_0}|(E_r)_i|^2\right)\cdot 2\pi r_1 \delta r_1,$$

or, scaled, with r and r_1 representing scaled radii,

$$\delta\bar{\beta} = -\frac{1}{2}\frac{(\bar{\epsilon}_1 - 1)(|E_z|^2 + |E_\varphi|^2 + \epsilon_1|(E_r)_i|^2)}{\int_0^{r_0} E_t^* \cdot \tilde{H}_r r\, dr}\cdot r_1 \delta r_1.$$

The formula (16) for the magnetic perturbation may now be written

$$\delta\bar{\beta} = \frac{\frac{1}{r_1}\frac{d\bar{\beta}}{dr_1}}{\bar{\epsilon}_1 - 1}\cdot\frac{\int_0^{r_1}\left[\left(\frac{\mu}{\mu_0} - 1\right)|\bar{H}_t|^2 - j\frac{\kappa}{\mu_0}(\bar{H}_t^* \cdot \tilde{H}_t)\right] r\, dr}{|E_z(r_1)|^2 + |E_\varphi(r_1)|^2 + \bar{\epsilon}_1|(E_r)_i|^2}.$$

Inside the rod, we may write

$$\bar{H}_z = \frac{\bar{H}_z(r_1)}{E_z(r_1)} \qquad E_z = jcE_z$$

and then

$$j\bar{\alpha}_1^2\bar{H}_r = jc\bar{\beta}\frac{\partial E_z}{\partial r} - \bar{\epsilon}_1\frac{j}{r}E_z,$$

$$j\bar{\alpha}_1^2\bar{H}_\varphi = jc\bar{\beta}\frac{j}{r}E_z + \bar{\epsilon}_1\frac{\partial E_z}{\partial r}.$$

The integrals are readily carried out and are as follows:

$$\int_0^{r_1}|\bar{H}_t|^2 r\, dr$$

$$= \frac{E_z^2(r_1)}{\bar{\alpha}_1^4}\left[(\bar{\epsilon}_1^2 + c^2\beta^2)\left(F(\bar{\alpha}_1 r_1) + \frac{F^2(\bar{\alpha}_1 r_1)}{2} + \frac{\bar{\alpha}_1^2 r_1^2 - 1}{2}\right) - 2c\bar{\beta}\bar{\epsilon}_1\right],$$

$$\int_0^{r_1}(\bar{H}_t^* \cdot \tilde{H}_t) r\, dr$$

$$= \frac{jE_z^2(r_1)}{\bar{\alpha}_1^4}\left[(\bar{\epsilon}_1^2 + c^2\bar{\beta}^2) - 2c\bar{\epsilon}_1\bar{\beta}\left(F(\bar{\alpha}_1 r_1) + \frac{F^2(\bar{\alpha}_1 r_1)}{2} + \frac{\bar{\alpha}_1^2\bar{r}_1^2 - 1}{2}\right)\right].$$

The term in the denominator may be evaluated by using

$$j\bar{\alpha}_1^2 E_r = -\frac{c}{r} E_z + \bar{\beta} \frac{\partial E_z}{\partial r},$$

$$j\bar{\alpha}_1^2 E_\varphi = -jc \frac{\partial E_z}{\partial r} + \frac{j\bar{\beta}}{r} E_z .$$

We obtain

$$| E_z(r_1) |^2 + | E_\theta(r_1) |^2 + \bar{\epsilon}_1 | (E_r)_1 |^2 = E_z^2(r_1) \left[\frac{[\bar{\beta} - cF(\bar{\alpha}_1 r_1)]^2 + \bar{\epsilon}_1[c - \bar{\beta}F(\bar{\alpha}_1\bar{r}_1)]^2}{r_1^2\, \bar{\alpha}_1^4} \right].$$

The perturbation may now be written:

$$\delta\bar{\beta} = \frac{r_1 \dfrac{\partial \bar{\beta}}{\partial r_1}}{\bar{\epsilon}_1 - 1} \cdot \frac{\left(\dfrac{\mu}{\mu_0} - 1\right)[A(\bar{\epsilon}_1^2 + c^2\bar{\beta}^2) - 2c\bar{\beta}\bar{\epsilon}_1] + \dfrac{\kappa}{\mu_0}[\bar{\epsilon}_1^2 + c^2\bar{\beta}^2 - 2Ac\bar{\epsilon}_1\bar{\beta}]}{r_1^2\bar{\alpha}_1^4 + [\bar{\beta} - cF(\bar{\alpha}_1 r_1)]^2 + \bar{\epsilon}_1[c - \bar{\beta}F(\bar{\alpha}_1 r_1)]^2}, \tag{17}$$

where

$$A = \frac{F^2(\bar{\alpha}_1 r_1)}{2} + F(\bar{\alpha}_1 r_1) + \frac{\bar{\alpha}_1^2 r_1^2}{2}$$

and c may be obtained from equation 15, and the definition of c

$$c = \frac{\dfrac{G(\bar{\alpha}_0 r_1)}{\bar{\alpha}_0^2} - \bar{\epsilon}_1 \dfrac{F(\bar{\alpha}_1 r_1)}{\bar{\alpha}_1^2}}{\bar{\beta}\left(\dfrac{1}{\bar{\alpha}_0^2} - \dfrac{1}{\bar{\alpha}_1^2}\right)} = \frac{\bar{\alpha}_1^2 G(\bar{\alpha}_0 r_1) - \bar{\epsilon}_1\bar{\alpha}_0^2 F(\bar{\alpha}_1 r_1)}{\bar{\beta}(\bar{\epsilon}_1 - 1)}$$

with

$$G(\bar{\alpha}_0 r_1) = \bar{\alpha}_0 r_1 \frac{N_1(\bar{\alpha}_0 r_0)J_1'(\bar{\alpha}_0 r_1) - J_1(\bar{\alpha}_0 r_0)N_1'(\bar{\alpha}_0 r_1)}{N_1(\bar{\alpha}_0 r_0)J_1(\bar{\alpha}_0 r_1) - J_1(\bar{\alpha}_0 r_0)N_1(\bar{\alpha}_0 r_1)} .$$

Fig. 1 shows the propagation constant as a function of the relative diameter, r_1/r_0, of the dielectric rod in the unmagnetized case, with $\bar{\epsilon}_1 = 10$. Fig. 2 shows the derivative $\dfrac{d\bar{\beta}}{d(r_1/r_0)}$ as a function of r_1/r_0. The guide radius is 0.4 times the free space wavelength. From equation (17) $\delta\bar{\beta}$ may be written as

$$\delta\bar{\beta} = P(\mu - \mu_0) + Q\kappa.$$

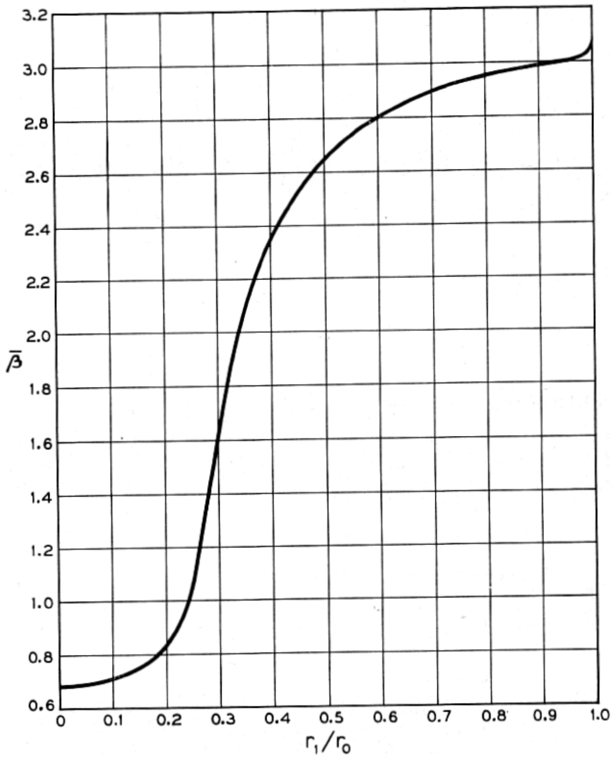


Fig. 1 — Propagation constant of a circular guide containing an unmagnetized coaxial, dielectric rod ($\epsilon/\epsilon_0 = 10$). r_1 is the radius of the rod and r_0 the radius of the guide. $r_0 = 0.4\lambda_0$, where λ_0 is the free space wavelength.

The computed values of Q and $P - Q$ are shown in Fig. 3 as a function of relative rod radius. $P - Q$ is plotted since P and Q are very nearly equal.*

1.3 *Perturbations Non-Uniform in z*

So far we have been concerned only with structures indefinitely extended in the direction of wave propagation. In practice the non-reciprocal element is, of course, finite, but end effects can frequently be neglected, since the element is matched at its ends (by tapering of the finite

* H. Seidel and Miss M. J. Brannon, at the suggestion of M. T. Weiss, have recently calculated the dielectric loss for the guides containing a dielectric rod described above. By combining such information with that obtained here it is possible to discuss figures of merit (degrees of rotation loss in db) for various pencil radii. Such an analysis is being made by M. T. Weiss and will appear in an article, by S. E. Miller, A. G. Fox and M. T. Weiss, in a forthcoming issue of the JOURNAL.

sample, for instance). The matching could be accomplished in such a way that the transition region, whose characteristics would be very difficult to compute, should contribute little to the overall non-reciprocal behavior. Therefore, in many cases, the theory for the indefinitely extended sample is adequate. For some special purposes, however, it is desirable to mismatch the sample deliberately. For instance, Rowen[3] has suggested that the change in Faraday rotation, due to internal reflections in an unmatched specimen, can offset to some extent the frequency dependence of the rotation which is implied by the Polder relations, broadening thereby the useful bandwidth of the device.

Consider an infinite slab of ferrite, magnetized in a direction normal to its two parallel plane bounding surfaces. A circularly polarized wave, normally incident on the slab, will be partially transmitted, and, since for such a wave, the medium behaves as though it had an ordinary scalar permeability, the phase and amplitude of the transmitted portion are readily calculated. It is clear that, as the result of multiple internal reflections, the phase of the emerging wave will differ from the value appropriate to a single trip through the slab (such as would be obtained were the slab perfectly matched). Both amplitude and phase of the transmitted wave will depend on the electrical thickness of the slab and its dielectric constant, and on the effective permeability. The latter

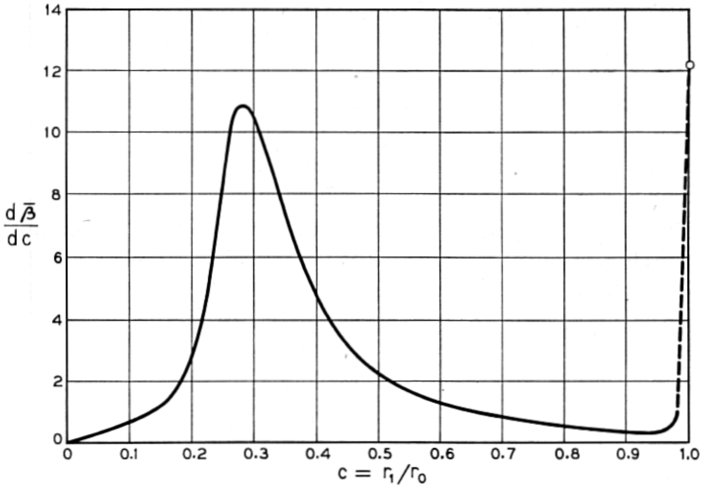


FIG. 2 — $\dfrac{d\beta}{d\left(\dfrac{r_1}{r_0}\right)}$ versus $\dfrac{r_1}{r_0}$.

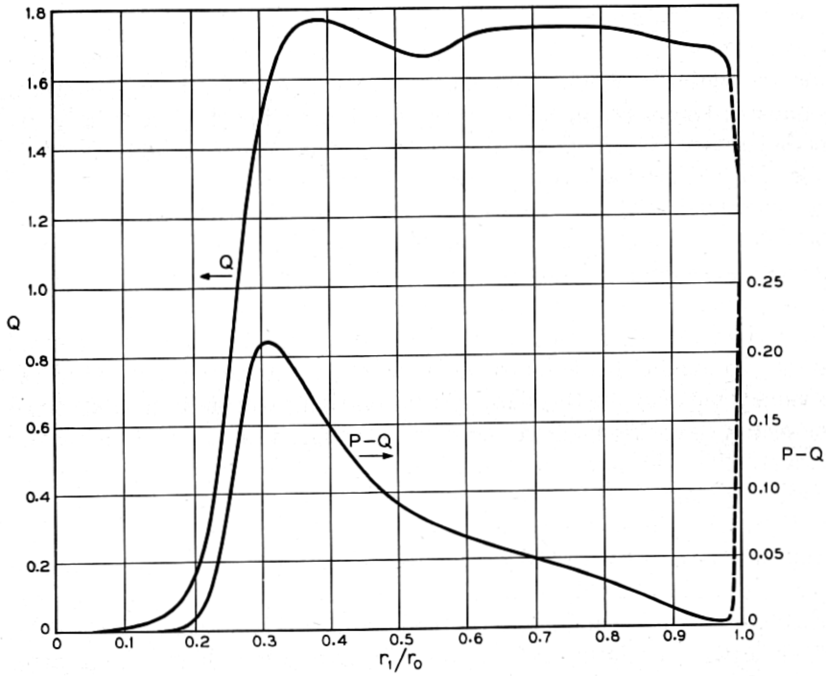


Fig. 3 — Q and $P - Q$ versus r_1/r_0 .

differs for right and left circular waves, so that a plane polarized signal (which is the sum of equal right and left circular components) will emerge, in general, elliptically polarized, with the major axis of the ellipse tilted from the polarization at incidence by an angle which differs from the single trip value as the result of internal reflections. It is clear that this change in rotation can be calculated in a very elementary way.

When the sample is confined to a waveguide a similar effect occurs, but its calculation becomes extremely involved, at least for arbitrarily large magnetizations. The reason, which should be clear from Part I of this paper, is that the circularly polarized modes no longer have the same field configurations inside and outside the sample.* Therefore any incident mode excites *all* of the normal modes of the ferrite; these, in turn, give rise to *all* the mode patterns of the air-filled portion of the guide. Even if all but one of these are cut off, the excitation modifies the phase and amplitude of the reflected and transmitted portions of the propagating mode.

* This is due to the fact that there is now no ordinary effective scalar permeability as for infinite geometry.

Thus all modes have to be included in the problem, which consequently takes the form of an infinite system of linear equations for the mode amplitudes. This can be solved only to some approximation whose general validity it would be hard to establish. The problem could also be stated as an integral equation involving a complicated Green's function, with no greater chance of complete solution.

We are therefore forced to restrict the problem to the ranges of magnetization, or of sample size, in which perturbation theory is applicable. However, we begin with a discussion of the infinite, plane, loss-free slab, a problem which can be solved completely, and which has some bearing on the perturbation problem for a slug of ferrite whose cross section completely fills the waveguide.

Let the plane boundaries of the slab be normal to the z-axis, which is also the direction of magnetization and the propagation direction of a circularly polarized wave incident on the boundary $z = 0$ (see Fig. 4). In terms of the parameters p and σ of Section 2.1, Part I, the effective permeability for a circularly polarized wave is

$$\mu \pm \kappa = \mu = \mu_0 \left(1 \pm \frac{p}{1 \pm \sigma}\right),$$

the upper sign referring to right circular polarization. The corresponding propagation constants in the slab are then

$$\beta_{\pm} = \omega\sqrt{\epsilon\mu_0}\sqrt{1 \pm \frac{p}{1 \pm \sigma}}$$

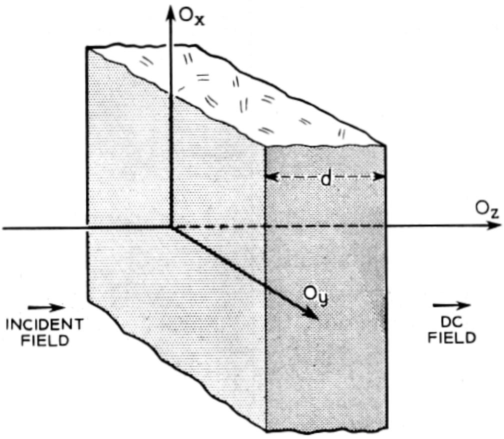


Fig. 4 — Normally magnetized ferrite slab.

where ϵ is the dielectric constant of the ferrite. If d is the thickness of the slab, the electrical thickness is

$$\theta_\pm = \beta_\pm d = \frac{2\pi d}{\lambda_\pm} = \theta_0 \sqrt{1 \pm \frac{p}{1 \pm \sigma}},$$

where $\theta_0 = \omega\sqrt{\mu_0 \epsilon}\, d$ is the electrical thickness of the unmagnetized sample. Let us now confine the discussion to the right circular wave. If the incident electric field is taken to be $e^{-j\beta_0 z}$, where β_0 is the free space propagation constant, $\omega \sqrt{\mu_0 \epsilon_0}$, the incident magnetic field will be

$$-\frac{\beta_0}{\omega\mu_0} e^{-j\beta_0 z},$$

and if the reflected electric field is $\rho e^{j\beta_0 z}$, the reflected magnetic field will be

$$\frac{\beta_0}{\omega\mu_0} \rho e^{j\beta_0 z}$$

since β_0 reverses sign. Inside the slab, the electric field consists of forward and backward travelling parts $\tau_1 e^{-j\beta_+ z}$ and $\tau_2 e^{j\beta_+ z}$, and the corresponding magnetic fields are

$$-\frac{\beta_+}{\omega\mu_+} \tau_1 e^{-j\beta_+ z}$$

and

$$\frac{\beta_+}{\omega\mu_+} \tau_2 e^{j\beta_+ z}.$$

Finally the transmitted electric and magnetic fields will be denoted by

$$\tau_3 e^{-j\beta_0(z-d)}$$

and

$$-\frac{\beta_0}{\omega\mu_0} \tau_3 e^{-j\beta_0(z-d)}$$

respectively. In general ρ, as well as the τ's, will be complex. To obtain τ_3, we write down the equations of continuity of all tangential fields across the boundaries $z = 0$ and $z = d$. Since the fields are confined to a plane normal to the z-direction, these equations are:

$$1 + \rho = \tau_1 + \tau_2,$$

$$\frac{\beta_0}{\omega\mu_0}(1 - \rho) = \frac{\beta_+}{\omega\mu_0}(\tau_1 - \tau_2)$$

and

$$\tau_3 = \tau_1 e^{-j\beta_+ d} + \tau_2 e^{-j\beta_+ d},$$

$$\frac{\beta_0}{\omega\mu_0}\tau_3 = \frac{\beta_+}{\omega\mu_0}(\tau_1 e^{-\beta_+ d} - \tau_2 e^{j\beta_+ d}).$$

These four equations in four unknowns are easily solved for τ_3. Writing

$$x_+ = \sqrt{1 + \frac{p}{1+\sigma}}$$

and noting that

$$\frac{\beta_+}{\mu_+} \bigg/ \frac{\beta_0}{\mu_0} = \sqrt{\frac{\epsilon}{\epsilon_0}} \bigg/ x_+ ,$$

$$= \frac{a}{x_+}, \qquad \text{say},$$

where

$$a = \sqrt{\frac{\epsilon}{\epsilon_0}},$$

one finds the solution to be

$$\tau_3 = \frac{1}{\cos\theta_0 x + \frac{j}{2}\left(\frac{a}{x_+} + \frac{x_+}{a}\right)\sin\theta_0 x_+},$$

$$= |\tau_3|_+ e^{-j\Phi_+},$$

where

$$\tan\Phi_+ = \frac{1}{2}\left(\frac{a}{x_+} + \frac{x_+}{a}\right)\tan\theta_0 x_+ \tag{18}$$

and

$$|\tau_3|_+ = \left[\cos^2\theta_0 x_+ + \frac{1}{4}\left(\frac{a}{x_+} + \frac{x_+}{a}\right)^2 \sin^2\theta_0 x_+\right]^{-1/2}. \tag{19}$$

Similar results apply to a left circular wave; it is necessary only to reverse the signs of σ, p in the expression for x_+. Equations (18) and (19) show that equal right and left circular incident waves emerge with different amplitudes and phases; hence an incident plane polarized wave emerges elliptically polarized with its major axis inclined to the polarization direction at incidence. The inclination and the ratio of minor to major axis are determined as follows: The right and left circular fields, upon emergence, may be written in terms of rectangular components (with the polarization of the total incident field along the x-axis)

$$E_x^+ - jE_y^+ = \tau_+ e^{j\omega t},$$
$$E_x^- + jE_y^- = \tau_- e^{j\omega t},$$

from which the resultant field in the x-direction is seen to be

$$\left.\begin{aligned} E_{xT} &= E_x^+ + E_x^- = |\tau_+| \cos(\omega t - \Phi_+) + |\tau_-| \cos(\omega t - \Phi_-) \\ &\text{and in the } y \text{ direction} \\ E_{yT} &= E_y^+ + E_y^- = |\tau_-| \sin(\omega t - \Phi_-) - |\tau_+| \sin(\omega t - \Phi_+). \end{aligned}\right\} \quad (20)$$

The amplitude at time t, $(E_{xT}^2 + E_{yT}^2)^{1/2}$, is thus given by

$$E_{xT}^2 + E_{yT}^2 = |\tau_+|^2 + |\tau_-|^2 + 2|\tau_+|\cdot|\tau_-| \cos[2\omega t - (\Phi_- + \Phi_+)] \quad (21)$$

The major axis of the ellipse is the maximum of $(E_{xT}^2 + E_{yT}^2)^{1/2}$ with respect to ωt. It equals $|\tau_+| + |\tau_-|$ and is attained at

$$\omega t = \tfrac{1}{2}(\Phi_- + \Phi_+).$$

Similarly the minor axis is the minimum and equals $|\tau_+| - |\tau_-|$. The ratio of minor to major axis is therefore

$$\frac{|\tau_+| - |\tau_-|}{|\tau_+| + |\tau_-|}.$$

The angle between the x-axis (the incident polarization direction) and the major axis is found by substituting $\omega t = \tfrac{1}{2}(\Phi_- + \Phi_+)$ in (20). This gives

$$E_{xT} = (|\tau_+| + |\tau_-|) \cos \frac{\Phi_+ - \Phi_-}{2},$$

$$E_{yT} = (|\tau_+| + |\tau_-|) \sin \frac{\Phi_+ - \Phi_-}{2},$$

which shows that the angle is

$$\frac{\Phi_+ - \Phi_-}{2}.$$

$|\tau|$ and Φ are plotted versus x in Fig. 5. τ_+, Φ_+ and τ_-, Φ_- at given

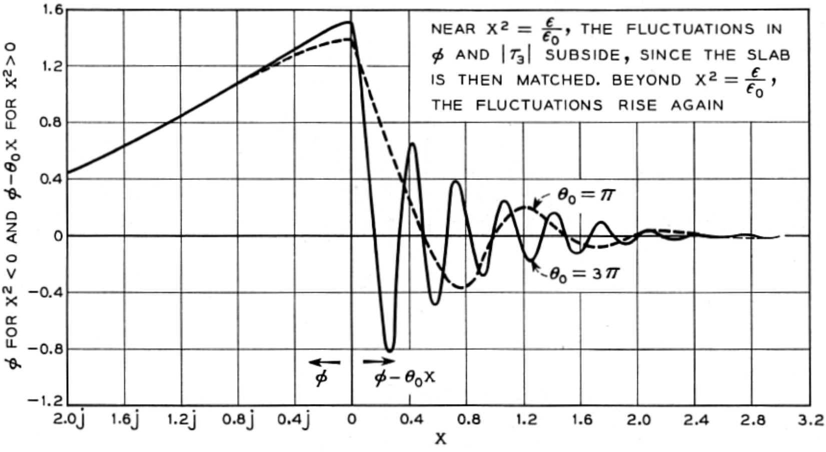


FIG. 5a

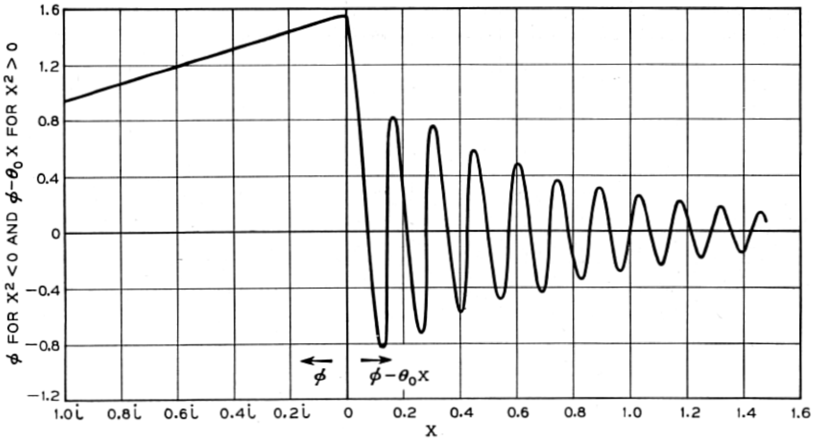


FIG. 5b

Fig. 5 — Phase and amplitude of transmitted circularly polarized wave as a function of $x = \sqrt{1 + \frac{p}{1 + \sigma}}$, with $\frac{\epsilon}{\epsilon_0} = 10$. (a), top, Φ for $\theta_0 = \pi$ and 3π; (b), bottom, Φ for $\theta_0 = 7\pi$; (c), $|\tau_3|$ for $\theta_0 = \pi$ and 3π; and (d), $|\tau_3|$ for $\theta_0 = 7\pi$.

$|\sigma|$, $|p|$ are found by choosing positive and negative signs for σ and p in

$$x = \sqrt{1 + \frac{p}{1 + \sigma}}.$$

Note that x can be imaginary corresponding to cut-off in the range $-1 < \sigma < -1 - p$ $(p < 0)$.

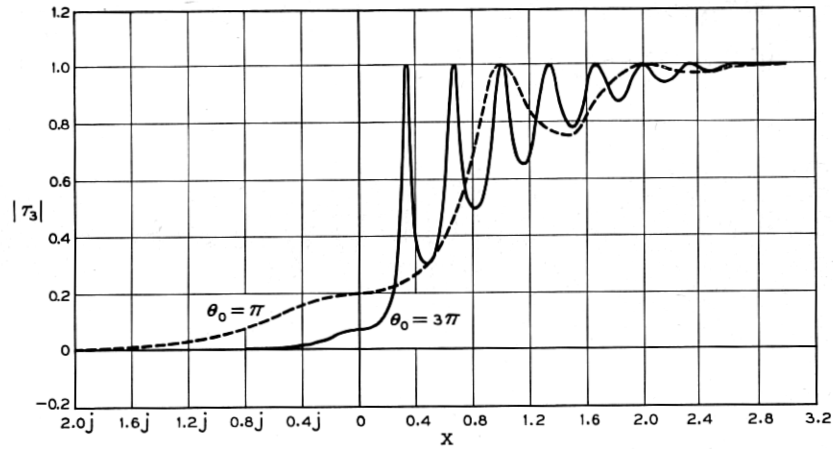


FIG. 5c

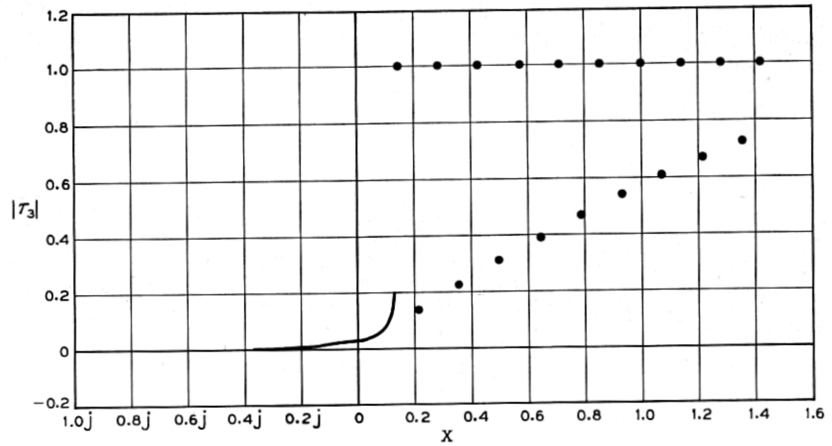


FIG. 5d

Fig. 5(c), top and (d), bottom — See Fig. 5.

Near $x = 1$ (corresponding either to small p or to sufficiently large σ), Φ differs from its value Φ_0 for the isotropic case by a small amount $\delta\Phi$. The rotation, to first order, is then just one half the difference between the $\delta\Phi$ for positive and negative p, σ. Writing

$$x_\pm = 1 + \delta x_\pm = 1 \pm \frac{1}{2} \frac{p}{1 \pm \sigma}$$

and expanding equation (18), we obtain

$$\delta\Phi_{\pm} = \theta_0 \delta x_{\pm} \frac{\cosh \Psi \sec^2 \theta_0 - \dfrac{\tan \theta_0}{\theta_0} \sinh \Psi}{1 + \cosh^2 \Psi \tan^2 \theta_0},$$

where Ψ is defined by

$$a = \sqrt{\frac{\epsilon}{\epsilon_0}} = e^{\Psi}.$$

[This result holds even near $\theta_0 = (n + \frac{1}{2})\pi$ where $\tan \theta_0 = \infty$, as can be seen by expanding the reciprocal of equation (18).] The quantity $\frac{1}{2}\theta_0(\delta x_+ - \delta x_-)$ is the rotation corresponding to a single trip through the sample. The actual rotation is $\frac{1}{2}(\delta\Phi_+ - \delta\Phi_-)$. Hence we may define a *rotation gain* as the ratio

$$\begin{aligned} g(\theta_0, a) &= \frac{\frac{1}{2}(\delta\Phi_+ - \delta\Phi_-)}{\frac{1}{2}\theta_0(\delta x_+ - \delta x_-)}, \\ &= \frac{\cosh \Psi \sec^2 \theta_0 - \dfrac{\tan \theta_0}{\theta_0} \sinh \Psi}{1 + \cosh^2 \Psi \tan^2 \theta_0}. \end{aligned} \tag{22}$$

In many cases, $\theta_0 \gg 1$, that is, the thickness of the sample is much greater than a reduced wavelength in the specimen. Then the second term in the numerator of g is always negligible compared with the first. g then simplifies to

$$g_1 = \frac{\cosh \Psi}{\cos^2 \theta_0 + \cosh^2 \Psi \sin^2 \theta_0}.$$

This expression is plotted in Fig. 6 as a function of θ for various $a = e^{\Psi}$. For given Ψ it has minima equal to $\dfrac{1}{\cosh \Psi}$ at $\theta_0 = \left(n + \dfrac{1}{2}\right)\pi$, and maxima equal to $\cosh \Psi$ at $\theta_0 = n\pi (n = 0, 1, 2, \cdots)$. When $a \gg 1$ ($a \sim 3$ for many ferrites), $\cosh \Psi$ is replaced by $\frac{1}{2}e^{\Psi} = \frac{1}{2}a$, and then

$$g_{1\,\max} = \frac{1}{2}a$$

$$g_{1\,\min} = \frac{2}{a}$$

It is to be noted that when $\theta = n\pi$, the condition for maximum g_1, the unperturbed reflected amplitude is zero, and the ellipticity vanishes to first order in δx.

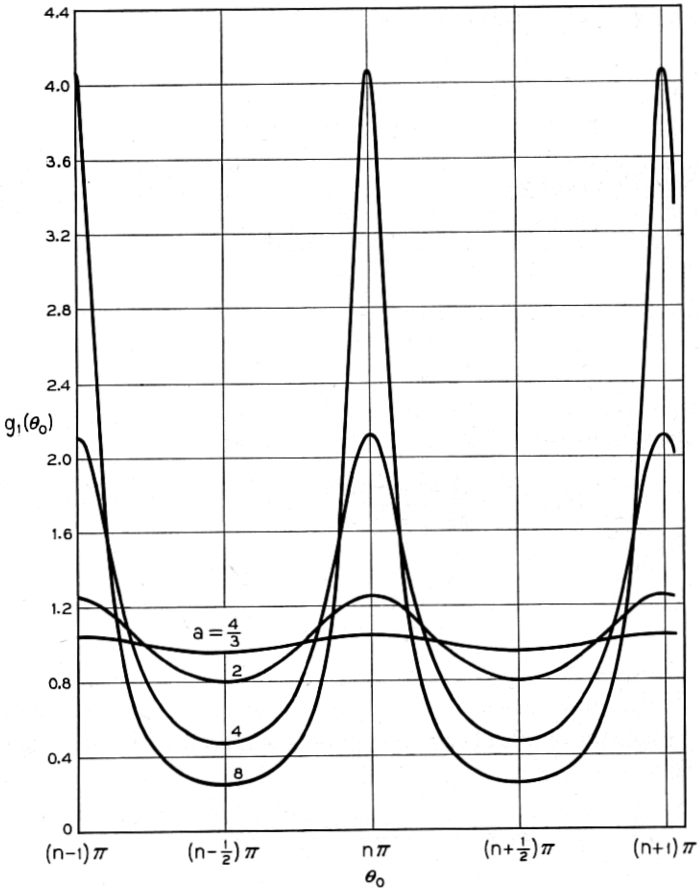


Fig. 6 — Rotation gain, $g_1(\theta_0)$, versus electrical thickness, θ_0 , for small magnetization.

Perturbation theory enables us to solve approximately, for weak magnetization, the problem just considered for the case in which a right circular cylinder of ferrite snugly fits a cylindrical waveguide. We will show that, with a suitable reinterpretation of the constants, equation (22) for the rotation gain of the extended slab continues to apply here.

Before magnetization, a particular right circularly polarized mode, say the m^{th}, is present in the sample. The small magnetization distorts the pattern of this mode and changes the propagation constant slightly. The distorted pattern can be expanded in the series of normal modes of the unperturbed material. In these expansions only the coefficient of the originally present m^{th} mode will be large; all others will at most be of the order of the perturbation. Denoting perturbed quantities by the superfix $+$, we have for the distorted fields

$$E_{tm}^{+} = e^{-j\beta_m{}^+z} \sum_{n=1}^{\infty} p_{mn}E_{tn} ,$$
$$H_{tm}^{+} = e^{-j\beta_m{}^+z} \sum_{n=1}^{\infty} q_{mn}H_{tn} ,$$

where p_{mm} and q_{mm} are large compared with the other coefficients. β_m^+ and the p's and q's are determined from equations (7a) and (7b). a_n in these equations is identified with $e^{-j\beta_m{}^+z}\, p_{mn}$, b_n with $e^{-j\beta_m{}^+z}\, q_{mn}$. Since all perturbation integrals involving E_t and H_z, E_z vanish in the present case, we obtain

$$\beta_n q_{mn} - \beta_m^+ p_{mn} = \frac{\omega}{\Delta_n}\left[\int (\mu - \mu_0) H_{tm}^+ \tilde{H}_{tn}\, dS - j \int \kappa H_{tm}^{+*} \tilde{H}_{tn}\, dS\right]$$

and

$$\beta_n p_{mn} - \beta_m^+ q_{mn} = 0.$$

In the first of these equations, H_{tm}^+, the perturbed magnetic field, consists of $q_{mm}H_{tm}$, plus an admixture of other modes with coefficients themselves of order $\mu - \mu_0$, κ. Therefore, to first order, it suffices to write

$$H_{tm}^+ = q_{mm}H_{tm}$$

in the integrand, with the result:

$$\beta_n q_{mn} - \beta_m^+ p_{mn} = I_{mn}^+ q_{mm} ,$$
$$p_{mn} = \frac{\beta_m^+}{\beta_n} q_{mn} ,$$

where

$$I_{mn}^+ = \frac{\omega}{\Delta_n} \int (\mu - \mu_0) H_{tm} \tilde{H}_{tn} - j\kappa H_{tm}^* \tilde{H}_{tn}]\, dS.$$

Elimination of p_{mn} gives

$$(\beta_n^2 - \beta_m^{+2}) q_{mn} = \beta_n I_{mn}^+ q_{mn} .$$

The case $n = m$ determines β_m^{+2}:

$$\beta_m^{+2} = \beta_m^2 (1 - I_{mm}^+/\beta_m). \tag{23}$$

All other cases give, to first order,

$$q_{nm} = \frac{\beta_n I_{mn}^+}{\beta_n^2 - \beta_m^2} q_{mm} ,$$

which, again to first order, equals

$$\frac{\beta_n I_{mn}^+}{\beta_n^2 - \beta_m^2} p_{mm} . \tag{24}$$

One of the quantities p_{mm}, q_{mm}, is still arbitrary. If it is required that the perturbed field is, to zero order in I_{mn}^+, normalized to the same value as the unperturbed field, it is readily found that one can take

$$p_{mm} = 1, \qquad q_{mm} = \frac{\beta_m}{\beta_m^+} .$$

We are now in a position to consider the problem of a right circular mode, say the r^{th}, incident upon the end plane, $z = 0$, of the ferrite cylinder extending to $z = d$. One simplifying feature of this problem is that the unperturbed modes inside, and the modes outside, the sample have the same dependence on radius since the sample fills the whole guide-cross-section. However the modes inside and outside may have different numerical coefficients. Thus, if we distinguish quantities outside the sample by primes, the TE_{sr} mode can be represented outside and inside the sample by

$$E'_{t_{sr}} = E_{t_{sr}} = \nabla^* \psi_{sr} ,$$

$$H'_{t_{sr}} = \frac{\beta_r' \operatorname{sgn} \beta_r'}{\omega\mu_0} \nabla\psi_{sr} ,$$

$$H_{t_{sr}} = \frac{\beta_r \operatorname{sgn} \beta_r}{\omega\mu_0} \nabla\psi_{sr}$$

Here sgn β means: sign of the propagation direction, +1 and −1 for forward and reverse respectively. The function ψ_{sr} is given by

$$\psi_{sr} = J_s\left(u_r \frac{r}{r_0}\right) e^{js\varphi},$$

where φ is the azimuthal angle, r_0 the guide radius, and u_r the r^{th} zero of $J_s'(x)$. Tor TM modes we have similarly

$$E'_{t_{sr}} = E_{t_{sr}} = \nabla\psi_{sr} ,$$

$$H_{t_{sr}} = -\frac{\omega\epsilon_0}{\beta_r'} (\Delta^* \psi_{sr}) \operatorname{sgn} \beta_r ,$$

$$H_{t_{sr}} = -\frac{\omega\epsilon_1}{\beta_r} (\nabla^* \psi_{sr}) \operatorname{sgn} \beta_r ,$$

where in the definition of the ψ, u_r is replaced by j_r, the r^{th} zero of $J_s(x)$. Since the perturbations considered here do not couple modes with different azimuthal number, s will be considered fixed hereafter, and only the suffix r will be retained.

The field at $z = -0$ (just outside the ferrite) will consist, not only of the incident field and its reflection ρ_r, but also of other modes excited by the perturbation. Taking the incident E_r' to have unit amplitude, we have, from the continuity of tangential electric fields,

$$\begin{aligned}(1 + \rho_r)E'_{tr} + \sum_{n \neq r} \rho_n E'_{tn} &= \sum_n (\tau_{n1} + \tau_{n2}) \sum_\ell p_{n\ell} E_{t\ell}, \\ &= \sum_{n,\ell} (\tau_{n1} + \tau_{n2}) p_{n\ell} E_{t\ell}, \end{aligned} \tag{25}$$

where the τ_{n1}, τ_{n2} are respectively the forward and backward traveling amplitudes of the n^{th} perturbed mode. In the same way, we have for the tangential magnetic field:

$$(1 - \rho_r)H'_{tr} - \sum_{n \neq r} \rho_n H'_{tn} = \sum_{n,\ell} (\tau_{n1} - \tau_{n2}) q_{n\ell} H_{t\ell}. \tag{26}$$

The changes in sign of the coefficients of the backward waves have already been explained in Section 1.1. Let us now suppose that the incident mode is the TE_r mode. Multiplying both sides of (25) by $\nabla^* \tilde{\psi}_r$ and both sides of (26) by $\nabla \tilde{\psi}_r$, and integrating over the cross-section we have, from the orthogonality relations of these functions,

$$1 + \rho_r = \sum_n (\tau_{n1} + \tau_{n2}) p_{nr},$$

$$\beta_r'(1 - \rho_r) = \sum_n \beta_r (\tau_{n1} - \tau_{n2}) q_{nr}.$$

But in these series, all τ except τ_r will be small, as will be all p, q except p_{rr}, q_{rr}. Hence all terms, except those for which $n = r$, will be small of second order, and can be neglected. Further,

$$p_{rr} = 1, \qquad q_{rr} = \frac{\beta_r}{\beta_r^+}.$$

Thus we obtain finally,

$$1 + \rho_r = \tau_{r1} + \tau_{r2},$$

$$1 - \rho_r = (\tau_{r1} - \tau_{r2}) \frac{\beta_r^2}{\beta_r' \beta_r^+}.$$

In view of equation (23) for β_r^+, these equations can be written, to first order in I_{rr}^{+},

$$\begin{aligned} 1 + \rho_r &= \tau_{r1} + \tau_{r2} \\ 1 - \rho_r &= \frac{\beta_r^{+}\mu_0}{\beta_r'\mu_{r+}}(\tau_{r1} - \tau_{r2}) \end{aligned} \tag{27}$$

where

$$\mu_{r+} = \mu_0\left(1 - \frac{I_{rr}^{+}}{\beta_r}\right).$$

Similarly at the output plane, $z = d$, the transmitted amplitude τ_{r3} satisfies

$$\begin{aligned} \tau_{r3} &= \tau_{r1}e^{-j\theta} + \tau_{r2}e^{j\theta}, \\ \tau_{r3} &= \frac{\beta_r^{+}\mu_0}{\beta_r'\mu_{r+}}(\tau_{r1}e^{-j\theta} - \tau_{r2}e^{j\theta}), \end{aligned} \tag{28}$$

where $\theta = \beta_r^{+} d$. Comparison of equations (27) and (28) with the corresponding equations for the infinite slab shows that they have the same form; however, equations (27) and (28) hold only for small magnetization. Hence only those results for the slab that relate to small magnetization can be generalized to the present case. One such result is that for the rotation gain. If we re-identify variables by

$$\begin{aligned} \delta x^{+} \rightarrow \delta x_r^{+} &= \frac{1}{2}\frac{I_{rr}^{+}}{\beta_r} = -\frac{1}{2}\frac{I_{rr}^{-}}{\beta_r}, \\ \theta_0 \rightarrow \theta_r &= \beta_r d, \\ a \rightarrow a_r &= \beta_r/\beta_r', \end{aligned}$$

we obtain, for the rotation gain,

$$g_r = \frac{\cosh\psi_r \sec^2\theta_r - \dfrac{\tan\theta_r}{\theta_r}\sinh\psi_r}{1 + \cosh^2\psi_r \tan^2\theta_r}, \tag{29}$$

where

$$e^{\psi_r} = a_r = \beta_r/\beta_r'.$$

The formulae and conditions for maximum and minimum rotation gain derived in the previous section, apply here also, in terms of the re-defined variables.

The conversion of part of the incident mode to the s^{th} mode in the transmitted wave can be examined by multiplying the matching equations by $\nabla^*\tilde{\psi}_s$ or $\nabla\tilde{\psi}_s$ and integrating over the cross-section. It is then found that the transmitted amplitude of the n^{th} mode is

$$\tau_{n3} = 2\gamma_{rn}Z_n \frac{(Z_r + Z_n)(\cos\theta_n - \cos\theta_r) + j(1 + Z_nZ_r)(\sin\theta_n - \sin\theta_r)}{(2Z_n\cos\theta_n + j(Z_n^2 + 1)\sin\theta_n)(2Z_r\cos\theta_r + j(Z_r^2 + 1)\sin\theta_r)},$$

where

$$\theta_r = \beta_r\, d, \qquad Z_{r,n} = \frac{\beta_r}{\beta_r'}$$

and

$$\gamma_{rn} = \frac{\beta_n I_{rn}^+}{\beta_n^2 - \beta_r^2}.$$

The perturbation theory just outlined assumed a guide closely fitted with a slug of ferrite of finite length, and slightly magnetized. The perturbation consisted in the small changes in permeability. Here we treat another kind of perturbation in which the ferrite does not fill the guide completely, and in which the magnetization can be arbitrarily large, but the sample is a thin lamina, mounted normally to the guide axis which only slightly perturbes the field pattern. Its shape can then be considered arbitrary; its thickness will be assumed uniform and very much smaller than a wavelength in the material.

Under these conditions, the equations for the perturbed amplitudes of the right-circularly polarized n^{th} radial mode are

$$\frac{\partial a_n}{\partial z} + j\beta_n b_n = j\frac{\omega}{\Delta_n}\left[\int (\mu - \mu_0)H_t\tilde{H}_{tn}\, dS - j\int \kappa H_t{}^*\tilde{H}_{tn}\, dS + \int (\epsilon - \epsilon_0)E_z\cdot\tilde{E}_{zn}\, dS\right], \quad (30)$$

$$\frac{\partial b_n}{\partial z} + j\beta_n a_n = j\frac{\omega}{\Delta_n}\left[\int (\epsilon - \epsilon_0)E_t\cdot\tilde{E}_{tn}\, dS\right],$$

in the usual notation. The terms on the right are assumed different from zero only in the range z, $z + \delta z$ occupied by the sample, and the integrals are extended over the cross section of the sample only. E_{tn}, H_{tn} are the mode functions for the empty guide. The fields H_t, E_t, E_z are the actual fields inside the sample. For the first-order calculation E_t and H_t are identical with the incident field and E_z (by continuity of the normal component of D_z) is $\frac{\epsilon_0}{\epsilon}$ times the incident E_z. Now the incident fields may be taken simply as

$$a_{n0}E_{tn}\,; \qquad a_{n0}H_{tn}\,, \qquad a_{n0}E_{zn}$$

(since, *in vacuo*, $b_{n0} = a_{n0} \operatorname{sgn} \beta_n$ and $\operatorname{sgn} \beta_n = +1$ for the incident field), and the amplitude a_{n0} may be taken to be unity. Therefore equation (30) may be written

$$\frac{\partial a_n}{\partial z} = j\frac{\omega}{\Delta_n}\left[\int (\mu - \mu_0) H_{tn}\tilde{H}_{tn}\, dS - j\int \kappa H_{tn}{}^*\tilde{H}_{tn}\, dS + \int \left(1 - \frac{\epsilon_0}{\epsilon}\right)\epsilon_0 E_{zn}\tilde{E}_{zn}\, dS\right] - j\beta_n b_n\,,$$

$$\frac{\partial b_n}{\partial z} = j\frac{\omega}{\Delta_n}\int (\epsilon - \epsilon_0) E_{tn}\tilde{E}_{tn}\, dS - j\beta_n a_n\,.$$

We solve these equations by integrating through the small thickness of the sample. Since we are interested only in results of first order in δz, a_n and b_n on the right hand side of these equations may be replaced by a_{n0}, b_{n0}; that is by 1, 1. Writing

$$I_A = \frac{\omega}{\Delta_n}\left[\int (\mu - \mu_0) H_{tn}\tilde{H}_{tn}\, dS - j\int \kappa H_{tn}{}^*\tilde{H}_{tn}\, dS + \int \left(1 - \frac{\epsilon_0}{\epsilon}\right)\epsilon_0 E_{zn}\tilde{E}_{zn}\, dS\right],$$

$$I_B = \frac{\omega}{\Delta_n}\int (\epsilon - \epsilon_0) E_{tn}\tilde{E}_{tn}\, dS,$$

we have, upon integration from the incidence plane (1) to the exit plane (2),

$$a_{n2} - a_{n1} = j(I_A - \beta_n)\delta z,$$

$$b_{n2} - b_{n1} = j(I_B - \beta_n)\delta z.$$

a_{n1}, the amplitude at the entrance plane consists of $a_{n0} = 1$, the incident amplitude, plus a small reflected amplitude ρ. Thus $a_{n1} = 1 + \rho$. Similarly b_n consists of $a_{n0} \operatorname{sgn} \beta_n = 1$, and the small reflected amplitude $\rho \operatorname{sgn} \beta_n = -\rho$ (corresponding to a backward wave). b_{n2} is equal to a_{n2} (since the transmitted wave is a forward wave). Hence we have

$$a_{n2} - (1 + \rho) = j(I_A - \beta_n)\,\delta z,$$

$$a_{n2} - (1 - \rho) = j(I_B - \beta_n)\,\delta z.$$

whose solutions for the transmitted and reflected amplitude are

$$a_{n2} = 1 - j\left(\beta_n - \frac{I_A + I_B}{2}\right)\delta z$$

and

$$\rho = j\,\frac{I_B - I_A}{2}\,\delta z.$$

Thus, to first order, the transmitted wave only undergoes a phase change, of amount

$$\delta\varphi_+ = \left(\beta_n - \frac{I_A + I_B}{2}\right)\delta z.$$

A similar result applies to a left circular wave; the only difference arises in the contribution

$$-\frac{\omega}{2\Delta_n}\int \kappa H_{tn}^* \cdot H_{tn}\, dS$$

to $\frac{I_A}{2}$ which will merely change sign. The remainder of I_A, and also I_B are unchanged. Thus the Faraday rotation of a plane polarized wave, which is one half the difference of the phase change for right and left circular waves, is equal to

$$\frac{\delta\varphi_+ - \delta\varphi_-}{2} = -\frac{\omega}{2\Delta_n}\, j \int_{\text{area of disc}} \kappa H_{tn}^* \tilde{H}_{tn}\, dS\delta z.$$

The integral on the right is real, and, when the disc is circular of radius r_1, is of exactly the type that has already been encountered in the case of the ferrite embedded in a material with the same dielectric constant (Section 1.2). Here, however, κ need not be small, so long as δz is sufficiently small. In using the results of Section 1.2 it must be remembered that the mode functions there related to a dielectric, while here they relate to air-filled guide. When this is taken into account, one obtains for the specific rotation of a TE-mode:

$$\frac{\delta\varphi}{\delta z} = \frac{\beta_m}{(u_{nm}^2 - 1)J_n^2(u_{nm})}\left[\frac{n\kappa}{\mu_0}\, J_n^2\left(u_{nm}\,\frac{r_1}{r_0}\right)\right], \tag{31a}$$

a TM-mode:

$$\frac{\delta\varphi}{\delta z} = \frac{\omega^2 \mu_0 \epsilon_0}{\beta_m j_{nm}^2 J_n'^2(j_{nm})}\left[\frac{n\kappa}{\mu_0}\, J_n^2\left(j_{nm}\,\frac{r_1}{r_0}\right)\right], \tag{31b}$$

results which show that the rotation in a thin disc is independent of the dielectric constant of the disc.

2. THE PARALLEL PLANE CABLE

Measurements on ferrites are sometimes made by a coaxial cable technique. The ferrite fills a section of the cable, and is magnetized axially. In this section we consider a simplified problem of the same kind: that of wave motion between parallel conducting planes bounding a ferrite slab magnetized in the propagation direction. The characteristic equation for the propagation constant has been established by a number of authors, particularly Van Trier,[1] but as in the case of the cylindrical waveguide, no analysis of the equation in terms of the laboratory variables appears to have been made. We shall make such an analysis by means of the methods already used in Part I for the cylindrical waveguide. The discussion will be limited to the dominant (TEM-limit) mode, and to the principal features of the incipient modes (shape resonances).

The distance between the two conducting boundaries will be denoted by 2a. The system of axes is as in Fig. 7; the y-axis is perpendicular to the walls (located at $y = \pm a$), the z-axis is the magnetization and propagation direction. The microwave fields are assumed not to vary along the x-axis.

The solution of Maxwell's equation proceeds just as in the case of the cylindrical waveguide (Part I, Section 3), with the simplifying feature that one of the coordinates (x) drops out of the problem. We shall use the reduced notation of Part I. β, the propagation constant will be measured in units of $\omega\sqrt{\mu_0\epsilon}$, the propagation constant of the unmagnetized, unbounded medium. $\bar{a}$ will denote $a\omega\sqrt{\mu_0\epsilon}$, the half-spacing measured in terms of the reduced wavelength $\dfrac{\lambda}{2\pi}$ of the unmagnetized, extended medium.

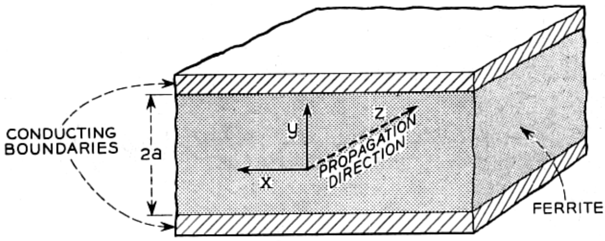


Fig. 7 — Parallel plane cable filled with ferrite magnetized along propagation direction.

It is found that (see Part I, Section 3) the longitudinal electric and magnetic fields are then given by

$$E_z = \frac{\Lambda_2\psi_1 - \Lambda_1\psi_2}{\Lambda_2 - \Lambda_1}; \qquad H_z = j\,\frac{\psi_1 - \psi_2}{\Lambda_2 - \Lambda_1},$$

where $\psi_{1,2}(y)$ are solutions of

$$\frac{\partial\psi_{1,2}}{\partial y^2} + \chi_{1,2}{}^2\psi_{1,2} = 0, \tag{32}$$

where $\Lambda_{1,2}$ are the roots of the quadratic (16), Part I, and where, finally, the χ's are defined in terms of the Λ by equations (17a) and (17b), Part I. The x-component of E is given by [see equation (22a), Part I]

$$(\Lambda_1 - \Lambda_2)\Omega E_x = \left[\beta\Lambda_1\rho_H - \rho_H^2\nu_H + \nu_H\left(1 - \frac{\beta^2}{\nu_H}\right)\right]\frac{\partial\psi_2}{\partial y},$$

or the same expression with suffixes 1, 2 interchanged. $\left(\text{Note that the } x \text{ component of } \nabla\psi \text{ is now zero, that of } \nabla^*\psi \text{ is } \frac{\partial\psi}{\partial y}\right)$. If the boundary conditions are to be satisfied at both $y = +a$ and $y = -a$, the solutions of equation (32) must be either even or odd functions of y. Since E_z must vanish at $y = \pm a$, we have for the symmetric (or even) case

$$\psi_1{}^s = \Lambda_1\,\frac{\cos\chi_1 y}{\cos\chi_1 a}; \qquad \psi_2{}^s = \Lambda_2\,\frac{\cos\chi_2 y}{\cos\chi_2 a},$$

and for the antisymmetric (odd) case

$$\psi_1{}^a = \Lambda_1\,\frac{\sin\chi_1 y}{\sin\chi_1 a}; \qquad \psi_2{}^a = \Lambda_2\,\frac{\sin\chi_2 y}{\sin\chi_2 a}.$$

The characteristic equation for these two sets now follows from the fact that $E_x = 0$ at $y = \pm a$. Expressing the Λ's in terms of the χ's by means of equation (17b), Part I, and writing $\beta\Lambda_{2,\,1} = \lambda_{1,\,2}$, we obtain

$$\frac{1}{\lambda_1\chi_1{}^2}\,\chi_1\bar{a}\tan\chi_1\bar{a} = \frac{1}{\lambda_2\chi_2{}^2}\,\chi_2\bar{a}\tan\chi_2\bar{a}$$

for the symmetric case, and

$$\lambda_1\chi_1{}^2\,\frac{\tan\chi_1\bar{a}}{\chi_1\bar{a}} = \lambda_2\chi_2{}^2\,\frac{\tan\chi_2\bar{a}}{\chi_2\bar{a}}$$

for the antisymmetric case. To bring these equations into a form suitable for graphical discussion, we express the χ in terms of λ, σ by means of the quadratic for λ, and by means of the relations between the tensor components of the permeability (see beginning of Section 4.11, Part I). We then obtain

$$\frac{1}{\lambda_1}\sqrt{\frac{1-\sigma\lambda_1}{1-\lambda_1^2}}\tan\bar{a}\sqrt{\frac{1-\lambda_1^2}{1-\sigma\lambda_1}} = \frac{1}{\lambda_2}\sqrt{\frac{1-\sigma\lambda_2}{1-\lambda_2^2}}\tan\bar{a}\sqrt{\frac{1-\lambda_2^2}{1-\sigma\lambda_2}} \tag{33}$$

for the symmetric modes, and

$$\lambda_1\sqrt{\frac{1-\lambda_1^2}{1-\sigma\lambda_1}}\tan\bar{a}\sqrt{\frac{1-\lambda_1^2}{1-\sigma\lambda_1}} = \lambda_2\sqrt{\frac{1-\lambda_2^2}{1-\sigma\lambda_2}}\tan\bar{a}\sqrt{\frac{1-\lambda_2^2}{1-\sigma\lambda_2}} \tag{34}$$

for the antisymmetric modes. When λ_1, λ_2 pairs satisfying one of these equations, and the Polder relation

$$\lambda_1 + \lambda_2 - \sigma\lambda_1\lambda_2 = \sigma + p \tag{35}$$

have been found, β^2 is given by

$$\beta^2 = -\lambda_1\lambda_2 .$$

Appearances to the contrary, equations (33) and (34) describe the ordinary TE and TM modes when $p = 0$, regardless of σ (see the discussion opposite Fig. 3. of Part I). When $\sigma = 0$, the Polder relation transforms $\frac{1-\lambda_2^2}{1-\sigma\lambda_2}$ into $\frac{1-\lambda_1^2}{1-\sigma\lambda_1}$, so that either of equations (33) and (34) can be satisfied only by

$$\frac{1-\lambda_1^2}{1-\sigma\lambda_1} = \frac{1-\lambda_2^2}{1-\sigma\lambda_2} = n^2\frac{\pi^2}{\bar{a}^2} \qquad \text{or} \qquad \left(n+\frac{1}{2}\right)^2\frac{\pi^2}{\bar{a}^2}$$

and the two alternatives respectively give

$$-\lambda_1\lambda_2 = \beta^2 = 1 - n^2\frac{\pi^2}{\bar{a}^2}$$

or

$$1 - \left(n+\frac{1}{2}\right)^2\frac{\pi^2}{\bar{a}^2}.$$

Since both equations (33) and (34) are satisfied by either the n-, or the $(n + \frac{1}{2})$- dependence, a problem of classification arises: "Which mode, TE or TM, is described, as $p \sim 0$, by the solutions of (33) and (34)?" Leaving aside the question of the TEM mode we note that the following degeneracy exists in the isotropic case:

A TE mode with antisymmetric H_z and a TM mode with symmetric E_z have the same propagation constant

$$\beta^2 = 1 - \frac{\pi^2}{\bar{a}^2}\left(n + \frac{1}{2}\right)^2 \qquad (n = 0, 1, 2 \cdots).$$

A TE mode with symmetric H_z and a TM mode with antisymmetric E_z have the same propagation constant

$$\beta^2 = 1 - \frac{n^2\pi^2}{\bar{a}^2} \qquad (n = 1, 2, \cdots).$$

Since application of a very small magnetization will not destroy the symmetry properties, it follows that the solution of equation (34), and the solution of equation (33), which in the limit $p = 0$ reduce to

$$\beta^2 = 1 - \frac{\pi^2}{\bar{a}^2}\left(n + \frac{1}{2}\right)^2,$$

correspond to a TE-limit and a TM-limit mode respectively. Likewise, the solutions of equation (33) and the solution of equation (34) which in the limit $p = 0$ reduce to

$$\beta^2 = 1 - n^2\frac{\pi^2}{\bar{a}^2}$$

correspond to a TE-limit and a TM-limit mode respectively. When $p \neq 0$, the new β^2 are thus given by different equations, so that the magnetization is seen to have removed the degeneracy of the isotropic case.

In the special case of the TEM limit mode ($\beta^2 = 1$ when $p = 0$) only one of equations (33) and (34) has real roots. For, when $p = 0$, the Polder relation gives

$$\begin{aligned}\frac{1 - \lambda_1^2}{1 - \sigma\lambda_1} &= \frac{1 - \lambda_2^2}{1 - \sigma\lambda_2}, \\ &= 0,\end{aligned} \qquad (36)$$

if β^2 is to equal unity. But equation (36) satisfies equation (34) only [equation (33) would demand $\lambda_1 = \lambda_2$ which is impossible for real β]. Thus the TEM mode exists as the limit of an antisymmetric mode only. In fact it is easily shown that for a value of $\bar{a}$ too small $\left(\text{less than } \frac{\pi}{2}\right)$ to admit any except the TEM mode in the isotropic medium, the only solutions of equation (33) for general σ, p describe incipient modes.

Thus for $\bar{a} < \frac{\pi}{2}$ the analysis may be confined to equation (34) which will give the course of the TEM limit mode. The graphical analysis follows closely that of the cylindrical waveguide (Part I, Section 4.1); the part played by the G-function there is now played by the function

$$L = \lambda \sqrt{\frac{1 - \lambda^2}{1 - \sigma\lambda_1}} \tan \bar{a} \sqrt{\frac{1 - \lambda^2}{1 - \sigma\lambda}}.$$

A contour map is drawn of the function L in the upper half of the λ-σ plane. Only the upper half plane need be considered since the L-equation (34) is unchanged by reflection in the origin. This, incidentally, means that the present situation is reciprocal, so that σ, p hereafter can be considered positive. Pairs of values $\lambda_1(>0)$ and $\lambda_2(<0)$ satisfying the L-equation can then immediately be read off, but while they will give $\beta^2 = \lambda_1\lambda_2$, they do not necessarily, for given σ, p, satisfy the Polder relation, equation (35). To ensure this, the quadrant, $\lambda < 0$, $\sigma > 0$, and the L-contours therein are transformed on to $\lambda > 0$, $\sigma > 0$ by the Polder relation for fixed p:

$$\lambda_2 = \frac{\sigma + p - \lambda_1}{1 - \sigma\lambda_1} = T(\lambda_1).$$

The surfaces $L_1 = L(\lambda_1, \sigma)$ and $L_2 = L(T(\lambda_1), \sigma)$ will intersect in a number of curves, along whose base curves in the λ-σ plane both the Polder equation and the L equation are satisfied, and along which $\beta^2 = -\lambda_1\lambda_2$ is known as a function of σ, p. The zero and infinity curves of $L(\lambda, \sigma)$ are denoted by 0, I when $\lambda > 0$, and by $0'$, I' when $\lambda < 0$. The transforms of $0'$, I' onto $\lambda > 0$ are denoted by $(0')_T$, $(I')_T$. The suffix n denotes the infinity (zero) curves corresponding to

$$\frac{1 - \lambda^2}{1 - \sigma\lambda} = \left(n + \frac{1}{2}\right)^2 \frac{\pi^2}{\bar{a}^2}; \qquad \left(\frac{1 - \lambda^2}{1 - \sigma\lambda} = n^2 \frac{\pi^2}{\bar{a}^2}\right) \quad n \text{ integral.}$$

The lines $\lambda = 0, +1, -1$, are all zero curves denoted by 0_A, 0_B, $0_B'$, respectively. The line $\lambda\sigma = 1$ is a conditional infinity curve, called I_c. It is an I curve when viewed from $\sigma\lambda > 1$ for $\lambda < 1$, and when viewed from $\sigma\lambda < 1$ for $\lambda > 1$. Otherwise (for $\sigma\lambda < 1$, $\lambda < 1$ and $\sigma\lambda > 1$, $\lambda > 1$) it is a limit curve of all possible curves L = const., where the constant takes on any value indefinitely many times. (See Part I, Section 4, where the curve $\sigma\lambda = 1$ is a conditional zero curve, 0_c.) Fig. 8, drawn for $\bar{a} \sim 1$, $p \sim 0.5$, shows the part of the first quadrant allowed by the Polder relation divided into regions of like and unlike sign of $L(\lambda, \sigma)$ and $L(T(\lambda_1, \sigma))$ by the various 0, I and $(0')_T$, $(I')_T$ curves. The unshaded regions are regions of like sign, and all these carry solution curves (dotted lines), by the same reasoning as was employed in Part I, Section 4.11. Two branches of the TEM limit mode exist; in the area bounded by $(0_A)_T$, 0_B, $(0_B')_T$ and in the area 0_B, I_1, $(0_B')_T$. The branch in the first region begins at $\sigma = 0$ with λ_1,λ_2 given by

$$\lambda_1\sqrt{1-\lambda_1^2}\tan\bar{a}\sqrt{1-\lambda_1^2} = \lambda_2\sqrt{1-\lambda_2^2}\tan\bar{a}\sqrt{1-\lambda_2^2},$$

$$\lambda_1 + \lambda_2 = p$$

and ends at the intersection of $(0_A)_T$, 0_B with $\sigma = 1 - p$ and $\beta^2 = 0$ (since $\lambda_2 = 0$ on $(0_A)_T$). The branch in the region 0_B, I_1, $(0_B')_T$ begins with $\beta_2 = \infty$, $\sigma = 1$ and proceeds towards $\sigma = \infty$, $\beta^2 = 1$.

The region bounded by I_1, $(0_A)_T$, I_c contains an infinity of solution curves, the incipient modes. The n^{th} of these begins at $\sigma = 1$, $\lambda_1 = 1$, the intersection of I_n and I_c (the line I_c is also the transform of $\lambda_2 = -\infty$,

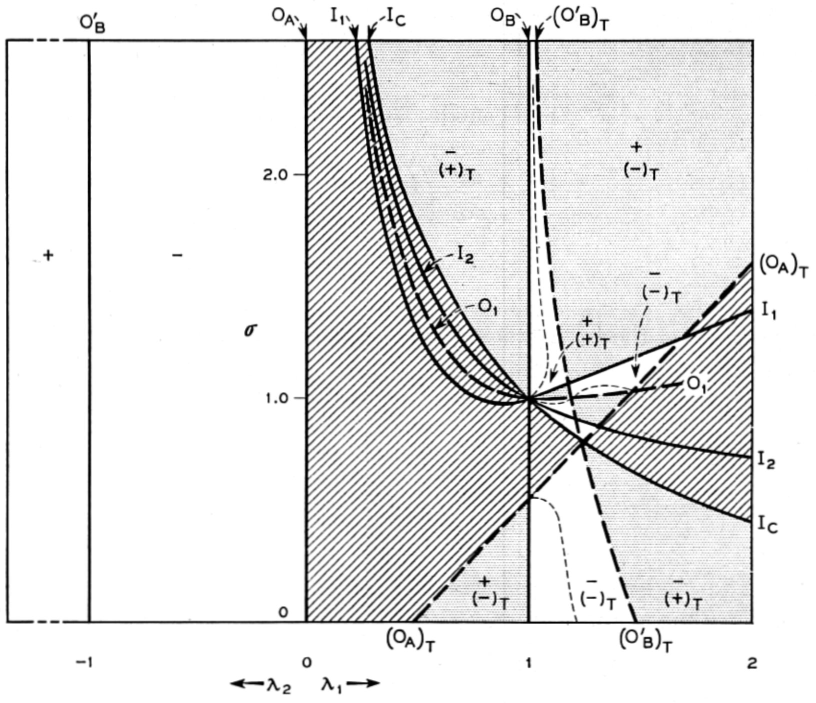


Fig. 8 — Zero and infinity contours of $L(\lambda_1, \sigma)$ and $L(T(\lambda_1), \sigma)$ in the first quadrant of the λ - σ plane. Cross-hatched regions are excluded by the Polder relation; shaded areas are regions of unlike sign of the two functions. Dotted curves are solution curves.

along which $L_2 = \infty$); crosses from one allowed region to the next through the intersection of 0_n with $(0_B')_T$ and proceeds to its cut-off point $(0_n - (0_A)_T)$.

Formulae giving β^2 in the neighborhood of special points on the solution curves, or for special values of the parameters, are obtained exactly as in Part I, by expansion of the L-equation. The results are tabulated below. They relate to antisymmetric modes only.

Cut-off of the TEM-limit mode ($\sigma = 1 - p$, $\beta^2 = 0$)

Near cut-off

$$\frac{d\beta^2}{d\sigma} = -\frac{1}{p\left(1 - \frac{\tan \bar{a}}{2\bar{a}}\right)} \tag{37}$$

For $p > 1$, the TEM mode has no lower branch.

Behavior near resonance ($\sigma = 1$):

$$\text{TEM mode: } \beta^2 = \frac{\frac{\pi^2}{8\bar{a}^2}}{1 - \sigma} p,$$

$$n^{\text{th}} \text{ incipient mode: } \beta^2 = \left(\frac{\left(n + \frac{1}{2}\right)^2 \pi^2}{2\bar{a}^2} - 1\right)\frac{p}{1 - \sigma}. \tag{38}$$

$$(n = 1, 2, \cdots)$$

Cut-off of the n^{th} incipient mode (parametric representation):

$$\lambda_1 = e^{\theta}; \qquad \sigma = e^{-\theta}\left(1 - \frac{\bar{a}^2}{n^2\pi^2}\right) + \frac{\bar{a}^2}{n^2\pi^2} e^{\theta}$$

$$p = \sigma - \lambda_1; \qquad \beta^2 = 0 \tag{39}$$

$$\frac{d\beta^2}{d\sigma} = \frac{m^2\pi^2}{\bar{a}^2}\left(\frac{m^2\pi^2}{\bar{a}^2} - 1\right) \coth \theta \Big/ \left[\left(\frac{m^2\pi^2}{\bar{a}^2} - 1\right)e^{-\theta} + \frac{2 \tan \bar{a}}{\bar{a}} e^{-\theta} - e^{\theta}\right].$$

(The reader may note the similarities of these formulas to those for the cylindrical waveguide).

Spot-point for the n^{th} incipient mode:

$$\sigma = \left[\left(1 - \frac{\bar{a}^2}{n^2\pi^2}\right) \Big/ \beta^2\right] + \frac{\bar{a}^2}{n^2\pi^2}\beta^2,$$

$$p = (1 + \sigma)(\beta^2 - 1). \tag{40}$$

Approximate formulae for all antisymmetric modes at small p, $\sigma \neq 1$:

$$\text{TM modes: } \beta^2 = \beta_{n,1}^2 \frac{\sigma p}{1 - \sigma^2}, \tag{41}$$

$$\text{TE modes: } \beta^2 = \beta_{n,2}^2 \left(1 - \frac{\sigma p}{1 - \sigma^2}\right), \tag{42}$$

where

$$\beta_{n,1}^2 = 1 - \frac{n^2\pi^2}{\bar{a}^2} \qquad n = 1, 2, \cdots$$

and

$$\beta_{n,2}^2 = 1 - \frac{\left(n + \frac{1}{2}\right)^2 \pi^2}{\bar{a}^2} \qquad n = 0, 1, 2, \cdots$$

TEM mode:

$$\beta^2 = 1 - \frac{\sigma p}{1 - \sigma^2}. \tag{43}$$

TEM mode for large σ:

$$\beta^2 = 1 + \frac{p}{\sigma} + O\left(\frac{1}{\sigma^3}\right). \tag{44}$$

TEM mode for very small $\bar{a}$:

for sufficiently small $\bar{a}$, the β^2 for the TEM mode is independent of $\bar{a}$. For general σ, p, except $\sigma = 1$,

$$\beta^2 = \frac{1 - (\sigma + p)^2}{1 - \sigma p - \sigma^2}. \tag{45}$$

Reference to the expression for μ and κ in terms of σ, p will show that the wave progresses as though the medium had a scalar permeability μ_{eff} given by

$$\frac{1}{\mu_{\text{eff}}} = \frac{1}{2}\left(\frac{1}{\mu + \kappa} + \frac{1}{\mu - \kappa}\right).$$

3. FIELD PATTERNS IN THE FILLED CIRCULAR GUIDE

In Part I we discussed propagation in a circular guide filled with ferrite and longitudinally magnetized. Formulae or the field components were given there, and it is of some interest to examine numerically or graphically the distribution of E and H. Since all field components vary (in complex notation) as $e^{j(n\theta+\omega t)}$, with $n = 1$ in the present case, it is clear that the field patterns are stationary in a system rotating with angular velocity, $-\omega$. Writing Φ for the angular variable, $\theta + \omega t$, in this system, one finds the following formulas:

$$\begin{aligned} E_x &= -E_- \sin 2\Phi, & E_r &= (E_+ - E_-) \sin \Phi, \\ E_y &= E_- \cos 2\Phi + E_+ , & E_\theta &= (E_+ + E_-) \cos \Phi, \\ E_z &= E_0 \cos \Phi, & & \\ H_x &= H_- \cos 2\Phi + H_+ , & H_r &= (H_+ + H_-) \cos \Phi, \\ H_y &= H_- \sin 2\Phi, & H_\theta &= (H_- - H_+) \sin \Phi, \\ H_z &= H_0 \sin \Phi, & & \end{aligned} \tag{46}$$

where

$$\begin{aligned} E_-(r) &= \frac{\beta\left(\dfrac{1}{\lambda_1} + 1\right)}{2\chi_1 J_1(\chi_1 r_0)} J_2(\chi_1 r) - \Big[\quad\Big]_2 , \\ E_+(r) &= \frac{\beta\left(1 - \dfrac{1}{\lambda_1}\right)}{2\chi_1 J_1(\chi_1 r_0)} J_0(\chi_1 r) - \Big[\quad\Big]_2 , \\ H_-(r) &= -\frac{1 + \dfrac{1 - \chi_1^2}{\lambda_1}}{2\chi_1 J_1(\chi_1 r_0)} J_2(\chi_1 r) - \Big[\quad\Big]_2 , \\ H_+(r) &= \frac{-1 + \dfrac{1 - \chi_1^2}{\chi_1}}{2\chi_1 J_1(\chi_1 r_0)} J_0(\chi_1 r) - \Big[\quad\Big]_2 , \\ E_0(r) &= \frac{J_1(\chi_1 r)}{J_1(\chi_1 r_0)} - \Big[\quad\Big]_2 , \\ H_0(r) &= -\frac{1}{\lambda_1} \frac{J_1(\chi_1 r)}{J_1(\chi_1 r_0)} - \Big[\quad\Big]_2 . \end{aligned} \tag{47}$$

The terms in square brackets are in each case the same as the corresponding unbracketed terms, λ_2 and χ_2 replacing λ_1 and χ_1. The quantities E_+ and E_- are the amplitudes of the left-handed and right-handed components of circular polarization into which the transverse E-field may be resolved at each point. H_+ and H_- have a similar significance for the transverse H-field. This may be readily verified by examining the vectors $(E_x + jE_y)e^{-j\omega t}$ and $(H_x + jH_y)e^{-j\omega t}$, which represent the transverse field vectors in the laboratory system. The transverse fields are elliptically polarized at any point and the ratio of minor to major axes of the ellipses are

$$\frac{||E_+| - |E_-||}{||E_+| + |E_-||} \quad \text{and} \quad \frac{||H_+| - |H_-||}{||H_+| + |H_-||}.$$

The fields so far are normalized only by the choice of a simple form for the function, $E_0(r)$; all components may, of course, be multiplied by the same arbitrary constant. There is some virtue to a normalization based upon power flow. The power flow is given in unreduced units by

$$\int_{\text{guide}} (E \times H)_z \, dS.$$

Using the scaled units of part I with

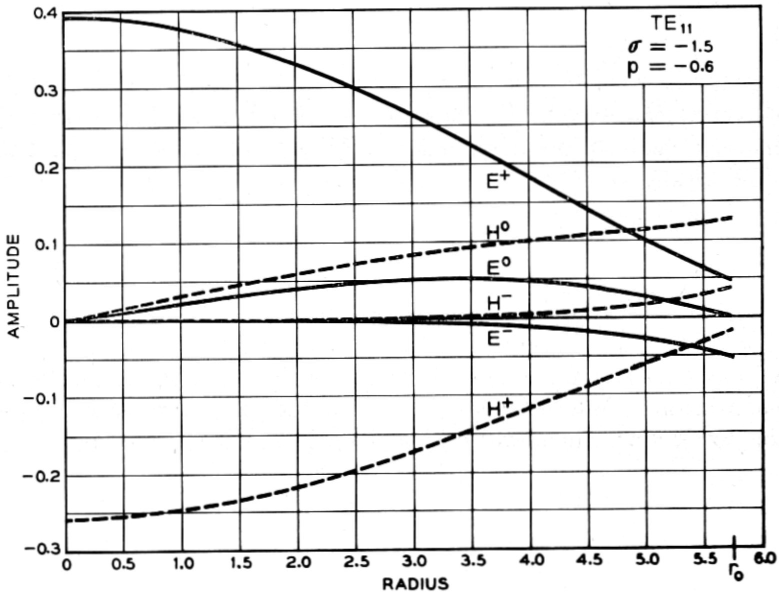
$$r_{\text{actual}} = \frac{r}{\omega\sqrt{\mu_0 \epsilon}}$$

and H replaced by $\sqrt{\frac{\mu_0}{\epsilon}}\, H$ to give it the same dimensions as E (ϵ is the dielectric constant of the ferrite), the power flow becomes in the present variables

$$P = -\sqrt{\frac{\epsilon}{\mu_0}}\, \frac{\pi}{\omega^2 \mu_0 \epsilon} \int (E_+ H_+ + E_- H_-)\, r\, dr,$$

$$= -\sqrt{\frac{\epsilon}{\mu_0}}\, \frac{\pi}{\omega^2 \mu_0 \epsilon}\, I.$$

We shall normalize the E and H fields by dividing the values given by equations (47) by $I^{1/2}$. This makes the power per (isotropic wavelength in the ferrite)^2 a constant. In Fig. 9 we show the normalized fields for the TE_{11}-limit and TM_{11}-limit modes as a function of r for the case $r_0 = 5.75$, $|p| = 0.6$ and several values of σ. The behavior of the modes as a function of σ and p for this radius is shown in Fig. 14 of Part I. We also show the amplitudes for the isotropic cases, $\sigma = p = 0$. It may be recalled from the discussion of cut-off points in Part I that, for the TM mode at this radius, when cut-off is reached at $\sigma = -0.4$, the amplitude of the H^+ field is overwhelmingly greater than that of the others. Even when normalized to the same power flow, the field amplitudes for a given σ are undetermined to a factor of ± 1. This factor has been



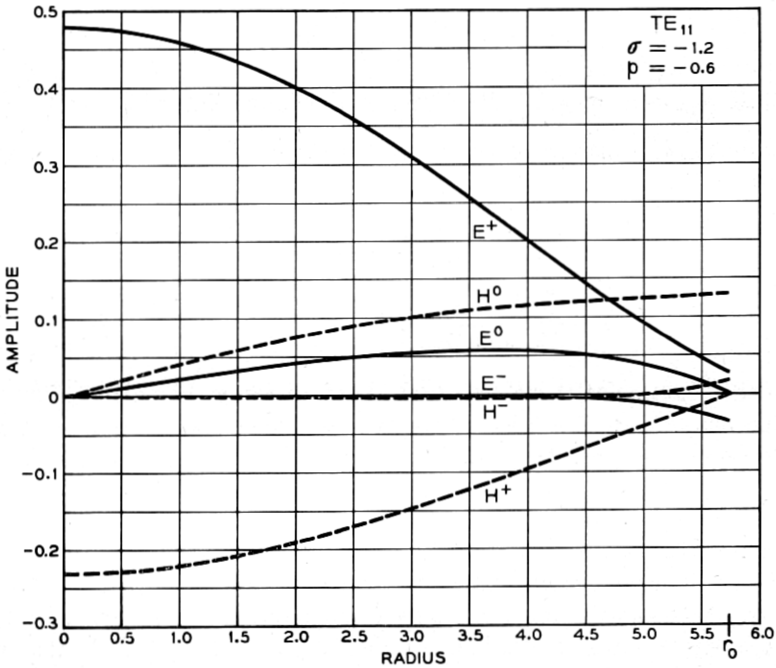


Fig. 9 — Amplitude of the field components in filled cylindrical guide, longitudinally magnetized, as a function of radius, for the TE_{11}-limit and TM_{11}-limit modes at several values of σ. $r_0 = 5.75$. $|p| = 0.6$, except for Figs. 9(f) and (p) which are the isotropic cases, $\sigma = p = 0$. The fields are normalized to a constant power flow per (free space wavelength)^2. (a), top, (b), bottom.

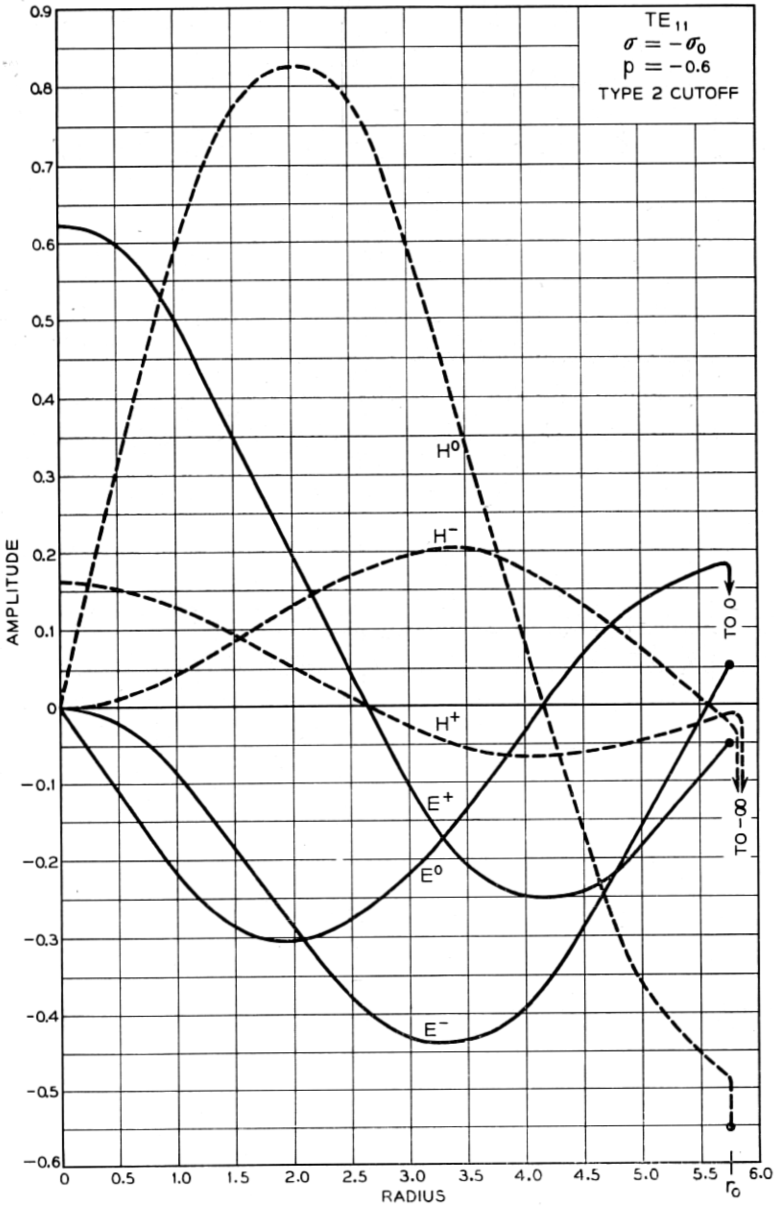


Fig. 9(c) — See Fig. 9.

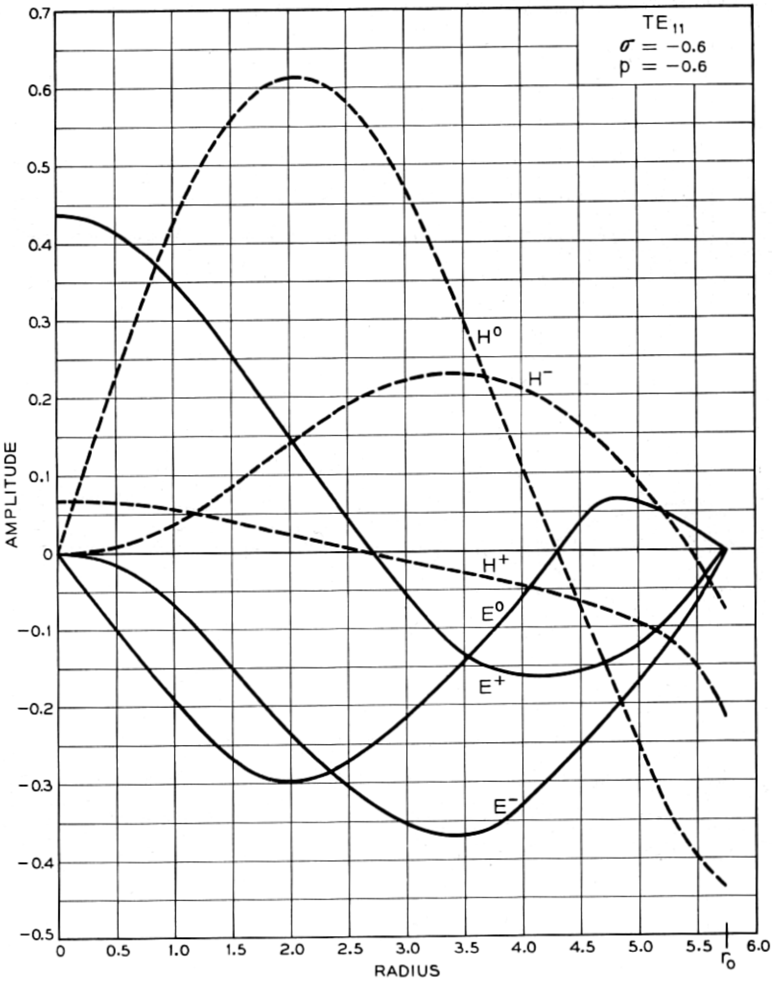
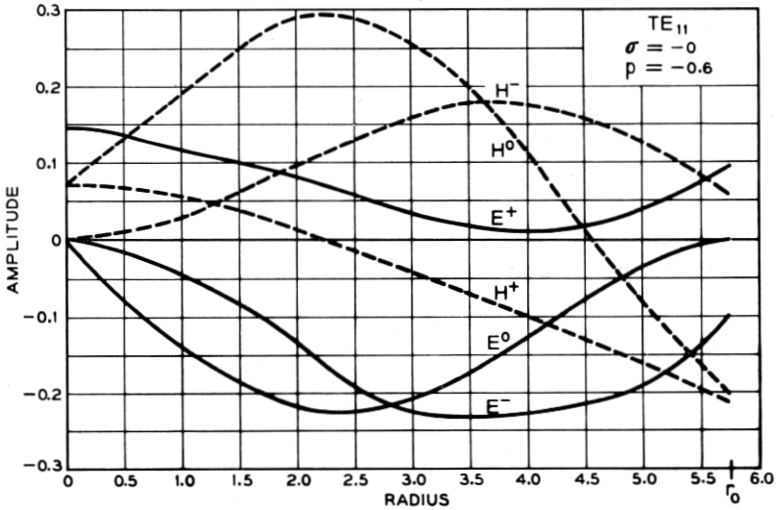


Fig. 9(d) — See Fig. 9.



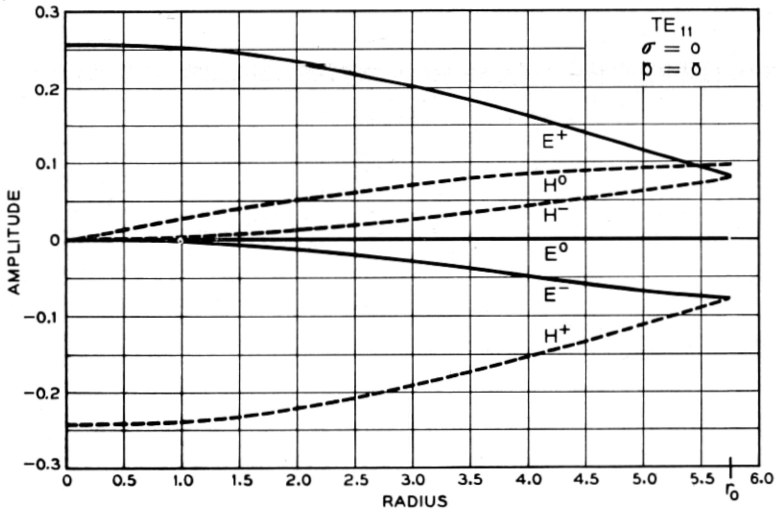
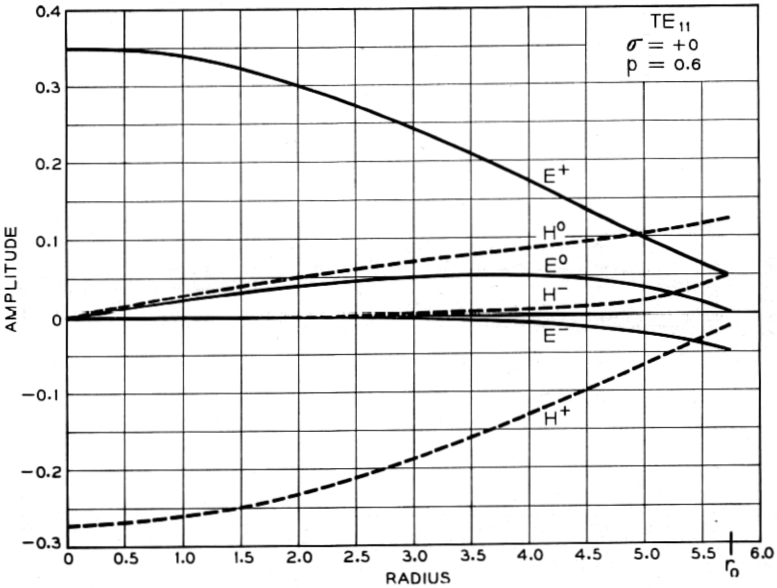


Fig. 9(e), top, and (f), bottom — See Fig. 9.



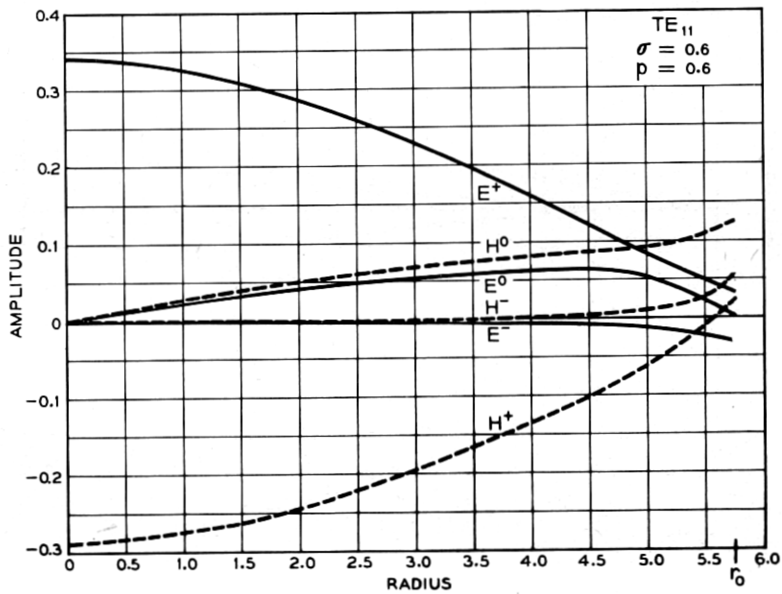


Fig. 9(g), top, and (h), bottom — See Fig. 9.

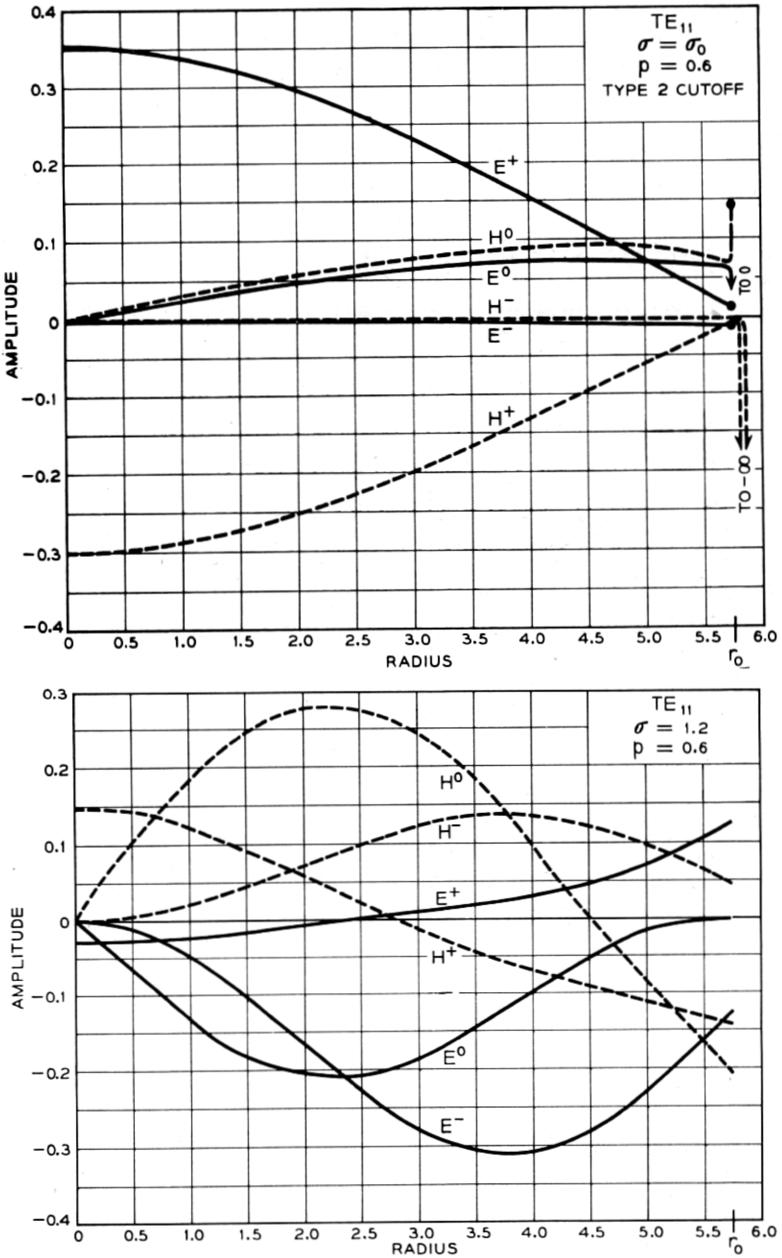
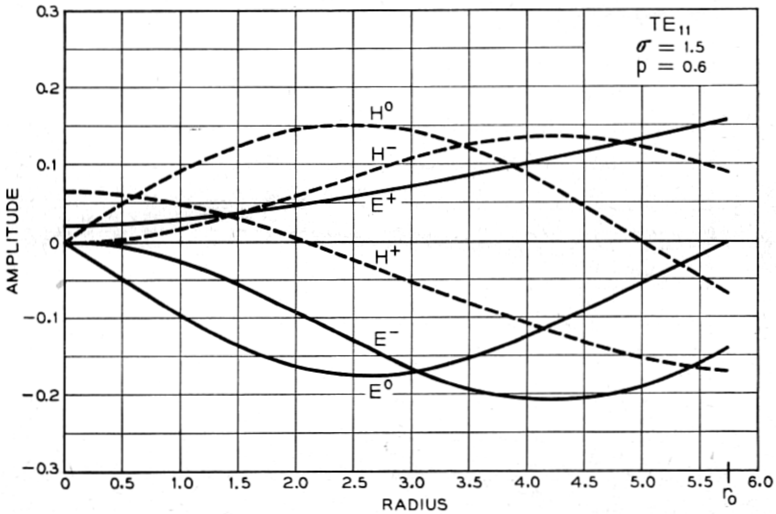


Fig. 9(i), top, and (j), bottom — See Fig. 9.



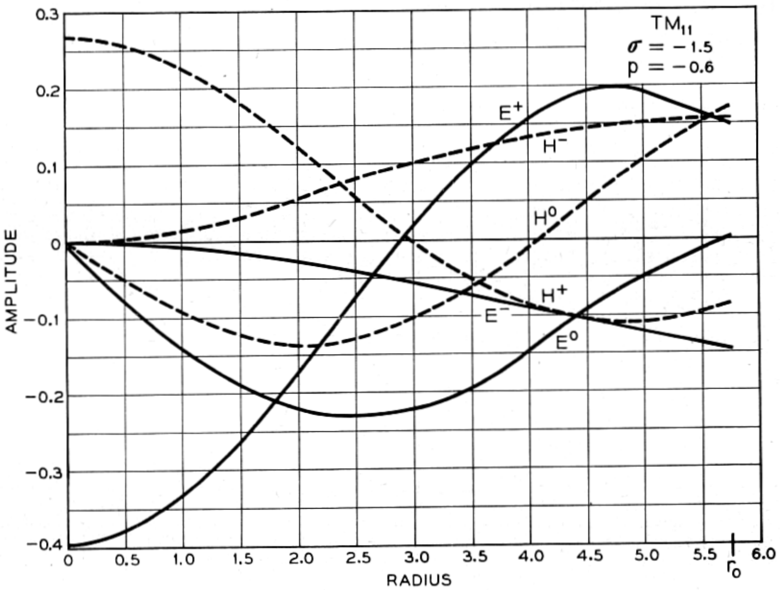


Fig. 9(k), top, and (l), bottom — See Fig. 9.

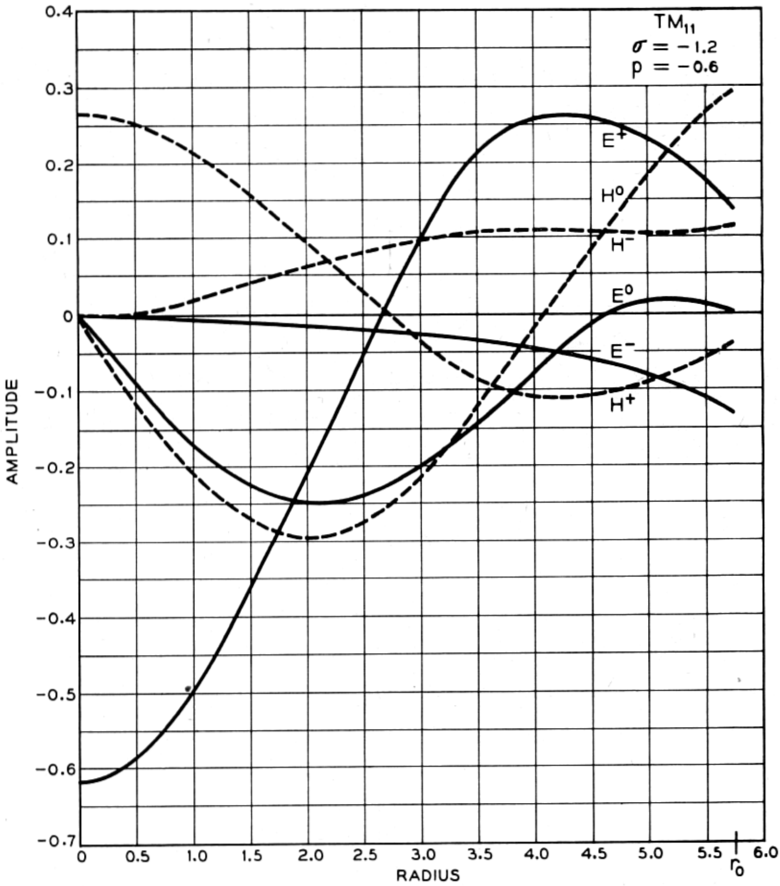


Fig. 9(m) — See Fig. 9.

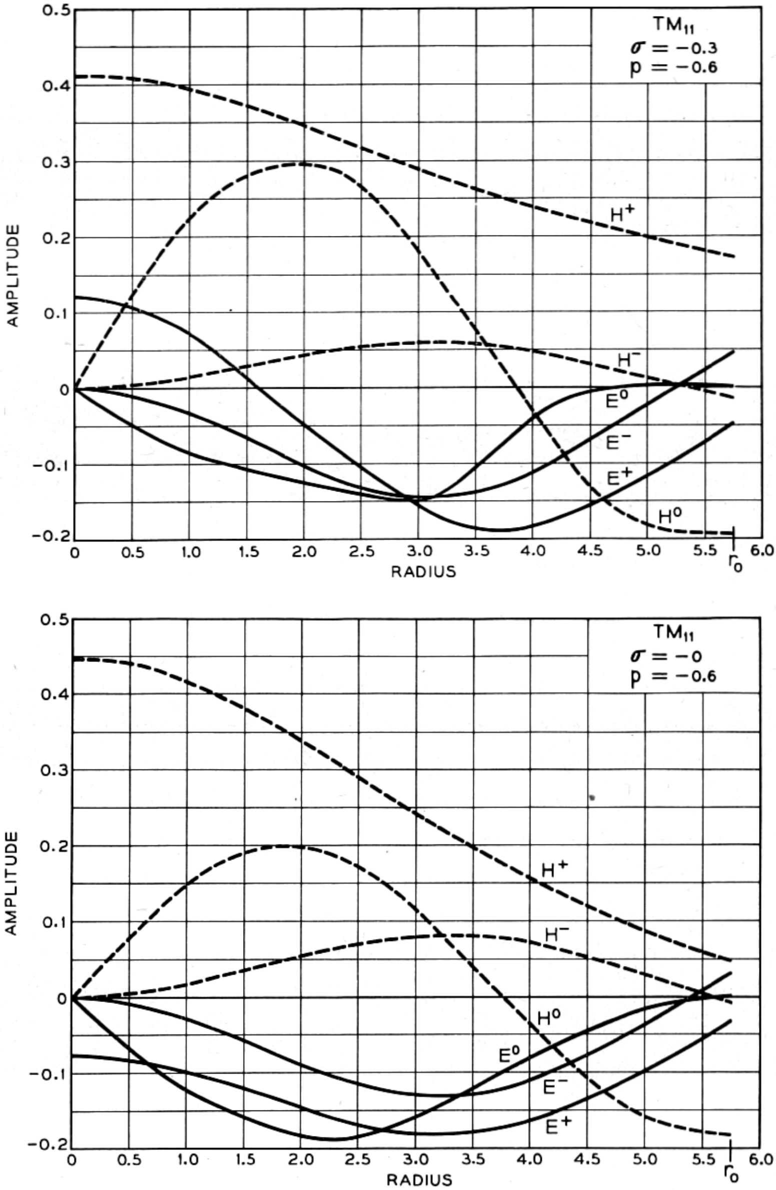


Fig. 9(n), top, and (o), bottom — See Fig. 9.

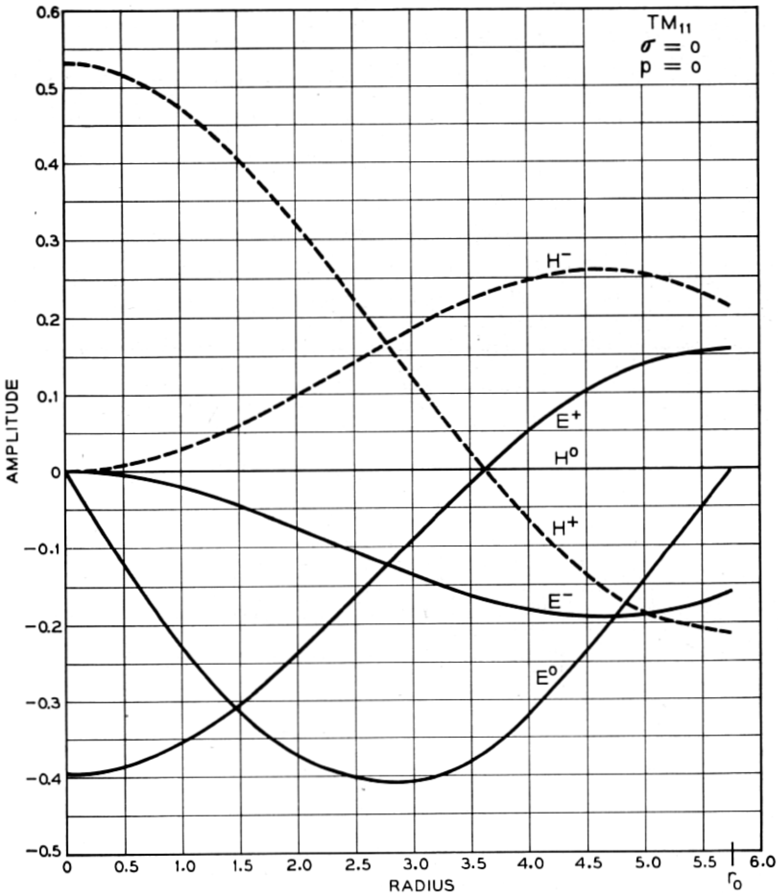


Fig. 9(p) — See Fig. 9.

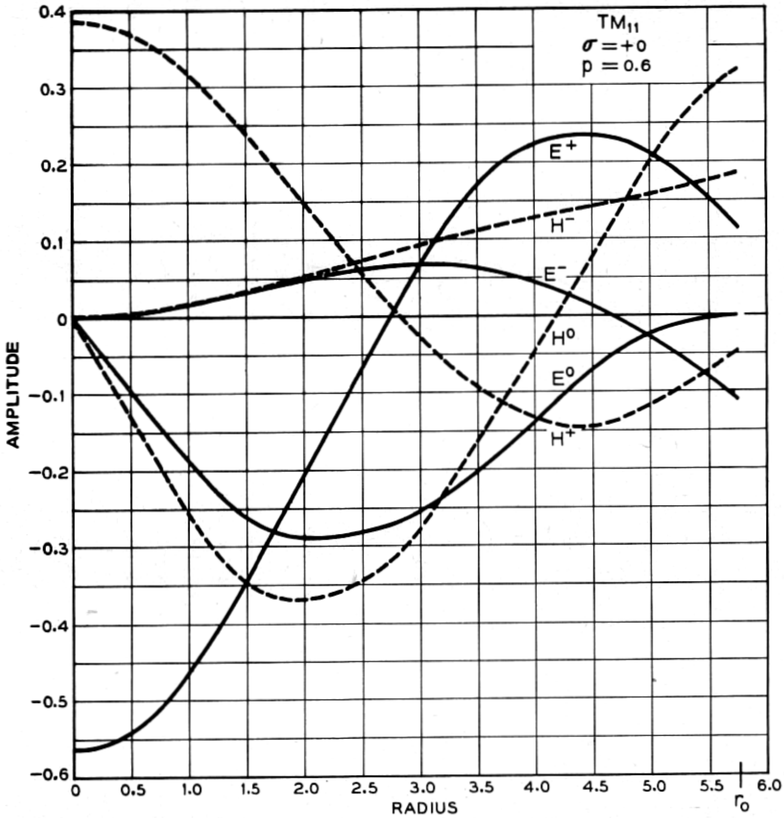


Fig. 9(q) — See Fig. 9.

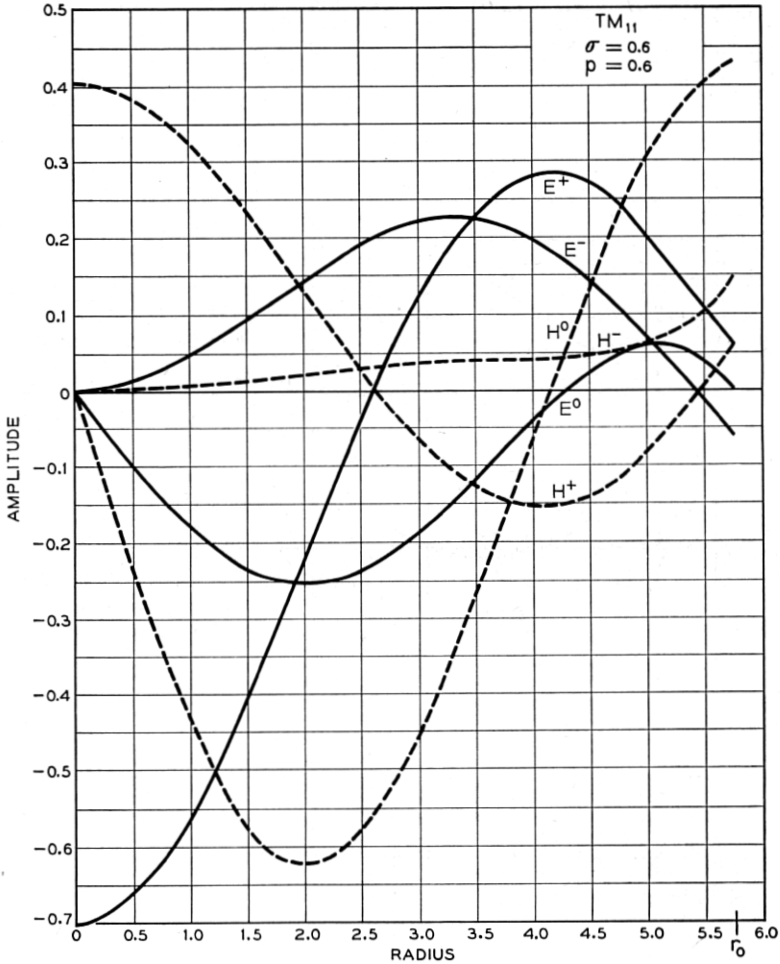


Fig. 9(r) — See Fig. 9.

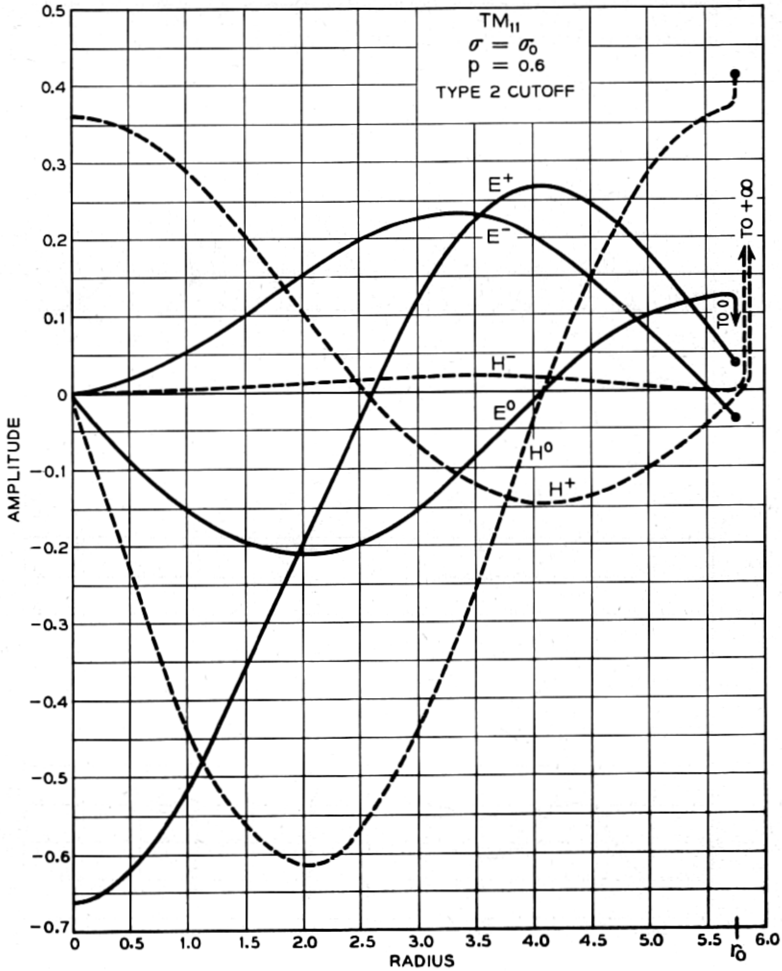
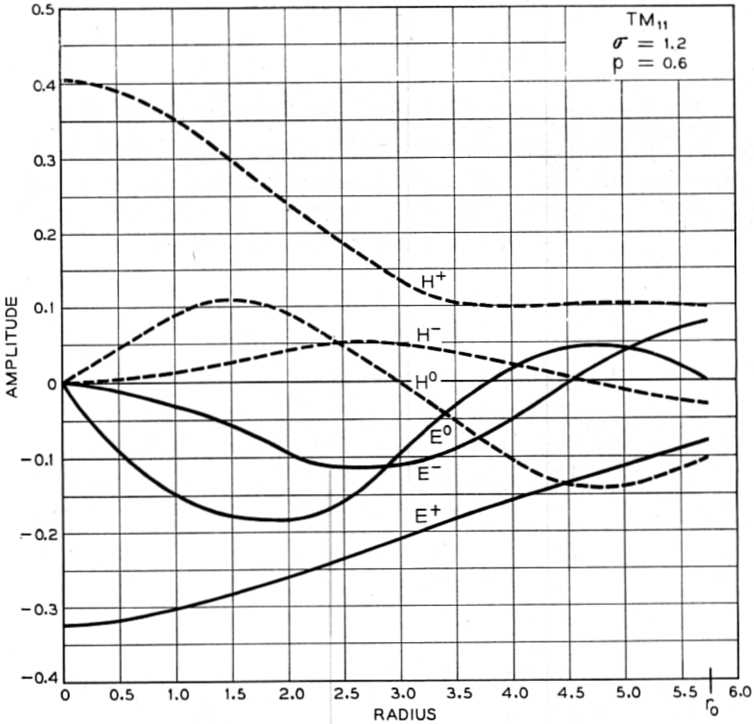


Fig. 9(s) — See Fig. 9.



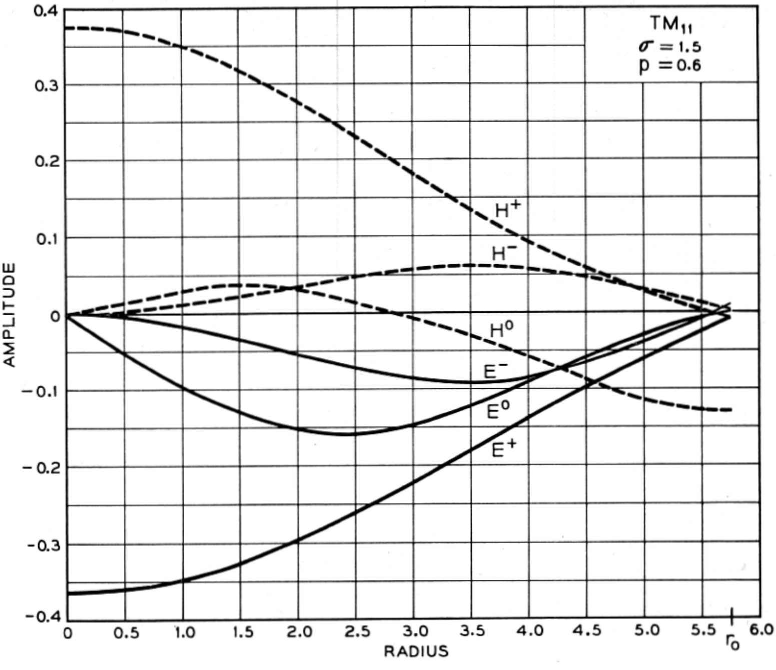


Fig. 9(t), top, and (u), bottom — See Fig. 9.

chosen for each pattern according to the following considerations. For large $|\sigma|$ the amplitudes should tend to those of the isotropic case; for $|\sigma| < \sigma_0$ the amplitudes should develop with increasing $|\sigma|$ away from the isotropic amplitudes. Thus, if the phase of the isotropic amplitudes is first fixed, that of all the others may be fixed by the conditions that

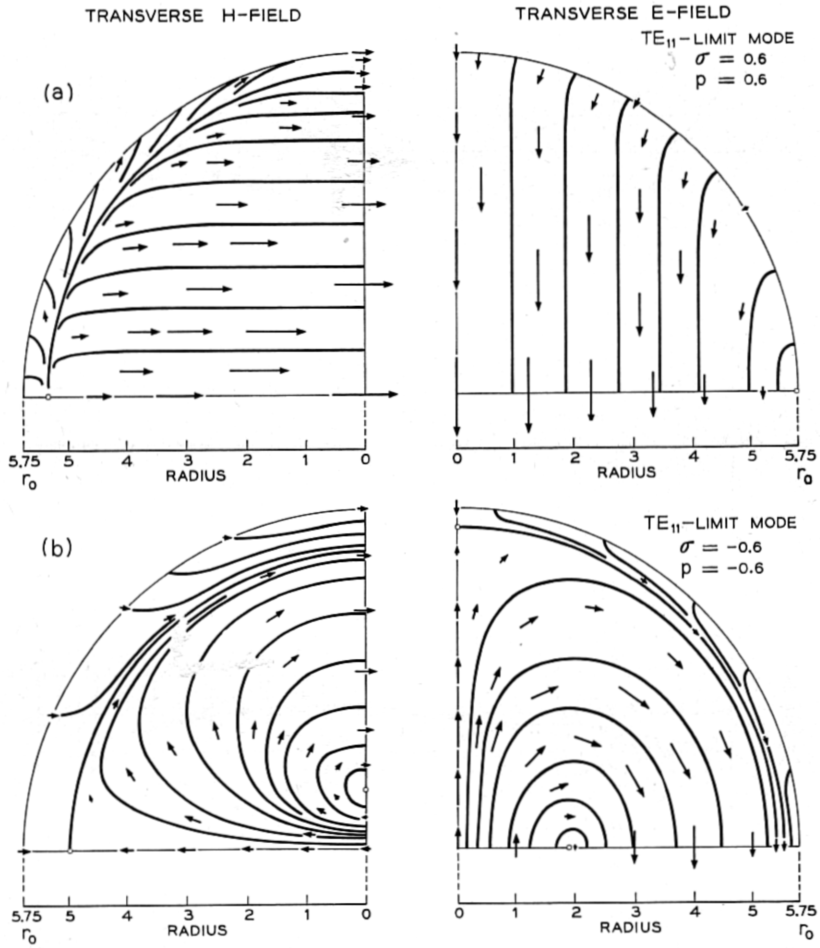


Fig. 10 — Transverse field patterns in a rotating system for four particular cases of Fig. 9. For any pair of E and H patterns the length of the arrows is proportional to field strength, but the patterns for different σ-values have not been given a common normalization. (a), top, TE_{11}-limit mode, $\alpha = 0.6$, $p = 0.6$; (b), bottom, TE_{11}-limit mode, $\sigma = -0.6$, $p = -0.6$; (c), TM_{11}-limit mode, $\sigma = 0.6$, $p = 0.6$; and (d), $\sigma = -0.3$, $p = -0.6$.

they shall fall into an orderly sequence. In Fig. (10) the actual transverse E- and H- field patterns are shown for some representative cases.

It will be noted from Fig. 9 that for both TE- and TM-limit modes, the amplitude patterns for $0 < \sigma < \sigma_0$ resemble those for $\sigma < -1$, while those for $-\sigma_0 < \sigma < 0$ resemble those for $1 < \sigma$. Further, the patterns in the first two ranges of σ are quite similar to the isotropic patterns; those of the latter ranges depart markedly from the isotropic. All of these similarities are more clearly seen for the TE modes than for

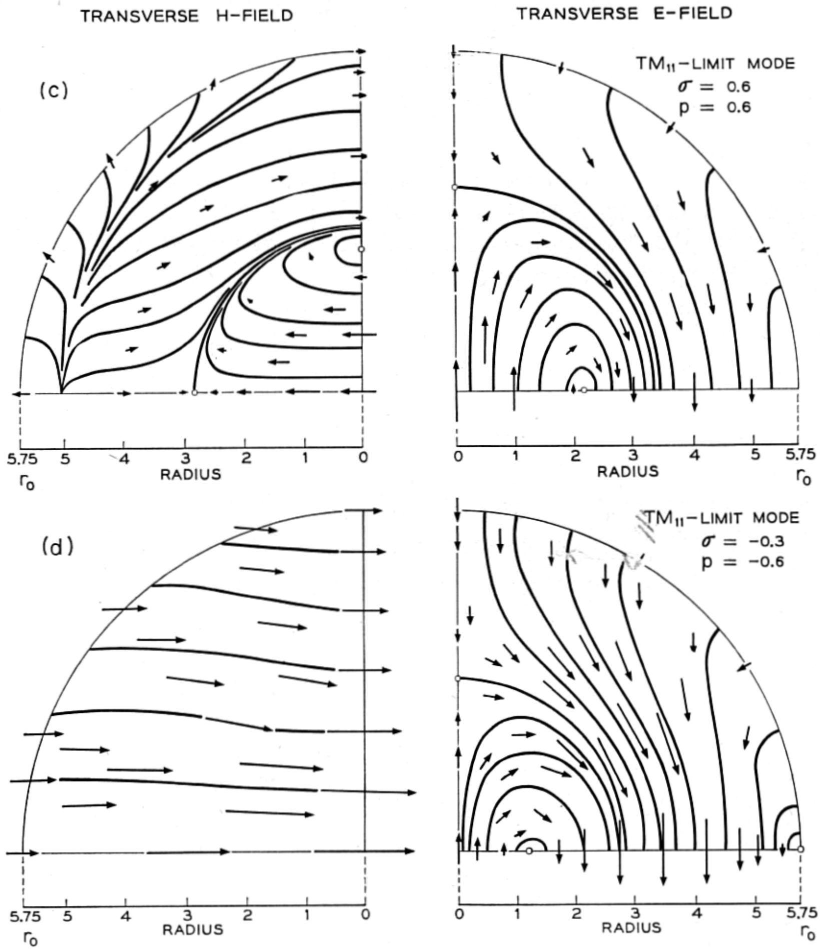


Fig. 10(c), top, and (d), bottom — See Fig. 10.

the TM modes. In Part I where the propagation in guides of very large radius was briefly examined, it may be recalled that in both of the ranges $0 < \sigma < \sigma_0$ and $\sigma < -1$ the field patterns approached those of a plane wave rotating in the same sense as the pattern as a whole. Again, it was found that for $-\sigma_0 < \sigma < 0$ or $\sigma > 1$, the field at most points in the guide was locally rotating in the opposite direction to that of the whole pattern. The points of similarity to the present case are clear and it is also evident that the TE mode more nearly approaches the large guide situation because $r_0 = 5.75$ is much further above the cut-off radius for this mode than it is beyond the cut-off for the TM-mode.

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