Theoretical Fundamentals of Pulse Transmission — I

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A compendium is presented of theoretical fundamentals relating to pulse transmission, for engineering applications. Emphasis is given to the consideration of various imperfections in transmission systems and resultant transmission impairments or limitations on transmission capacity.

In Part I of this paper, Sections 1 to 11, fundamental properties of transmission-frequency characteristics are discussed, together with general relations between frequency and pulse transmission characteristics and special transmission characteristics of importance in pulse systems. This is followed by a presentation of engineering methods of evaluating pulse distortion from various types of gain and phase deviations.

In Part II, Sections 12–16, transmission limitations imposed by characteristic distortion will be discussed.

PART I

PART II

INTRODUCTION

Pulse transmission is a basic concept in communication theory and certain methods of modulating pulses to carry information approach in their characteristics the ideal performance allowed by nature. In certain applications, such as telegraphy, pulse signalling and data transmission, it has the advantage of great accuracy, since the information is transmitted in digital form by "on-off" pulses. This at the same time facilitates regeneration of pulses to avoid accumulation of distortion from noise and other system imperfections, together with the storing, automatic checking and ciphering of messages, as well as their translation into different digital systems or transmission at different speeds, as may be required in extensive communication systems. Another characteristic of pulse systems is that improved signal-to-noise ratio can be secured in exchange for increased bandwidth, as in pulse code, pulse position and certain other methods of pulse modulation. Finally, pulse modulation systems permit multiplexing of communication channels on a time division basis, which under appropriate conditions may have appreciable advantages over frequency division in the design of multiplex terminals.

In pulse modulation systems, pulses are applied at the transmitting end in various combinations, or in varying amplitude, duration or position, depending on the type of system. Pulses thus modulated to carry information may be transmitted in various ways, or undergo a second modulation process suitable to the transmission medium. The received pulses will differ in shape from the transmitted pulses because of bandwidth limitations, noise and other system imperfections. The performance of the system in the absence of noise can be predicted if the "pulse transmission characteristic" is known, that is, the shape of a received pulse for a given applied pulse.

Although the pulse-transmission characteristic suffices for determination of system performance it is customary for various reasons to relate it to the "transmission-frequency characteristic," that is, the steady-state transmission response expressed as a function of frequency. For one thing the transmission-frequency characteristics of various existing facilities and their components are known, and for new facilities can be determined more readily by calculation or measurements than the pulse-transmission characteristic. But the more fundamental reason is that the transmission-frequency characteristics of various system components connected in tandem or parallel can readily be combined to obtain the over-all transmission characteristic, while this is not the case for pulse transmisssion characteristics. It is thus possible to analyze complicated systems with the transmission-frequency characteristic as a basic parameter, and to specify requirements that must be imposed on the transmission-frequency characteristic of the system and its components for a given transmission performance.

A fundamental problem in pulse modulation systems is transmission distortion of pulses by system imperfections in the form of phase and gain deviations over the transmission band or a low-frequency cut-off, usually referred to as "characteristic distortion," which may give rise to excessive interference between pulses and resultant crosstalk noise or errors in reception, depending on the type of system. Because of such interference, characteristic distortion limits the number of pulse amplitudes permissible in the transmission of information or messages over a given channel, and may reduce the rate at which pulses can be transmitted in systems employing only two pulse amplitudes, the minimum number. It thus places a limitation on channel capacity which, unlike signal distortion by noise, cannot be overcome by increasing the signal power.

Characteristic distortion is an important consideration particularly in wire systems where there is a low-frequency cut-off caused by transformers, and where the transmission band may extend over several octaves with substantial variation in attenuation and phase shift, or may be sharply confined by filters. In wire systems there are also fine structure deviations from a smooth attenuation and phase characteristic of a more or less random nature, resulting from small random impedance variations and mismatches along the lines. Gain and phase deviations remaining even after fairly elaborate equalization may be appreciable and difficult to overcome, especially in systems comprising a large number of repeater sections.

The purpose of this paper is to present a compendium of theoretical fundamentals on pulse transmission in a form suitable for engineering applications, both from the standpoint of design of new pulse transmission systems and pulse transmission over existing facilities. Emphasis is placed on considerations of various system imperfections, because of their importance from the standpoint of transmission performance, and since literature on this question is rather limited. Certain fundamental properties of transmission-frequency characteristics are discussed, together with general relations between frequency and pulse transmission characteristics and special transmission characteristics of importance in pulse systems. This is followed by a presentation of methods of evaluating pulse distortion from various types of gain and phase deviations, together with resultant transmission impairments or limitations on pulse transmission rates in low-pass, symmetrical and asymmetrical sideband systems. Conversely, these methods may be used in the design of pulse modulation systems to evaluate requirements imposed on the transmission characteristics for a given transmission performance.

Transmission impairments may result from system imperfections other than characteristic distortion, which require a different theoretical approach and are not considered here. Among them are erratic timing of pulses, thermal and other noise within the transmission system and interference from outside sources, such as other communication systems or atmospheric disturbances.

1. PROPERTIES OF TRANSMISSION-FREQUENCY CHARACTERISTICS

A basic parameter of transmission systems is the transmission-frequency characteristic

$$T(i\omega) = A(\omega)e^{-i\psi(\omega)}, \tag{1.01}$$

in which $\omega = 2\pi f$ is the radian frequency, $A(\omega)$ is the amplitude and $\psi(\omega)$ the phase characteristic. The transmission-frequency characteristic may designate the ratio of received voltage to transmitted current, of received current to transmitted voltage, of received to transmitted current or of received to transmitted voltage. The two latter ratios are not the same except for symmetrical networks with impedance matching at both ends. For symmetrical structures having appreciable attenuation, such as transmission lines between repeaters, the ratios are virtually the same with impedance matching at the receiving end. In the following, $T(i\omega)$ will designate any of the above ratios, as the case may be.

When a number of networks are connected in series, as is usually the case in transmission systems, the resultant transmission characteristic is

$$\begin{aligned} T(i\omega) &= T_1(i\omega)\, T_2(i\omega) \cdots T_n(i\omega), \\ &= (A_1\, A_2 \cdots A_n)\, e^{-i(\psi_1+\psi_2+ \cdots +\psi_n)}, \end{aligned} \tag{1.02}$$

where T_1, $T_2 \cdots T_n$ are the transmission characteristics of the individual networks with the same impedance terminations as encountered in the series arrangement, i.e. as measured in place or with equivalent terminations.

The phase characteristic ψ can in general be regarded as the sum of three components. The first is the minimum phase shift component, ψ^0, which has a definite relation to the amplitude characteristic of the system, and is of particular interest in connection with phase distortion with different types of amplitude characteristics. The second is a linear component $\omega\tau_d$, which represents a constant transmission delay τ_d for all frequencies, as in the case of an ideal delay network. Ladder type structures and transmission lines have phase characteristics which can be represented by the above two components. The third component can be represented by a lattice structure with constant amplitude characteristic but varying phase. Such a network component may be present in a transmission system or may be inserted intentionally for phase equalization, i.e. to supplement the first component above so as to secure a linear phase characteristic without altering the amplitude characteristic of the system.

The following discussion is concerned with the relationship of the first component to the amplitude characteristic of the system, or conversely.

The natural logarithm of the transmission-frequency characteristic given by (1.01) is

$$\ell n T(i\omega) = \ell n A(\omega) - i\psi(\omega). \qquad (1.03)$$

The component $\ell n A(\omega)$ is referred to as the attenuation characteristic, and when expressed in decibels equals 8.69 $ln A(\omega)$.

The following relations exist between the attenuation and phase characteristics of minimum phase shift systems or system components:[1,2]

$$\ell n A(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\psi^0(u)}{\omega - u}\,du = \frac{2}{\pi}\int_0^{\infty}\frac{u\psi^0(u)}{u^2 - \omega^2}\,du, \qquad (1.04)$$

and

$$\psi^0(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\ell n A(u)}{\omega - u}\,du = -\frac{2}{\pi}\int_0^{\infty}\frac{\omega\,\ell n A(u)}{u^2 - \omega^2}\,du. \qquad (1.05)$$

In the evaluation of these integrals, the principal values are to be used, i.e., results of the form $\ell n(-u)$ are to be taken as $ln\,|-u|$ rather than $\ell n\,|u| + i\pi$.

As an example consider an attenuation characteristic as shown in Fig. 1, with $A(\omega) = A_0$ between $\omega = 0$ and ω_c and A_1 between $\omega = \omega_c$ and ∞. Equation (1.05) then becomes

$$\begin{aligned}\psi^0(\omega) &= -\frac{2\omega}{\pi}\left[\ell n A_0\int_0^{\omega_c}\frac{du}{u^2 - \omega^2} + \ell n A_1\int_{\omega_c}^{\infty}\frac{du}{u^2 - \omega^2}\right],\\ &= \frac{1}{\pi}\,\ell n(A_0/A_1)\,\ell n\left|\frac{\omega_c + \omega}{\omega_c - \omega}\right|.\end{aligned} \qquad (1.06)$$

In Fig. 1 is shown the phase characteristic for $A_0/A_1 = 100$, corresponding to a 40 db cutoff at $\omega = \omega_c$.

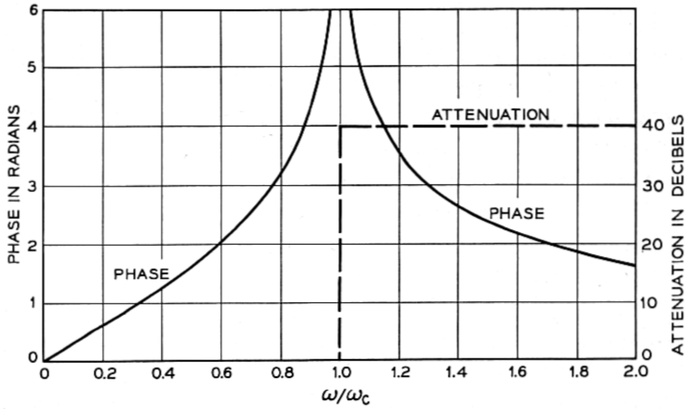


Fig. 1 — Low-pass transmission frequency characteristic with sharp cut-off.

In Fig. 2 the attenuation and phase characteristics are shown as a function of ω/ω_c for $\omega < \omega_c$ and as a function of the inverse ratio ω_c/ω for $\omega > \omega_c$. It will be noticed that for the above case the phase characteristic is infinite for $\omega/\omega_c = 1$ and has even symmetry about this point, while the attenuation characteristic has odd symmetry with respect to the midpoint of the amplitude discontinuity. The phase characteristic may be modified by a gradual cutoff in the attenuation characteristic, as illustrated in the figure. It is possible to shape the attenuation characteristic to obtain a linear phase characteristic in the transmission band, i.e. between $\omega/\omega_c = 0$ and 1. Since transmission systems with a

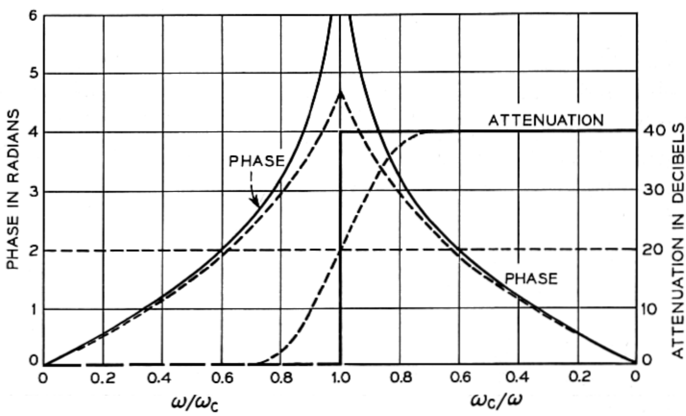


Fig. 2 — Solid curves same as in Fig. 1, but with inverse scale for $\omega/\omega_c > 1$. Dashed curves illustrate modification in phase characteristic with gradual cut-off in attenuation (not computed).

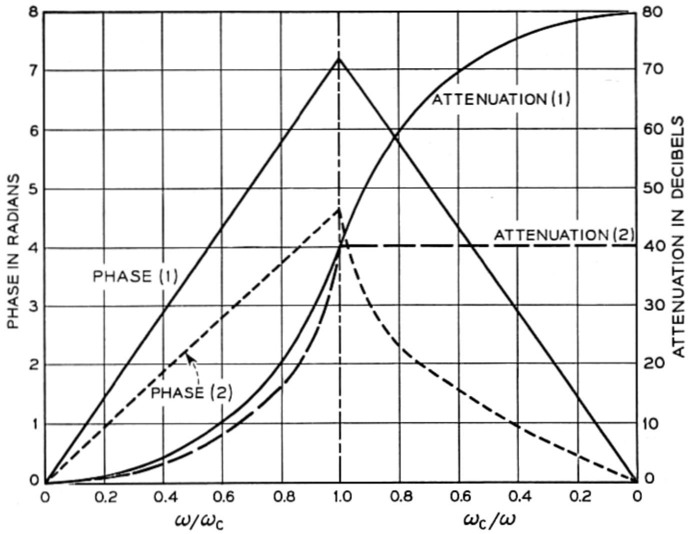


Fig. 3 — Low-pass transmission frequency characteristics with natural linear phase shift for $\omega/\omega_c < 1$.

linear phase characteristic in this range are of particular importance in pulse transmission, this case will be considered further.

It will be assumed that the phase characteristic has even symmetry when expressed in the scales of Fig. 2, in which case the phase characteristic as shown by the solid lines in Fig. 3 is given by

$$\begin{aligned} \psi^0(\omega) &= \omega\tau & \omega/\omega_c < 1, \\ &= \omega_c^2\tau/\omega & \omega/\omega_c > 1. \end{aligned} \tag{1.07}$$

With these expressions in (1.04) the attenuation characteristic becomes:

$$\ell n A(\omega) = \frac{2\omega_c\tau}{\pi}\left[1 + \frac{1}{2}\left(\frac{\omega_c}{\omega} - \frac{\omega}{\omega_c}\right)\ell n\,\frac{1 + \omega/\omega_c}{1 - \omega/\omega_c}\right].$$

For $\omega = 0$, the latter expression approaches the limit $\ell n A(0) = 4\omega_c\tau/\pi$, so that

$$\ell n A(\omega)/A(0) = -\frac{2\omega_c\tau}{\pi}\left[1 + \frac{1}{2}\left(\frac{\omega}{\omega_c} - \frac{\omega_c}{\omega}\right)\ell n\,\frac{1 + \omega/\omega_c}{1 - \omega/\omega_c}\right]. \tag{1.08}$$

which is the attenuation characteristic shown in Fig. 3.

Other attenuation characteristics with a linear phase characteristic between $\omega/\omega_c = 0$ and 1 are possible with other types of variations in the attenuation or phase characteristic for $\omega/\omega_c > 1$ than assumed above. For example, the attenuation characteristics may be assumed constant for $\omega/\omega_c > 1$, in which case the attenuation characteristic will be somewhat different for $\omega/\omega_c < 1$ and the phase characteristic different for $\omega/\omega_c > 1$, as illustrated in Fig. 3. (The solution for the latter case is given in Reference 2.) It will be noticed that there is a comparatively minor difference between the attenuation characteristics for $\omega/\omega_c < 1$ in the above cases, so that the attenuation characteristic for $\omega/\omega_c > 1$ has a relatively minor effect, provided there is no discontinuity near $\omega/\omega_c = 1$. The transmission loss characteristics shown in Fig. 3 represent a close approximation to the type of characteristic employed in pulse transmission systems, as will be shown later.

In the above examples low-pass characteristics were assumed. For high-pass characteristics the algebraic sign of the phase is reversed with respect to the amplitude characteristic as indicated in Fig. 4, which also illustrates relationships for band-pass characteristics. The band-pass characteristics are obtained by connecting low-pass and high-pass networks in tandem. The resultant attenuation and phase characteristics are obtained by adding the low and high-pass attenuation and phase characteristics, as illustrated in the figure. In the second case shown in the figure, the band-pass characteristic is assumed to have a linear phase characteristic in the transmission band, in which case the attenuation characteristic will not be symmetrical about the midband frequency, unless the latter is high in relation to the bandwidth. The third case illustrates the type of band-pass characteristic encountered in wire systems with a low-frequency cutoff. There will then be phase distortion at the low end of the band, since it is not feasible with a fairly sharp low-frequency cutoff to obtain a linear phase characteristic in the transmission band.

If the amplitude or attenuation characteristic of a transmission system is modified, it will be accompanied by a modification in the phase characteristic. Of basic importance are cosine modifications in the attenuation and amplitude characteristics. Let the modified amplitude characteristic be of the form

$$A(\omega) = A_0(\omega)\, e^{a \cos \omega\tau}, \tag{1.09}$$

where $A_0(\omega)$ is the original amplitude characteristic. The modified attenuation characteristic is then

$$\ell n A(\omega) = \ell n A_0(\omega) + a \cos \omega\tau. \tag{1.10}$$

In accordance with (1.05) the modified phase characteristic becomes,

$$\begin{aligned} \psi^0(\omega) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ell n A_0(\omega)}{\omega - u}\, du + \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{\cos \omega\tau}{\omega - u}\, du, \\ &= \psi_0(\omega) + a \sin \omega\tau, \end{aligned} \tag{1.11}$$

where $\psi_0(\omega)$ is the phase characteristic of the original amplitude characteristic $A_0(\omega)$.

Thus, for any consine modification in the attenuation characteristic there is a corresponding sine modification in the phase characteristic, and for any sine modification in the phase characteristic a corresponding cosine modification in the attenuation characteristic. In general any modification in the attenuation characteristic may be represented by a Fourier cosine series, in which case the modification in the phase characteristic will be the corresponding Fourier sine series.

With a cosine modification in the amplitude rather than in the at-

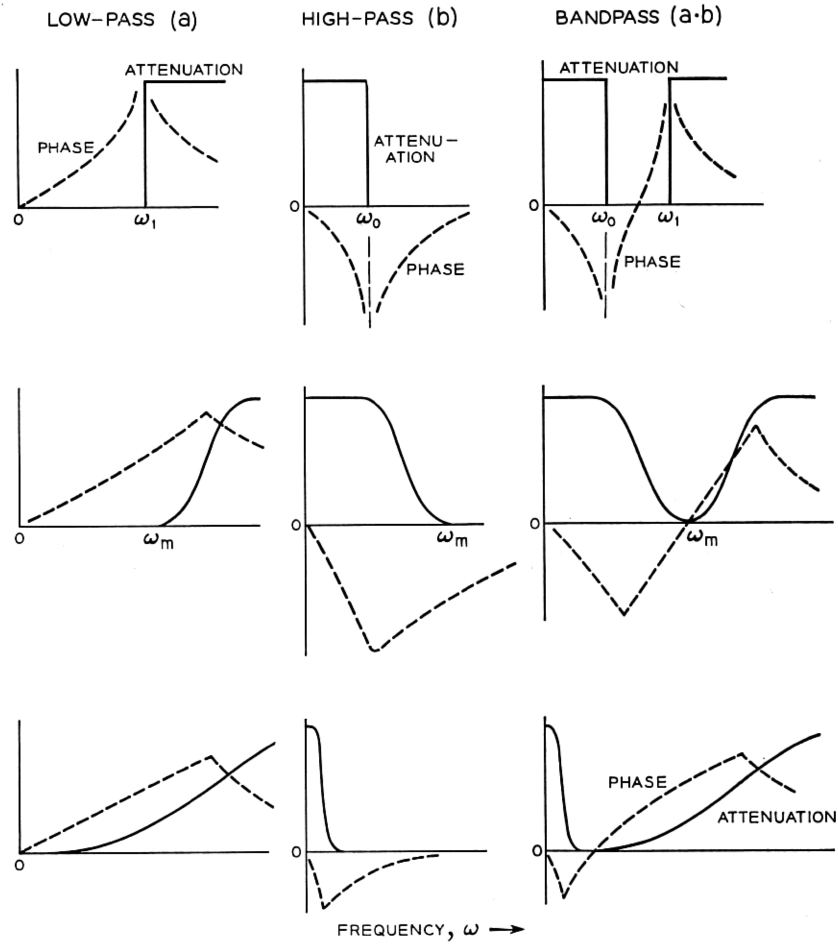


Fig. 4 — Attenuation and phase shift for various types of transmission frequency characteristics.

tenuation characteristic

$$A(\omega) = A_0(\omega)\,[1 + a \cos \omega\tau], \tag{1.12}$$

and the corresponding phase characteristic becomes

$$\begin{aligned} \psi^0(\omega) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ell n A_0(u)[1 + a \cos u\tau]}{\omega - u}\, du, \\ &= \psi_0(\omega) + 2 \tan^{-1} \frac{r \sin \omega\tau}{1 + r \cos \omega\tau}, \\ &= \psi_0(\omega) + 2[r \sin \omega\tau + \frac{r^2}{2} \sin 2\,\omega\tau \\ &\qquad + \frac{r^3}{3} \sin 3\,\omega\tau + \dots \end{aligned} \tag{1.13}$$

where

$$r = \frac{1}{a}\,[1 \mp \sqrt{1 - a^2}], \tag{1.14}$$

and the minus sign is to be used.

Thus, a cosine modification in the amplitude characteristic is accompanied by an infinite series of sine deviations in the phase characteristic. For sufficiently small values of a, $r \cong a/2$ and (1.13) reduces to (1.11).

2. FREQUENCY AND IMPULSE TRANSMISSION CHARACTERISTICS

In dealing with pulse transmission, it is customary to consider three basic types of time variations of currents and electromotive forces, a cisoidal variation, a unit impulse and a unit step. The cisoidal variation, $e^{i\omega t}$, is basic in the solution of network and transmission problems in terms of complex impedances and admittances. The unit impulse is a current or electromotive force of very high intensity and short duration, such that the area under the impulse is unity. The unit step is a current or electromotive force which is zero for $t < 0$ and unity thereafter.

The time responses of networks or transmission systems to these three basic time functions are interrelated so that each may be obtained when one of the others is known. Furthermore, the time responses for electromotive forces or currents of arbitrary wave shape may be obtained from the response characteristic for any one of these basic time functions.

The pulses applied in pulse systems can usually be approximated by impulses. Furthermore, with impulses certain simple relationships can be established which are either obscured or more complicated when a unit step is assumed. For these reasons, only the transmission characteristic for impulses will be considered here, or for pulses of sufficiently short duration to be regarded as impulses.

Corresponding to any transmission-frequency characteristic is an impulse transmission characteristic, $P(t)$, which designates the received pulse as a function of time for a transmitted unit impulse. The impulse and transmission frequency characteristics are interrelated by the following Fourier integral relations

$$P(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} T(i\omega)e^{i\omega t}\, d\omega, \qquad (2.01)$$

$$T(i\omega) = \int_{-\infty}^{\infty} P(t)e^{-i\omega t}\, dt. \qquad (2.02)$$

The transmission characteristic for an applied pulse or signal of arbitrary shape $G(t)$ is given by

$$H(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} T(i\omega)S(i\omega)e^{i\omega t}\, d\omega, \qquad (2.03)$$

where $S(i\omega)$ is the frequency spectrum of the applied pulse and is given by

$$S(i\omega) = \int_{-\infty}^{\infty} G(t)e^{-i\omega t}\, dt. \qquad (2.04)$$

In the case of a symmetrical pulse $S(i\omega)$ is a real function.

In view of (1.01), expression (2.03) may also be written

$$H(t) = \frac{1}{\pi} \int_{0}^{\infty} A(\omega)S(\omega) \cos [\omega t - \psi(\omega)]\, d\omega, \qquad (2.05)$$

where the relations $A(-\omega) = A(\omega)$, $S(-\omega) = S(\omega)$, $\psi(-\omega) = -\psi(\omega)$ have been used, and it is assumed that $S(i\omega) = S(\omega)$ is a real function, as for a symmetrical pulse.

In most pulse transmission systems, the applied pulses can be approximated by short rectangular pulses. Rectangular pulses of unit amplitude and duration δ have a frequency spectrum

$$S(\omega) = \delta \frac{\sin \omega\delta/2}{\omega\delta/2}. \qquad (2.06)$$

The same pulse transmission characteristic as when an impulse is applied is obtained with a rectangular pulse if $A(\omega)$ is modified by the factor $(\omega\delta/2)/\sin(\omega\delta/2)$. In the following it will be assumed that the applied pulses are of sufficiently short duration to be regarded as im-

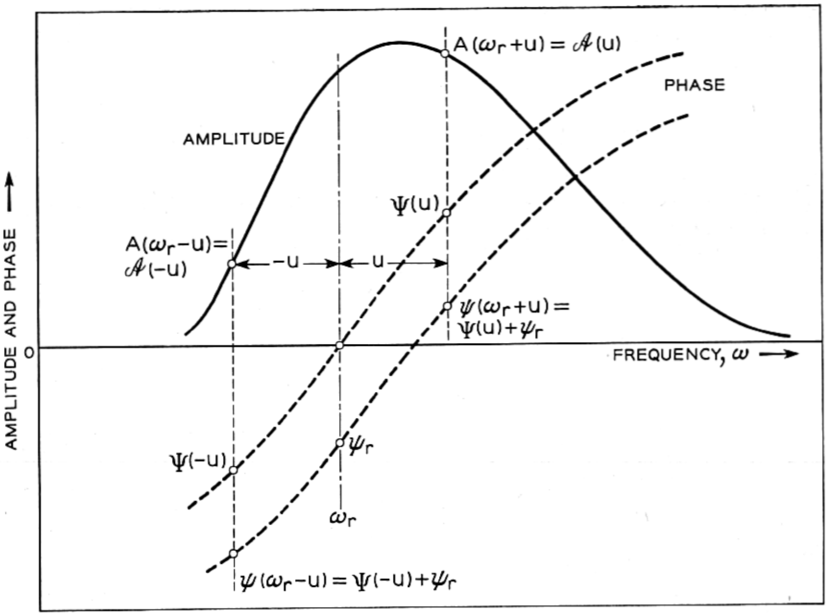


Fig. 5 — Transfer of reference frequency from $\omega = 0$ to $\omega = \omega_r$.

pulses or that otherwise the above modification is applied, in which case

$$P(t) = \frac{\delta}{\pi} \int_0^\infty A(\omega) \cos [\omega t - \psi(\omega)] \, d\omega. \tag{2.07}$$

In the latter equation $A(\omega)$ can also be regarded as the frequency spectrum of a pulse applied to a transmission system having a constant amplitude characteristic and a phase characteristic $\psi(\omega)$ over the band of the pulse spectum.

Equation (2.07) applies to any type of transmission-frequency characteristic and is convenient in this form for low-pass characteristics. For band-pass characteristics as shown in Fig. 5 however, it is convenient from the standpoint of general analysis as well as for numerical evaluation to use a reference frequency ω_r within the tranmsission band, that is, to employ the transformation $\omega = \omega_r + u \quad d\omega = du$.

With the notation

$$\begin{aligned} \mathfrak{A}(u) &= A(\omega) = A(u + \omega_r), \\ \Psi(u) &= \psi(\omega) - \psi(\omega_r) = \psi(\omega) - \psi_r, \end{aligned} \tag{2.08}$$

equation (2.07) can be written:

$$P(t) = \cos(\omega_r t - \psi_r)[R_-(t) + R_+(t)] + \sin(\omega_r t - \psi_r)[Q_-(t) - Q_+(t)]. \qquad (2.09)$$

$$R_- = \frac{\delta}{\pi}\int_0^{\omega_r} \mathfrak{A}(-u)\cos[ut + \Psi(-u)]\,du,$$
$$R_+ = \frac{\delta}{\pi}\int_0^{\infty} \mathfrak{A}(u)\cos[ut - \Psi(u)]\,du, \qquad (2.10)$$

$$Q_- = \frac{\delta}{\pi}\int_0^{\omega_r} \mathfrak{A}(-u)\sin[ut + \Psi(-u)]\,du, \quad \text{and}$$
$$Q_+ = \frac{\delta}{\pi}\int_0^{\infty} \mathfrak{A}(u)\sin[ut - \Psi(u)]\,du. \qquad (2.11)$$

The envelope $\overline{P}(t)$ of the impulse transmission characteristic is given by

$$\overline{P}(t) = [(R_- + R_+)^2 + (Q_- - Q_+)^2]^{1/2}. \qquad (2.12)$$

Comparison of (2.09) with (2.07) shows that R_- and R_+ can be identified with the impulse characteristics of low-pass systems having the same frequency characteristics as the bandpass system below and above ω_r. The impulse characteristics Q_- and Q_+ which arise from asymmetry in the transmission characteristic with respect to ω_r are not present in low-pass systems, since by definition the amplitude characteristic has even symmetry and the phase characteristic odd symmetry with respect to zero frequency.

The first and second components of (2.09) are referred to as the in-phase and quadrature components of the impulse characteristic of band-pass systems.[3] The transmission-frequency characteristic may correspondingly be regarded as made up of a component with even symmetry and another component with odd symmetry about ω_r, as indicated in Fig. 6. These two components, together with the in-phase and quadrature components, will depend on the choice of ω_r. However, $P(t)$ as given by (2.09) and the envelope as given by (2.12), will remain the same, since a single impulse characteristic is associated with a given transmission-frequency characteristic.

With the customary pulse transmission methods, the reference frequency ω_r may be identified with a modulating or carrier frequency, which has a special significance when the envelope of a sequence of received pulses is considered. Although for a single pulse the envelope is always the same, for a sequence of pulses the resultant envelope of the received pulse train will depend on the in-phase and quadrature components.[4] The reason for this is that one has even and the other odd symmetry about the peak amplitude of the envelope for a single pulse, when the phase characteristic is linear.

In order to compare the transmission performance as the reference or carrier frequency is changed, it is necessary to determine the in-phase and quadrature components for each carrier frequency under considera-

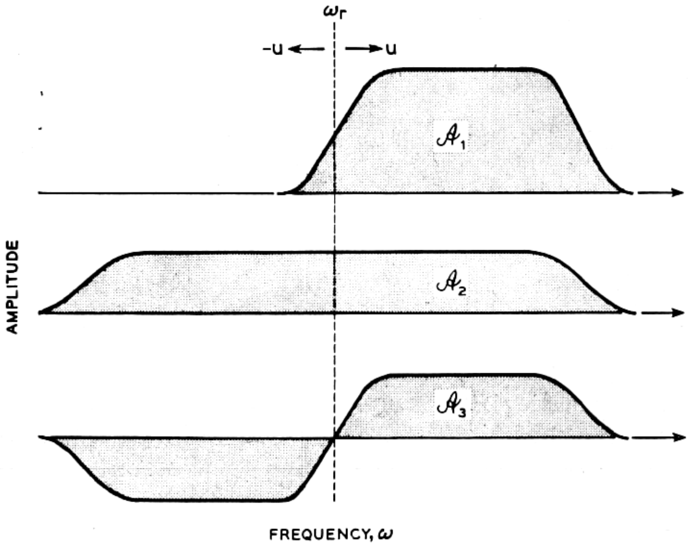


Fig. 6 — Decomposition of amplitude characteristic $\mathcal{A}_1$ asymmetrical with respect to ω_r into a component $\mathcal{A}_2$ of even symmetry and a component $\mathcal{A}_3$ of odd symmetry about ω_r. When the phase shift is linear, $\mathcal{A}_1 = \mathcal{A}_2 + \mathcal{A}_3$.

tion. One method is to evaluate integrals (2.10) and (2.11) for each carrie frequency, which may be facilitated by resolving the transmission-frequency characteristic into symmetrical and anti-symmetrical components as indicated in Fig. 6. This, however, is a rather elaborate procedure which can be avoided with the aid of a simple translation from one reference or carrier frequency to another, as shown below, provided the in-phase and quadrature components or the envelope has been determined for one reference frequency.

Equation (2.09) may also be written, with $\varphi = \varphi(t)$:

$$\begin{aligned} P(t) &= \cos(\omega_r t - \psi_r - \varphi)\, \overline{P}(t), \\ &= \cos(\omega_r t - \psi_r) \cos \varphi\, \overline{P}(t) + \sin(\omega_r t - \psi_r) \sin \varphi\, \overline{P}(t). \end{aligned} \tag{2.13}$$

Comparison of (2.13) with (2.09) shows that:

$$\begin{aligned} R_- + R_+ &= \cos\varphi\, \overline{P}(t), \\ Q_- - Q_+ &= \sin\varphi\, \overline{P}(t), \end{aligned} \tag{2.14}$$

$$\tan\varphi = (Q_- - Q_+)/(R_- + R_+). \tag{2.15}$$

To find the corresponding components when ω_r is changed to ω_r', equation (2.13) may be written

$$\begin{aligned} P(t) &= \cos[\omega_r't - \psi_r' - (\omega_r' - \omega_r)t + (\psi_r' - \psi_r) - \varphi]\, \overline{P}(t) \\ &= \cos(\omega_r't - \psi_r' - \varphi')\, \overline{P}(t), \end{aligned} \tag{2.16}$$

where $\varphi' = \varphi'(t)$ is given by:

$$\begin{aligned} \varphi' &= \varphi + (\omega_r' - \omega_r)t - (\psi_r' - \psi_r), \\ &= \varphi + \omega_y t - \psi_y\,. \end{aligned} \tag{2.17}$$

Thus, when the reference frequency is changed by ω_y and its phase by ψ_y, the corresponding in-phase and quadrature components become:

$$\begin{aligned} R_-' + R_+' &= \cos(\varphi + \omega_y t - \psi_y)\, \overline{P}(t), \qquad \text{and} \\ Q_-' - Q_+' &= \sin(\varphi + \omega_y t - \psi_y)\, \overline{P}(t) \end{aligned} \tag{2.18}$$

To summarize, when the in-phase and quadrature components have been determined for any reference frequency ω_r from (2.10) and (2.11), and the envelope $\overline{P}$ together with the function φ from (2.12) and (2.14), the in-phase and quadrature components for another reference frequency ω_r' can readily be determined with the aid of (2.18). In the particular case where the amplitude characteristic has even and the phase characteristic odd symmetry with respect to the midband frequency, the quadrature component disappears with respect to the midband frequency, so that $\varphi = 0$ and (2.18) simplifies to

$$\begin{aligned} R_-' + R_+' &= \cos(\omega_y t - \psi_y)\, \overline{P}(t), \qquad \text{and} \\ Q_-' - Q_+' &= \sin(\omega_y t - \psi_y)\, \overline{P}(t). \end{aligned} \tag{2.19}$$

The above relations (2.18) and (2.19) facilitate comparison of transmission performance as the reference or carrier frequency is changed, for example the comparison of double with vestigial sideband transmission, as illustrated in section 14.

3. IDEALIZED CHARACTERISTICS WITH SHARP CUTOFF

In pulse transmission theory, particularly in dealing with transmission capacity of idealized transmission systems, an ideal low-pass transmission frequency characteristic is ordinarily assumed, with constant amplitude and delay in the transmission band together with an abrupt cutoff at the top frequency and zero amplitude beyond, as shown in Fig. 7. As is evident from Fig. 1, this type of characteristic is an abstraction which cannot be physically realized since it will have phase distortion and infinite transmission delay. It can, however, be approached with sufficiently elaborate phase equalization.

For the above type of characteristic, $A(\omega) = 1$ between $\omega = 0$ and ω_1, while $\psi(\omega) = \omega\tau_d$, where τ_d is the transmission delay. With these values in (2.07):

$$P(t) = \frac{\delta\omega_1}{\pi} \frac{\sin \omega_1 t_0}{\omega_1 t_0}, \tag{3.01}$$

where $t_0 = t - \tau_d$ is the time referred to the peak amplitude of the received pulse.

The resultant pulse transmission characteristic is shown in Fig. 7, with the factor $\delta\omega_1/\pi$ omitted. The peak amplitude is attained after an infinite time, since the above type of characteristic can be realized only with $\tau_d \rightarrow \infty$. The impulse characteristic is zero when $\omega_1 t_0 = \pm n\pi$, or $t_0 = \pm\tau_1, \pm 2\tau_1, \cdots \pm n\tau_1$ where

$$\tau_1 = \frac{1}{2f_1}. \tag{3.02}$$

Impulses can thus be transmitted at the latter intervals without

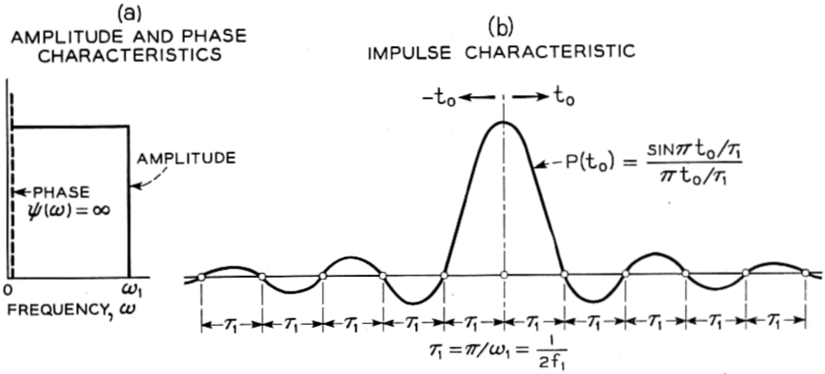


Fig. 7 — Idealized low-pass characteristic with sharp complete cut-off.

mutual interference between the peaks of the received pulses. This is a basic theorem underlying the determination of the transmission capacity of idealized systems.[3]

For an idealized bandpass characteristic between ω_0 and ω_1, it follows from (2.09) with $\Psi(u) = u\tau_d$ and $\Psi(-u) = -u\tau_d$ that the impulse characteristic with respect to the midband frequency $\omega_r = \omega_m$ is

$$P(t) = 2 \cos[\omega_m t_0 - \psi_0]\, \overline{P}(t), \tag{3.03}$$

where $\overline{P}(t)$ is given by (3.01) and $\psi_0 = \Psi_m - \omega_m \tau_d$ is the phase intercept at zero frequency. For the transmission characteristic to be ideal in the sense that the peak pulse amplitude occurs when $t_0 = t - \tau_d = 0$, it is necessary that $\psi_0 = \pm n\pi$, where n is an integer. This is not necessary if the bandwidth is small in relation to the midband frequency. There will then be a large number of cycles of the modulating frequency ω_m within the envelope $\overline{P}(t)$, and the latter can be recovered by envelope detection regardless of the phase of the modulating frequency.

With $\psi_0 = \pm n\pi$,

$$P(t) = \frac{2\omega_s \delta}{\pi} \cos \omega_m t_0 \frac{\sin \omega_s t_0}{\omega_s t_0}, \tag{3.04}$$

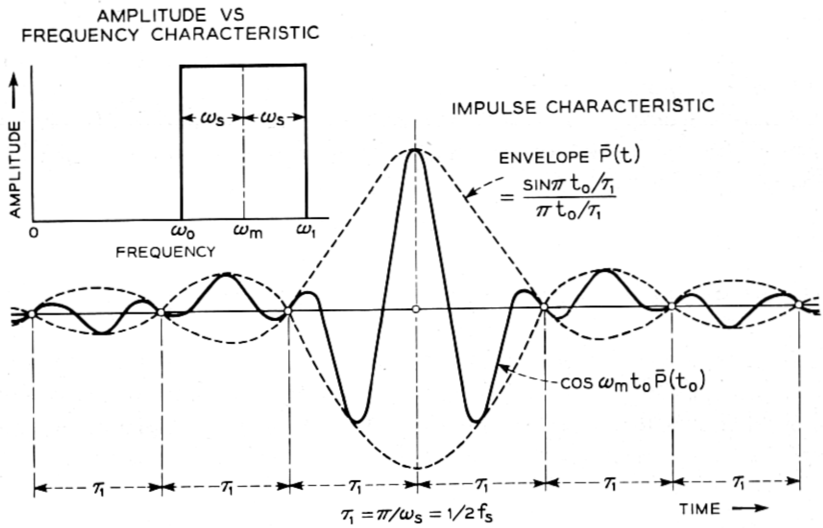
$$= \frac{\omega_1 \delta}{\pi} \frac{\sin \omega_1 t_0}{\omega_1 t_0} - \frac{\omega_0 \delta}{\pi} \frac{\sin \omega_0 t_0}{\omega_0 t_0}, \tag{3.05}$$

where $\omega_m = (\omega_0 + \omega_1)/2$ and $\omega_s = (\omega_1 - \omega_0)/2$.

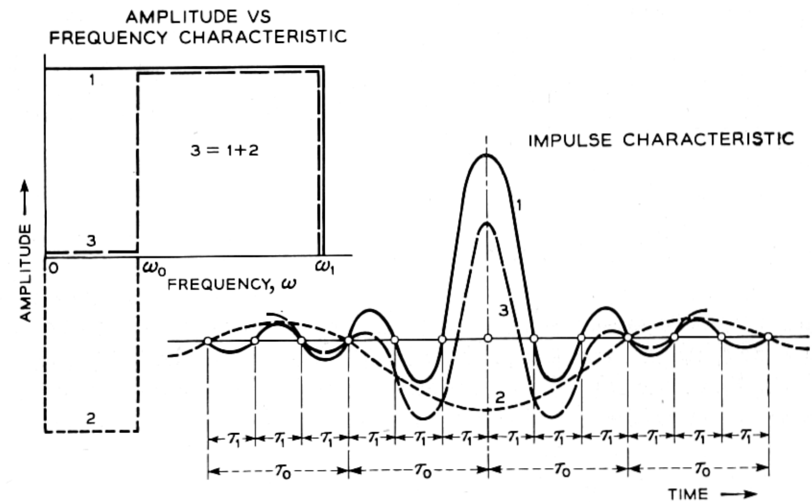
The shape of the impulse characteristic as given by (3.04) is illustrated in the upper half of Fig. 8. Alternately the impulse characteristic may be regarded as made up of two components in accordance with (3.05). The first component corresponds to a low-pass characteristic of bandwidth ω_1, the second component to a negative low-pass characteristic of bandwidth ω_0, as indicated in the lower part of the Fig. 8.

The factor $\sin \omega_s t_0/\omega_s t_0$ in (3.04) is zero at the same intervals as for a low-pass characteristic of bandwidth ω_s, as shown in Fig. 8, so that pulses may be transmitted at the same rate without mutual interference between pulse peaks. The bandwidth in the present case, however, is $2\omega_s = \omega_1 - \omega_0$, so that for the same bandwidth the pulse transmission rate is half as great as for a low-pass characteristic.

An exception to this is the particular case when $\omega_1 = 2\omega_0$, so that the total bandwidth is ω_0. The factor $\sin \omega_0 t_0/\omega_0 t_0$ in (3.05) is then zero at intervals $\tau_0 = 1/2f_0$, while the factor $\sin \omega_1 t_0/\omega_1 t_0$ is zero at intervals $1/2f_1 = 1/4f_0$, as shown in Fig. 9. Pulses may accordingly in principle be



(a) REPRESENTATION OF IMPULSE CHARACTERISTIC AS ENVELOPE MODULATED BY MIDBAND FREQUENCY



(b) REPRESENTATION OF AMPLITUDE VS FREQUENCY AND IMPULSE CHARACTERISTICS, 3, AS THE SUM OF A POSITIVE LOW-PASS CHARACTERISTIC, 1, AND A NEGATIVE LOW-PASS CHARACTERISTIC, 2

Fig. 8 — Idealized band-pass characteristics and corresponding impulse transmission characteristics.

transmitted without mutual interference at the same rate as for a low-pass characteristic of bandwidth ω_0, or at the same rate as with single sideband transmission over a band-pass system of bandwidth ω_0. More generally, pulses can in principle be transmitted without mutual interference between pulse peaks at the same rate as for a low-pass characteristic of bandwidth $\omega_1 - \omega_0 = 2\omega_s$ if ω_0 is a multiple of $\omega_1 - \omega_0$. It should be noted however, that this pulse transmission rate cannot actually be realized since the phase characteristic will have infinite slope, so that the transmission delay will be infinite. In addition, the zero frequency phase intercept ψ_0 must be $\pm n\pi$, a condition which cannot be attained or remain stable in view of the infinite slope of the phase characteristic.

With the envelope given by the factor $\sin \omega_s t_0/\omega_s t_0$ in (3.04), the in-phase and quadrature components for any reference frequency can be determined with the aid of (2.19). If the lower band-edge is selected, i.e. $\omega_r = \omega_0$, then $\omega_y = \omega_s$. With a linear phase characteristic $\psi_y = \omega\tau_d$, so that in (2.19) $\omega_y t - \psi_y = \omega_s t_0$. The in-phase and quadrature components are accordingly obtained by multiplying the envelope by $\cos \omega_s t_0$ and $\sin \omega_s t_0$, respectively.

As an alternate method, the two components can be obtained from

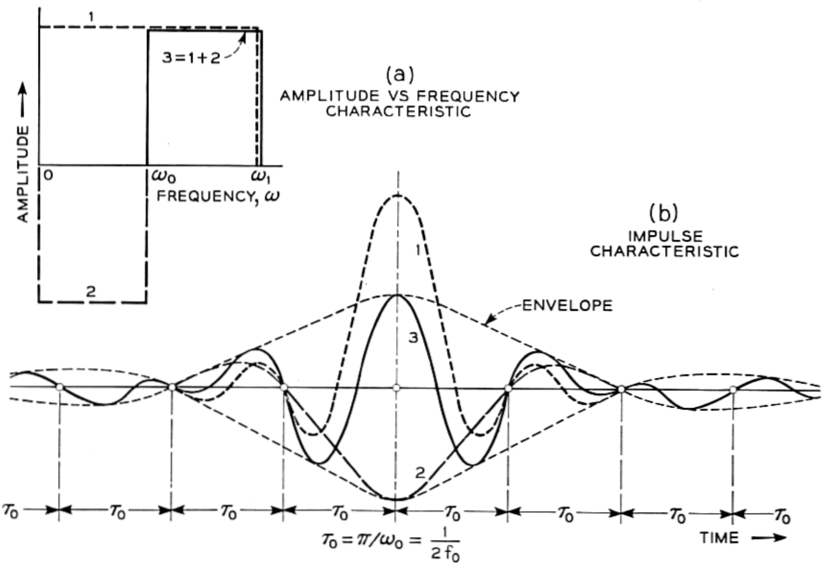


Fig. 9 — Special case of idealized band-pass characteristic in which $\omega_1 = 2\omega_0$ and resultant impulse characteristic is zero at intervals $\tau_0 = \dfrac{1}{2f_0}$.

(2.09), which with $R_- = 0$, $Q_- = 0$ becomes:

$$P(t) = \cos \omega_0 t_0 R_+(t) + \sin \omega_0 t_0 Q_+(t), \tag{3.06}$$

with

$$\begin{aligned} R_+ &= \frac{\delta}{\pi} \int_0^{\omega_b} \cos ut_0 \, du, \\ &= \frac{\delta \omega_b}{\pi} \frac{\sin \omega_b t_0}{\omega_b t_0} = \frac{2\delta\omega_s}{\pi} \cos \omega_s t_0 \frac{\sin \omega_s t_0}{\omega_s t_0}, \end{aligned} \tag{3.07}$$

and

$$\begin{aligned} Q_+ &= \frac{\delta}{\pi} \int_0^{\omega_b} \sin ut_0 \, du, \\ &= \frac{\delta \omega_b}{\pi} \frac{1 - \cos \omega_b t_0}{\omega_b t_0} = \frac{2\delta\omega_s}{\pi} \sin \omega_s t_0 \frac{\sin \omega_s t_0}{\omega_s t_0}, \end{aligned} \tag{3.08}$$

where $\omega_b = 2\omega_s$ is the bandwidth. It will be noticed that R_+ and Q_+ are obtained by multiplying the envelope by $\cos \omega_s t_0$ and $\sin \omega_s t_0$ in accordance with (2.19).

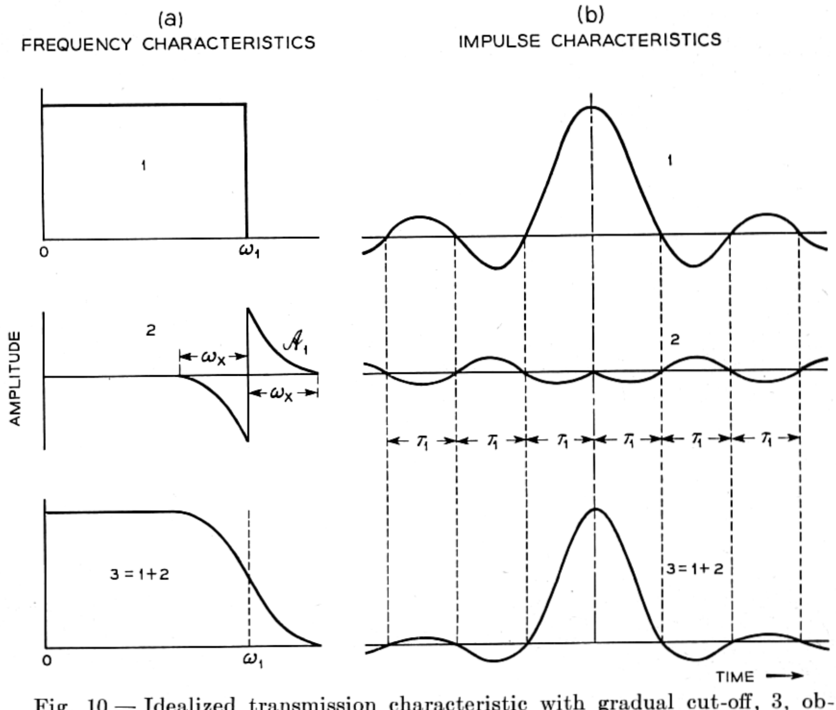


Fig. 10 — Idealized transmission characteristic with gradual cut-off, 3, obtained by superposition of characteristic with sharp cut-off, 1, and characteristic, 2, with odd symmetry about ω_1. Linear phase shift assumed.

4. IDEALIZED CHARACTERISTICS WITH GRADUAL CUTOFF

The idealized transmission characteristics discussed above are of principal interest in that they indicate the physical limitations on pulse transmission rates for a given bandwidth. Even if these impulse characteristics could be realized without undue difficulties from the standpoint of phase equalization, they would be impracticable in most applications. Their oscillatory nature would entail the use of discrete pulse positions and precise synchronized sampling at fixed intervals, and would preclude certain methods of pulse modulation and detection.

The non-linearity in the phase characteristic as well as the oscillations in the impulse characteristic can be reduced with a gradual rather than a sharp cut-off, as illustrated in Fig. 10. It is assumed that an ideal characteristic with a sharp cutoff is supplemented by an amplitude characteristic $\mathfrak{A}_1$ which has odd symmetry about the cutoff frequency ω_1, i.e., $\mathfrak{A}_1(-u) = -\mathfrak{A}_1(u)$.

If the latter component alone is considered, and a linear phase characteristic assumed, it follows from (2.09) with $\omega_1 = \omega_r$ that the effect of this component on the pulse transmission characteristic is given by

$$P_1(t) = -Q_1 \sin \omega_1 t_0 , \tag{4.01}$$

where $t_0 = t - \tau_d$ and

$$Q_1 = \frac{2\delta}{\pi} \int_0^{\omega_x} \mathfrak{A}_1(u) \sin ut_0 \, du. \tag{4.02}$$

The function $P_1(t)$ will be zero at the same points as the original pulse transmission characteristic with a sharp cut-off at ω_1 and under certain conditions also at other points. It will modify the original impulse characteristic by reducing the oscillatory tail, as illustrated in Fig. 10, but the zero points remain unchanged.[3]

With the above modification, the resultant impulse characteristic obtained by superposition of (3.01) and (4.02) becomes

$$\begin{aligned} P(t) &= \frac{\delta}{\pi} \sin \omega_1 t_0 \left(\frac{1}{t_0} - 2 \int_0^{\omega_x} \mathfrak{A}_1(u) \sin ut_0 \, du \right), \\ &= \frac{\delta}{\pi} \sin \omega_1 t_0 F(t), \end{aligned} \tag{4.03}$$

where

$$F(t) = \left[\frac{1}{t_0} - 2 \int_0^{\omega_x} \mathfrak{A}_1(u) \sin ut_0 \, du \right]. \tag{4.04}$$

In the following the expression for $F(t)$ is given for the case when the band-edge is modified by a supplementary characteristic of the form

$$\begin{aligned} \mathfrak{A}_1(u) &= \tfrac{1}{2}(1 - \sin \pi u/2\omega_x) \qquad u < \omega_x \,, \\ &= 0 \qquad\qquad\qquad\qquad\quad u > \omega_x \,. \end{aligned} \tag{4.05}$$

This form of $\mathfrak{A}_1$ represents a close approximation to actual modifications of band-edges by a gradual cutoff and also results in rather simple expressions for the modified impulse characteristic

With (4.05) in (4.04),

$$\begin{aligned} F(t) &= \frac{1}{t_0} - \int_0^{\omega_x} (1 - \sin \pi u/2\omega_x) \sin ut_0 \, du, \\ &= \frac{1}{t_0} - \omega_x \left[\frac{1 - \cos \omega_x t_0}{\omega_x t_0} + \frac{\cos \omega_x t_0}{\pi + 2\omega_x t_0} - \frac{\cos \omega_x t_0}{\pi - 2\omega_x t_0} \right], \\ &= \omega_x \cos \omega_x t_0 \left[\frac{1}{\omega_x t_0} + \frac{1}{\pi - 2\omega_x t_0} - \frac{1}{\pi + 2\omega_x t_0} \right], \\ &= \frac{1}{t_0} \frac{\cos \omega_x t_0}{1 - (2\omega_x t_0/\pi)^2}. \end{aligned} \tag{4.06}$$

The impulse characteristic obtained from (4.03) is

$$P(t) = \frac{\delta\omega_1}{\pi} \frac{\sin \omega_1 t_0}{\omega_1 t_0} \frac{\cos \omega_x t_0}{1 - (2\omega_x t_0/\pi)^2} \tag{4.07}$$

For the particular case shown in Fig. 11 the value of ω_x is taken to be $\omega_1/2$.

For a symmetrical bandpass characteristic, as shown in Fig. 12,

$$P(t) = 2 \cos (\omega_m t_0 - \psi_0) \, \overline{P}(t). \tag{4.08}$$

$\overline{P}(t)$ is obtained by replacing ω_1 by ω_s in (4.07), and ψ_0 is the phase intercept at zero frequency as in connection with (3.03). This gives

$$\overline{P}(t) = \frac{\delta\omega_s}{\pi} \frac{\sin \omega_s t_0}{\omega_s t_0} \frac{\cos \omega_x t_0}{1 - (2\omega_x t_0/\pi)^2}. \tag{4.09}$$

For the particular case shown in Fig. 12, the value of ω_x is taken to be $\omega_s/2$.

The in-phase and quadrature components with respect to any frequency are obtained from (2.19) with $\psi_y = \omega\tau_d$ and are shown in Fig. 12 for the particular case in which the reference frequency is displaced from the midband frequency by $\omega_y = \omega_s$.

5. IDEALIZED CHARACTERISTICS WITH NATURAL LINEAR PHASE SHIFT

With the type of amplitude characteristics discussed above it is necessary to employ phase equalization to obtain a linear phase characteristic. Furthermore, oscillations of appreciable amplitude remain in the impulse characteristic. A virtually linear phase characteristic together with a reduction of these oscillations can be attained by a further extension of the gradual cut-off in Fig. 10, such that $\omega_x = \omega_1$. An amplitude characteristic of this type, together with the corresponding impulse

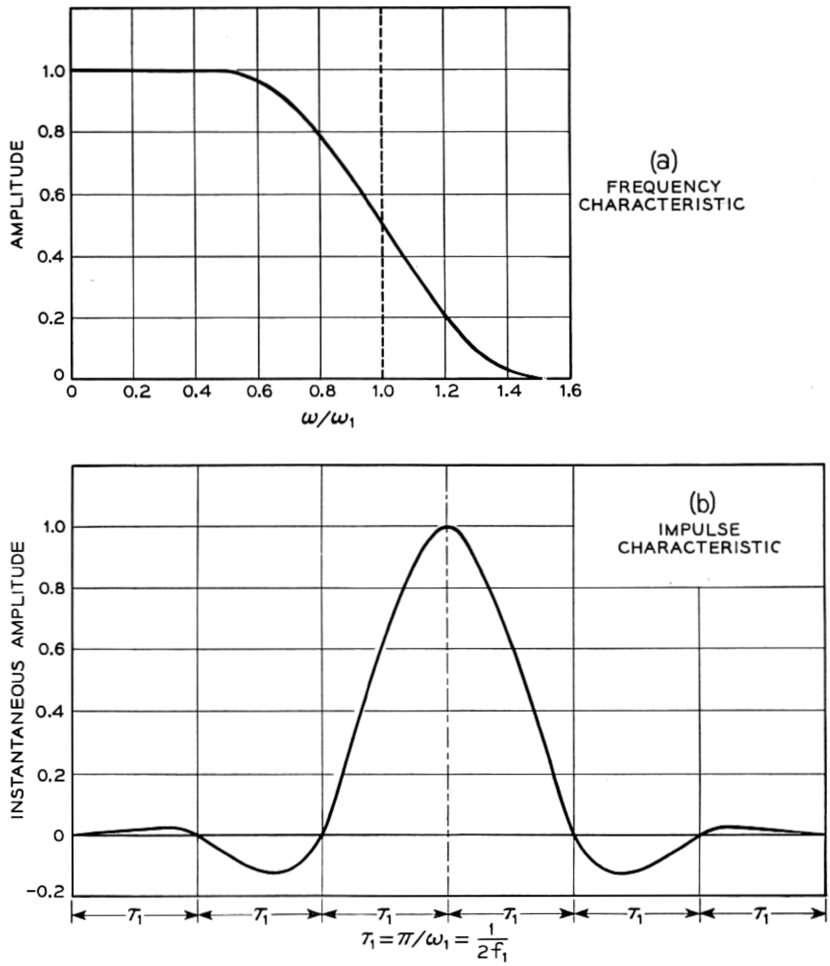


Fig. 11 — Low-pass characteristic with gradual cut-off and associated impulse characteristic. Linear phase characteristic assumed.

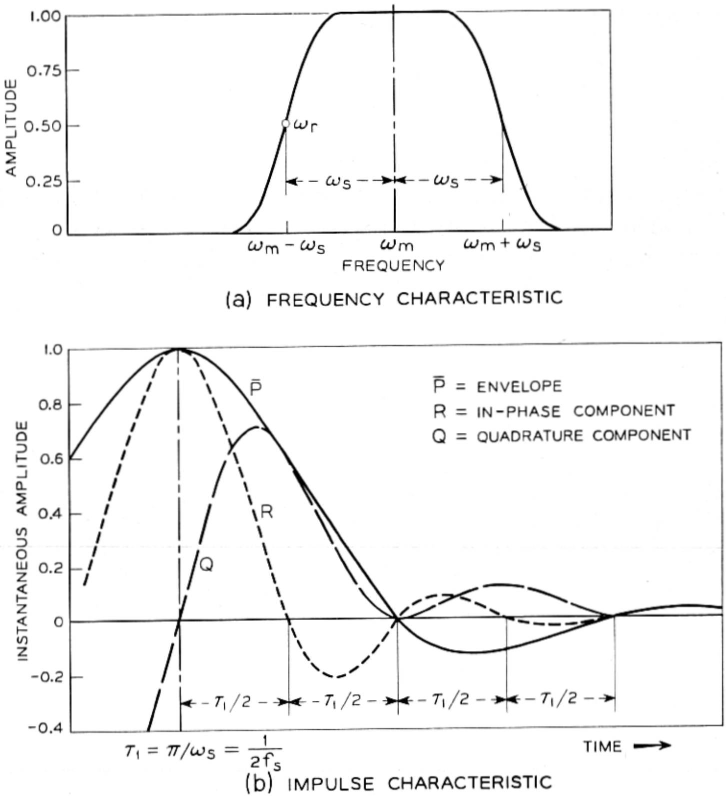


Fig. 12 — Symmetrical band-pass characteristic with gradual cut-off and associated impulse characteristic. In-phase and quadrature components shown with respect to $\omega_r = \omega_m - \omega_s$.

characteristic is shown in Fig. 13. The supplementary amplitude characteristic and the impulse characteristic are obtained by making $\omega_x = \omega_1$ in (4.05) and 4.07).

The resultant amplitude characteristic between $\omega = 0$ and $\omega = 2\omega_1$ in this case becomes

$$A(\omega) = \frac{1}{2}\left[1 + \cos\frac{\pi\omega}{2\omega_1}\right] = \cos^2\frac{\pi\omega}{4\omega_1}, \tag{5.01}$$

and the impulse characteristic:

$$P(t) = \frac{\mathcal{E}\omega_1}{\pi}\frac{\sin 2\omega_1 t_0}{2\omega_1 t_0[1 - (2\omega_1 t_0/\pi)^2]}, \tag{5.02}$$

where ω_1 is the bandwidth to the half-amplitude point on the transmission frequency characteristic and $2\omega_1$ the bandwidth to the point of zero amplitude.

In Fig. 13 is also shown the amplitude characteristic given by (1.08), which will have a linear phase characteristic in the transmission band, i.e. from $\omega = 0$ to $2\omega_1$. Because of the close approximation of (5.01) to the proper type of amplitude characteristic as regards phase linearity, the phase characteristic associated with (5.01) may for practical purposes be regarded as linear.

For a symmetrical band-pass characteristic as shown in Fig. 14, the impulse characteristic is given by (4.08) and the envelope by (4.09) with $\omega_x = \omega_s$, or

$$\overline{P}(t) = \frac{i\omega_s}{\pi} \frac{\sin 2\omega_s t_0}{2\omega_s t_0[1 - (2\omega_s t_0/\pi)^2]}. \tag{5.03}$$

The in-phase and quadrature components shown in Fig. 14 with

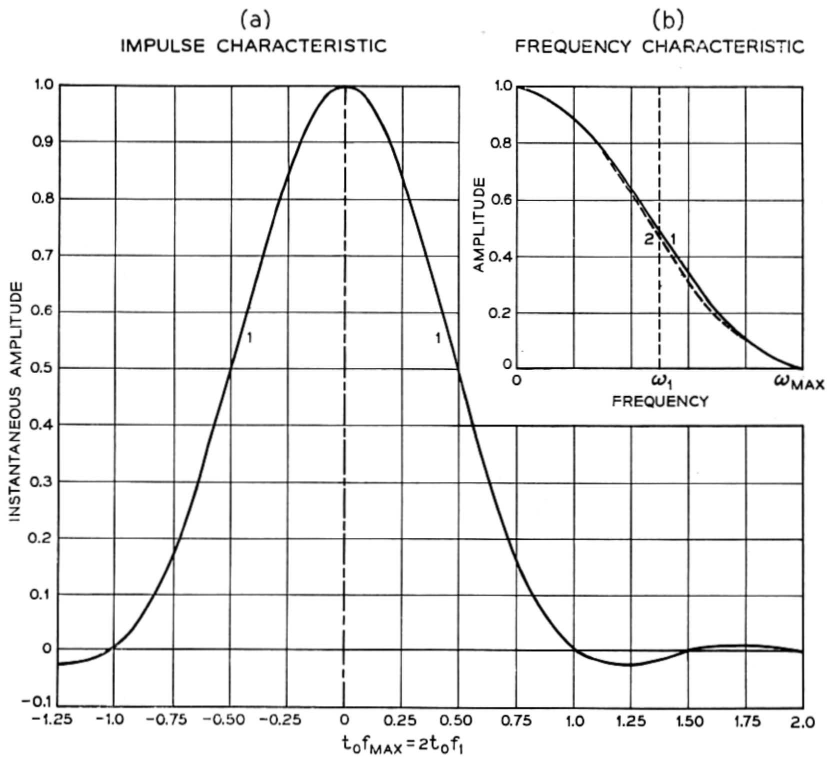


Fig. 13 — Low-pass transmission frequency characteristic, 1, and associated impulse characteristic. Frequency characteristic, 2, is same as shown by solid lines in Fig. 3 and has a linear phase characteristic between $\omega = 0$ and ω_{max}.

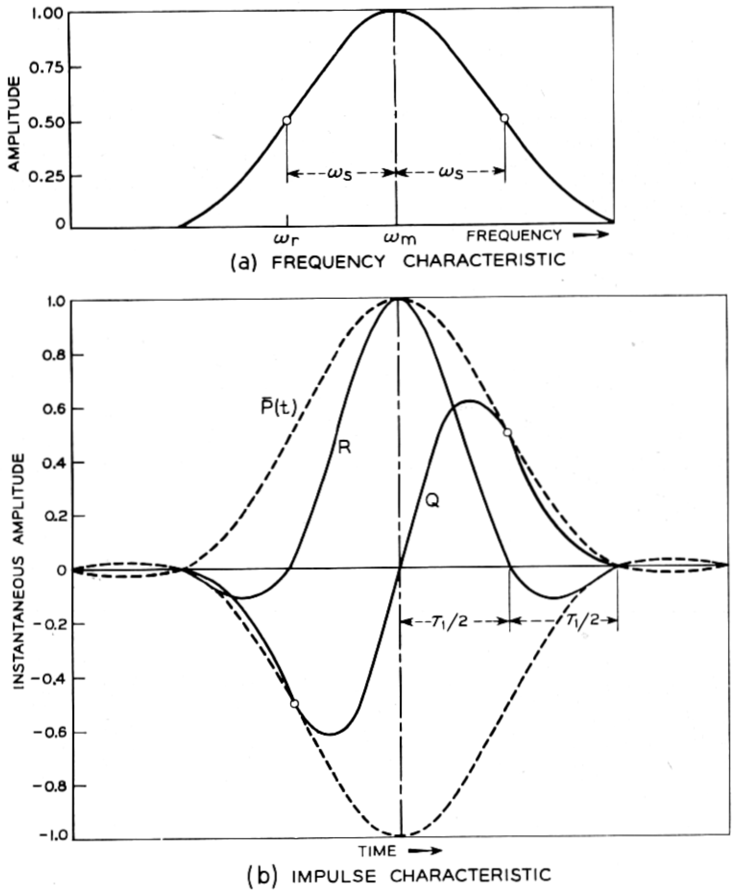


Fig. 14 — Symmetrical band-pass characteristic with linear phase shift and corresponding impulse characteristic. In-phase and quadrature components shown with respect to $\omega_r = \omega_m - \omega_s$.

respect to a reference frequency at the midpoint of the band-edge are obtained from (2.19) with $\omega_y = \omega_s$. This gives $\overline{P}(t)$ cos $\omega_s t_0$ for the in-phase and $\overline{P}$ sin $\omega_s t_0$ for the quadrature component.

In Fig. 15 is shown a special case of a band-pass characteristic, which corresponds to that illustrated in Fig. 9 with $\omega_1 = 2\omega_0$, shown for comparison by dashed lines in Fig. 15. In this particular case $\omega_m = 3\ \omega_0/2$ and $\omega_x = \omega_s = \omega_0/2$. With $\psi_0 = n\pi$, equation (4.08) in conjunction with (4.09) gives

$$P(t) = \frac{\delta\omega_0}{2\pi} \cos\ (\omega_0 t_0/2)\ \frac{\sin 2\omega_0 t_0 - \sin \omega_0 t_0}{\omega_0 t[1 - (\omega_0 t_0/\pi)^2]}\,. \qquad (5.04)$$

This expression is zero when $\sin 2\omega_0 t_0 - \sin \omega_0 t_0 = 0$, and also when $\cos \omega_0 t_0/2 = 0$. Zero points in the impulse characteristic will occur at uniform intervals $\tau_0 = \pi/\omega_0 = 1/2f_0$. Pulses can accordingly be transmitted at these intervals without mutual interference, or at the same rate as for a low-pass characteristic with the bandwidth to the half-amplitude point equal to ω_0. This is the same pulse transmission rate as is possible in principle with an ideal band-pass characteristic as shown

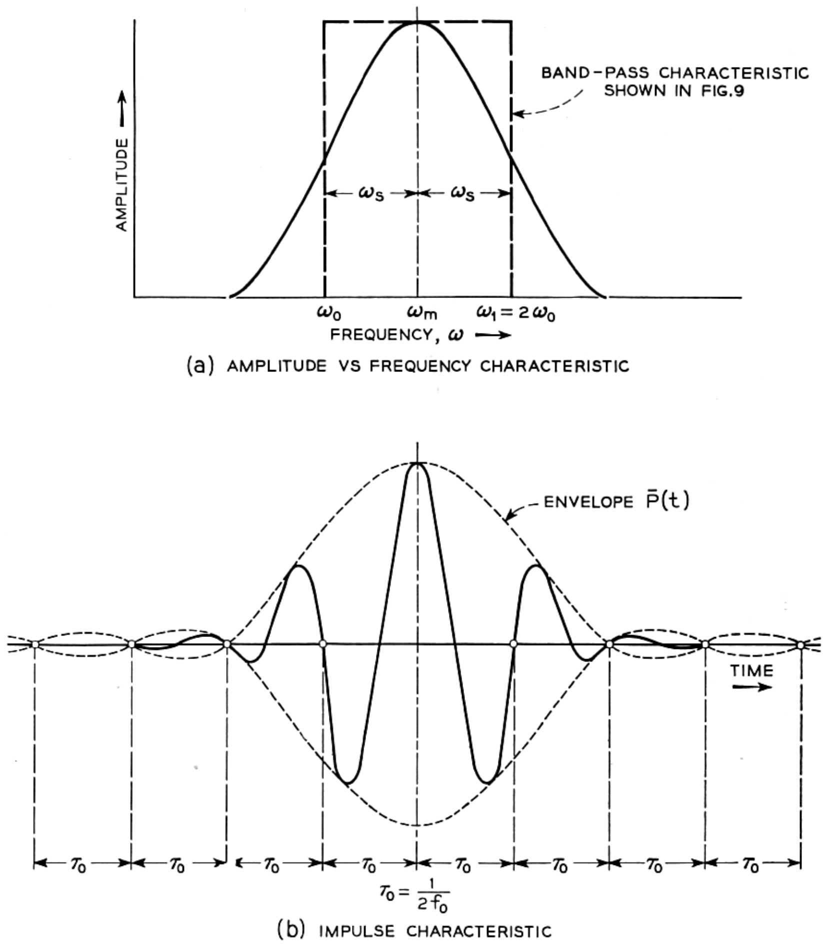


Fig. 15 — Particular case of band-pass characteristic with gradual cut-off in which impulse characteristic is zero at intervals $\tau_0 = \dfrac{1}{2f_0}$.

by the dashed lines in Fig. 15. With a gradual cut-off, however, the phase characteristic will be nearly linear and have a finite slope, so that the above pulse transmission rate can be realized provided $\psi_0 = \pm n\pi$. The same pulse transmission rate can also be attained with vestigial side-band transmission, discussed in section 14.

Another particular case of interest is that shown in Fig. 16, in which $\omega_m = 2\omega_s$. In this case (4.08) becomes with $\psi_0 = \pm n\pi$ and with $\bar{P}(t)$ as

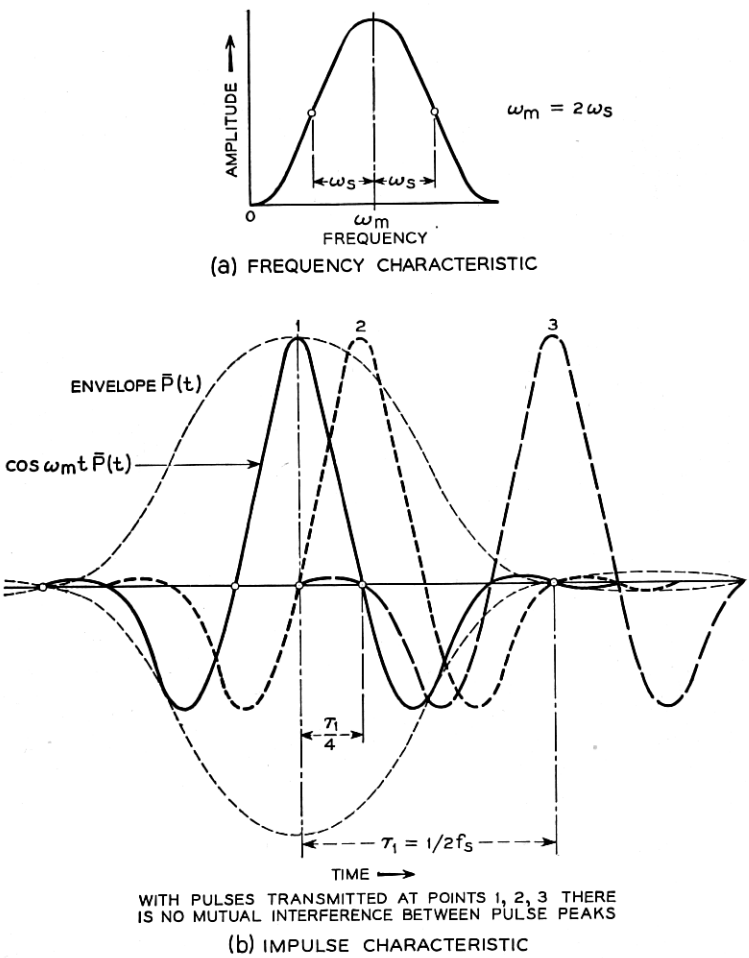


Fig. 16 — Particular case of symmetrical band-pass characteristic for which $\omega_m = 2\omega_s$.

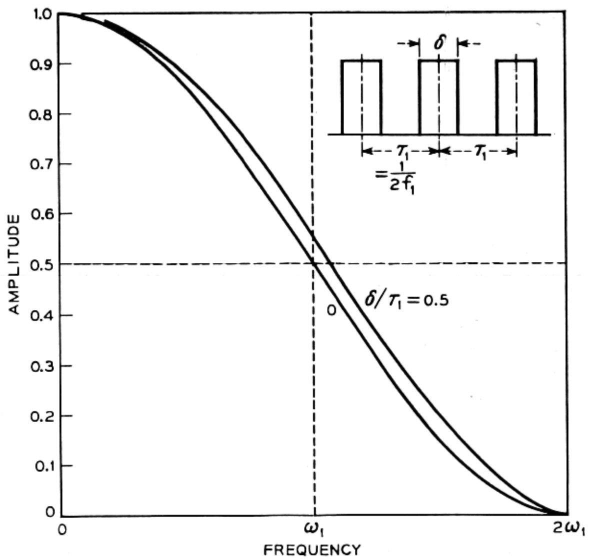


Fig. 17 — Modification of frequency characteristic to obtain same response as for impulses, when pulse duration is prolonged to half the pulse interval.

given by (5.03)

$$P(t) = \frac{\delta\omega_m}{\pi}\cos\omega_m t_0 \frac{\sin\omega_m t_0}{\omega_m t_0[1 - (\omega_m t_0/\pi)^2]}, \qquad (5.05)$$

$$= \frac{\delta\omega_m}{\pi}\frac{\sin 2\omega_m t_0}{2\omega_m t_0[1 - (\omega_m t_0/\pi)^2]}.$$

Pulses can in this case be transmitted without mutual interference between the pulse peaks at the points shown in the above figure. The effective pulse transmission rate is the same as for a low-pass characteristic between $\omega = 0$ and $\omega = 2\omega_m$ with half amplitude at ω_m.*

As mentioned in Section 2, when pulses of finite duration are employed, the same response as for impulses is obtained if the amplitude characteristic is modified by the factor $(\omega\delta/2)/\sin(\omega\delta/2)$. In Fig. 17 is shown the resultant minor modification in the amplitude characteristic (5.01) when the duration of the pulses is equal to half the pulse interval.

The low-pass and band-pass amplitude characteristics considered above can also be regarded as the spectra of pulses applied to a transmission system having a constant amplitude characteristic over the

* W. R. Bennett and C. B. Feldman originally proposed this type of characteristic in an unpublished memorandum, as a means of matching the bandwidth economy of baseband transmission without inclusion of frequencies near zero.

band of the spectra. If the phase characteristic of the system is linear over this band, the received pulses will have the same shape as the impulse characteristics. It should be recognized, however, that there may be appreciable phase distortion within the transmission band or pulse spectrum, if there are amplitude discontinuities beyond the band resulting from a sharp cut-off by filters. Nevertheless, the type of amplitude characteristic or frequency spectrum considered above has decisive advantages from the standpoint of transmission distortion of the pulses, as shown later, since appreciable phase distortion will ordinarily be confined to the edges of the band where the frequency components of the pulse spectrum have low amplitudes.

Another type of amplitude characteristic resembling that shown in Fig. 13 and frequently considered in connection with pulse transmission is a Gaussian characteristic:

$$A(\omega) = e^{-\sigma\omega^2}. \tag{5.06}$$

The corresponding impulse characteristic is

$$P(t) = \frac{\delta}{2(\pi\sigma)^{1/2}} e^{-t_0^2/4\sigma}. \tag{5.07}$$

If it is assumed that the amplitude is reduced to 1 per cent of the peak value after an interval $t_0 = \pi/\omega_1$, corresponding to the first zero point of an ideal impulse characteristic, it is necessary that $t_0^2/4\sigma = 4.6$, or $\sigma = .54/\omega_1^2$. The corresponding amplitude and impulse characteristics are

$$A(\omega) = e^{-0.54(\omega/\omega_1)^2}, \tag{5.08}$$

and

$$P(t) = \frac{\delta\omega_1}{0.83\pi} e^{-0.46(t_0\omega_1)^2}. \tag{5.09}$$

In Fig. 18 a comparison is made of the two frequency characteristics (5.01) and (5.08) considered above, and of the corresponding impulse characteristics (5.02) and (5.09). The comparison shows that for the same pulse transmission rate and with negligible intersymbol interference, a somewhat wider band must be provided with a Gaussian amplitude characteristic. This is a disadvantage, particularly when the band is restricted within prescribed limits by considerations of interference in adjacent transmission bands, as radio pulse systems.

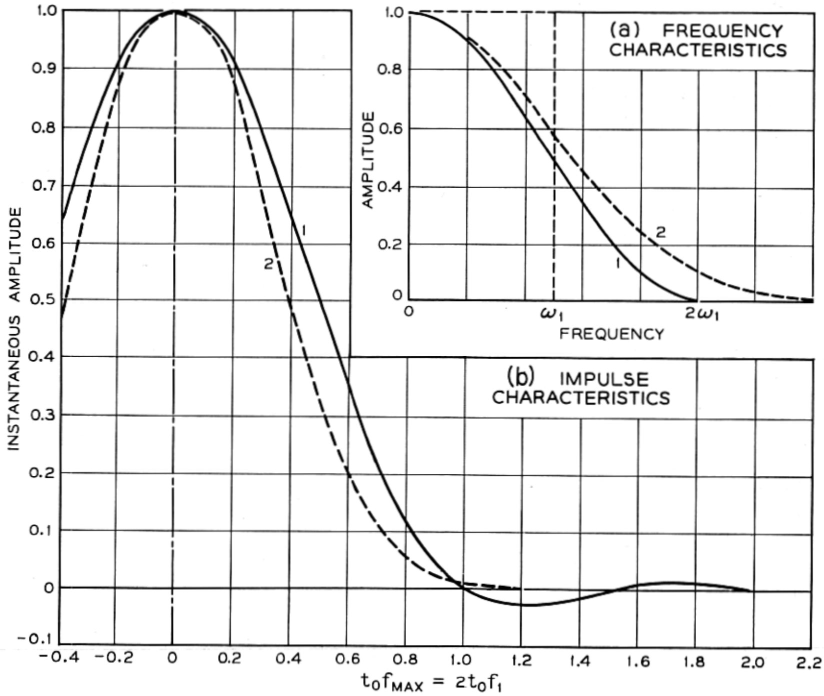


Fig. 18 — Comparison of two representative frequency and impulse transmission characteristics. Frequency characteristic 1: $T(\omega) = \frac{1}{2}[1 + \cos \pi\omega/2\omega_1]$. Frequency characteristic 2: $T(\omega) = \exp - 0.54(\omega/\omega_1)^2$.

Amplitude characteristic 1 of Fig. 18 has certain properties, aside from the linearity of the associated phase characteristic, which makes it preferable to a Gaussian as well as other types of amplitude characteristics for most pulse systems. The corresponding impulse characteristic has zero points at intervals $\tau_1 = 1/2f_1$ with the minimum possible oscillation consistent with this property for a given bandwidth. This permits the use of this impulse characteristic for pulse systems with discrete pulse positions with minimum intersymbol interference and considerable tolerance on synchronization. Since the oscillation in the impulse characteristic is inappreciable, it can also be used for pulse systems without discrete pulse positions and with other methods of detection than synchronized instantaneous sampling. In view of these attributes, an amplitude characteristic of the above type, rather than a constant amplitude characteristic with sharp cut-off, may be regarded as ideal when various physical requirements for practicable pulse systems are taken into consideration.

6. PULSE ECHOES FROM PHASE DISTORTION

For any transmission—frequency characteristic the corresponding impulse characteristics can be determined from the Fourier integral relation (2.01). This, however, may involve the evaluation of complicated integrals, which in general would require numerical integration and would be a rather elaborate procedure. A preferable method of sufficient accuracy in most engineering applications is to employ the theoretical solutions given previously for various ideal transmission characteristics with a linear phase shift as a point of departure or first approximation. A satisfactory second approximation can in many instances be secured by evaluating the transmission distortion resulting from a sinusoidal deviation in the phase characteristic. Furthermore, any type of deviation in the phase characteristic can in principle be represented by a Fourier series in terms of harmonic sinusoidal deviations.

Aside from the circumstance that in many cases a sine deviation in the phase characteristic affords a fairly satisfactory approximation to actual phase distortion it has the advantage in theoretical formulation that it permits determination of the resultant pulse distortion by the method of "paired echoes." In the usual application of this method only small phase deviations are considered resulting in a single pair of pulse or signal echoes of small amplitude, and the method is then particularly simple.[5, 6] When delay distortion is appreciable, however, as is frequently the case in wire circuits, it becomes necessary to consider a large number of pulse or signal echoes of considerable amplitude. Since the amplitudes of the pulse echoes may be obtained from available tables of Bessel Functions, the determination of the echoes is, nevertheless, simple in procedure and the determination of the shape of the distorted pulses or other signals not too elaborate.

A given amplitude characteristic within the transmission band may be associated with various phase characteristics, depending on the shape of the amplitude characteristic outside the transmission band and also on whether or not a minimum phase shift system is involved. It is therefore permissible to consider the effect of various departures from a given phase characteristic independent of the amplitude characteristic within the transmission band.

With a sinusoidal departure from a given phase characteristic $\psi_0(\omega)$ as shown in Fig. 19, the modified phase function becomes

$$\psi(\omega) = \psi_0(\omega) - b \sin \omega\tau. \tag{6.01}$$

With

$$T_0(i\omega) = A_0(\omega)\, e^{-i\psi_0(\omega)}$$

the modified transmission-frequency characteristic becomes

$$T(i\omega) = T_0(i\omega)\, e^{ib \sin \omega\tau}, \tag{6.02}$$

which, inserted in (2.01) gives

$$P(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} T_0(i\omega) e^{ib \sin \omega\tau} e^{i\omega t}\, d\omega. \tag{6.03}$$

The following relation (Jacobi's expansion) in which J_1, $J_2 \cdots$ are Bessel Functions in their usual notation can now be employed[7]

$$\begin{aligned} e^{ib \sin \omega\tau} = J_0(b) &+ J_1(b)[e^{i\omega\tau} - e^{-i\omega\tau}] \\ &+ J_2(b)[e^{2i\omega\tau} + e^{-2i\omega\tau}] \\ &+ J_3(b)[e^{3i\omega\tau} - e^{-3i\omega\tau}] \\ &+ J_4(b)[e^{4i\omega\tau} + e^{-4i\omega\tau}] + \cdots \end{aligned} \tag{6.04}$$

Let $P_0(t)$ designate the shape of the received pulse or other signal for a transmission frequency characteristic $T_0(i\omega)$ obtained from (6.03) with $b = 0$. In view of (6.04) the solution of (6.03) may then be written

$$\begin{aligned} P(t) = J_0(b)P_0(t) &+ J_1(b)[P_0(t + \tau) - P_0(t - \tau)] \\ &+ J_2(b)[P_0(t + 2\tau) + P_0(t - 2\tau)] \\ &+ J_3(b)[P_0(t + 3\tau) - P_0(t - 3\tau)] \\ &+ J_4(b)[P_0(t + 4\tau) + P_0(t - 4\tau)] + \cdots \end{aligned} \tag{6.05}$$

The shape of the received pulse or signal $P(t)$ is thus obtained by superposing an infinite sequence of pulses or signals of shape $P_0(t)$. The peak amplitudes of the pulse or signal echoes and the times at which they occur with respect to $t = 0$ are given in the following table. The reference point $t = 0$ is arbitrarily selected to coincide with the peak of the pulse $P_0(t)$:

Time	-3τ	-2τ	$-\tau$	0	τ	2τ	3τ
Amplitude	$J_3(b)$	$J_2(b)$	$J_1(b)$	$J_0(b)$	$-J_1(b)$	$J_2(b)$	$-J_3(b)$

A sufficient number of echoes must be considered until their peak amplitudes become negligible.

The superposition of echoes to obtain the resultant pulse is illustrated in Fig. 20. Instead of plotting the various echoes and combining them into a resultant pulse or signal as in Fig. 20(c) the equivalent and less laborious method shown in (d) can be employed. With the latter method the pulse P_0 is plotted with reversed time scale and its peak made to coincide with the point for which the amplitude of the resultant pulse P is to be determined. The amplitude of P is determined as indicated in the figure. In the particular case where the original phase characteristic ψ_0 is linear, the pulses $P_0(t)$ will be symmetrical with respect to their peak amplitude, and this assumption will be made in the following applications.

For amplitudes $b \ll 1$, the Bessel Functions become negligible except for J_0 and J_1, which are given by $J_0(b) \cong 1$ and $J_1(b) \cong b/2$, so that (6.05) becomes

$$P(t) = P_0(t) + \frac{b}{2} P_0(t + \tau) - \frac{b}{2} P_0(t - \tau). \tag{6.06}$$

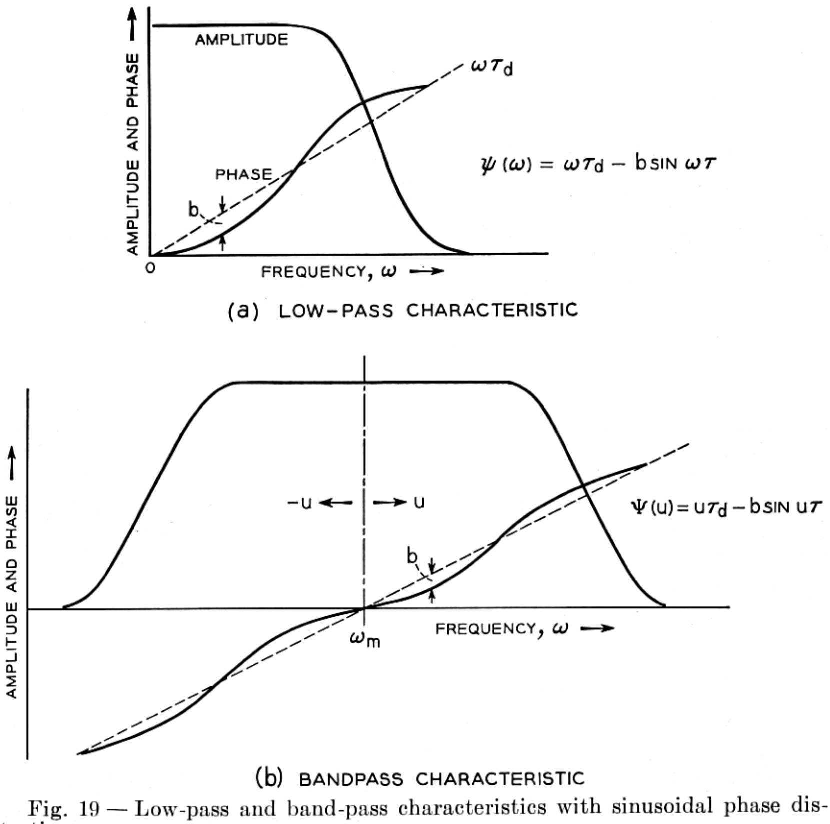


Fig. 19 — Low-pass and band-pass characteristics with sinusoidal phase distortion.

For amplitudes $b > 1$, it is necessary to consider a greater number of echoes, as will be evident from Table I for $b = 1, 2, 5, 10$ and 15 radians.[8]

The preceding equations apply to low-pass characteristics and also to symmetrical bandpass characteristics, as shown in Fig. 19. In the latter case $\mathcal{A}(u) = \mathcal{A}(-u)$ and $\Psi(-u) = -\Psi(u)$ in (2.10) and (2.11) so that $R_+ = R_-$ and $Q_+ = Q_-$ and (2.09) becomes

$$P(t) = \cos(\omega_r t - \psi_r)\, R(t), \tag{6.07}$$

where $R(t) = R_+ + R_-$ and $\omega_r = \omega_m =$ midband frequency. The envelope $R(t)$ is accordingly obtained by replacing $P_0(t)$ in (6.05) by $R_0(t)$, the envelope in the absence of phase distortion.

In Fig. 21 is shown a particular case of a sine deviation in the phase characteristic and the corresponding delay distortion, which approximates that encountered in many instances. For a low-pass system the phase and delay distortion would be as shown for $u > 0$. In this particu-

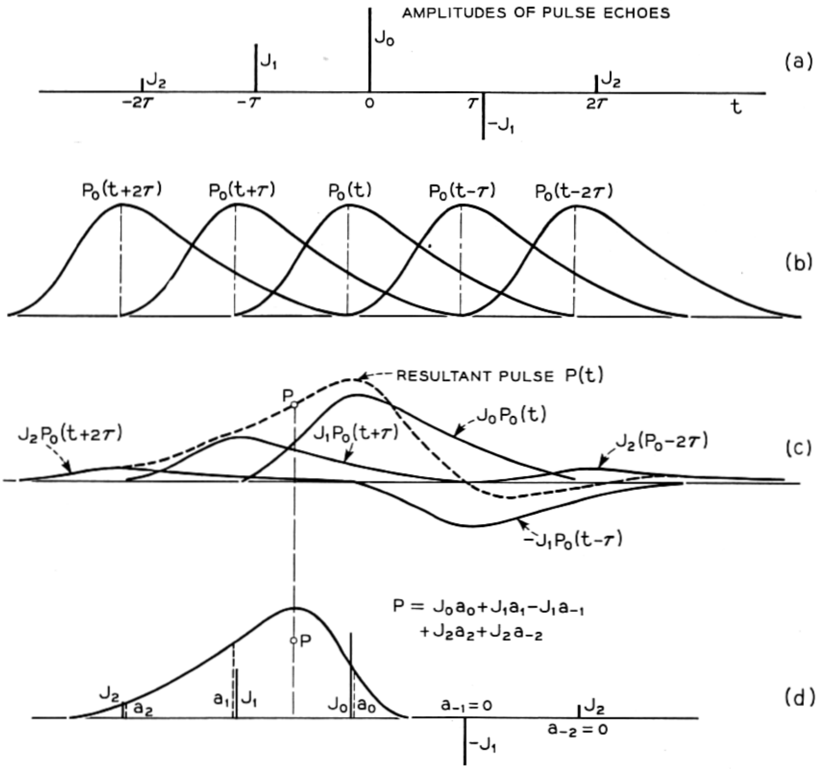


Fig. 20 — Determination of resultant pulse by superposition of pulse echoes.

TABLE I — AMPLITUDES OF ECHOES, $J_n(b)$

	$b = 1$	2	5	10	15
$n = 0$	0.7652	0.2239	−0.1776	−0.2459	−0.0142
1	0.4401	0.5767	−0.3276	0.0434	0.2051
2	0.1149	0.3528	0.0466	0.2546	0.0416
3	0.0196	0.1289	0.3648	0.0584	−0.1940
4		0.0340	0.3912	−0.2196	−0.1192
5			0.2611	−0.2341	0.1305
6			0.1310	−0.0145	0.2061
7			0.0534	0.2167	0.0345
8			0.0184	0.3179	−0.1740
9			0.0055	0.2919	−0.2200
10				0.2075	−0.0901
11				0.1231	0.1000
12				0.0634	0.2367
13				0.0290	0.2787
14				0.0120	0.2464
15				0.0045	0.1813
16					0.1162
17					0.0665
18					0.0346
19					0.0166
20					0.0073

lar case the maximum amplitude b is at the maximum frequency $\omega_{max} = 2\omega_1$, so that $\sin \omega\tau = 1$, or $\omega\tau = \pi/2$, for $\omega = 2\omega_1$. Hence the interval between pulse echoes is $\tau = \pi/4\omega_1 = 1/8f_1$. The interval τ is accordingly ¼ the interval $\tau_1 = 1/2f_1$ required for the pulse $P_0(t)$ to reach zero amplitude in the absence of phase distortion.

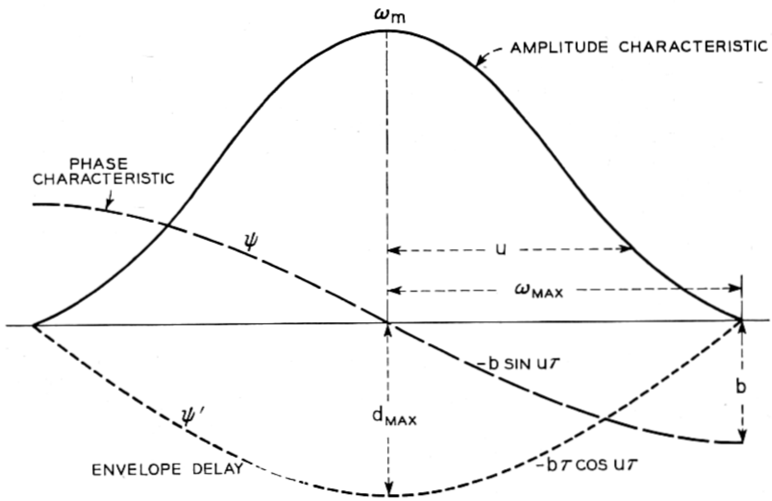


Fig. 21 — Particular case of sinusoidal phase deviation.

For the particular case illustrated in the above figure, the delay distortion is given by

$$d\psi/d\omega = -b\tau \cos \omega\tau.$$

When $\omega = 0$

$$d\psi/d\omega = -b\tau = -d_{\max} . \tag{6.08}$$

When $\omega\tau = \tau\omega_{\max} = \pi/2$

$$d\psi/d\omega = 0.$$

Hence

$$d\psi/d\omega = -d_{\max} \cos (\omega\pi/2\omega_{\max}). \tag{6.09}$$

With $\tau = \pi/2\omega_{\max}$ and $b\tau = d_{\max}$, the following relation is obtained

$$b = \frac{2}{\pi} \omega_{\max} d_{\max} = 4f_{\max} d_{\max} . \tag{6.10}$$

In Fig. 22 are shown the positions of the pulse echoes for the above case on a numerical scale $t \cdot f_{\max}$, together with their amplitudes for $b = 5$ radians. On this scale the interval between pulse echoes $\tau = 1/4f_{\max}$ is $\frac{1}{4}$. In the same figure is shown an assumed pulse shape in the absence of phase distortion, which is the same as shown in Fig. 13, except that the small tail has been neglected. The peak of the pulse is taken at $tf_{\max} = -0.75$, and the amplitude of the resultant pulse at the

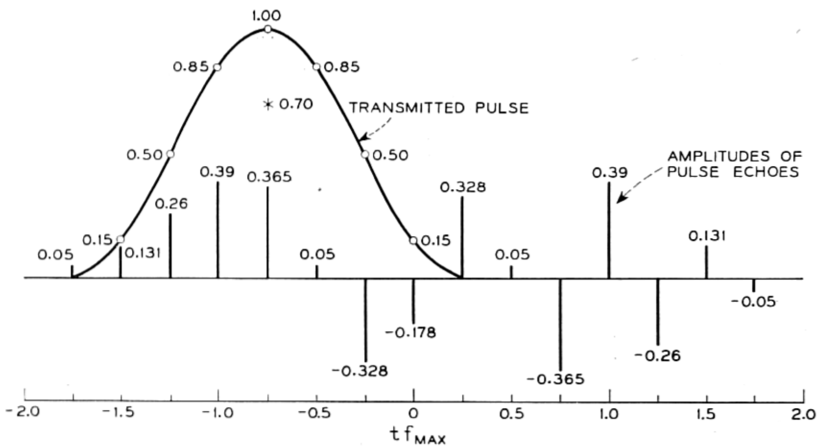


Fig. 22 — Illustrative example of calculation of impulse characteristic shown in Fig. 23, by method illustrated in Fig. 20(d).

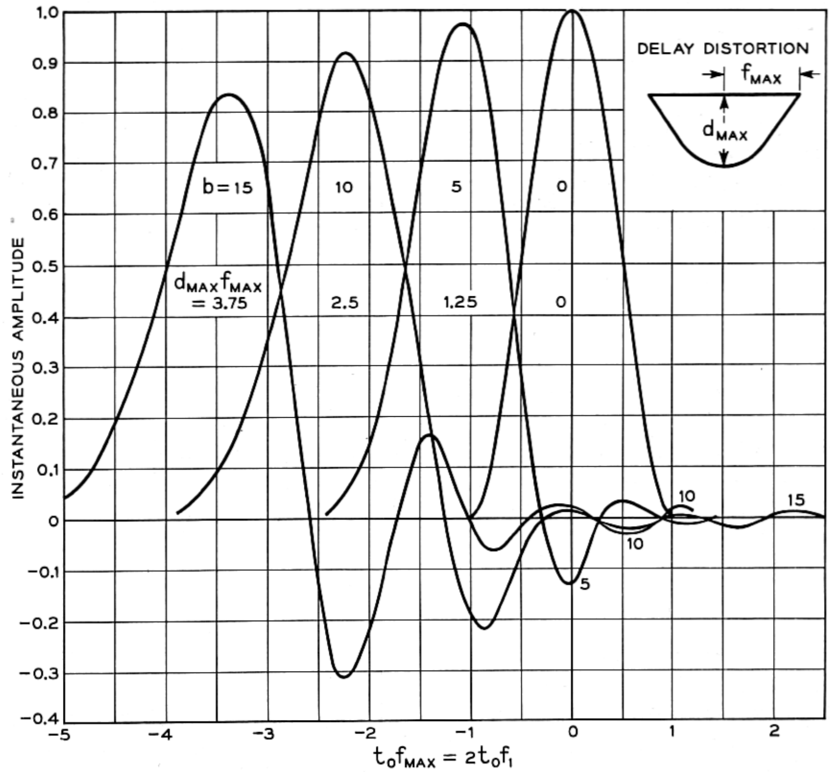


Fig. 23 — Impulse transmission characteristics with cosine variation in delay.

corresponding point obtained by the method illustrated in *d* of Fig. 20 is

$$P = 1 \cdot 0.365 + 0.85\ (0.39 + 0.05) + 0.5\ (0.26 - 0.328) + 0.15\ (0.131 - 0.178) = 0.70.$$

In Fig. 23 are shown the resultant pulses obtained by the above method for various values of b and the corresponding values of $d_{max}f_{max}$. Since the interval between pulse echoes is small in relation to the duration of the pulse $P_0(t)$, as seen from Fig. 22, the individual pulse echoes cannot be discerned in the resultant pulses shown in Fig. 23. It will be noticed that as b increases, the pulses are received with decreasing transmission delay, which is due to the choice of reference delay in the delay distortion curve. That is, as d_{max} or b is increased, the delay becomes increasingly negative with respect to $d_{max} = 0$ used for reference. The curves apply to a band-pass system as indicated in the figure, and also to a low-pass system having the delay distortion shown above the midband frequency of the band-pass system.

An improved approximation to phase distortion is sometimes obtained by considering two sine deviations in the phase characteristic.

If the phase characteristic is given by

$$\psi(\omega) = \psi_0(\omega) - b' \sin \omega\tau - b'' \sin \omega\tau', \tag{6.11}$$

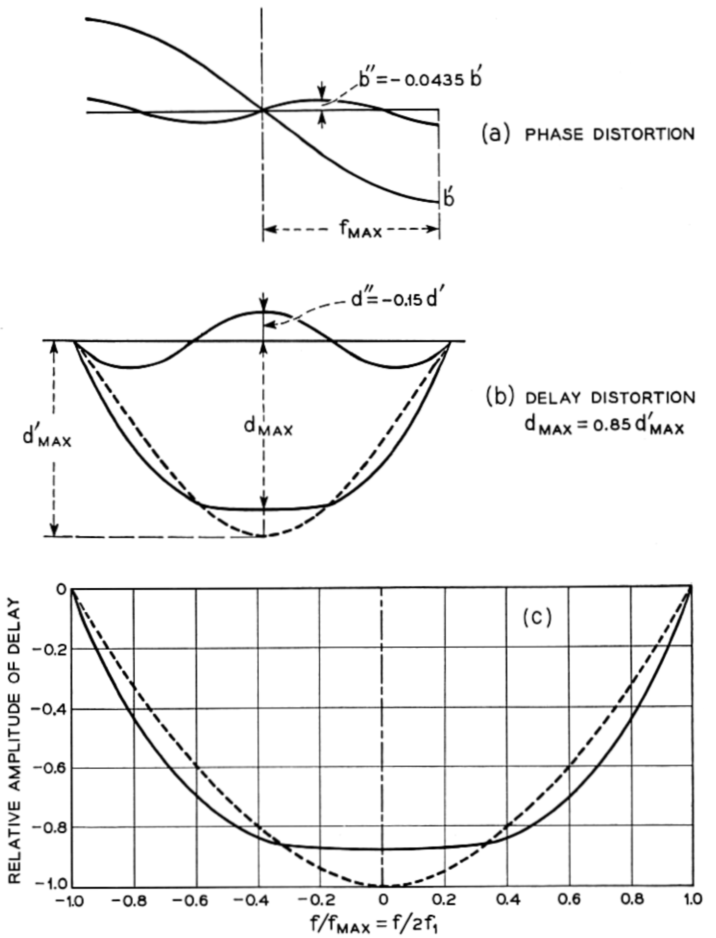


Fig. 24 — Shape of delay distortion with combined fundamental and third harmonic cosine variation in delay.

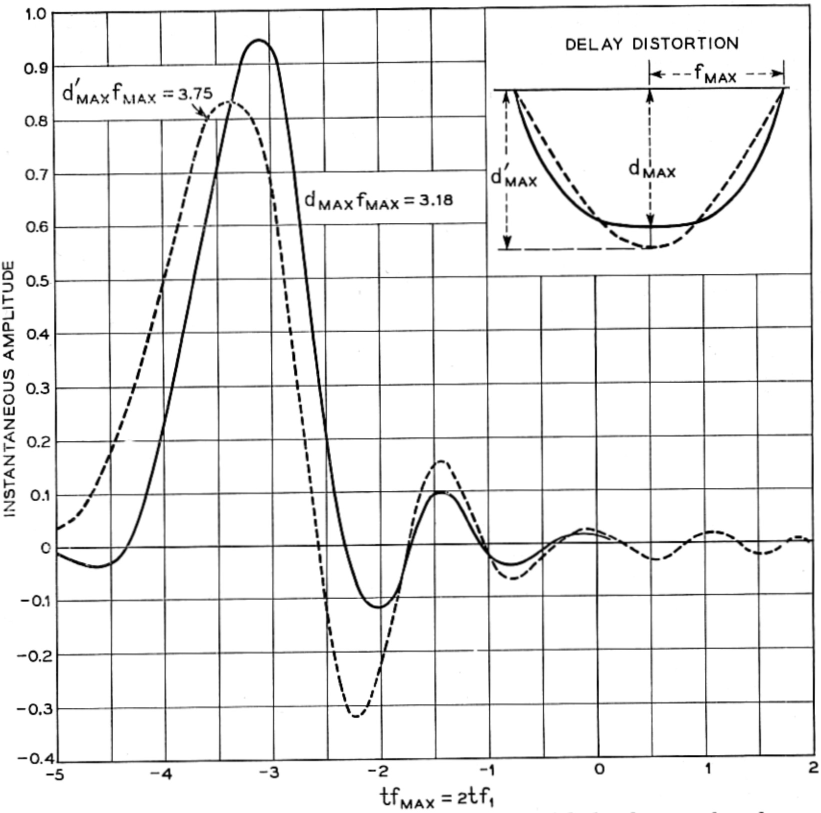


Fig. 25 — Comparison of impulse characteristics with fundamental and combined fundamental and third harmonic cosine variation in delay as in Fig. 24.

the combined effect of the two sine deviations is obtained by first determining the effect of $-b' \sin \omega\tau'$ from (6.05). The value of $P(t) = P_1(t)$ thus obtained from (6.05) with $b = b'$ and $\tau = \tau'$ is next substituted for $P_0(t)$ in (6.05), with $b = b''$ and $\tau = \tau''$ to evaluate the effect of $-b''$ $\sin \omega\tau''$. That is, the system is considered to consist of a tandem arrangement of two components, the first with a phase distortion $-b' \sin \omega\tau$ and the second with phase distortion $-b'' \sin \omega\tau''$.

In Fig. 24 is shown a particular case in which the second component is a triple harmonic of the first with amplitude $b'' = -0.0435b'$. This results in an improved approximation to the delay distortion encountered in certain wire facilities, where the band is sharply confined by filters. In Fig. 25 is shown the pulse shape for this case with $b' = 15$ radians, together with that for a single sine deviation of $b = 15$ radians.

It will be recognized from the above that as the number of sine components required to represent a given phase distortion increases, the determination of the resultant pulse becomes rather laborious, unless the sine deviations are all small in amplitude. In the latter case each sine deviation corresponds in a first approximation to a single pair of echoes, so that the effect of a number of sine deviations can be obtained by direct superposition.

7. PULSE ECHOES FROM AMPLITUDE DISTORTION

Departures from a given amplitude characteristic may in certain cases be approximated by a single cosine variation, as illustrated in Fig. 26. Since the amplitude characteristic is an even function of ω, any departure from a given amplitude characteristic may be represented by a cosine Fourier series. The effect of a cosine variation in the amplitude characteristic is therefore of basic interest.

A cosine variation will in general be accompanied by a change in the phase characteristic, as discussed in Section 1, but it will first be assumed that phase correction is employed to maintain a fixed phase characteristic.

Let $A_0(\omega)$ be the original amplitude characteristic and let the modified amplitude characteristic be of the form

$$A(\omega) = A_0(\omega)[1 + a \cos \omega\tau]. \tag{7.01}$$

Equation (2.01) for the impulse transmission characteristic then becomes, with $T_0(i\omega) = A_0(\omega)e^{-i\psi_0(\omega)}$,

$$\begin{aligned} P(t) &= \frac{1}{2\pi}\int_{\infty}^{\infty} T_0(i\omega)\left[1 + \frac{a}{2}\,(e^{i\omega\tau} + e^{-i\omega\tau})\right] e^{i\omega t}\, d\omega, \\ &= P_0(t) + \frac{a}{2}\,P_0(t+\tau) + \frac{a}{2}\,P_0(t-\tau). \end{aligned} \tag{7.02}$$

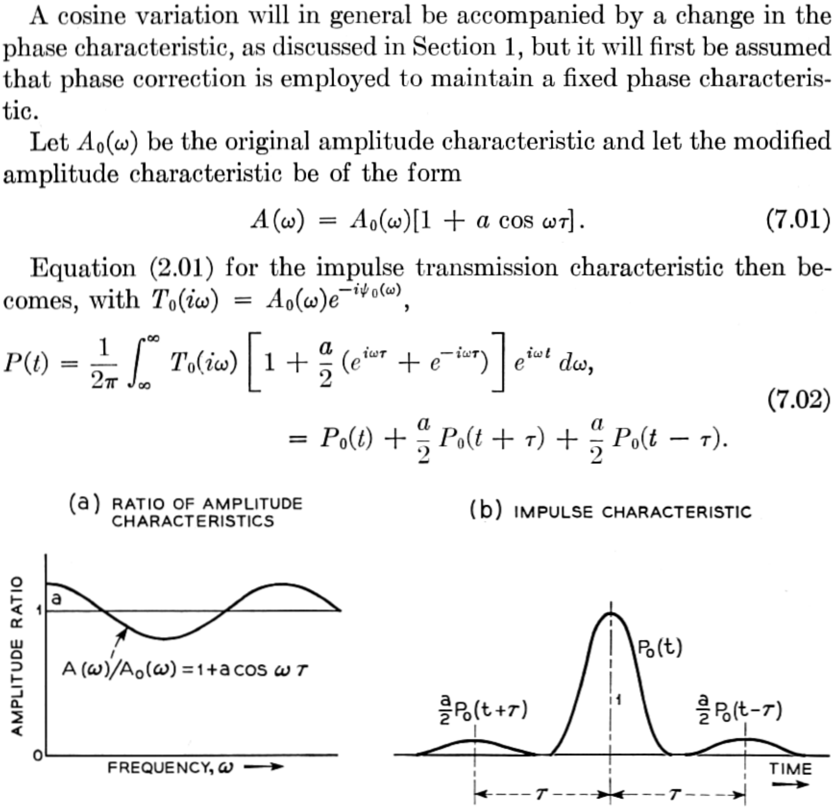


Fig. 26 — Pulse echoes from cosine variation in amplitude characteristic without change in phase characteristic.

There will thus be pulse or signal echoes of amplitude $a/2$ at the time τ before and after the main pulse as illustrated in Fig. 26.

With a cosine variation in the attenuation rather than in the amplitude characteristic, the modified amplitude characteristic becomes

$$A(\omega) = A_0(\omega)\, e^{a \cos \omega\tau}, \tag{7.03}$$

and the modified impulse characteristic

$$P(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} T_0(i\omega) e^{a \cos \omega\tau} e^{i\omega t}\, d\omega. \tag{7.04}$$

The expansion corresponding to (6.04) is in this case:

$$\begin{aligned} e^{a \cos \omega\tau} = I_0(a) &+ I_1(a)(e^{i\omega\tau} + e^{-i\omega\tau}) \\ &+ I_2(a)(e^{2i\omega\tau} + e^{-2i\omega\tau}) \\ &+ I_3(a)(e^{3i\omega\tau} + e^{-3i\omega\tau}), + \cdots \end{aligned} \tag{7.05}$$

where I_1, $I_2 \cdots$ are Bessel functions for imaginary arguments in their usual notation.

The resultant modified impulse characteristic in this case becomes

$$\begin{aligned} P(t) = I_0(a)P_0(t) &+ I_1(a)[P_0(t + \tau) + P_0(t - \tau)] \\ &+ I_2(a)[P_0(t + 2\tau) + P_0(t - 2\tau)] \\ &+ I_3(a)[P_0(t + 3\tau) + P_0(t - 3\tau)] + \cdots \end{aligned} \tag{7.06}$$

which can be interpreted in a similar way as discussed for (6.05). For small values of a, $I_0(a) \cong 1$, $I_1(a) \cong a/2$ and the remaining terms in (7.06) negligible, so that (7.02) is obtained.

As discussed in Section 1, when the amplitude characteristic is modified in accordance with (7.01), the resultant modification in the phase characteristic is in accordance with (1.13)

$$\psi_1 = 2 \tan^{-1} \frac{r \sin \omega\tau}{1 + r^2 \cos \omega\tau}. \tag{7.07}$$

The modified transmission-frequency characteristic is in this case

$$T(i\omega) = T_0(i\omega)(1 + a \cos \omega\tau)e^{-i\psi_1}, \tag{7.08}$$

which can be transformed into

$$\begin{aligned} T(i\omega) &= T_0(i\omega) \frac{1}{1 + r^2} (1 + re^{-i\omega\tau})^2, \\ &= T_0(i\omega) \frac{1}{1 + r^2} (1 + 2re^{-i\omega\tau} + r^2 e^{-2i\omega\tau}). \end{aligned} \tag{7.09}$$

Thus, with a cosine variation in the amplitude characteristic in accordance with (7.01), accompanied by a minimum phase shift change in the phase characteristic in accordance with (7.07), the modified impulse characteristic becomes

$$P(t) = \frac{1}{1 + r^2} [P_0(t) + 2rP_0(t - \tau) + r^2P_0(t - 2\tau)], \qquad (7.10)$$

where

$$r = \frac{1}{a} [1 - \sqrt{1 - a^2}]. \qquad (7.11)$$

The received pulse or signal $P(t)$ will thus consist of three components each having the same shape as the pulse or signal $P_0(t)$, but differing in amplitude and displaced in time, as indicated in Fig. 27.

For small values of the amplitude a of the cosine deviation, $r \cong a/2$ and $1 + r^2 \cong 1$, so that

$$P(t) = P_0(t) + aP_0(t - \tau) + \frac{a^2}{4} P_0(t - 2\tau). \qquad (7.11)$$

The solution for a somewhat similar case given elsewhere,[9] has an infinite number of echoes, with the second echo given by $a^2P_0(t - 2\tau)$ rather than $(a^2/4)P_0(t - 2\tau)$ as above. In the case referred to, the amplitude deviation is in a first approximation $a \cos \omega\tau$, but there are additional terms in $\cos 2\omega\tau$, $\cos 3\omega\tau$ etc, which are responsible for the different amplitude of the second echo and for the infinite sequence of echoes.

With a cosine modification in the attenuation characteristic as given by (7.03), there will be a corresponding sine modification in the phase characteristic in accordance with (1.11). The modified transmission-

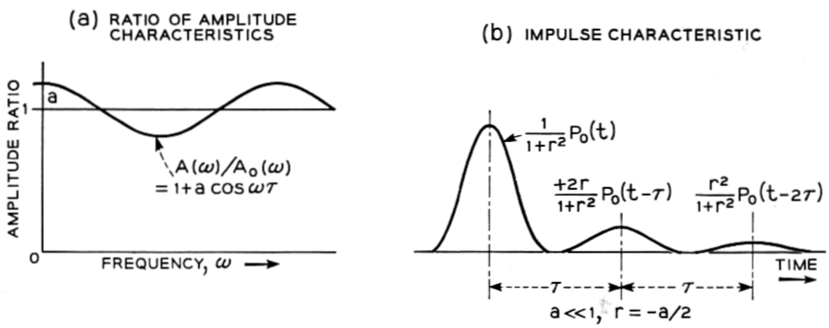


Fig. 27 — Pulse echoes from cosine variation in amplitude characteristic with associated minimum phase shift variation in phase characteristic.

frequency characteristic is in this case

$$\begin{aligned} T(i\omega) &= T_0(i\omega)e^{a\,(\cos\omega\tau - i\sin\omega\tau)}, \\ &= T_0(i\omega)e^{ae^{-i\omega\tau}}, \\ &= T_0(i\omega)\left[1 + ae^{-i\omega\tau} + \frac{a^2}{2!}\,e^{-2i\omega\tau} + \frac{a^3}{3!}\,e^{-3i\omega\tau} + \cdots\right]. \end{aligned} \tag{7.12}$$

The modified impulse characteristics is in this case

$$\begin{aligned} P(t) = P_0(t) + aP_0(t - \tau) + \frac{a^2}{2!}\,P_0(t - 2\tau) \\ + \frac{a^3}{3!}\,P_0(t - 3\tau) + \cdots \end{aligned} \tag{7.13}$$

For small values of a both (7.11) and (7.13) give for the modification in the impulse characteristic resulting from a small cosine deviation in the amplitude or attenuation characteristics accompanied by changes in the phase characteristic:

$$P(t) = P_0(t) + a\,P_0(t - \tau). \tag{7.14}$$

In certain applications it is convenient to regard $P_0(t)$ as a pulse or signal applied to a transmission line and $P(t)$ as the received pulse or signal with a cosine deviation in the amplitude characteristic of the transmission line.

In the lower part of Fig. 28 is shown the modification in the received pulses resulting from a slow pronounced cosine deviation in the amplitude characteristic shown at the top. In Fig. 29 is shown the effect of positive and negative cosine variations when the amplitude at zero frequency is held constant, a condition which may be approximated in wire systems as a result of variation in attenuation over the transmission band with temperature. Curve 1 would correspond to a 3.5 db smaller loss at the maximum frequency $2\omega_1$ than at zero frequency, and curve 2 to a 6 db greater loss at the maximum frequency. It will be noticed that pulse distortion as well as the variation in the peak amplitude of the pulses is greater under the first condition, i.e. curve 1. Pulse overlaps can in both cases be avoided by a moderate increase in pulse spacing, and in the first case can be substantially reduced also by a decrease in pulse spacing.

8. FINE STRUCTURE IMPERFECTIONS IN TRANSMISSION CHARACTERISTICS

As a result of imperfections in the transmission medium and in equalization there may be fine structure departures from a nominal transmission characteristic, as illustrated in Fig. 30. They are often caused by

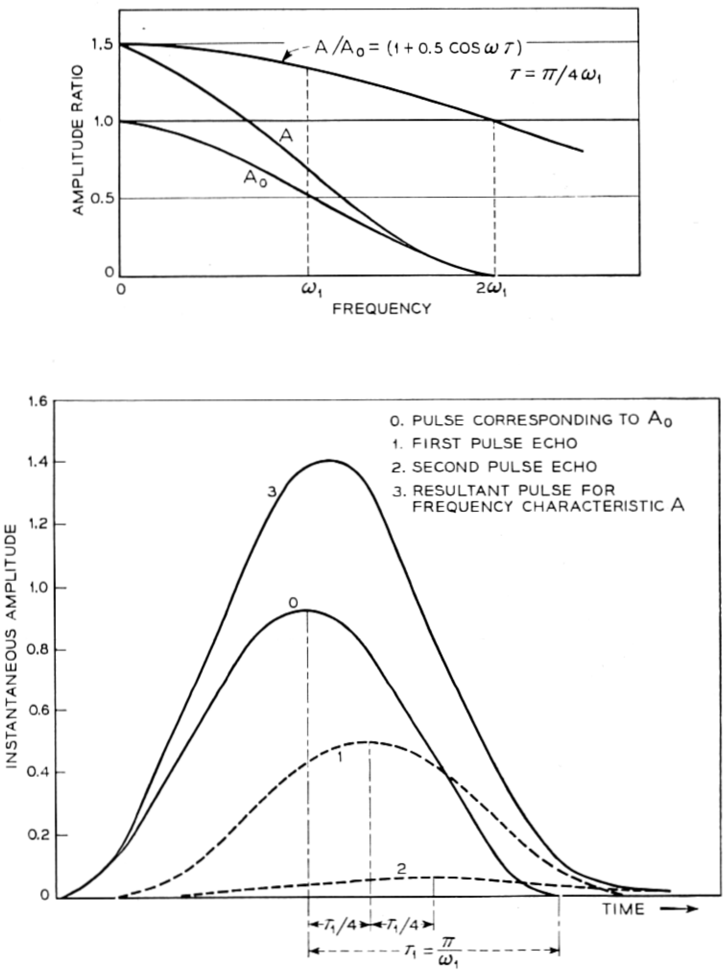


Fig. 28 — Modification of impulses characteristic by slow cosine variation in amplitude characteristic.

echoes in very long lines resulting from impedance mismatches. Fine structure deviations from a specified amplitude characteristic may in principle be represented by a cosine Fourier series, since the amplitude function is an even function of ω. Thus, if the specified amplitude characteristic is $A_0(\omega)$, the actual amplitude characteristic $A(\omega)$ may be represented by an infinite cosine Fourier series as:

$$A(\omega) = A_0(\omega)[1 + a_1 \cos \omega\tau + a_2 \cos 2\omega\tau + \cdots + a_m \cos m\omega\tau + \cdots]. \tag{8.01}$$

The coefficients a_1, $a_2 \cdots a_m \cdots$ are determined in the usual manner by Fourier series analysis to represent the function

$$f(\omega) = \frac{A(\omega)}{A_0(\omega)} = 1 + \alpha(\omega) \qquad (8.02)$$

over the frequency band. If $A_0(\omega)$ closely approaches $A(\omega)$ the fine

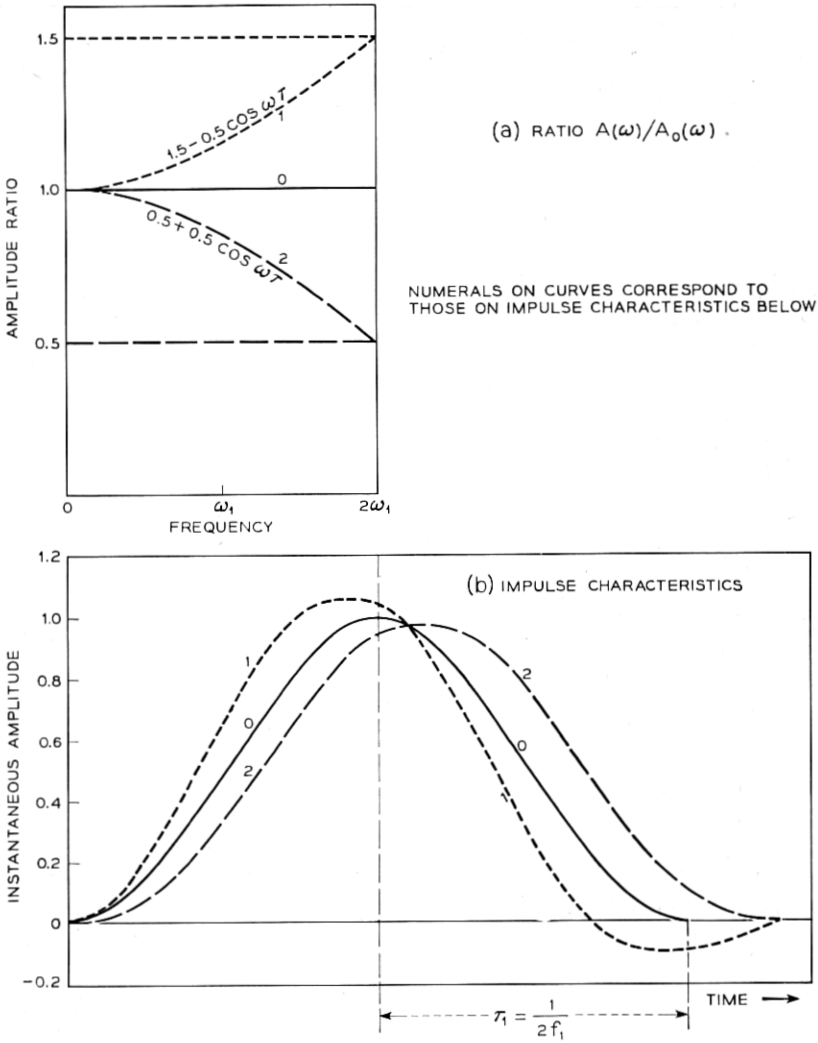


Fig. 29 — Effect of slow cosine variation in amplitude characteristic when amplitude at zero frequency is held constant.

structure departures $\alpha(\omega)$ in the transmission characteristic and hence the coefficients $a_1, a_2 \cdots a_m \cdots$ will be small.

In the above representation $A_0(\omega)$ can also be regarded as the amplitude characteristic of a terminal network or as the frequency spectrum of a pulse applied to a transmission system with an amplitude characteristic $f(\omega) = 1 + \alpha(\omega)$.

In a Fourier series analysis of the deviation in the amplitude characteristic, the fundamental period of the amplitude variation would be selected so that there is one complete cycle between $-\omega_1$ and ω_1, the cutoff frequency, in which case $\omega_1\tau = \pi$ or

$$\tau = \frac{\pi}{\omega_1}.$$

This is the interval between pulse echoes when the amplitude characteristic is represented by (8.01). It is identical with the interval τ_1 given by (3.02) at which pulses can be transmitted without mutual interference with a constant amplitude transmission frequency characteristic.

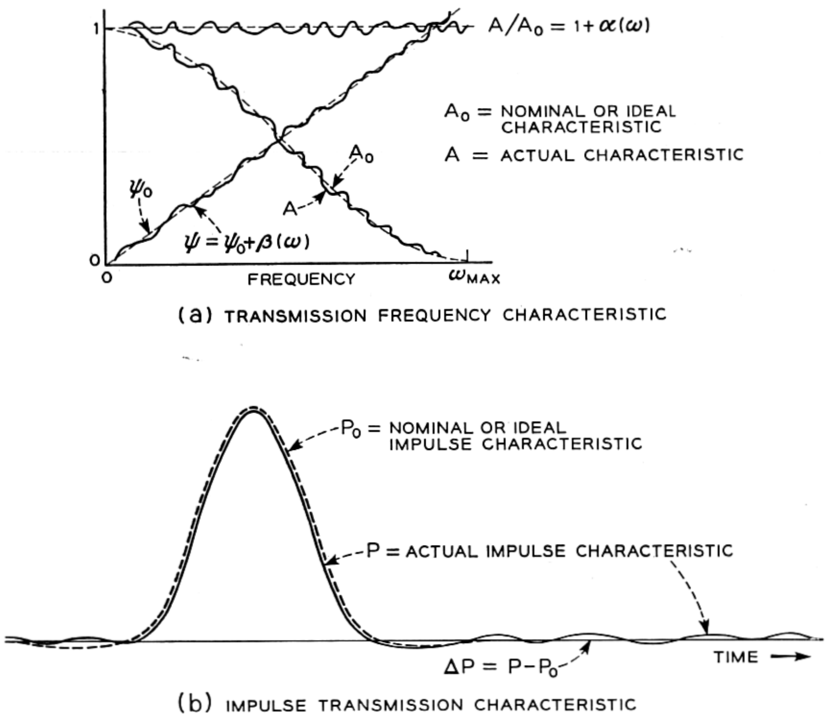
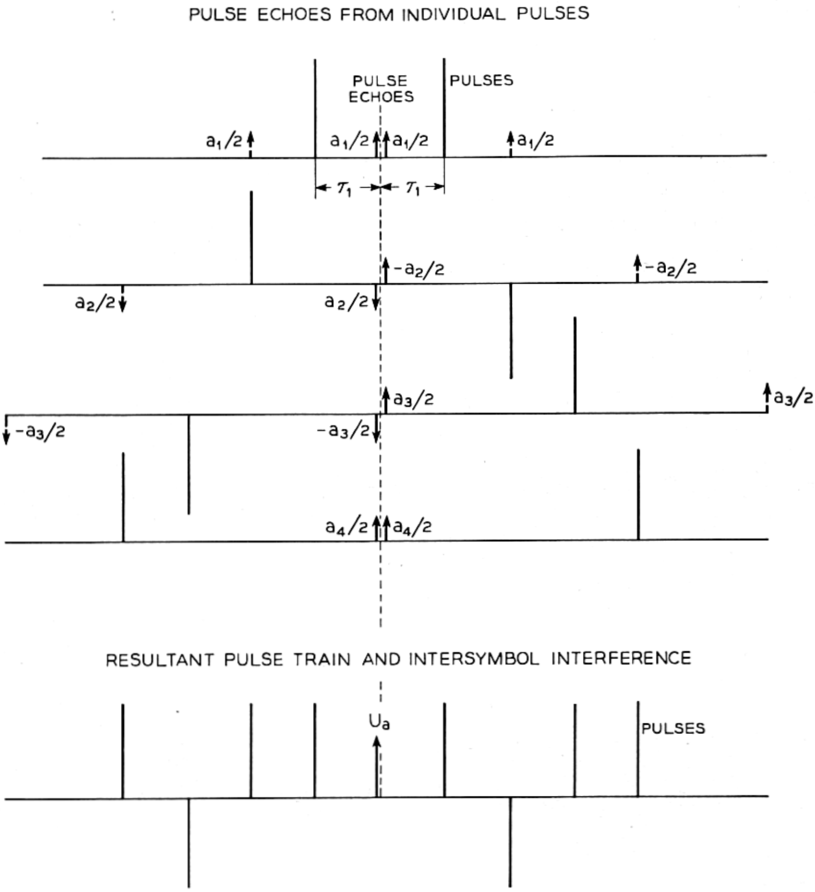


Fig. 30 — Fine structure imperfections in transmission frequency characteristic and resultant prolongation of impulse characteristic.

Assume that pulses of unit peak amplitude but varying polarity are transmitted at intervals $\tau = \tau_1$ and consider the interference with a given pulse from all pulses. As illustrated in Fig. 31, the first preceding and following pulses will in accordance with (7.02) give rise to a pulse echo $\pm a_1/2$ and the second preceding and following pulses to a pulse echo $\pm a_2/2$ etc., where the signs of the echoes depend on the polarity of the pulses and on the signs of the coefficients a_1, a_2. The resultant intersymbol interference $U_a(t)$ will depend on the polarity of the various



$$U_a = \frac{a_1}{2} + \frac{a_1}{2} + \frac{a_2}{2} - \frac{a_2}{2} - \frac{a_3}{2} + \frac{a_3}{2} + \frac{a_4}{2} + \frac{a_4}{2} = a_1 + a_4$$

Fig. 31 — Combination of pulse echoes into intersymbol interference for a particular case.

pulses and will thus vary with time. It can have any value assumed by the expression

$$U_a(t) = \pm\frac{a_1}{2} \pm \frac{a_1}{2} \pm \frac{a_2}{2} \pm \frac{a_2}{2} \cdots \pm \frac{a_m}{2} \pm \frac{a_m}{2} + \cdots \qquad (8.03)$$

The maximum possible intersymbol interference will thus be the sum of the absolute values of the coefficients a_m.

$$\hat{U}_a = |\, a_1 \,| + |\, a_2 \,| + |\, a_3 \,| + \cdots + |\, a_m \,| + \cdots \qquad (8.04)$$

In certain pulse systems, such as PAM time division systems, rms intersymbol interference is of main importance, while in others, such as PCM or telegraph systems, peak intersymbol interference is of principal interest. If the fine structure imperfections are regarded as of random nature, in the sense that they are not predictable and vary between systems having the same nominal transmission characteristics, peak intersymbol interference can be estimated from rms interference by applying a peak factor of about 4. With random variation in the amplitude of intersymbol interference, the probability of exceeding 4 times the rms value is in accordance with the normal law about 5×10^{-5}. Peaks in excess of 4 times the rms value will thus be so rare that they can for practical purposes be neglected.

The rms intersymbol interference is equal to the root mean square of all the different values which can be assumed by expression (8.03). This turns out to be equal to the root sum square of the amplitudes $a_m/2$ and $-a_m/2$ of the pulse echoes, or

$$\underline{U}_a = \left[\sum_1^{\infty} \left(\frac{a_m}{2}\right)^2 + \sum_1^{\infty} \left(\frac{-a_m}{2}\right)^2\right]^{1/2} = \left(\frac{1}{2}\sum_1^{\infty} a_m^{\,2}\right)^{1/2}. \qquad (8.05)$$

When a_m are the various coefficients in the Fourier representation of $\alpha(\omega)$ over the frequency band from $-\omega_1$ to ω_1, the following relation holds.

$$\frac{1}{2}\sum_1^{\infty} a_m^{\,2} = \frac{1}{2\omega_1}\int_{-\omega_1}^{\omega_1} \alpha^2(\omega)\, d\omega = \frac{1}{\omega_1}\int_0^{\omega_1} \alpha^2(\omega)\, d\omega \qquad (8.06)$$

where $\alpha(\omega)$ in the present case is given by (8.02) and represents the departure in the ratio $A(\omega)/A_0(\omega)$ from unity.

With (8.06) in (8.05) the following expression is obtained for rms intersymbol interference due to amplitude deviations $\alpha(\omega)$ not accompanied by phase deviations

$$\underline{U}_a = \underline{a} \qquad (8.07)$$

where $\underline{a}$ is the rms deviation in $\alpha(\omega)$ over the transmission band as given by

$$\begin{aligned}\underline{a} &= \left[\frac{1}{\omega_1}\int_0^{\omega_1} \alpha^2(\omega)\,d\omega\right]^{1/2} \\ &= \left[\frac{1}{f_1}\int_0^{f_1} \alpha^2(f)\,df\right]^{1/2}\end{aligned} \qquad (8.08)$$

The rms amplitude deviation expressed in db is

$$\begin{aligned}\underline{a}' &= 20 \log_{10}(1 + \underline{a}) \\ &\cong 8.69\,\underline{a} \qquad \text{when } \underline{a} < 0.1\end{aligned} \qquad (8.09)$$

A corresponding analysis can be made for fine structure imperfections in the phase characteristic. The deviation $\beta(\omega) = \psi(\omega) - \psi_0(\omega)$ from a prescribed phase characteristic $\psi_0(\omega)$ may in this case be represented by a sine Fourier series since the phase characteristic is an odd function of ω:

$$\beta(\omega) = b_1 \sin \omega\tau + b_2 \sin 2\omega\tau + \cdots + b_m \sin m\omega\tau + \cdots \qquad (8.10)$$

The resultant peak intersymbol interference becomes

$$\hat{U}_b = |\,b_1\,| + |\,b_2\,| + \cdots + |\,b_m\,| + \cdots \qquad (8.11)$$

and the rms intersymbol interference

$$\underline{U}_b = \left[\frac{1}{2}\sum_1^\infty b_m^2\right]^{1/2} = \underline{b}, \qquad (8.12)$$

where $\underline{b}$ is the rms phase deviation in radians as given by

$$\begin{aligned}\underline{b} &= \left[\frac{1}{\omega_1}\int_0^{\omega_1} \beta^2(\omega)\,d\omega\right]^{1/2}, \\ &= \left[\frac{1}{f_1}\int_0^{f_1} \beta^2(f)\,df\right]^{1/2}.\end{aligned} \qquad (8.13)$$

In the above derivation, the amplitude and phase deviations were assumed independent of each other. The resultant rms intersymbol interference from both is in this case

$$\underline{U} = (\underline{U}_a^2 + \underline{U}_b^2)^{1/2} = (\underline{a}^2 + \underline{b}^2)^{1/2}. \qquad (8.14)$$

This relationship, applying to an ideal transmission characteristic, has been established by a different method in a paper by W. R. Bennett.[10]

From (7.05) it will be seen that with minimum phase shift relationships a small cosine deviation of amplitude a_m in the amplitude characteristic will be accompanied by a phase deviation $b_m = a_m$. Hence in this case (8.14) gives

$$\underline{U} = 2^{1/2} \underline{a} \tag{8.15}$$

This also follows when it is considered that in this case all the pulse echoes occur after the main pulse, and have amplitudes $a_1, a_2 \cdots a_m$. The root sum square of the amplitudes is in this case $[\sum_1^\infty a_m^2]^{1/2}$, which is greater than $\underline{U}_a$ as given by (8.05) by the factor $2^{1/2}$.

The above analysis was based on an infinite sequence of pulse echoes, which combine to give the proper pulse distortion but may be regarded as fictitious in nature. The assumption of an infinite sequence of pulse echoes can be avoided by a different method of analysis outlined below, which does not involve the assumption that the coefficients are known from a Fourier series analysis, and furthermore, does not assume an ideal amplitude characteristic with a sharp cut-off as above.

Let $Ae^{-i\psi}$ and $A_0e^{-i\Psi_0}$ designate two transmission—frequency characteristics, where A, A_0, ψ and ψ_0 are functions of ω, which for convenience is omitted in the following. The squared absolute value of the difference in the transmission frequency characteristics is then

$$| A\ e^{-i\psi} - A_0 e^{-i\psi_0} |^2 = A_0^2[2(1 - \cos\beta)\ (1 + \alpha) + \alpha^2], \tag{8.16}$$

where $\alpha = \alpha(\omega) = (A - A_0)/A_0$ represents the deviation in the ratio of the amplitude characteristics from unity and $\beta = \beta(\omega) = \psi - \psi_0$ the deviation in the phase characteristic.

Let P and P_0 designate the impulse characteristics corresponding to the above transmission frequency characteristics, and let $\Delta P = P - P_0$. Assume that unit impulses of varying polarity are transmitted at uniform intervals τ_1. The rms value of ΔP over the interval τ_1 in relation to the maximum amplitude $P(0)$ of the received pulses, or the rms intersymbol interference $\underline{U}$, is then given by

$$\begin{aligned} \underline{U} &= \frac{1}{P(0)} \left[\frac{1}{\tau_1} \int_{-\infty}^{\infty} (\Delta P)^2\, dt) \right]^{1/2}, \\ &= \frac{1}{P(0)} \left[\frac{1}{\pi\tau_1} \int_0^{\infty} A^2[2(1 - \cos\beta)(1 + \alpha) + \alpha^2]\, d\omega \right]^{1/2}. \end{aligned} \tag{8.17}$$

For small values of α and β, this expression becomes

$$\underline{U} = \frac{1}{P(0)} \left[\frac{1}{\pi\tau_1} \int_0^{\infty} A_0^2(\alpha^2 + \beta^2)\, d\omega \right]^{1/2}. \tag{8.18}$$

If α and β are random variables representing fine structure deviations uniformly distributed over the transmission band, it is permissible to simplify (8.18) to:

$$\underline{U} = \eta \left(\frac{1}{\omega_1 \tau_1}\right)^{1/2} (\underline{a}^2 + \underline{b}^2)^{1/2}, \tag{8.19}$$

where

$$\eta = \frac{1}{\pi P(0)} \left(\omega_1 \int_0^{\omega_{\max}} A_0^2 \, d\omega\right)^{1/2}, \tag{8.20}$$

$$\underline{a} = \left(\frac{1}{\omega_{\max}} \int_0^{\omega_{\max}} \alpha^2 \, d\omega\right)^{1/2}, \quad \text{and} \tag{8.21}$$

$$\underline{b} = \left(\frac{1}{\omega_{\max}} \int_0^{\omega_{\max}} \beta^2 \, d\omega\right)^{1/2}, \tag{8.22}$$

where $\omega_{\max}$ is defined as in Fig. 30 and ω_1 is the bandwidth at the half amplitude point.

For a transmission characteristic with linear phase shift, aside from small random imperfections as considered here:

$$P(0) = \frac{1}{\pi} \int_0^{\omega_{\max}} A_0 \, d\omega. \tag{8.23}$$

For the particular case of a transmission characteristic with constant amplitude between $\omega = 0$ and $\omega_1 = \omega_{\max}$, $\eta = 1$. Pulses would in this case be transmitted at intervals $\tau_1 = \pi/\omega_1$ so that $\pi/\omega_1\tau_1 = 1$ and (8.19) is identical with (8.14).

For a transmission characteristic of the type shown in Fig. 13, pulses would also be transmitted at intervals $\tau_1 = \pi/\omega_1$ so that $\pi/\omega_1\tau_1 = 1$. In this case $\omega_{\max} = 2\omega_1$, and evaluation of (8.20) gives $\eta = 3^{1/2}/2 = 0.866$. Rms intersymbol interference is thus reduced by the factor 0.866, for the same values of $\underline{a}$ and $\underline{b}$. However, these are now the rms deviations taken over a band which is twice as great as with a sharp cut-off at ω_1.

Expressions (8.14) and (8.19) can also be applied to localized imperfections in the amplitude and phase characteristics confined to a narrow portion of the transmission band. This follows when it is considered that such deviations can be represented by Fourier series containing a large number of coefficients, so that the resultant intersymbol interference can attain a great number of different values depending on the sequence of transmitted pulses. A particular case of a localized imperfection in the amplitude characteristic in the form of a low-frequency cut-off is considered in the following section.

9. TRANSMISSION DISTORTION BY LOW FREQUENCY CUT-OFF

A low-frequency cut-off in the transmission frequency characteristic of wire systems is unavoidable with transformers as employed for increased transmission efficiency or other reasons. In single sideband frequency division systems, there is a low-frequency cut-off in individual channels caused by elimination of the carrier and part of the desired sideband. The effect of a low-frequency cut-off can be avoided by employing a symmetrical band-pass characteristic as illustrated in Fig. 16, or more generally by double sideband transmission with a two-fold increase in bandwidth as compared to a low-pass system. It can also be overcome by vestigial sideband transmission with inappreciable bandwidth penalty, but with complications in terminal instrumentation. The effect of a low-frequency cut-off can, furthermore, be reduced without frequency translation as involved in double or vestigial sideband transmission, by certain methods of shaping or transmission of pulses, as discussed in the following, and by certain methods of compensation at the receiving end or at points of pulse regeneration not considered here.

The nature of the pulse distortion resulting from a low-frequency cut-off is illustrated in Fig. 32. If the phase characteristic is assumed linear, the amplitude characteristic may be regarded as made up of two components, in accordance with the following identity:

$$A(\omega) = A_0(\omega) + [A(\omega) - A_0(\omega)], \tag{9.01}$$

where $A_0(\omega)$ is the amplitude characteristic without a low-frequency cut-off and $[A(\omega) - A_0(\omega)]$ a supplementary characteristic of negative amplitude, as indicated in Fig. 32.

The impulse characteristic may correspondingly be written

$$P(t) = P_0(t) + [P(t) - P_0(t)]. \tag{9.02}$$

If the cut-off is confined to rather low frequencies, the impulse characteristic $\Delta P(t) = P(t) - P_0(t)$ will extend over time intervals substantially longer than the duration of $P_0(t)$ or the interval at which pulses are transmitted. The total area under the resultant pulse is always zero.

When a sufficiently long sequence of pulses of one polarity is transmitted, the cumulative effect of the pulse overlaps resulting from the modification $P(t) - P_0(t)$ in the impulse characteristic will be a displacement of the received pulse train, as illustrated in Fig. 33 for various intervals between the pulses. This apparent displacement of the zero line, often referred to as "zero wander," will reduce the margin for distinction between the presence and absence of pulses in a random pulse train. In the particular case when pulses are transmitted at the minimum interval $\tau_1 = 1/2f_1$ possible without intersymbol interference in the absence of a low-frequency cut-off, the pulse train will ultimately vanish when an infinite sequence of pulses of one polarity is transmitted, as illustrated for the last case in Fig. 33.

The number of pulses of one polarity, or nearly all of the same polarity, which can be transmitted before the limiting condition illustrated in Fig. 33 is approached depends on the extent of the low-frequency cut-off. If the low-frequency cut-off is inappreciable, this number may be sufficiently great so that the probability of encountering such a sequence in a random pulse train and resultant errors in reception may be so small that it can be disregarded. The requirement of the low-frequency cut-off which is necessary to this end is evaluated below for pulses transmitted at intervals $\tau_1 = 1/2f_1$.

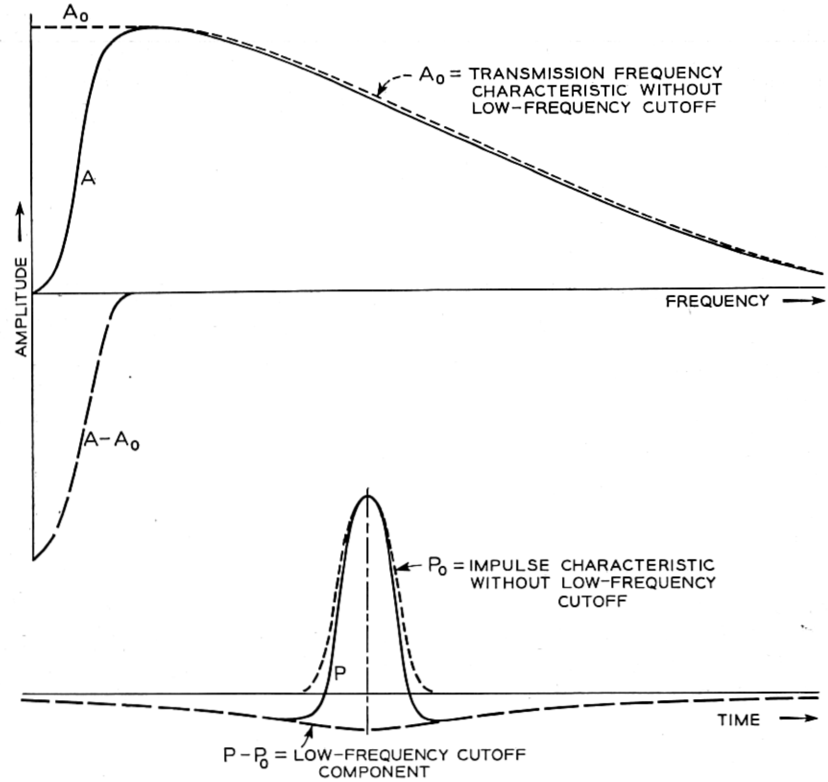


Fig. 32 — Separation of low-frequency cut-off componente $A–A_0$ and $P–P_0$ in transmission frequency and impulse characteristics.

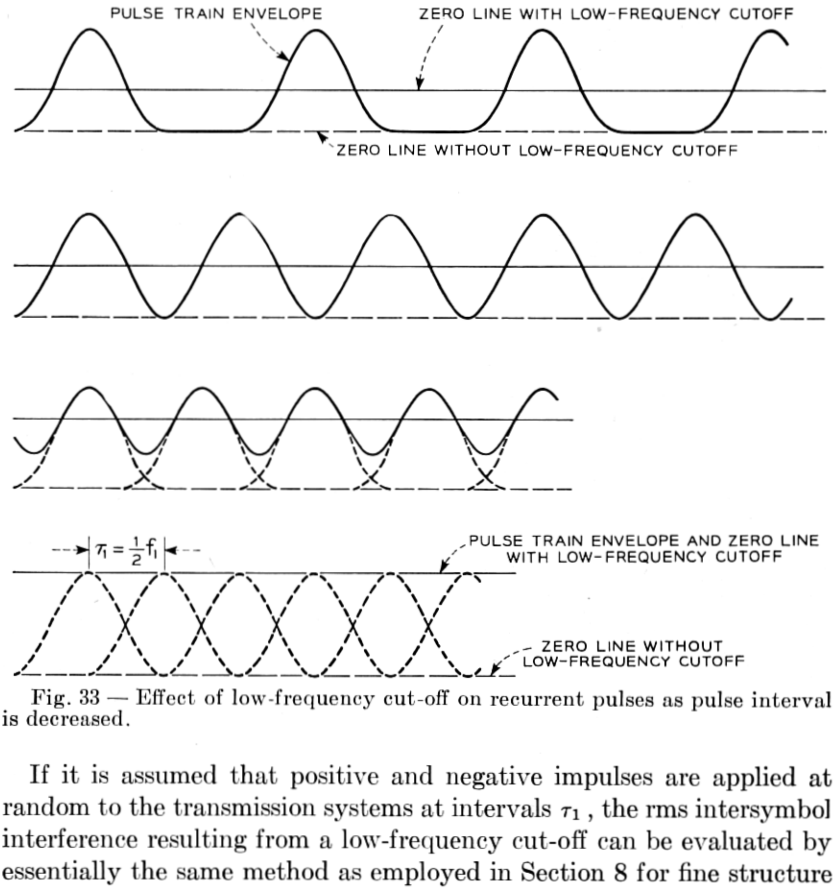


Fig. 33 — Effect of low-frequency cut-off on recurrent pulses as pulse interval is decreased.

If it is assumed that positive and negative impulses are applied at random to the transmission systems at intervals τ_1, the rms intersymbol interference resulting from a low-frequency cut-off can be evaluated by essentially the same method as employed in Section 8 for fine structure imperfections in the transmission characteristic, provided ω_0 is much smaller than ω_1. On this basis, rms intersymbol interference in relation to the peak amplitude $P_0(0)$ of the pulses in the absence of a low-frequency cut-off becomes:

$$\underline{U} = \frac{1}{P_0(0)} \left(\frac{1}{\tau_1} \int_{-\infty}^{\infty} [P(t) - P_0(t)]^2 \, dt \right)^{1/2}, \tag{9.03}$$

$$= \frac{1}{P_0(0)} \left(\frac{1}{\pi \tau_1} \int_0^{\infty} [A(\omega) - A_0(\omega)]^2 \, d\omega \right)^{1/2}. \tag{9.04}$$

For a transmission characteristic with linear phase shift

$$P_0(0) = \frac{1}{\pi} \int_0^{\infty} A_0(\omega) \, d\omega. \tag{9.05}$$

For the particular case of sharp cut-offs at ω_0 and ω_1

$$A_0(\omega) = 1 \qquad 0 < \omega < \omega_1 ,$$

$$A(\omega) - A_0(\omega) = -1 \qquad 0 < \omega < \omega_0 , \text{ and}$$

$$P_0(0) = \omega_1/\pi \qquad \tau_1 = \pi/\omega_1 ,$$

and

$$\underline{U} = \frac{\pi}{\omega_1}\left(\frac{\omega_1\omega_0}{\pi^2}\right)^{1/2} = \left(\frac{\omega_0}{\omega_1}\right)^{1/2}. \tag{9.06}$$

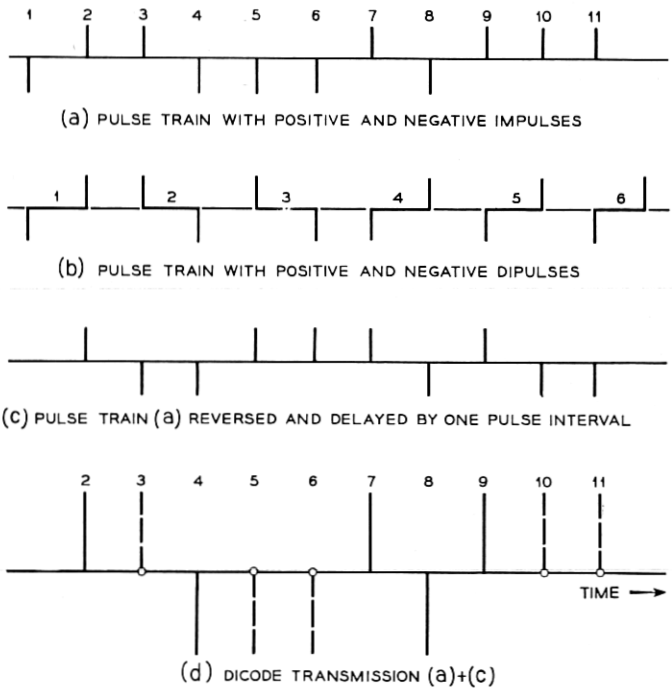
It will be noticed that the same result is obtained from (8.07) with the amplitude deviation $\alpha = [A(\omega) - A_0(\omega)] = -1$ between 0 and ω_0.

In actual systems, the low-frequency cut-off will be gradual between $\omega = 0$ and ω_0, rather than abrupt as assumed above. With a linear variation in the amplitude characteristic between 0 and ω_0, $A(\omega) - A_0(\omega) = (-1 + \omega/\omega_0)A_0(0)$ and $\underline{U} = (\omega_0/3\omega_1)^{1/2}$.

If a sufficient number of pulses of one polarity is transmitted in succession at intervals $\tau_1 = 1/2f_1$ the received pulses will as noted before in the limit be reduced to zero amplitude by the low-frequency cut-off. The maximum pulse distortion resulting from pulse overlaps when a train of pulses as transmitted is thus equal and opposite to the amplitude $P_0(0)$ of the received pulses in the absence of a low-frequency cut-off, so that peak intersymbol interference $\hat{U} = -1$. If rms intersymbol interference is held at one-quarter the peak value, i.e., $\underline{U} = 0.25$, the probability of encountering the maximum tolerable intersymbol interference and resultant errors in reception is low enough to be disregarded. On this basis the ratio ω_0/ω_1 would in accordance with (9.06) have to be less than 0.0625. Actually a substantially smaller ratio would be required because of intersymbol interference from other imperfections in the transmission characteristic and noise. Furthermore, a low-frequency cut-off will be accompanied by phase distortion at the low end of the transmission band, disregarded in the above evaluation. The requirements imposed on the low-frequency cut-off will thus be rather severe for a pulse system as assumed above in which random sequences of pulses are transmitted at intervals $\tau_1 = 1/2f_1$. Two pulse amplitudes were assumed above, and with a greater number of amplitudes the requirements would be more severe.

From Fig. 33 it is evident that the effect of a low-frequency cut-off on a received pulse train can be reduced by transmitting pulses at longer intervals than $\tau_1 = 1/2f_1$ considered above. For example, with a two-fold increase in the pulse interval, as represented by the second case in Fig. 33, the maximum displacement of the zero line would be half the peak amplitude of the pulses. There would then be a 50 per cent reduction in the margin for distinction between the presence and absence of a pulse in a random pulse train, rather than a complete elimination of the margin for an infinite train of pulses of the same polarity transmitted at intervals $\tau_1 = 1/2f_1$. This improvement would be achieved at the expense of a two-fold increase in bandwidth for a given pulse transmission rate. A further improvement, for the same two-fold increase in bandwidth, can be achieved by "dipulse" transmission, as discussed below.

In dipulse transmission a positive pulse followed by a negative pulse in the next pulse position would be transmitted to indicate "on," and a negative pulse followed by a positive pulse to indicate "off," as indicated in Fig. 34. There will then be a substantial reduction in the pulse



THE PULSES AND ZEROS IN THE RECEIVED PULSE TRAIN (d) HAVE THE FOLLOWING RELATIONS TO THE ORIGINAL PULSES (a)

1. POSITIVE AND NEGATIVE PULSES IN (d) REPRESENT CORRESPONDING PULSES IN (a)
2. POINTS ON PULSE TRAIN IN (d) REPRESENT A REPETITION OF PREVIOUS PULSE, AS INDICATED BY DASHED LINES

Fig. 34 — Dipulse and dicode pulse transmission methods.

overlaps resulting from a low-frequency cut-off, as illustrated in Fig. 35, and in peak intersymbol interference.

If $\Delta P(t) = P(t) - P_0(t)$ is the modification in the impulse characteristic shown in Fig. 32, the modification in the dipulse transmission characteristic resulting from a low-frequency cut-off becomes

$$\Delta_1 P(t) = \Delta P(t) - \Delta P(t - \tau_1), \tag{9.07}$$

where τ_1 is the interval between the positive and negative dipulse components.

The difference given by (9.07) represents the differential in the curve $P(t) - P_0(t)$ shown in Fig. 32 over an interval τ_1. It can be shown that the maximum cumulative effect or peak intersymbol interference for a long pulse train is represented by the sum of the differentials given by (9.07) and is approximately equal to

$$\hat{U} \cong \Delta P(\tau_1) = P(\tau_1) - P_0(\tau_1). \tag{9.08}$$

As an example, if the shape of $A - A_0$ in Fig. 32 were about the same as that of A_0, $\Delta P(t)$ would have the same shape as $P_0(t)$ but would be lower in peak amplitude by the factor f_0/f_1 and would have the time scale increased by the factor f_1/f_0. Peak intersymbol interference as

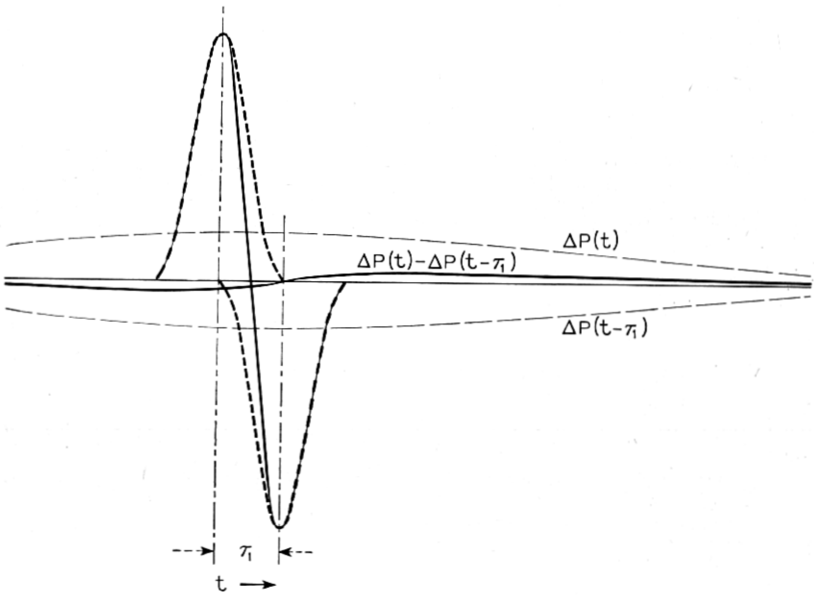


Fig. 35 — Low-frequency cut-off effects $\Delta P(t)$ and $\Delta P(t - \tau_1)$ for positive and negative pulses and resultant effect $\Delta_1 P(t) = \Delta P(t) - \Delta P(t - \tau_1)$ for a dipulse.

obtained from (9.08) would then be about $\hat{U} = f_0/f_1$ and thus in the order of 10 per cent of the peak pulse amplitude for $f_0/f_1 = 0.10$.

The bandwidth penalty incurred in dipulse transmission can be avoided by transmitting two identical pulse trains, one of which is delayed by one pulse interval and reversed in polarity with respect to the other.* The combined pulse train will then be as indicated in Fig. 34, and one or the other of the two original component pulse trains can be restored at the receiving end by suitable conversion equipment. In the combined pulse train, a pulse of one polarity is always followed by a pulse of opposite polarity, but not necessarily in the next pulse position. For this reason the low-frequency cut-off compensation with the above method of "dicode" transmission is not quite as effective as with dipulse transmission. Furthermore, since it is necessary to distinguish between three pulse amplitudes (1, 0, -1), in the received pulse train, the maximum tolerable pulse distortion in relation to the peak pulse amplitude is only half as great as with two pulse amplitudes (1, -1) in an ordinary code.

10. TRANSMISSION DISTORTION FROM BAND-EDGE PHASE DEVIATIONS

In pulse transmission systems where phase equalization is employed, it may be impracticable or unnecessary to equalize over the entire transmission band. There will then be residual phase distortion near the band-edges, as indicated in Fig. 36. This type of phase deviation will give rise to pulse distortion extending over appreciable time intervals if the band-edge phase deviations are large, as indicated in the above figure, for the reason that the frequency components outside the linear phase range will be received with increased transmission delay. Evaluation of the pulse shape is in this case a rather elaborate procedure, but rms pulse distortion or intersymbol interference resulting from such phase distortion can readily be determined as outlined below. In certain pulse modulation systems, such as PAM time division systems, rms intersymbol interference is of principal interest. In other systems where peak intersymbol interference is controlling, it may usually be estimated with engineering accuracy by applying a peak factor.

When the pulse shape is known, peak intersymbol interference may be determined by methods outlined in Section 13. Comparison of peak intersymbol interference evaluated in this manner with rms pulse distortion, for some cases in which the pulse shapes in the presence of phase

* L. A. Meacham originally proposed this method is an unpublished memorandum.

distortion were determined, indicates that the peak factor is about 3 when phase distortion is appreciable and the pulses are substantially prolonged in duration.

Returning to equation (8.17) and assuming $\alpha = 0$, the following relationship is obtained for rms intersymbol interference due to phase deviations

$$\underline{U} = \frac{1}{P(0)} \left(\frac{1}{\pi \tau_1} \right)^{1/2} \left[\int_0^\infty 2A_0^2(1 - \cos \beta) \, d\omega \right]^{1/2}, \tag{10.01}$$

where $\beta = \beta(\omega)$ is the deviation from a linear phase characteristic.

For transmission systems with a linear phase shift, the peak amplitude of the pulses is given by (8.23) and with this relation in (10.01)

$$\underline{U} = \left(\frac{\pi}{\omega_1 \tau_1} \right)^{1/2} \lambda, \tag{10.02}$$

where

$$\lambda = \left[\omega_1 \int_0^{\omega_{\max}} 2A_0^2(1 - \cos \beta) \, d\omega \right]^{1/2} \Big/ \left[\int_0^{\omega_{\max}} A_0 \, d\omega \right]. \tag{10.03}$$

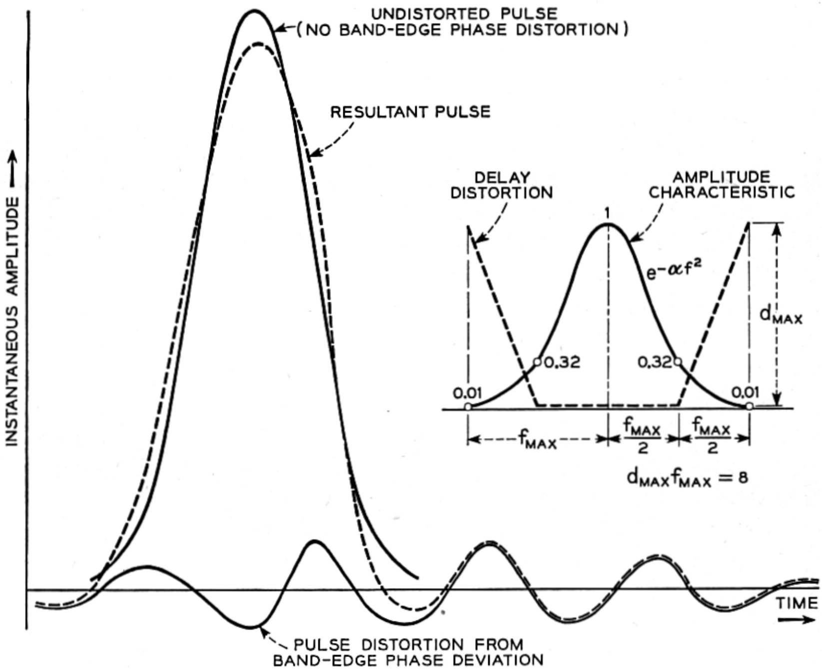


Fig. 36 — Pulse distortion from band-edge phase deviation for particular case of linear band-edge delay distortion.

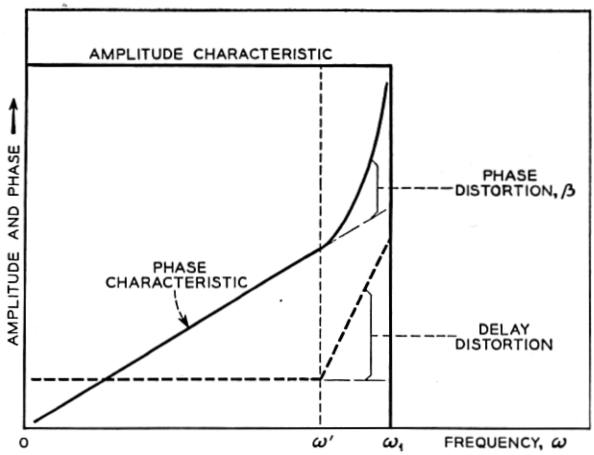


Fig. 37 — Constant amplitude characteristic with band-edge phase distortion.

If there is no phase distortion, i.e., $\beta = 0$, between $\omega = 0$ and ω', equation (10.03) becomes

$$\lambda = \left[\omega_1 \int_{\omega'}^{\omega_{max}} 2A_0^2(1 - \cos \beta\, d\omega\right]^{1/2} \Bigg/ \left[\int_0^{\omega_{max}} A_0\, d\omega\right]. \quad (10.04)$$

As an example, consider a parabolic deviation from a linear phase characteristic between ω' and ω_{max}, in which case delay distortion would vary linearily in this band, as indicated in Fig. 37 for a constant amplitude characteristic for which $\omega_{max} = \omega_1$. In this case

$$\beta = \beta_1 \left(\frac{\omega - \omega'}{\omega_1 - \omega'}\right)^2, \quad (10.05)$$

where β_1 is the maximum phase deviation, obtained for $\omega = \omega_1$.

Equation (10.04) in the above case becomes:

$$\begin{aligned} \lambda^2 &= \frac{2}{\omega_1} \int_{\omega'}^{\omega_1} \left[1 - \cos \beta_1 \left(\frac{\omega - \omega'}{\omega_1 - \omega'}\right)^2\right] d\omega, \\ &= \frac{2(\omega_1 - \omega')}{\omega_1} \left[1 - \beta_1^{-1/2} \int_0^{\beta_1^{1/2}} \cos u^2\, du\right], \quad (10.06) \\ &= \frac{2(\omega_1 - \omega')}{\omega_1} \left[1 - \frac{1}{2}\left(\frac{\pi}{2\beta_1}\right)^{1/2} (R + X)\right], \end{aligned}$$

where $R + iX = \text{erf}\,(\beta_1^{1/2} e^{i\pi/4})$ in which erf is the error function.

For a constant amplitude transmission characteristic as assumed above, $(\pi/\omega_1\tau_1) = 1$, so that (10.02) becomes $\underline{U} = \lambda$, which may also be written:

$$\underline{U} = \left(\frac{\omega_1 - \omega'}{\omega_1}\right)^{1/2} \cdot F(\beta_1), \qquad \text{and} \tag{10.07}$$

$$F(\beta_1) = 2^{1/2}\left[1 - \frac{1}{2}\left(\frac{\pi}{2\beta_1}\right)^{1/2}(R + X)\right]^{1/2}. \tag{10.08}$$

For various values of the maximum phase deviation β_1 in radians the function F becomes:

β_1	0	0.25	1	4	∞
F	0	0.14	0.43	1.24	1.42

If, for example, phase distortion were confined to 10 per cent of the transmission band, then $(\omega_1 - \omega')/\omega_1 = 0.1$. For a maximum phase deviation of 1 radian at the edge of the transmission band, $F = 0.43$ and $\underline{U} = 0.135$. For a maximum phase distortion of 4 radians, $F = 1.24$ and $\underline{U} = 0.39$. Since peak intersymbol interference may exceed the above rms values by a factor of about 3, and the maximum tolerable peak intersymbol interference in a system employing two pulse amplitudes would be less than 1, it is evident that band-edge phase deviations must be held at rather small values, less than about 3 radians, in the upper 10 per cent of the transmission band.

The above severe tolerances on band-edge phase distortion can be overcome by employing a transmission frequency characteristic of the type shown in Fig. 38 and previously discussed in Section 5. If the phase characteristic is linear between $\omega = 0$ and ω_1, and phase distortion between ω_1 and $2\omega_1$ varies as

$$\beta = \beta_1\left(\frac{\omega - \omega_1}{2\omega_1 - \omega_1}\right)^2 = \beta_1\left(1 - \frac{\omega}{\omega_1}\right)^2, \tag{10.09}$$

equation (10.04) can be written

$$\begin{aligned}\lambda^2 &= \frac{1}{\omega_1}\int_{\omega_1}^{2\omega_1}\left(1 + \cos\frac{\pi\omega}{2\omega_1}\right)^2\left[1 - \cos\beta_1\left(1 - \frac{\omega}{\omega_1}\right)^2\right]d\omega, \\ &= \int_0^1\left(1 - \sin\frac{\pi}{2}u\right)^2(1 - \cos\beta_1 u^2)\,du.\end{aligned} \tag{10.10}$$

Pulses may also in this case be transmitted at intervals $\tau_1 = \pi/\omega_1$ without intersymbol interference in the absence of phase distortion, so that (10.02) becomes $\underline{U} = \lambda$ or

$$\underline{U} = \left[\frac{1}{2}\int_0^1 \left(1 - \sin\frac{\pi}{2}u\right)^2 (1 - \cos\beta_1 u^2)\, du\right]^{1/2}. \qquad (10.11)$$

The maximum delay distortion at the edge of the transmission band, i.e., $\omega = 2\omega_1$, is $d_{max} = 2\beta_1/\omega_1$. The product of this delay distortion with the maximum frequency $f_{max} = 2f_1$ is $d_{max}f_{max} = 2\beta_1/\pi$. For various values of maximum phase distortion β_1 and the corresponding product $d_{max}f_{max}$, the following values of rms intersymbol interference are obtained by numerical integration of (10.11). (This integral can be expressed in terms of a number of Fresnel integrals, but numerical integration is simpler and sufficiently accurate for the present purpose.)

β_1	π	2π	4π	∞
$d_{max}\, f_{max}$	2	4	8	∞
$\underline{U}$	0.070	0.120	0.185	0.330

The particular case $d_{max}f_{max} = 8$ is similar to that shown in Fig. 36, except that this figure applies to a Gaussian characteristic, for which the amplitude at $\omega = \omega_1$ has been taken as 0.32 rather than 0.5 in the case considered here. For this reason rms intersymbol interference from phase distortion would be greater in the present case.

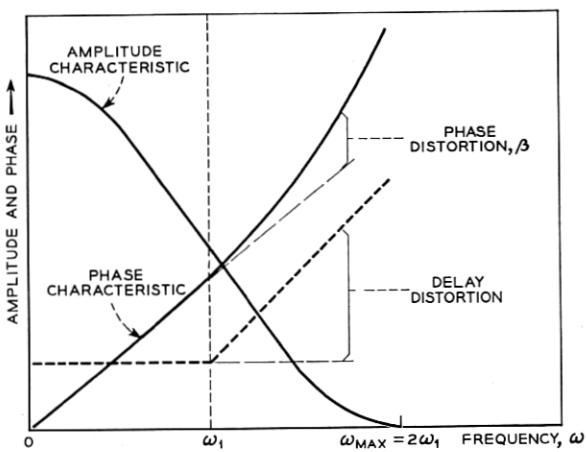


Fig. 38 — Typical transmission frequency characteristic with phase equalization over 50 per cent of transmission band.

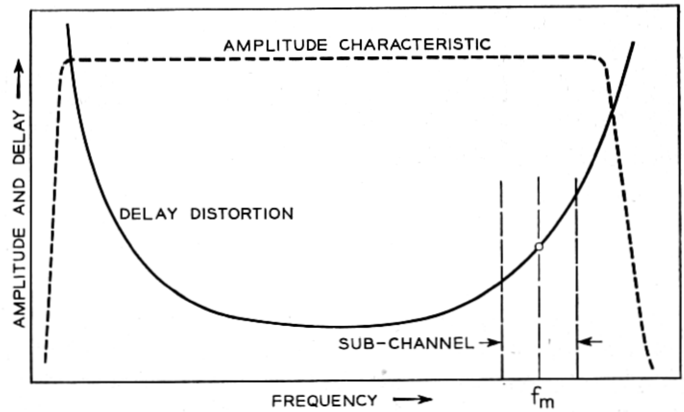


Fig. 39 — Sub-channel with nearly linear delay distortion.

Peak intersymbol interference may exceed the above rms values by a factor of about 3. In a system employing two pulse amplitude (1 and −1), the maximum tolerable intersymbol interference is 1. This value would thus be attained in the above case for $d_{max}f_{max} = \infty$. Hence, in a system employing two pulse amplitudes, and in the absence of noise and intersymbol interference from other sources, there would be no limitation on phase distortion for $\omega > \omega_1$, provided the phase characteristic is linear between $\omega = 0$ and ω_1.

11. BAND-PASS CHARACTERISTICS WITH LINEAR DELAY DISTORTION

In Fig. 39 is shown a transmission frequency characteristic together with an assumed delay distortion $d\psi/d\omega$. When a portion of the transmission band is employed for pulse transmission, as for example in pulse signalling, data or telegraph transmission over portion of a voice channel, there may be an appreciable component of substantially linear delay distortion, as indicated in the above figure. The departure from a linear variation can usually be approximated by a cosine variation in delay, and the system can then be regarded as made up of two components in tandem, one with linear the other with cosine variation in delay. The effect of the latter can be evaluated by the methods outlined in Section 6, and the effect of a linear variation by methods established in this section.

In Fig. 40 is shown a symmetrical amplitude characteristic with linear delay distortion over the transmission band. Phase distortion with respect to the midband frequency is in this case

$$\Psi(u) = \beta u^2 \qquad \text{and} \qquad \Psi(-u) = \beta u^2, \tag{11.01}$$

and delay distortion

$$d\Psi(u)/du = 2\beta u, \; d\Psi(-u)/du = -2\beta u. \tag{11.02}$$

(The symbol β, together with α, η, a and b used later in this section do not have the same meaning as in earlier sections.) With (11.01) in (2.10) and (2.11), the in-phase and quadrature components in (2.09) become

$$\begin{aligned} R_- + R_+ &= \frac{2\delta}{\pi} \int_0^\infty \mathfrak{a}(u) \cos ut \cos \beta u^2, \quad \text{and} \\ Q_- + Q_+ &= \frac{2\delta}{\pi} \int_0^\infty \mathfrak{a}(u) \cos ut \sin \beta u^2. \end{aligned} \tag{11.03}$$

The in-phase and quadrature components can accordingly be identified with the real and the negative imaginary component of the integral

$$J = \frac{2\delta}{\pi} \int_0^\infty \mathfrak{a}(u) \cos ut e^{-i\beta u^2} \, du. \tag{11.04}$$

The solution of this integral is rather simple for the particular case of a Gaussian transmission characteristic

$$\mathfrak{a}(u) = e^{-\alpha u^2}, \tag{11.05}$$

in which case

$$\begin{aligned} J &= \frac{2\delta}{\pi} \int_0^\infty e^{-(\alpha + i\beta) u^2} \cos ut \, du, \\ &= \frac{\delta}{[\pi(\alpha + i\beta)]^{1/2}} e^{-t^2/4(\alpha + i\beta)}. \end{aligned} \tag{11.06}$$

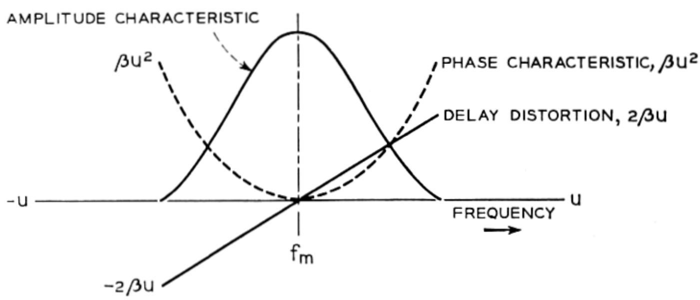


Fig. 40 — Symmetrical band-pass amplitude characteristic with linear delay distortion.

The real and negative imaginary components of this expression are

$$R_- + R_+ = 2\delta\left(\frac{c}{\pi}\right)^{1/2} e^{-at^2} \cos(\Theta - bt^2), \qquad \text{and}$$
$$Q_- - Q_+ = 2\delta\left(\frac{c}{\pi}\right)^{1/2} e^{-at^2} \sin(\Theta - bt^2), \tag{11.07}$$

where

$$a = \frac{\alpha}{4(\alpha^2 + \beta^2)} \qquad b = \frac{\beta}{4(\alpha^2 + \beta^2)} \qquad c = (\alpha^2 + \beta^2)^{1/2}$$
$$\tan 2\Theta = \beta/\alpha = b/a$$

The impulse characteristic obtained with (11.07) in (2.09) becomes

$$P(t) = 2\delta\left(\frac{c}{\pi}\right)^{1/2} e^{-at^2} [\cos(\omega_r t - \psi_r)\cos(\Theta - bt^2) + \sin(\omega_r t - \psi r)\sin(\Theta - bt^2)]. \tag{11.08}$$

From (11.08) it is seen that the envelope is

$$\overline{P}(t) = 2\delta\left(\frac{c}{\pi}\right)^{1/2} e^{-at^2}. \tag{11.09}$$

The peak of the envelope obtained with $t = 0$ is smaller than without delay distortion ($\beta = 0$) by the factor

$$\eta = \frac{1}{[1 + (\beta/\alpha)^2]^{1/2}}. \tag{11.10}$$

The constant a is smaller than without delay distortion by the factor η^4. If t_0 designates the time required for the instantaneous amplitude of a pulse to decay from its peak to a given value without delay distortion, the time t_1 to reach the same amplitude with delay distortion is

$$t_1 = t_0/\eta^2 = t_0[1 + (\beta/\alpha)^2]^{1/2}. \tag{11.11}$$

If ω_{max} indicates the frequency at the 40 db down point on the transmission frequency characteristic, $\alpha\omega_{max}^2 = 4.6$. The corresponding delay distortion is $d_{max} = 2\omega_{max}\beta$. Thus $\beta/\alpha = .68\ d_{max}f_{max}$ so that (11.11) becomes:

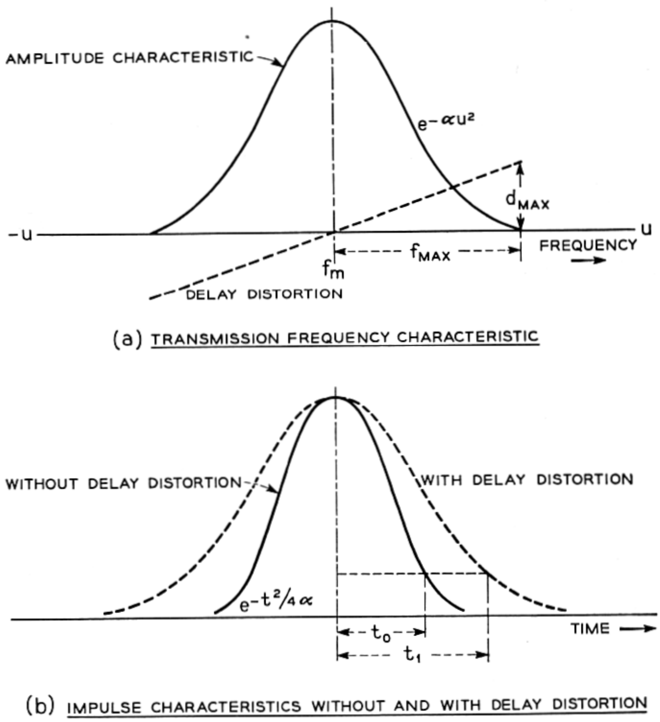
$$t_1 = t_0[1 + 0.46\,(d_{max}f_{max})^2]^{1/2}. \tag{11.12}$$

The effect of a linear delay distortion across the transmission band is thus to disperse or broaden the envelope of the received pulses, as illustrated in Fig. 41. For a specified pulse overlap or intersymbol interference the pulse spacing must accordingly be increased by the factor t_1/t_0, so that for a given transmission performance the transmission capacity is reduced by the factor t_0/t_1. About the same effect would be expected for other pulse shapes or amplitude characteristics resembling the Gaussian shape assumed in the above derivation.

Comparison of (11.08) with (2.13) shows that the function $\varphi(t)$ with respect to the midband frequency is

$$\varphi(t) = \theta - bt^2. \tag{11.13}$$

If the reference or carrier frequency is displaced from the midband by



$t_1 = t_0[1 + 0.46(d_{MAX}\, f_{MAX})^2]^{\frac{1}{2}}$

f_{MAX} = FREQUENCY FROM MIDBAND AT WHICH AMPLITUDE OF TRANSMISSION FREQUENCY CHARACTERISTIC IS REDUCED 40 DECIBELS

d_{MAX} = DELAY DISTORTION AT f_{MAX} IN SECONDS

Fig. 41 — Lengthening of impulse envelope by linear delay distortion for Gaussian transmission characteristic.

ω_y, the in-phase and quadrative components are in accordance with (2.18)

$$\begin{aligned} R_-' + R_+' &= \cos(\theta - bt^2 + \omega_y t - \psi_y)\,\overline{P}(t), \qquad \text{and} \\ Q_-' - Q_+' &= \sin(\theta - bt^2 + \omega_y t - \psi_y)\,\overline{P}(t), \end{aligned} \tag{11.14}$$

where $\psi_y = \beta\omega_y^2$ and $\overline{P}(t)$ is given by (11.09).

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