Mathematical Theory of Laminated Transmission Lines—Part I

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A mathematical analysis is given of the low-loss, broad-band, laminated transmission lines proposed by A. M. Clogston, including both idealized parallel-plane lines and coaxial cables. Part I deals with "Clogston 1" lines, which have laminated conductors with a dielectric, chosen to provide the proper phase velocity for waves on the line, filling the space between the conductors. Part II will treat lines having an arbitrary fraction of their total volume filled with laminations and the rest with dielectric, and will be concerned in particular with "Clogston 2" lines, in which the entire propagation space is occupied by laminated material.

The electromagnetic problem is first formulated in general terms, and then specialized to yield detailed results. The major theoretical questions treated include the determination of the propagation constants and the fields of the principal mode and the higher modes in laminated transmission lines, the choice of optimum proportions for these lines, the calculation of the frequency dependence of attenuation due to the finite thickness of the laminae, the increase in loss caused by improper phase velocity (dielectric mismatch) in Clogston 1 lines and by nonuniformity of the laminated material in Clogston 2 lines, and the effects of dielectric and magnetic dissipation.

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I. INTRODUCTION

A recent theoretical paper[1] by A. M. Clogston presents the very interesting discovery that under certain conditions skin effect losses in the conductors of a transmission line at elevated frequencies can be much reduced by laminating the conducting surfaces, parallel to the direction of current flow, with alternate thin layers of conducting and insulating material. The requirements are that the thickness of each conducting layer must be considerably smaller than the skin depth in the conductor, and the phase velocity of waves on the transmission line must be held very close to a certain critical value, which depends on the relative thicknesses and the electrical properties of the conducting and insulating layers. Under these conditions the "effective skin depth" of the laminated surface is greatly increased; in other words, the eddy currents induced by a high-frequency alternating field will penetrate much farther into such a laminated structure than into a solid conductor, with consequent marked reduction of ohmic losses in the metal. The metal losses can also be made to vary much less with frequency, over a fixed band, than the ordinary skin effect losses, which are known to be very nearly proportional to the square root of frequency.

Clogston goes on to show that a laminated material composed of alternate thin conducting and insulating layers may itself be regarded as a transmission medium. For example, if the space in a coaxial cable which is ordinarily occupied by air or other dielectric be filled with a large number of coaxial cylindrical tubes which are alternately conducting and insulating, the cable will propagate various transmission modes, and under the proper circumstances some of these modes will exhibit lower attenuation constants than the transmission mode in a conventional coaxial cable of the same size at the same frequency.

Experimental verification of Clogston's theory of laminated conductors has been obtained[2] at the Bell Telephone Laboratories, and the transmission properties of a line filled with laminated material have also been measured at these Laboratories and found in reasonable agreement with theory. However experiments with structures as complex as those proposed by Clogston are by no means simple, and the experimental work on laminated conductors is still in an early, exploratory stage. Inasmuch as the experiments are necessarily time-consuming, it has been thought

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[1] A. M. Clogston, *Proc. Inst. Radio Engrs.*, **39,** 767 (1951), and *Bell System Tech. J.*, **30,** 491 (1951). References will be to the *Bell System Technical Journal* article, although except for equation numbers the two papers are identical.

[2] H. S. Black, C. O. Mallinckrodt, and S. P. Morgan, Jr., Proc. Inst. Radio Engrs., **40,** p. 902 (1952).

desirable to carry out simultaneously as complete a theoretical treatment of Clogston-type transmission lines as possible. Clogston's original paper brought out the fundamental ideas by analysis of idealized transmission lines bounded by infinite parallel planes. The present paper considerably extends the theoretical analysis of parallel-plane systems, and also treats laminated transmission lines bounded by coaxial circular cylinders, which are of course the structures of practical engineering interest.

Part I of this paper deals with both plane and coaxial lines having laminated conductors and having the space between the conductors filled with a suitable main dielectric, which may so far as the theory is concerned also be a nonconducting magnetic material. Structures of this type are called "Clogston 1" transmission lines. Although in principle the total space may be divided between the main dielectric and the laminated stacks in any desired ratio, we suppose in Part I that the width of the main dielectric is several times the total thickness of the laminations. When this is true, the principal mode fields in the main dielectric are almost identical to the fields of the transverse electromagnetic (TEM) mode between perfectly conducting planes or cylinders. The phase velocity is controlled by the properties of the main dielectric, while the attenuation constant is determined by the surface impedances of the laminated boundaries (and the dissipation, if any, in the main dielectric). The calculation of the surface impedance of a laminated plane or cylindrical stack is reduced, using the generalized impedance concept developed by Schelkunoff, to the calculation of the input impedance of a chain of transducers with known impedance elements, the chain also being terminated in a known impedance. We are thus able to employ the language and the results of one-dimensional transmission theory to solve our three-dimensional field problem.

In the remaining sections of Part I we introduce various simplifying approximations and special assumptions into the general equations in order to obtain simple and explicit results. We first calculate the propagation constant and the field components of the principal mode under the assumption that the individual conducting laminae are extremely thin compared to the skin depth at the operating frequency, and show that the attenuation constant is substantially independent of frequency so long as this assumption is valid. We then give formulas for the reduction of the effective skin depth in the stacks and the consequent increase of attenuation with frequency when the laminae are of finite thickness. Next we investigate the effect of varying the phase velocity of the line away from the optimum value given by Clogston; and in the last section we discuss losses due to imperfect dielectrics and lossy magnetic materials.

Part II will be largely devoted to transmission lines of the so-called "Clogston 2" type, in which the entire propagation space is filled with the laminated medium, though to a lesser extent we shall also consider transmission lines having an arbitrary fraction of their total volume filled with laminations and the rest with dielectric. We shall first derive expressions for the propagation constant and the fields of the lowest Clogston 2 mode assuming infinitesimally thin laminae, so that the attenuation constant is essentially independent of frequency, and then go on to investigate the transition of the lowest Clogston 1 mode into the lowest Clogston 2 mode as the space occupied by the main dielectric is gradually filled with laminations. We shall also discuss the higher modes which can exist in Clogston 1 and Clogston 2 lines with infinitesimally thin laminae. Next the effect of finite lamina thickness on the variation of attenuation with frequency in a Clogston 2 will be investigated, and then the important question of the influence of nonuniformity of the laminated medium on the transmission properties of the line. We shall conclude with a short section on dielectric and magnetic losses.

Insofar as possible, plane and coaxial lines will be treated together throughout the paper. Since however Bessel functions are not so easy to manipulate as hyperbolic functions, there will be a few cases where explicit formulas are not yet available for the cylindrical geometry. In these cases the formulas derived for the parallel-plane geometry usually provide reasonably good approximations, or if greater accuracy is desired specific examples may be worked out numerically from the fundamental equations in cylindrical coordinates.

The purpose of the present paper is to set up a general mathematical framework for the analysis of laminated transmission lines, and to treat the major theoretical questions which arise in connection with these lines. In view of the length of the mathematical analysis, we have not devoted much space to numerical examples, although a large number of specific formulas are given which may be used to calculate the theoretical performance of almost any Clogston-type line that happens to be of interest. A considerable part of our work is directed toward evaluating the effects of deviations from the ideal Clogston structure. Both theoretical and experimental results suggest that the limitations on the ultimate applications of the Clogston cable are likely to be imposed by practical problems of manufacture. These limitations, however, depend upon engineering questions which we shall not consider here.

II. WAVE PROPAGATION BETWEEN PLANE AND CYLINDRICAL IMPEDANCE SHEETS

We shall consider waves in a homogeneous, isotropic medium of dielectric constant ϵ, permeability μ, and conductivity g (rationalized MKS units). When convenient we shall also describe the medium in terms of the secondary electromagnetic constants σ and η, defined by

$$\sigma = \sqrt{i\omega\mu(g + i\omega\epsilon)}, \qquad \eta = \sqrt{i\omega\mu/(g + i\omega\epsilon)}. \tag{1}$$

The quantity σ is called the intrinsic propagation constant and η the intrinsic impedance of the medium.

We begin by considering structures bounded by infinite planes parallel to the x-z coordinate plane, and we confine our attention to transverse magnetic waves propagating in the z-direction. We assume that the only non-vanishing component of magnetic field is H_x, and that all the fields are independent of x. Then the non-zero field components, written to indicate their dependence on the spatial coordinates, are $H_x(y, z)$, $E_y(y, z)$ and $E_z(y, z)$, the time dependence $e^{i\omega t}$ being understood throughout. The field components are shown in Fig. 1.

The field vectors are connected by Maxwell's two curl equations, which reduce in the present case to

$$\begin{aligned} \partial H_x/\partial z &= (g + i\omega\epsilon)E_y, \\ \partial H_x/\partial y &= -(g + i\omega\epsilon)E_z, \end{aligned} \tag{2}$$

and

$$\partial E_y/\partial z - \partial E_z/\partial y = i\omega\mu H_x. \tag{3}$$

If we eliminate E_y and E_z we get

$$\partial^2 H_x/\partial y^2 + \partial^2 H_x/\partial z^2 = \sigma^2 H_x, \tag{4}$$

where σ is the intrinsic propagation constant defined above. It is easy to see that (4) is satisfied by a wave function of exponential form, say

$$H_x = e^{-\kappa y - \gamma z}, \tag{5}$$

provided that the constants κ and γ are such that

$$\kappa^2 + \gamma^2 = \sigma^2. \tag{6}$$

We may regard κ and γ as the (possibly complex) propagation constants in the y- and z-directions respectively. Either may be chosen at will and the other is then determined by the condition (6). The electric field components corresponding to any particular H_x are easily obtained from equations (2).

A concept important in what follows is that of wave impedances[3] at a point. For a wave whose field components are H_x, E_y, E_z, the wave impedances looking in the positive and negative y- and z-directions at a typical point are defined to be, respectively,

$$\begin{aligned} Z_y^+ &= E_z/H_x, \qquad & Z_z^+ &= -E_y/H_x, \\ Z_y^- &= -E_z/H_x, \qquad & Z_z^- &= E_y/H_x. \end{aligned} \tag{7}$$

For waves of the type that we consider, Z_y^+ and Z_y^- are functions of y only, so that if two media having different electrical properties are separated by the plane $y = y_0$, the continuity of the tangential compo-

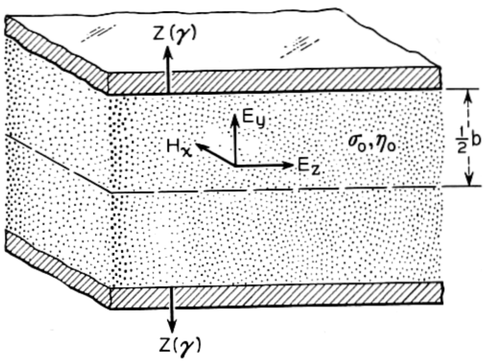

Fig. 1—Transmission line bounded by parallel impedance sheets.

nents of **E** and **H** across the boundary can be assured by merely requiring the continuity of Z_y^+ (say) at $y = y_0$. This is equivalent to the requirement that the sum of the impedances Z_y^+ and Z_y^- looking into the media on opposite sides of the boundary be zero. A similar condition holds for the impedances Z_z^+ and Z_z^- at a boundary $z = z_0$.

As an example of the use of the wave impedance concept, we shall consider the propagation of a transverse magnetic wave between parallel impedance sheets[4] which are separated by a distance b. For the moment nothing is specified about the structure of the sheets except that the normal surface impedance looking into each is $Z(\gamma)$, for a wave whose propagation constant in the z-direction is γ. The fact that in general Z will depend upon γ should be noted, since in some cases this dependence

[3] S. A. Schelkunoff, *Electromagnetic Waves*, D. van Nostrand Co., Inc., New York, 1943, pp. 249–251. Since in our problem three field components vanish identically, we need only two of the six impedances which are defined in the general case.

[4] Reference 3, pp. 484–489.

is quite important. The sheets are located at $y = \pm\frac{1}{2}b$, as shown in Fig. 1, and the space between them is filled with a medium whose electrical constants are ϵ_0, μ_0, g_0 (or σ_0, η_0, if we wish to use the derived constants).

From the symmetry of the boundary conditions it is evident that for any particular mode H_x must be either an even function or an odd function of y about the plane $y = 0$. Taking the even case first, we have

$$\begin{aligned} H_x &= \operatorname{ch} \kappa_0 y\, e^{-\gamma z}, \\ E_y &= -\frac{\gamma}{g_0 + i\omega\epsilon_0} \operatorname{ch} \kappa_0 y\, e^{-\gamma z}, \\ E_z &= -\frac{\kappa_0}{g_0 + i\omega\epsilon_0} \operatorname{sh} \kappa_0 y\, e^{-\gamma z}, \end{aligned} \tag{8}$$

where

$$\kappa_0^2 + \gamma^2 = \sigma_0^2 . \tag{9}$$

If we replace $g_0 + i\omega\epsilon_0$ by σ_0/η_0 and κ_0 by $(\sigma_0^2 - \gamma^2)^{\frac{1}{2}}$, the boundary condition at $y=\frac{1}{2}b$, namely

$$Z_y^+ = Z(\gamma), \tag{10}$$

becomes

$$\tfrac{1}{2}(\sigma_0^2 - \gamma^2)^{\frac{1}{2}}b \tanh \tfrac{1}{2}(\sigma_0^2 - \gamma^2)^{\frac{1}{2}}b = -\frac{\sigma_0 b}{2\eta_0} Z(\gamma). \tag{11}$$

Similarly, the odd case gives

$$\begin{aligned} H_x &= \operatorname{sh} \kappa_0 y\, e^{-\gamma z}, \\ E_y &= -\frac{\gamma}{g_0 + i\omega\epsilon_0} \operatorname{sh} \kappa_0 y\, e^{-\gamma z}, \\ E_z &= -\frac{\kappa_0}{g_0 + i\omega\epsilon_0} \operatorname{ch} \kappa_0 y\, e^{-\gamma z}; \end{aligned} \tag{12}$$

and the boundary condition becomes

$$\tfrac{1}{2}(\sigma_0^2 - \gamma^2)^{\frac{1}{2}}b \coth \tfrac{1}{2}(\sigma_0^2 - \gamma^2)^{\frac{1}{2}}b = -\frac{\sigma_0 b}{2\eta_0} Z(\gamma). \tag{13}$$

The transcendental equations (11) and (13) are satisfied by the propagation constants of the various even and odd modes; presumably each has an infinite number of roots, which we could find, at least in principle, if we knew the explicit form of the function $Z(\gamma)$. We shall confine ourselves here to deriving an approximate expression for the propagation constant of the principal mode (lowest even mode) when the walls are very good conductors.

If the walls were perfectly conducting we should have $Z(\gamma) = 0$, and the lowest root γ_0 of (11) would be given by

$$(\sigma_0^2 - \gamma_0^2)^{\frac{1}{2}} b = 0, \qquad \text{or} \qquad \gamma_0 = \sigma_0 . \tag{14}$$

The principal mode between perfectly conducting sheets is just an undisturbed slice of the plane TEM wave which could propagate in an unbounded medium. If $Z(\gamma_0)$ is not rigorously zero, but still so small that

$$\left| \frac{\sigma_0 b Z(\gamma_0)}{2\eta_0} \right| \ll 1, \tag{15}$$

and if $Z(\gamma)$ does not vary rapidly with γ in the neighborhood of γ_0, then the lowest root of (11) is given approximately by

$$\gamma^2 = \sigma_0^2 + 2\sigma_0 Z(\gamma_0)/\eta_0 b. \tag{16}$$

If $Z(\gamma_0)$ is so small that we have the further inequality

$$\frac{1}{2} \left| \frac{Z(\gamma_0)}{\sigma_0 b \eta_0} \right|^2 \ll 1, \tag{17}$$

then (16) yields the approximation

$$\gamma = \sigma_0 + Z(\gamma_0)/\eta_0 b, \tag{18}$$

where the second term is the first-order change in γ due to the finite impedance of the walls. If we formally set $g_0 = 0$ (this does not actually restrict us to perfect dielectrics since we could still assume ϵ_0 or μ_0 to be complex), we have

$$\sigma = i\omega\sqrt{\mu_0\epsilon_0}, \qquad \eta = \sqrt{\mu_0/\epsilon_0} . \tag{19}$$

If the medium between the sheets is lossless, the attenuation and phase constants of the principal mode become

$$\alpha = \text{Re}\,\gamma = \text{Re}\,Z(\gamma_0)/\eta_0 b, \tag{20}$$

$$\beta = \text{Im}\,\gamma = \omega\sqrt{\mu_0\epsilon_0} + \text{Im}\,Z(\gamma_0)/\eta_0 b. \tag{21}$$

Although the fields of the principal mode between perfectly conducting walls are entirely transverse to the direction of propagation, if the walls are not perfectly conducting there will also be a small longitudinal component E_z of electric field associated with this mode. The leading terms in the expressions for the field components, as obtained from equations (8), (9), and (16), are

$$
\begin{aligned}
H_x &\approx H_0 e^{-\gamma z}, \\
E_y &\approx -\eta_0 H_0 e^{-\gamma z}, \\
E_z &\approx \frac{2Z_0(\gamma_0) H_0 y}{b} \, e^{-\gamma z},
\end{aligned}
\tag{22}
$$

where H_0 is an arbitrary amplitude factor.

As an example of the use of (20) and (21), suppose that the impedance sheets in Fig. 1 are electrically thick metal walls of permeability μ_1 and (high) conductivity g_1. Then to a very good approximation at all engineering frequencies and for all ordinary dielectrics between the walls, the surface impedance is

$$Z(\gamma_0) = (1 + i)/g_1\delta_1 , \tag{23}$$

where

$$\delta_1 = \sqrt{2/\omega\mu_1 g_1} \tag{24}$$

is the skin depth in the metal. We thus obtain from (20) and (21) the familiar formulas

$$\alpha = 1/\eta_0 b g_1 \delta_1 , \tag{25}$$

$$\beta = \omega\sqrt{\mu_0\epsilon_0} + 1/\eta_0 b g_1 \delta_1 . \tag{26}$$

It should be noted that in practical cases the inequality (17) on which we based the approximations (20) and (21) does not hold down to the mathematical limit of zero frequency. In the present paper, however, when we speak of "low frequencies" we shall mean frequencies still high enough so that the approximations (20) and (21) for α and β are valid. Generally this will be equivalent to the assumption that the attenuation per radian is small. In our applications this assumption will usually be justified down to frequencies of the order of a few $\text{kc}\cdot\text{sec}^{-1}$.

Now let us consider transmission lines bounded by coaxial circular cylinders and confine our attention to circular transverse magnetic waves propagating in the z-direction. For these waves the fields are independent of the angle ϕ, and the only non-vanishing field components are $H_\phi(\rho, z)$, $E_\rho(\rho, z)$, and $E_z(\rho, z)$. The field components are shown in Fig. 2.

For circular transverse magnetic fields Maxwell's curl equations in a homogeneous, isoptropic medium reduce to

$$
\begin{aligned}
\partial H_\phi/\partial z &= -(g + i\omega\epsilon)E_\rho , \\
\partial(\rho H_\phi)/\partial\rho &= (g + i\omega\epsilon)\rho E_z ,
\end{aligned}
\tag{27}
$$

and

$$\partial E_z/\partial \rho - \partial E_\rho/\partial z = i\omega\mu H_\phi , \tag{28}$$

from which we can eliminate $E\rho$ and E_z to obtain

$$\frac{\partial^2 H\phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial H_\phi}{\partial \rho} - \frac{H_\phi}{\rho^2} + \frac{\partial^2 H_\phi}{\partial z^2} = \sigma^2 H_\phi. \tag{29}$$

If we assume a wave traveling in the positive z-direction with propagation constant γ and write

$$H_\phi(\rho, z) = R(\rho)e^{-\gamma z}, \tag{30}$$

we find that (29) becomes

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) - \left(\kappa^2 + \frac{1}{\rho^2}\right)R = 0, \tag{31}$$

where κ is given by (6) as before. But (31) is just the equation satisfied by modified Bessel functions of order one and argument $\kappa\rho$, so

$$R(\rho) = AI_1(\kappa\rho) + BK_1(\kappa\rho), \tag{32}$$

where A and B are arbitrary constants. The other field components can be obtained from H_ϕ using (27); the results are

$$\begin{aligned} H_\phi &= [AI_1(\kappa\rho) + BK_1(\kappa\rho)]e^{-\gamma z}, \\ E\rho &= \frac{\gamma}{g + i\omega\epsilon}[AI_1(\kappa\rho) + BK_1(\kappa\rho)]e^{-\gamma z}, \\ E_z &= \frac{\kappa}{g + i\omega\epsilon}[AI_0(\kappa\rho) - BK_0(\kappa\rho)]e^{-\gamma z}. \end{aligned} \tag{33}$$

For cylindrical fields of the type that we are considering, the wave impedances looking in the positive and negative ρ- and z-directions at a typical point are defined to be, respectively,

$$\begin{aligned} Z_\rho^+ &= -E_z/H_\phi , & Z_z^+ &= E_\rho/H_\phi , \\ Z_\rho^- &= E_z/H_\phi , & Z_z^- &= -E_\rho/H_\phi . \end{aligned} \tag{34}$$

We shall now discuss the propagation of circular transverse magnetic waves in a homogeneous region of space whose electrical constants are ϵ_0, μ_0, g_0 (or σ_0, η_0), and which is bounded by coaxial cylinders of radii ρ_1 and ρ_2, where $\rho_2 > \rho_1$, as shown in Fig. 2. We suppose that the radial impedances looking from the main dielectric into the inner and outer cylinders are, respectively,

$$Z_\rho^-\big|_{\rho=\rho_1} = Z_1(\gamma), \qquad Z_\rho^+\big|_{\rho=\rho_2} = Z_2(\gamma). \tag{35}$$

Then from (33) and (34) the boundary conditions are

$$\begin{aligned} \eta_{0\rho} \frac{AI_0(\kappa_0\rho_1) - BK_0(\kappa_0\rho_1)}{AI_1(\kappa_0\rho_1) + BK_1(\kappa_0\rho_1)} &= Z_1(\gamma), \\ \eta_{0\rho} \frac{AI_0(\kappa_0\rho_2) - BK_0(\kappa_0\rho_2)}{AI_1(\kappa_0\rho_2) + BK_1(\kappa_0\rho_2)} &= -Z_2(\gamma), \end{aligned} \tag{36}$$

where

$$\kappa_0 = (\sigma_0^2 - \gamma^2)^{\frac{1}{2}}, \qquad \eta_{0\rho} = \frac{\kappa_0}{g_0 + i\omega\epsilon_0} = \eta_0(1 - \gamma^2/\sigma_0^2)^{\frac{1}{2}}. \tag{37}$$

If equations (36) are to be satisfied by values of A and B which are not both zero, it is easily shown that a necessary and sufficient condition is

$$\frac{\eta_{0\rho}K_0(\kappa_0\rho_1) + Z_1(\gamma)K_1(\kappa_0\rho_1)}{\eta_{0\rho}I_0(\kappa_0\rho_1) - Z_1(\gamma)I_1(\kappa_0\rho_1)} = \frac{\eta_{0\rho}K_0(\kappa_0\rho_2) - Z_2(\gamma)K_1(\kappa_0\rho_2)}{\eta_{0\rho}I_0(\kappa_0\rho_2) + Z_2(\gamma)I_1(\kappa_0\rho_2)}, \tag{38}$$

and (38) is a transcendental equation for the determination of the propagation constants of all the circular magnetic modes in the coaxial line.

As in the discussion of the parallel-plane line, we shall confine our attention to the principal mode and shall assume forthwith that the wall losses are small.[5] Since for the principal mode we expect that γ will be nearly equal to σ_0, we may write γ_0 for σ_0 and evaluate Z_1 and Z_2 at γ_0; and we may replace the modified Bessel functions in (38) by their ap-

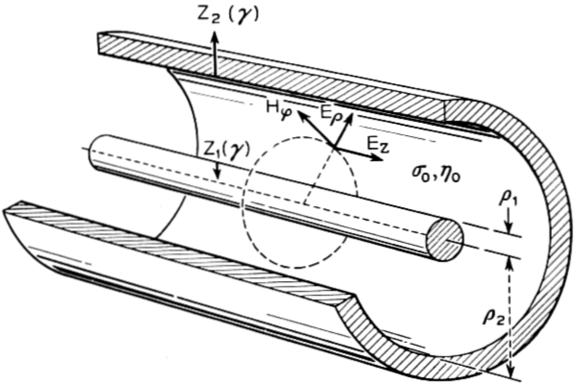

Fig. 2—Transmission line bounded by coaxial impedance cylinders.

[5] J. A. Stratton, *Electromagnetic Theory*, McGraw-Hill, New York, 1941, pp. 551–554, gives a similar treatment of the principal mode in an ordinary coaxial cable with solid metal walls.

proximate values for small argument. From the series given in Dwight[6] 813.1, 813.2, 815.1, and 815.2, we have

$$\begin{aligned} I_0(x) &\approx 1, \\ I_1(x) &\approx \tfrac{1}{2}x, \\ K_0(x) &\approx -(0.5772 + \log \tfrac{1}{2}x) = -\log 0.8905x, \\ K_1(x) &\approx \frac{1}{x} + \tfrac{1}{2}x \log 0.8905x, \end{aligned} \tag{39}$$

for $|x| \ll 1$, where log represents the natural logarithm. If we put these approximations into (38), and if we suppose that the wall impedances are so small that

$$|\sigma_0 \rho_1 Z_1(\gamma_0)/2\eta_0| \ll 1, \qquad |\sigma_0 \rho_2 Z_2(\gamma_0)/2\eta_0| \ll 1, \tag{40}$$

we obtain, after a little algebra,

$$\kappa_0^2 = \sigma_0^2 - \gamma^2 = -\frac{\sigma_0[Z_1(\gamma_0)/\rho_1 + Z_2(\gamma_0)/\rho_2]}{\eta_0 \log(\rho_2/\rho_1)}. \tag{41}$$

Now further assuming that

$$\frac{1}{8}\left|\frac{Z_1(\gamma_0)/\rho_1 + Z_2(\gamma_0)/\rho_2}{\sigma_0\eta_0 \log(\rho_2/\rho_1)}\right|^2 \ll 1, \tag{42}$$

we get by the binomial theorem

$$\gamma = \sigma_0 + \frac{Z_1(\gamma_0)/\rho_1 + Z_2(\gamma_0)/\rho_2}{2\eta_0 \log(\rho_2/\rho_1)}. \tag{43}$$

If we formally set $g_0 = 0$, we find that the attenuation and phase constants of the principal mode in a coaxial line with low-loss walls and no dissipation in the main dielectric are

$$\alpha = \operatorname{Re}\gamma = \operatorname{Re}\frac{Z_1(\gamma_0)/\rho_1 + Z_2(\gamma_0)/\rho_2}{2\eta_0 \log(\rho_2/\rho_1)}, \tag{44}$$

$$\beta = \operatorname{Im}\gamma = \omega\sqrt{\mu_0\epsilon_0} + \operatorname{Im}\frac{Z_1(\gamma_0)/\rho_1 + Z_2(\gamma_0)/\rho_2}{2\eta_0 \log(\rho_2/\rho_1)}. \tag{45}$$

As before, these approximations for α and β will ultimately break down as the frequency approaches zero, but they will certainly be valid over the frequency range in which we are interested in the present paper.

[6] H. B. Dwight, *Tables of Integrals and Other Mathematical Data*, Revised Edition, Macmillan, New York, 1947. We shall refer to Dwight for a number of standard series expansions.

The magnetic field lines of the principal mode will of course be circles and the electric field will be largely radial, but with a small longitudinal component unless the wall impedances are rigorously zero. The general expressions (33) for the fields may be reduced to simple approximate formulas if we use the fact that κ_0^2 is given by (41) and $\kappa_0\rho$ is small compared to unity. The ratio A/B may be obtained from either of equations (36). Introducing the approximations (39) for the Bessel functions and carrying out a little algebra, we get the following approximate expressions for the fields:

$$\begin{aligned} H_\phi &\approx \frac{I}{2\pi\rho}\, e^{-\gamma z}, \\ E_\rho &\approx \frac{\eta_0 I}{2\pi\rho}\, e^{-\gamma z}, \\ E_z &\approx \frac{I}{2\pi \log(\rho_2/\rho_1)} \left[\frac{Z_1(\gamma_0)}{\rho_1} \log\frac{\rho_2}{\rho} + \frac{Z_2(\gamma_0)}{\rho_2}\log\frac{\rho_1}{\rho}\right] e^{-\gamma z}, \end{aligned} \tag{46}$$

where the amplitude factor I is equal to the total current flowing in the inner cylinder. Incidentally we note that the above results might have been derived from more elementary arguments if we had started with the fields in a coaxial line with perfectly conducting walls and treated the effect of finite wall impedance as a small perturbation.

If we consider an ordinary coaxial cable with solid metal walls at a frequency high enough so that there is a well-developed skin effect on both conductors, then to a good approximation

$$Z_1(\gamma_0) = Z_2(\gamma_0) = (1 + i)/g_1\delta_1\,, \tag{47}$$

where g_1 and δ_1 are the conductivity and the skin thickness of the metal; and the attenuation and phase constants are given by the well-known expressions

$$\alpha = \frac{1/\rho_1 + 1/\rho_2}{2\eta_0 g_1 \delta_1 \log(\rho_2/\rho_1)}\,, \tag{48}$$

$$\beta = \omega\sqrt{\mu_0\epsilon_0} + \frac{1/\rho_1 + 1/\rho_2}{2\eta_0 g_1 \delta_1 \log(\rho_2/\rho_1)}\,. \tag{49}$$

If necessary we may take account of dissipation in the main dielectric of either a plane or a coaxial transmission line by assigning complex values[7] to ϵ_0 and μ_0, say

[7] See, for example, C. G. Montgomery, *Principles of Microwave Circuits*, M. I. T. Rad. Lab. Series, **8**, McGraw-Hill, New York, 1948, pp. 365–369 and 382–385.

$$\begin{aligned} \epsilon_0 &= \epsilon_0' - i\epsilon_0'' = \epsilon_0'(1 - i \tan \phi_0), \\ \mu_0 &= \mu_0' - i\mu_0'' = \mu_0'(1 - i \tan \zeta_0), \end{aligned} \tag{50}$$

where $\tan \phi_0$ is the dielectric loss tangent and $\tan \zeta_0$ is the magnetic loss tangent (if any). Inserting (50) into (18) or (43), we find for the attenuation due to dielectric and magnetic losses,

$$\begin{aligned} \alpha_d &= \text{Re } \sigma = \text{Re } i\omega\sqrt{\mu_0'\epsilon_0'(1 - i \tan \phi_0)(1 - i \tan \zeta_0)} \\ &= \tfrac{1}{2}\omega\sqrt{\mu_0'\epsilon_0'}\ (\tan \phi_0 + \tan \zeta_0), \end{aligned} \tag{51}$$

provided that $\tan \phi_0$ and $\tan \zeta_0$ are both small compared to unity, as they will always be in practice. We shall neglect second-order effects and so regard the dielectric losses, the magnetic losses, and the wall losses as additive.

III. SURFACE IMPEDANCE OF A LAMINATED BOUNDARY

The main problem in the theory of Clogston 1 transmission lines is the computation of the surface impedance of a laminated plane or cylindrical boundary having alternate thin layers of conductor and dielectric. Portions of such laminated structures are shown schematically in Figs. 3 and 4. We shall begin with an analysis, similar to Clogston's,[8] of the plane stack. This will lead to a convenient point of view for the treatment of the mathematically more complicated coaxial stack.

Let us consider a wave with field components H_x, E_y, E_z, propagating

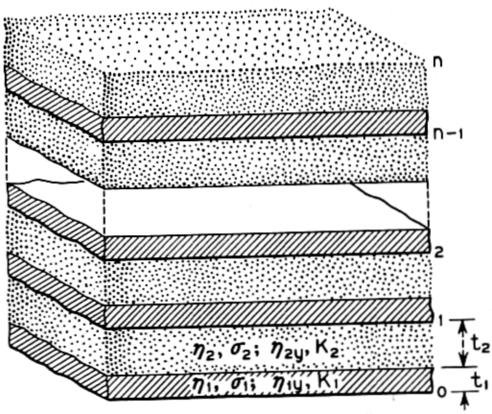

Fig. 3—Portion of laminated plane stack.

[8] Reference 1, Section III.

in a layer of homogeneous, isotropic material whose electrical constants are ϵ, μ, g (or σ, η), and which is bounded by planes perpendicular to the y-axis. Henceforth we shall always assume that the z-dependence of every field component is given by the factor $e^{-\gamma z}$, where the complex quantity γ, whose value may or may not be known a priori, is the propagation constant of the wave in the z-direction. Then the first of Maxwell's equations (2) yields

$$E_y = -[\gamma/(g + i\omega\epsilon)]H_x\,, \tag{52}$$

and on eliminating E_y from the other Maxwell equations, we get

$$\begin{aligned} \partial H_x/\partial y &= -(g + i\omega\epsilon)E_z\,, \\ \partial E_z/\partial y &= -[\kappa^2/(g + i\omega\epsilon)]H_x\,, \end{aligned} \tag{53}$$

where κ^2 is defined by equation (6).

Now if we formally identify H_x with "current" and E_z with "voltage", equations (53) are just the equations of a uniform one-dimensional transmission line extending in the y-direction, with series impedance $\kappa^2/(g + i\omega\epsilon)$ per unit length and shunt admittance $(g + i\omega\epsilon)$ per unit length; in other words a transmission line whose propagation constant is κ and whose characteristic impedance is η_y, where

$$\kappa = \sigma(1 - \gamma^2/\sigma^2)^{\frac{1}{2}}, \qquad \eta_y = \kappa/(g + i\omega\epsilon) = \eta(1 - \gamma^2/\sigma^2)^{\frac{1}{2}}. \tag{54}$$

Hence we can apply the whole theory of one-dimensional transmission lines with the assurance that in so doing we shall not violate the field equations. For example, if $E(0)$, $H(0)$ and $E(t)$, $H(t)$ represent the tangential field components E_z, H_x at two planes separated by a dis-

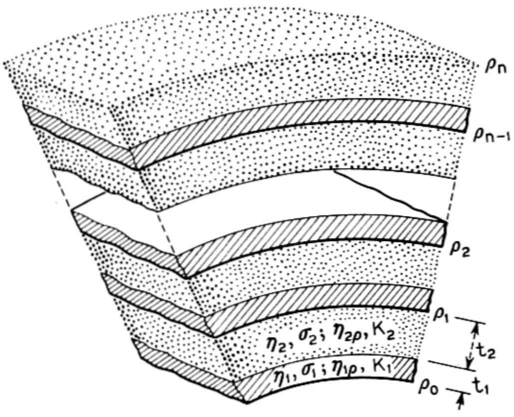

Fig. 4—Portion of laminated coaxial stack.

tance t, these fields are related by the general circuit parameter matrix of a uniform line, namely

$$\begin{pmatrix} E(0) \\ H(0) \end{pmatrix} = \begin{pmatrix} \text{ch } \kappa t & \eta_y \text{ sh } \kappa t \\ \dfrac{\text{sh } \kappa t}{\eta_y} & \text{ch } \kappa t \end{pmatrix} \begin{pmatrix} E(t) \\ H(t) \end{pmatrix}. \tag{55}$$

We are now in a position to determine the surface impedance normal to a laminated plane structure composed of layers of which every other one has thickness t_1 and electrical constants σ_1, η_1, while the intervening layers each have thickness t_2 and electrical constants σ_2, η_2. Fig. 3 shows the cross section of such a stack in which the total number of double layers is n ($2n$ single layers), while Fig. 4 represents the corresponding coaxial stack. Ultimately we shall assume the layers of thickness t_1 to be good conductors and those of thickness t_2 to be good insulators, but these assumptions need not be brought in immediately.

If the fields in the plane stack all vary with z according to $e^{-\gamma z}$, then when we look in the direction of increasing y each double layer may be regarded as a four-terminal network formed by two sections of uniform transmission line of lengths t_1 and t_2, the propagation constants and characteristic impedances of the two sections being given respectively by

$$\begin{aligned} \kappa_1 &= \sigma_1(1 - \gamma^2/\sigma_1^2)^{\frac{1}{2}}, & \eta_{1y} &= \eta_1(1 - \gamma^2/\sigma_1^2)^{\frac{1}{2}}, \\ \kappa_2 &= \sigma_2(1 - \gamma^2/\sigma_2^2)^{\frac{1}{2}}, & \eta_{2y} &= \eta_2(1 - \gamma^2/\sigma_2^2)^{\frac{1}{2}}. \end{aligned} \tag{56}$$

The matrix of the double layer is the product of the matrices of the two single layers in the proper order. Thus if the tangential field components are E_0, H_0 at the lower surface of the first layer and E_1, H_1 at the upper surface of the second layer, we have

$$\begin{pmatrix} E_0 \\ H_0 \end{pmatrix} = \begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{pmatrix} \begin{pmatrix} E_1 \\ H_1 \end{pmatrix}, \tag{57}$$

where

$$\begin{aligned} \mathfrak{A} &= \text{ch } \kappa_1 t_1 \text{ ch } \kappa_2 t_2 + \frac{\eta_{1y}}{\eta_{2y}} \text{ sh } \kappa_1 t_1 \text{ sh } \kappa_2 t_2, \\ \mathfrak{B} &= \eta_{2y} \text{ ch } \kappa_1 t_1 \text{ sh } \kappa_2 t_2 + \eta_{1y} \text{ sh } \kappa_1 t_1 \text{ ch } \kappa_2 t_2, \\ \mathfrak{C} &= \frac{1}{\eta_{1y}} \text{ sh } \kappa_1 t_1 \text{ ch } \kappa_2 t_2 + \frac{1}{\eta_{2y}} \text{ ch } \kappa_1 t_1 \text{ sh } \kappa_2 t_2, \\ \mathfrak{D} &= \frac{\eta_{2y}}{\eta_{1y}} \text{ sh } \kappa_1 t_1 \text{ sh } \kappa_2 t_2 + \text{ch } \kappa_1 t_1 \text{ ch } \kappa_2 t_2. \end{aligned} \tag{58}$$

The stack of double layers may be regarded as a chain of iterated four-poles; such chains have an extensive literature.[9] The relation between the tangential fields E_n, H_n at the upper surface of the nth double layer and E_0, H_0 at the lower surface of the first double layer is

$$\begin{pmatrix} E_0 \\ H_0 \end{pmatrix} = \mathbf{M}^n \begin{pmatrix} E_n \\ H_n \end{pmatrix}, \tag{59}$$

where $\mathbf{M}$ is the $\mathfrak{ABCD}$-matrix appearing in equation (57). However there is a simple expression[10] for the nth power of a square matrix of order two, namely

$$\mathbf{M}^n = M^{\frac{1}{2}(n-1)} \frac{\text{sh } n\Gamma}{\text{sh } \Gamma} \mathbf{M} - M^{\frac{1}{2}n} \frac{\text{sh } (n-1)\Gamma}{\text{sh } \Gamma} \mathbf{I}, \tag{60}$$

where $\mathbf{I}$ is the unit matrix of order two, Γ is the propagation constant per section of the chain of four-poles, defined by

$$\text{ch } \Gamma = (\mathfrak{A} + \mathfrak{D})/2M^{\frac{1}{2}}, \tag{61}$$

and M is the determinant of the matrix $\mathbf{M}$, that is,

$$M = \mathfrak{A}\mathfrak{D} - \mathfrak{B}\mathfrak{C}. \tag{62}$$

The determinant of the matrix whose elements are given by (58) is unity, as may easily be verified; but this may not be the case for all the matrices which occur in our study of cylindrical structures. M will therefore be carried explicitly in the following equations.

We now introduce the iterative impedances K_1 and K_2, defined by

$$\begin{aligned} K_1 &= \frac{(\mathfrak{A} - \mathfrak{D}) + \sqrt{(\mathfrak{A} + \mathfrak{D})^2 - 4M}}{2\mathfrak{C}}, \\ K_2 &= \frac{-(\mathfrak{A} - \mathfrak{D}) + \sqrt{(\mathfrak{A} + \mathfrak{D})^2 - 4M}}{2\mathfrak{C}}. \end{aligned} \tag{63}$$

K_1 is the impedance seen when we look into a semi-infinite stack of double layers if the first layer is of type 1, while K_2 is the impedance seen if the first layer is of type 2. In calculations relating to Clogston 1 lines with dissipative walls, the real parts of K_1 and K_2 will both be positive. By a straightforward procedure we may express the matrix elements $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ in terms of K_1, K_2, Γ, and M, and then transform equation

[9] See, for example, E. A. Guillemin, *Communication Networks*, **2,** Wiley, New York, 1935, pp. 161–166.

[10] F. Abelès, *Comptes Rendus*, **226,** 1872 (1948). This result was called to the author's attention by Mr. J. G. Kreer.

(60) into

$$\mathbf{M}^n = \frac{2M^{\frac{1}{2}n}}{(K_1 + K_2)} \begin{pmatrix} \frac{1}{2}(K_1 e^{n\Gamma} + K_2 e^{-n\Gamma}) & K_1 K_2 \operatorname{sh} n\Gamma \\ \operatorname{sh} n\Gamma & \frac{1}{2}(K_1 e^{-n\Gamma} + K_2 e^{n\Gamma}) \end{pmatrix}. \tag{64}$$

Finally we obtain from (59) and (64) an expression for the impedance Z_0 looking into a plane stack of n double layers when the nth layer is backed by a surface whose impedance is Z_n, namely

$$Z_0 = \frac{E_0}{H_0} = \frac{\frac{1}{2}Z_n(K_1 e^{n\Gamma} + K_2 e^{-n\Gamma}) + K_1 K_2 \operatorname{sh} n\Gamma}{Z_n \operatorname{sh} n\Gamma + \frac{1}{2}(K_1 e^{-n\Gamma} + K_2 e^{n\Gamma})}. \tag{65}$$

For the cylindrical geometry, matters are a good deal more complicated. If we consider waves having field components H_ϕ, E_ρ, E_z in a homogeneous, isotropic shell bounded by coaxial cylindrical surfaces, and assume a propagation factor $e^{-\gamma z}$, Maxwell's equations (27) and (28) may be written

$$E_\rho = [\gamma/(g + i\omega\epsilon)]H_\phi, \tag{66}$$

and

$$\begin{aligned} \partial(-\rho H_\phi)/\partial\rho &= -(g + i\omega\epsilon)\rho E_z, \\ \partial E_z/\partial\rho &= -[\kappa^2/(g + i\omega\epsilon)\rho](-\rho H_\phi). \end{aligned} \tag{67}$$

If desired, we might identify E_z with "voltage" and $-\rho H_\phi$ with "current" and regard equations (67) as describing a nonuniform radial transmission line, having series impedance $\kappa^2/(g + i\omega\epsilon)\rho$ per unit length and shunt admittance $(g + i\omega\epsilon)\rho$ per unit length. Since, however, in equations (34) we have already defined the radial wave impedance to be a field ratio without the extra factor of ρ, we shall carry out the analysis of the present paper directly in terms of the field components E_z and $-H_\phi$.

From the general expressions (33) for the fields in cylindrical coordinates, we can show that the matrix relation between the tangential field components E_z, $-H_\phi$ at two radii ρ_1 and ρ_2 is given by

$$\begin{pmatrix} E(\rho_1) \\ -H(\rho_1) \end{pmatrix} = \begin{pmatrix} \kappa\rho_2(K_{01}I_{12} + K_{12}I_{01}) & \eta_\rho\kappa\rho_2(K_{01}I_{02} - K_{02}I_{01}) \\ \frac{\kappa\rho_2}{\eta_\rho}(K_{11}I_{12} - K_{12}I_{11}) & \kappa\rho_2(K_{11}I_{02} + K_{02}I_{11}) \end{pmatrix} \begin{pmatrix} E(\rho_2) \\ -H(\rho_2) \end{pmatrix}, \tag{68}$$

where

$$\kappa = (\sigma^2 - \gamma^2)^{\frac{1}{2}}, \qquad \eta_\rho = \eta(1 - \gamma^2/\sigma^2)^{\frac{1}{2}}, \tag{69}$$

and we have used the abbreviations

$$I_{rs} = I_r(\kappa\rho_s), \qquad K_{rs} = K_r(\kappa\rho_s). \tag{70}$$

It may be verified that the determinant M of the square matrix appearing in (68) is simply

$$M = \rho_2/\rho_1 . \tag{71}$$

In principle equation (68) permits us to determine by matrix multiplication the relation between the tangential fields at the inner and outer surfaces of a coaxial double layer, or of a laminated stack of any number of double layers, such as is shown in Fig. 4. The difficulty is that the elements of the matrix of a single layer are not functions only of the electrical properties of the layer and its thickness, but depend in a more complicated way on the inner and outer radii separately. Whereas in the plane case we had merely to take the nth power of a single matrix, we are now faced with the problem of multiplying together n matrices, each of which differs more or less from all the others. An exact expression for the result is practically out of the question; but we can make some reasonable approximations if we assume that each individual layer is thin compared to its mean radius, so that the matrix elements do not change much from one layer to the next.

If the thickness t $(= \rho_2 - \rho_1)$ of a single layer is small compared to ρ_1, then the Bessel function combinations appearing in (68) may be expanded in series, as shown in Appendix I, and the circuit parameter matrix takes the following approximate form,

$$\begin{pmatrix} \left[1 + \dfrac{t}{2\rho_1}\right] \operatorname{ch} \kappa t - \dfrac{1}{2\kappa\rho_1} \operatorname{sh} \kappa t & \eta_\rho \left[1 + \dfrac{t}{2\rho_1}\right] \operatorname{sh} \kappa t \\ \dfrac{1}{\eta_\rho} \left[1 + \dfrac{t}{2\rho_1}\right] \operatorname{sh} \kappa t & \left[1 + \dfrac{t}{2\rho_1}\right] \operatorname{ch} \kappa t + \dfrac{1}{2\kappa\rho_1} \operatorname{sh} \kappa t \end{pmatrix}, \tag{72}$$

where terms of the order of t/ρ_1 represent the first-order curvature corrections. If we use the same value of ρ_1, say $\bar{\rho}$, for both parts of a double layer, then up to first order the elements of the matrix of the double layer become

$$\begin{aligned}
\mathcal{A} &= \left[1 + \frac{t_1 + t_2}{2\bar{\rho}}\right]\left[\text{ch } \kappa_1 t_1 \text{ ch } \kappa_2 t_2 + \frac{\eta_{1\rho}}{\eta_{2\rho}} \text{ sh } \kappa_1 t_1 \text{ sh } \kappa_2 t_2\right] \\
&\qquad - \left[\frac{1}{2\kappa_1\bar{\rho}} \text{ sh } \kappa_1 t_1 \text{ ch } \kappa_2 t_2 + \frac{1}{2\kappa_2\bar{\rho}} \text{ ch } \kappa_1 t_1 \text{ sh } \kappa_2 t_2\right], \\
\mathcal{B} &= \left[1 + \frac{t_1 + t_2}{2\bar{\rho}}\right]\left[\eta_{2\rho} \text{ ch } \kappa_1 t_1 \text{ sh } \kappa_2 t_2 + \eta_{1\rho} \text{ sh } \kappa_1 t_1 \text{ ch } \kappa_2 t_2\right] \\
&\qquad + \left[\frac{\eta_{1\rho}}{2\kappa_2\bar{\rho}} - \frac{\eta_{2\rho}}{2\kappa_1\bar{\rho}}\right] \text{sh } \kappa_1 t_1 \text{ sh } \kappa_2 t_2, \\
\mathcal{C} &= \left[1 + \frac{t_1 + t_2}{2\bar{\rho}}\right]\left[\frac{1}{\eta_{1\rho}} \text{ sh } \kappa_1 t_1 \text{ ch } \kappa_2 t_2 + \frac{1}{\eta_{2\rho}} \text{ ch } \kappa_1 t_1 \text{ sh } \kappa_2 t_2\right] \\
&\qquad + \left[\frac{1}{2\eta_{2\rho}\kappa_1\bar{\rho}} - \frac{1}{2\eta_{1\rho}\kappa_2\bar{\rho}}\right] \text{sh } \kappa_1 t_1 \text{ sh } \kappa_2 t_2, \\
\mathcal{D} &= \left[1 + \frac{t_1 + t_2}{2\bar{\rho}}\right]\left[\frac{\eta_{2\rho}}{\eta_{1\rho}} \text{ sh } \kappa_1 t_1 \text{ sh } \kappa_2 t_2 + \text{ch } \kappa_1 t_1 \text{ ch } \kappa_2 t_2\right] \\
&\qquad + \left[\frac{1}{2\kappa_1\bar{\rho}} \text{ sh } \kappa_1 t_1 \text{ ch } \kappa_2 t_2 + \frac{1}{2\kappa_2\bar{\rho}} \text{ ch } \kappa_1 t_1 \text{ sh } \kappa_2 t_2\right].
\end{aligned} \tag{73}$$

As in the analogous equations (58) for a plane double layer, the subscripts 1 and 2 refer to the first and second layers respectively.

If we have a stack of double layers in which all the layers of the same kind have the same thickness and same electrical constants, then the only term in (73) which varies from one double layer to the next is the mean radius $\bar{\rho}$. Depending on the circumstances, we may wish to use a single value of $\bar{\rho}$ for the whole stack, or a few different values, or even, if high-speed computing machinery is available to carry out the matrix multiplications, a different value of $\bar{\rho}$ for each double layer. The matrix of the whole stack then becomes a product of powers of as many different matrices as we have chosen values of $\bar{\rho}$. Obviously this method is better adapted to the numerical analysis of special cases than to the general theoretical treatment of a stack whose ratio of outer radius to inner radius is unspecified.

In principle we are now able to compute the normal surface impedance of any laminated plane or coaxial stack at a given frequency provided that we know the electrical constants and the thickness of each layer, the number of layers, the propagation constant γ in the z-direction, and the normal impedance Z_n of the material behind the last layer. Since the general formulas even for plane stacks are quite complicated, however, we shall introduce at this point some very good approximations which will be valid for all of the following work.

Henceforth we shall take the layers of thickness t_1 to be such good conductors that the ratio $\omega\epsilon_1/g_1$ of displacement current to conduction current is negligible in comparison with unity. For metals like copper this is an excellent approximation at even the highest engineering frequencies. Then on introducing the characteristic skin thickness δ_1, we have for the conducting layers,

$$\begin{aligned} \sigma_1 &= \sqrt{i\omega\mu_1 g_1} = (1 + i)/\delta_1, \\ \eta_1 &= \sqrt{i\omega\mu_1/g_1} = (1 + i)/g_1\delta_1, \end{aligned} \tag{74}$$

where

$$\delta_1 = \sqrt{2/\omega\mu_1 g_1}\,. \tag{75}$$

For pure copper the permeability and conductivity are

$$\begin{aligned} \mu_1 &= 1.257 \times 10^{-6} \text{ henrys}\cdot\text{meter}^{-1}, \\ g_1 &= 5.800 \times 10^{7} \text{ mhos}\cdot\text{meter}^{-1}, \end{aligned} \tag{76}$$

from which we obtain the numerical values

$$\begin{aligned} \sigma_1 &= 1.513 \times 10^{4}\,(1 + i)\sqrt{f_{\mathrm{Mc}}} \text{ meters}^{-1}, \\ \eta_1 &= 2.609 \times 10^{-4}\,(1 + i)\sqrt{f_{\mathrm{Mc}}} \text{ ohms}, \end{aligned} \tag{77}$$

and

$$\delta_1 = \frac{6.609 \times 10^{-5}}{\sqrt{f_{\mathrm{Mc}}}} \text{ meters} = \frac{2.602}{\sqrt{f_{\mathrm{Mc}}}} \text{ mils}, \tag{78}$$

where f_{Mc} is the frequency in $\mathrm{Mc}\cdot\mathrm{sec}^{-1}$. Referring to equations (56) and (69) and bearing in mind the above numerical values, we see that for the conducting layers we have

$$\begin{aligned} \kappa_1 &\approx \sigma_1 = (1 + i)/\delta_1, \\ \eta_{1y} &= \eta_{1\rho} \approx \eta_1 = (1 + i)/g_1\delta_1, \end{aligned} \tag{79}$$

to a very good approximation, since in our applications the quantity γ will always be of the order of $2\pi i/\lambda_v$, where the vacuum wavelength λ_v is at least a few meters, while the skin thickness δ_1 will be at most a small fraction of a centimeter.

For the insulating layers of thickness t_2 we shall set the conductivity g_2 equal to zero, so that

$$\sigma_2 = i\omega\sqrt{\mu_2\epsilon_2}\,, \qquad \eta_2 = \sqrt{\mu_2/\epsilon_2}\,. \tag{80}$$

We denote the *relative* dielectric constant and permeability by ϵ_{2r} and μ_{2r} respectively; dissipation in the insulating layers may be included if necessary by making ϵ_{2r} and/or μ_{2r} complex. In MKS units we have

$$\epsilon_2 = \epsilon_{2r}\epsilon_v , \qquad \mu_2 = \mu_{2r}\mu_v , \tag{81}$$

where the electrical constants of vacuum are

$$\begin{aligned} \epsilon_v &= 8.854 \times 10^{-12} \text{ farads}\cdot\text{meter}^{-1}, \\ \mu_v &= 1.257 \times 10^{-6} \text{ henrys}\cdot\text{meter}^{-1}. \end{aligned} \tag{82}$$

It follows that

$$\begin{aligned} \sigma_2 &= \sigma_v\sqrt{\mu_{2r}\epsilon_{2r}} = \frac{2\pi i\sqrt{\mu_{2r}\epsilon_{2r}}}{\lambda_v} = \frac{2\pi i f_{\text{Mc}}\sqrt{\mu_{2r}\epsilon_{2r}}}{299.8} \text{ meters}^{-1}, \\ \eta_2 &= \eta_v\sqrt{\mu_{2r}/\epsilon_{2r}} = 376.7\sqrt{\mu_{2r}/\epsilon_{2r}} \text{ ohms}, \end{aligned} \tag{83}$$

where as usual the subscript v refers to vacuum. It is clear that unless we deal with ferromagnetics, the quantities σ_2 and η_2 will be of roughly the same order of magnitude as σ_v and η_v . From (56) and (69) we have

$$\begin{aligned} \kappa_2 &= \sigma_2(1 - \gamma^2/\sigma_2^2)^{\frac{1}{2}}, \\ \eta_{2y} &= \eta_{2p} = \eta_2(1 - \gamma^2/\sigma_2^2)^{\frac{1}{2}}, \end{aligned} \tag{84}$$

where since σ_2 and γ are both of the same order of magnitude as $2\pi i/\lambda_v$, in general no further approximations can be made.

In all of what follows we shall assume that the thickness t_2 of each insulating layer is very small compared to the vacuum wavelength at the highest operating frequency; in practice t_2 will be at most a few mils and λ_v at least a few meters. Then the quantity $|\kappa_2 t_2|$, which is of the order of $2\pi t_2/\lambda_v$, will be so small that to an excellent approximation we may set sh $\kappa_2 t_2 = \kappa_2 t_2$ and ch $\kappa_2 t_2 = 1$. Using this simplification, together with the fact that $\eta_{1y} \ll \eta_{2y}$ for all frequencies which may conceivably be of interest, it is not difficult to show from (58) that the matrix elements of the plane double layer reduce to

$$\begin{aligned} \mathcal{A} &= \text{ch } \kappa_1 t_1 , \\ \mathcal{B} &= \eta_{2y}\kappa_2 t_2 \text{ ch } \kappa_1 t_1 + \eta_{1y} \text{ sh } \kappa_1 t_1 , \\ \mathcal{C} &= \frac{1}{\eta_{1y}} \text{ sh } \kappa_1 t_1 , \\ \mathcal{D} &= \frac{\eta_{2y}\kappa_2 t_2}{\eta_{1y}} \text{ sh } \kappa_1 t_1 + \text{ch } \kappa_1 t_1 . \end{aligned} \tag{85}$$

The determinant of the matrix is unity, and from (61) the propagation constant per section is defined by

$$\text{ch}\,\Gamma = \frac{\eta_{2y}\kappa_2 t_2}{2\eta_{1y}}\,\text{sh}\,\kappa_1 t_1 + \text{ch}\,\kappa_1 t_1\,, \tag{86}$$

while from (63) the iterative impedances are

$$\begin{aligned} K_1 &= -\tfrac{1}{2}\eta_{2y}\kappa_2 t_2 + \sqrt{(\tfrac{1}{2}\eta_{2y}\kappa_2 t_2)^2 + \eta_{1y}\eta_{2y}\kappa_2 t_2 \coth \kappa_1 t_1 + \eta_{1y}^2}\,, \\ K_2 &= +\tfrac{1}{2}\eta_{2y}\kappa_2 t_2 + \sqrt{(\tfrac{1}{2}\eta_{2y}\kappa_2 t_2)^2 + \eta_{1y}\eta_{2y}\kappa_2 t_2 \coth \kappa_1 t_1 + \eta_{1y}^2}\,. \end{aligned} \tag{87}$$

If we make the same simplifications in the approximate expressions (73) for the matrix elements of a coaxial double layer, we obtain

$$\begin{aligned} \mathfrak{A} &= \left[1 + \frac{t_1}{2\bar{\rho}}\right]\text{ch}\,\kappa_1 t_1 - \frac{1}{2\kappa_1\bar{\rho}}\,\text{sh}\,\kappa_1 t_1\,, \\ \mathfrak{B} &= \left[1 + \frac{t_1 + t_2}{2\bar{\rho}}\right]\eta_{2\rho}\kappa_2 t_2\,\text{ch}\,\kappa_1 t_1 \\ &\qquad + \left[1 + \frac{t_1}{2\bar{\rho}} + \left(2 - \frac{\eta_{2\rho}\kappa_2}{\eta_{1\rho}\kappa_1}\right)\frac{t_2}{2\bar{\rho}}\right]\eta_{1\rho}\,\text{sh}\,\kappa_1 t_1\,, \\ \mathfrak{C} &= \left[1 + \frac{t_1}{2\bar{\rho}}\right]\frac{1}{\eta_{1\rho}}\,\text{sh}\,\kappa_1 t_1\,, \\ \mathfrak{D} &= \left[1 + \frac{t_1 + t_2}{2\bar{\rho}} + \frac{\eta_{1\rho}}{2\eta_{2\rho}\kappa_1\kappa_2 t_2\bar{\rho}}\right]\frac{\eta_{2\rho}\kappa_2 t_2}{\eta_{1\rho}}\,\text{sh}\,\kappa_1 t_1 \\ &\qquad + \left[1 + \frac{t_1 + 2t_2}{2\bar{\rho}}\right]\text{ch}\,\kappa_1 t_1\,. \end{aligned} \tag{88}$$

In the preceding equations no restrictions have been laid on the thicknesses t_1 and t_2 except the trivial requirement that t_2 shall be small compared to a wavelength. We shall now consider the limiting case in which both t_1 and t_2 are infinitesimally small. When we make this last and most drastic approximation we do not expect that the idealized structure thus obtained will show all of the features which are of interest in a physical transmission line with finite layers; but the results of the simplified analysis will be useful in some cases nevertheless. It need scarcely be pointed out that we are dealing here only with a mathematical limiting process, in which we assume that each layer, no matter how thin, always exhibits the same electrical properties as the bulk material. If this assumption be regarded as unrealistic, it may be observed that the quantity which we actually allow to tend to zero is the ratio of layer thickness to skin depth. The skin depth may be made as large as desired by lowering the frequency, so that the formulas which we derive by letting t_1 and t_2 approach zero at a finite frequency will also hold for finite thicknesses if the frequency is sufficiently low.

We shall let θ denote the fraction of the stack which is occupied by conducting material, so that

$$\theta = t_1/(t_1 + t_2), \tag{89}$$

where at present t_1 and t_2 are both infinitesimal. Then the stack may be regarded as a homogeneous, anisotropic medium, characterized by an average dielectric constant $\bar{\epsilon}$ perpendicular to the layers, an average permeability $\bar{\mu}$ parallel to the layers, and an average conductivity $\bar{g}$ parallel to the layers. Sakurai[11] has treated such an artificial anisotropic medium, and from his formulas we find that when the layers are alternately conductors and insulators, the average electrical constants are, to a very good approximation,

$$\begin{aligned} \bar{\epsilon} &= \epsilon_2/(1 - \theta), \\ \bar{\mu} &= \theta\mu_1 + (1 - \theta)\mu_2, \\ \bar{g} &= \theta g_1 . \end{aligned} \tag{90}$$

Sakurai has also shown that the average values of the electrical constants may be used in Maxwell's equations for the average (macroscopic) fields, due regard being paid to the orientations of the field vectors with respect to the laminae.

For the plane stack, these equations read

$$\begin{aligned} \partial\bar{H}_x/\partial z &= i\omega\bar{\epsilon}\bar{E}_y , \\ \partial\bar{H}_x/\partial y &= -\bar{g}\bar{E}_z , \\ \partial\bar{E}_y/\partial z - \partial\bar{E}_z/\partial y &= i\omega\bar{\mu}\bar{H}_x , \end{aligned} \tag{91}$$

where the bars denote average values. By analysis exactly similar to that carried out at the beginning of this section for a homogeneous, isotropic medium, we may find the relation between the tangential field components E_z, H_x at the two surfaces of a stack of infinitesimally thin layers. (The bars representing average values may be omitted, since the tangential components of **E** and **H** are continuous across the boundaries of the layers.) We obtain a matrix relation analogous to (55), namely

$$\begin{pmatrix} E(0) \\ \\ H(0) \end{pmatrix} = \begin{pmatrix} \text{ch } \Gamma_t s & K \text{ sh } \Gamma_t s \\ \\ \dfrac{1}{K} \text{ sh } \Gamma_t s & \text{ch } \Gamma_t s \end{pmatrix} \begin{pmatrix} E(s) \\ \\ H(s) \end{pmatrix}, \tag{92}$$

[11] T. Sakurai, *J. Phys. Soc. Japan*, **5**, 394 (1950), especially Section 3.

where s is the thickness of the stack. The propagation constant Γ_ℓ per unit distance normal to the stack and the characteristic impedance K of the stack are given by

$$\Gamma_\ell = \left[\frac{i\bar{g}}{\omega\bar{\epsilon}}\,(\omega^2\bar{\mu}\bar{\epsilon} + \gamma^2)\right]^{\frac{1}{2}}, \tag{93}$$

$$K = \Gamma_\ell/\bar{g} = \left[\frac{i}{\omega\bar{\epsilon}\bar{g}}\,(\omega^2\bar{\mu}\bar{\epsilon} + \gamma^2)\right]^{\frac{1}{2}}. \tag{94}$$

Γ_ℓ and K may also be derived from equations (86) and (87) by limiting processes; we have

$$\Gamma_\ell = \lim_{t_1+t_2\to 0} \Gamma/(t_1 + t_2), \tag{95}$$

$$K = \lim_{t_1+t_2\to 0} K_1 = \lim_{t_1+t_2\to 0} K_2\,. \tag{96}$$

It should perhaps be noted that terms of the order of $\omega\epsilon_1/g_1$ and $\omega\epsilon_2/g_1$ compared to unity were omitted in the expressions (90) for $\bar{\epsilon}$ and $\bar{g}$, and in the derivations of Γ_ℓ and K. Since, however, under all practical circumstances the omitted terms appear to be insignificant, we shall not take space to write out the formally more complicated results which would be obtained by keeping them.[12]

In a cylindrical stack of infinitesimal layers, the average fields satisfy

$$\begin{aligned} \partial\bar{H}_\phi/\partial z &= -i\omega\bar{\epsilon}\bar{E}_\rho\,, \\ \partial(\rho\bar{H}_\phi)/\partial\rho &= \bar{g}\rho\bar{E}_z\,, \\ \partial\bar{E}_z/\partial\rho - \partial\bar{E}_\rho/\partial z &= i\omega\bar{\mu}\bar{H}_\phi\,. \end{aligned} \tag{97}$$

The relation between the tangential field components E_z, $-H_\phi$ at two radii ρ_0 and ρ_n is expressed by a matrix equation analogous to (68), namely

$$\begin{pmatrix} E(\rho_0) \\ -H(\rho_0) \end{pmatrix} = \begin{pmatrix} \Gamma_\ell\rho_n(K_{00}I_{1n} + K_{1n}I_{00}) & K\Gamma_\ell\rho_n(K_{00}I_{0n} - K_{0n}I_{00}) \\ \dfrac{\Gamma_\ell\rho_n}{K}\,(K_{10}I_{1n} - K_{1n}I_{10}) & \Gamma_\ell\rho_n(K_{10}I_{0n} + K_{0n}I_{10}) \end{pmatrix} \begin{pmatrix} E(\rho_n) \\ -H(\rho_n) \end{pmatrix}, \tag{98}$$

[12] In Reference 1, equations (II–17) through (II-26) give examples of equations in which these small terms have been retained.

where

$$I_{rs} = I_r(\Gamma_t \rho_s), \qquad K_{rs} = K_r(\Gamma_t \rho_s), \tag{99}$$

and Γ_t and K are given, as in the plane case, by (93) and (94).

IV. PRINCIPAL MODE IN CLOGSTON 1 LINES WITH INFINITESIMALLY THIN LAMINAE

An idealized parallel-plane Clogston 1 transmission line is shown schematically in Fig. 5. It consists of a slab of dielectric of thickness b, with electrical constants μ_0, ϵ_0, bounded above and below by laminated stacks each of thickness s. Outside each stack there may be an insulating or a conducting sheath, of which nothing more will be assumed at present than that its normal surface impedance $Z_n(\gamma)$ is known. The total distance between the sheaths will be denoted by a, where $a = b + 2s$.

The corresponding Clogston 1 coaxial line is shown in Fig. 6. We denote the thickness of the inner and outer stacks by s_1 and s_2 respectively, while a is the radius of the inner core (if any), and b is the inner radius of the sheath around the outer stack. The inner and outer radii of the main dielectric are $\rho_1 = a + s_1$ and $\rho_2 = b - s_2$, respectively. In practice the core may be a dielectric rod and the sheath may be a conducting shield, but in the present theoretical analysis we shall merely assume that the radial impedances $Z_a(\gamma)$ and $Z_b(\gamma)$ looking into the core and the sheath are known.

In Part I of this paper we shall deal with "extreme" Clogston 1 lines, in which the space occupied by the stacks is small compared to the space occupied by the main dielectric. We may then regard the laminated boundaries as impedance sheets guiding waves whose phase velocity is

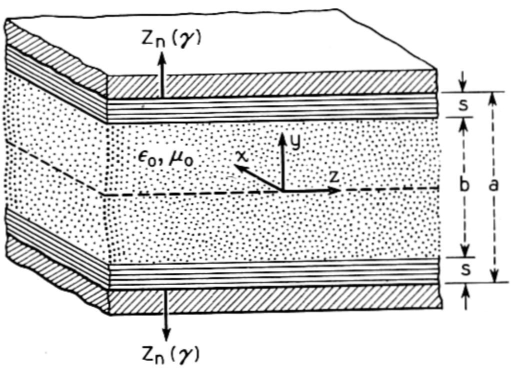

Fig. 5—Parallel-plane Clogston 1 transmission line.

determined by the properties of the main dielectric, as discussed in Section II, and we may use the intrinsic propagation constant of the main dielectric in calculating the surface impedance of the boundaries. This approximation simplifies the analysis of Clogston 1 lines a great deal. We shall treat the general case, in which an arbitrary fraction of the total space is filled with laminations, in Section IX of Part II, as a part of our study of Clogston 2 lines.

In this section we shall assume that the laminae are infinitesimally thin, so that the stacks may be completely characterized by their average properties $\bar{\epsilon}$, $\bar{\mu}$, and $\bar{g}$. The case of finite laminae will be taken up in the next section. We shall also assume throughout that dielectric and magnetic dissipation may be neglected except, as in Section VII, where the contrary is explicitly stated.

In general the current density and the other field quantities in a plane stack of infinitesimally thin layers will be linear combinations of the functions sh $\Gamma_\ell y$ and ch $\Gamma_\ell y$, where y is distance measured into the stack, and Γ_ℓ is the propagation constant per unit distance, as given by (93). The qualitative behavior of the fields in a cylindrical stack will be similar. In particular, if the stack is thick enough the current density and the fields will fall off as $e^{-\Gamma_\ell y}$, and we can define an "effective skin depth" Δ by

$$\Delta = 1/(\mathrm{Re}\ \Gamma_\ell)\,. \tag{100}$$

Clogston's fundamental observation was that in order to minimize the

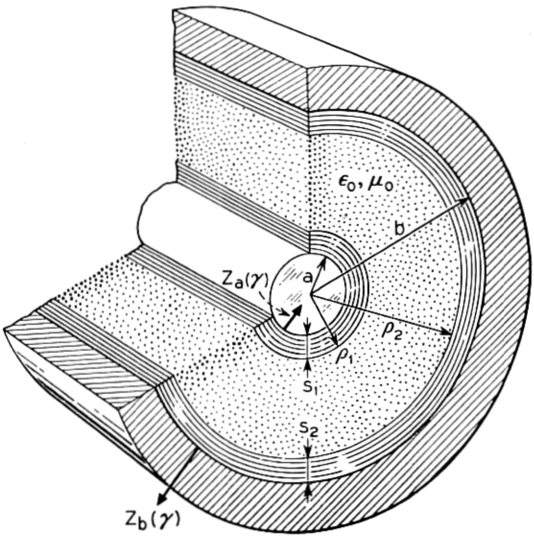

Fig. 6—Coaxial Clogston 1 transmission line.

ohmic losses in a stack carrying a fixed total current the current density should be uniform across the stack, and that we can achieve uniform current density by adjusting the $\mu_0\epsilon_0$ product of the main dielectric so as to make Γ_ℓ equal to zero. If in equation (93) we set

$$\gamma = \gamma_0 = i\omega\sqrt{\mu_0\epsilon_0}\,, \tag{101}$$

then Γ_ℓ will be zero if

$$\mu_0\epsilon_0 = \bar{\mu}\bar{\epsilon} = [\theta\mu_1 + (1 - \theta)\mu_2][\epsilon_2/(1 - \theta)]. \tag{102}$$

Equation (102) will be referred to henceforth as *Clogston's condition*. If the permeabilities of the various materials are all equal, the condition reduces to

$$\epsilon_0 = \bar{\epsilon} = \epsilon_2/(1 - \theta)\,, \tag{103}$$

which is the form employed by Clogston in Reference 1.

When Clogston's condition is satisfied, $\Gamma_\ell = 0$ and the effective skin depth of the stack is infinite;[13] that is, the current density is uniform in any stack of finite total thickness. The quantities Γ_ℓ and K vanish simultaneously, but the limiting value of their ratio is finite; and the matrix of the plane stack, as given by (92), takes the form

$$\begin{pmatrix} 1 & 0 \\ \bar{g}s & 1 \end{pmatrix}. \tag{104}$$

Accordingly we obtain, for the surface impedance $Z_0(\gamma_0)$ of the stack,

$$Z_0(\gamma_0) = \frac{1}{\bar{g}s + 1/Z_n(\gamma_0)}\,, \tag{105}$$

which is, as might have been expected, just the impedance between opposite edges of a unit square of material of conductivity $\bar{g}$ and thickness s through which the current density is uniform, in parallel with the sheath impedance $Z_n(\gamma_0)$. It follows from equations (20) and (21) of Section II that the attenuation and phase constants of the principal mode in a plane Clogston 1 line with infinitesimally thin laminae, Clogston's condition being satisfied exactly, are

[13] This statement is certainly accurate enough for all practical purposes, although an exact calculation which takes into account the small terms that were neglected in the approximate formula (93) for Γ_ℓ shows that the effective skin depth is $\lambda_0/2\pi\theta$, where λ_0 is the length of a free wave in the main dielectric. The exact result is derived by Clogston in Reference 1, equation (II-26). In practice, finite lamina thickness will restrict us to effective skin depths much smaller than this theoretical limit.

$$\alpha = \mathrm{Re}\, \frac{1}{\eta_0 b[\bar{g}s + 1/Z_n(\gamma_0)]}, \tag{106}$$

$$\beta = \omega\sqrt{\mu_0\epsilon_0} + \mathrm{Im}\, \frac{1}{\eta_0 b[\bar{g}s + 1/Z_n(\gamma_0)]}. \tag{107}$$

In general the sheath impedance $Z_n(\gamma_0)$ will be large compared to the impedance $1/\bar{g}s$ of the stack, since even if the sheath is an electrically thick metal plate of the same material as the conducting layers, its impedance is

$$Z_n(\gamma_0) = (1 + i)/g_1\delta_1, \tag{108}$$

whereas θs will usually be several times the skin thickness δ_1 in the frequency range of interest. If the sheath is free space, its impedance is a fortiori much greater than $1/\bar{g}s$, since then it may be shown that

$$Z_n(\gamma_0) = -i\eta_v(\mu_{0r}\epsilon_{0r} - 1)^{\frac{1}{2}}, \tag{109}$$

where $\eta_v = 376.7$ ohms is the intrinsic impedance of free space, and μ_{0r} and ϵ_{0r} are the relative permeability and relative dielectric constant of the main dielectric. Under most circumstances, therefore, we may neglect $1/Z_n(\gamma_0)$ in comparison with $\bar{g}s$, and obtain the very simple results,

$$\alpha = 1/\eta_0 b\bar{g}s, \tag{110}$$

$$\beta = \omega\sqrt{\mu_0\epsilon_0}. \tag{111}$$

To this approximation the line exhibits neither amplitude nor phase distortion.

For a coaxial stack of infinitesimally thin layers with Clogston's condition satisfied, the stack matrix given in (98) reduces to

$$\begin{pmatrix} 1 & 0 \\ \frac{\bar{g}}{2\rho_0}(\rho_n^2 - \rho_0^2) & \frac{\rho_n}{\rho_0} \end{pmatrix}, \tag{112}$$

where ρ_0 and ρ_n denote the inner and outer radii of the stack. It follows from (112) that

$$\begin{aligned} \frac{Z_1(\gamma_0)}{\rho_1} &= \frac{1}{\frac{1}{2}\bar{g}(\rho_1^2 - a^2) + a/Z_a(\gamma_0)} = \frac{1}{\bar{g}s_1(a + \frac{1}{2}s_1) + a/Z_a(\gamma_0)}, \\ \frac{Z_2(\gamma_0)}{\rho_2} &= \frac{1}{\frac{1}{2}\bar{g}(b^2 - \rho_2^2) + b/Z_b(\gamma_0)} = \frac{1}{\bar{g}s_2(b - \frac{1}{2}s_2) + b/Z_b(\gamma_0)}, \end{aligned} \tag{113}$$

where $Z_1(\gamma_0)$ and $Z_2(\gamma_0)$ are the radial impedances looking into the stacks at ρ_1 and ρ_2 respectively, and $Z_a(\gamma_0)$ and $Z_b(\gamma_0)$ are the radial impedances looking into the core and the outer sheath. From equations (44) and (45) of Section II, the attenuation and phase constants of a coaxial Clogston 1 cable with infinitesimally thin layers, Clogston's condition being satisfied exactly, are

$$\alpha = \text{Re}\,\frac{Z_1(\gamma_0)/\rho_1 + Z_2(\gamma_0)/\rho_2}{2\eta_0 \log(\rho_2/\rho_1)}, \tag{114}$$

$$\beta = \omega\sqrt{\mu_0\epsilon_0} + \text{Im}\,\frac{Z_1(\gamma_0)/\rho_1 + Z_2(\gamma_0)/\rho_2}{2\eta_0 \log(\rho_2/\rho_1)}, \tag{115}$$

where $Z_1(\gamma_0)/\rho_1$ and $Z_2(\gamma_0)/\rho_2$ are given by (113).

The impedances $Z_a(\gamma_0)$ and $Z_b(\gamma_0)$ may be computed if we know the structure of the core and the sheath. For a solid, homogeneous core and a homogeneous sheath of effectively infinite thickness, we have

$$Z_a(\gamma_0) = \frac{\eta\kappa}{\sigma}\frac{I_0(\kappa a)}{I_1(\kappa a)}, \qquad Z_b(\gamma_0) = \frac{\eta\kappa}{\sigma}\frac{K_0(\kappa b)}{K_1(\kappa b)}, \tag{116}$$

where

$$\kappa = \sqrt{\sigma^2 - \gamma_0^2}, \tag{117}$$

but of course the intrinsic propagation constant σ and the intrinsic impedance η need not be the same for the core and the sheath. If the sheath is of finite electrical thickness or has a laminated structure (alternate layers of copper and iron, for example, to provide effective shielding), its surface impedance may be calculated by a straightforward but longer procedure. We shall not go into this matter here, but shall merely observe that in many cases of interest $Z_a(\gamma_0)$ and $Z_b(\gamma_0)$ are so large that we may neglect the terms containing their reciprocals in (113). This means that we neglect the total conduction and displacement currents flowing in the core and the sheath, compared to the conduction currents in the stacks. Then the expressions for the attenuation and phase constants become

$$\alpha = \frac{1}{2\eta_0\bar{g}\log(\rho_2/\rho_1)}\left[\frac{1}{s_1(a + \frac{1}{2}s_1)} + \frac{1}{s_2(b - \frac{1}{2}s_2)}\right], \tag{118}$$

$$\beta = \omega\sqrt{\mu_0\epsilon_0}, \tag{119}$$

and again to this approximation there is neither amplitude nor phase distortion.

The formulas which have just been derived on the assumption of infinitesimally thin laminae approach validity for laminae of finite thickness as the frequency is reduced, provided of course that we do not go to such extremely low frequencies that the attenuation per wavelength becomes large. We shall show in the next section that the effect of finite lamina thickness is to introduce a frequency dependence into the attenuation and phase constants, in addition to the variations (if any) which arise from the frequency dependence of the core and sheath impedances.

We next write down approximate expressions for the field components in a plane Clogston 1 line with infinitesimally thin laminae. In the main dielectric we have, from equations (22) of Section II,

$$\begin{aligned} H_x &\approx H_0 e^{-\gamma z}, \\ E_y &\approx -\sqrt{\frac{\mu_0}{\epsilon_0}}\, H_0 e^{-\gamma z}, \\ E_z &\approx \frac{2Z_0(\gamma_0)H_0 y}{b}\, e^{-\gamma z}, \end{aligned} \tag{120}$$

for $-\frac{1}{2}b \leqq y \leqq \frac{1}{2}b$, where H_0 is an arbitrary amplitude factor and $Z_0(\gamma_0)$ is given by (105). In the stacks the fields are

$$\begin{aligned} H_x &\approx H_0[1 + \bar{g}Z_0(\gamma_0)(\tfrac{1}{2}b \mp y)]e^{-\gamma z}, \\ \bar{E}_y &\approx -\sqrt{\frac{\bar{\mu}}{\bar{\epsilon}}}\, H_0[1 + \bar{g}Z_0(\gamma_0)(\tfrac{1}{2}b \mp y)]e^{-\gamma z}, \\ E_z &\approx \pm Z_0(\gamma_0)H_0 e^{-\gamma z}, \end{aligned} \tag{121}$$

for $\frac{1}{2}b \leqq |y| \leqq \frac{1}{2}a$, where in cases of ambiguous sign the upper sign refers to the upper stack ($y > 0$) and the lower sign to the lower stack ($y < 0$). It should be noted that whereas the tangential field components H_x and E_z are continuous through the stack, the normal field component E_y is discontinuous at layer boundaries. From equation (52) we have, in the conducting layers,

$$E_y = -(\gamma/g_1)H_x\,, \tag{122}$$

while in the insulating layers,

$$E_y = -(\gamma/i\omega\epsilon_2)H_x\,. \tag{123}$$

To our approximation, therefore, the only contributions to the average field $\bar{E}_y$ come from the insulating layers.

The average current density $\bar{J}_z$ in either stack is uniform, being given by

$$\bar{J}_z = \bar{g}E_z = \pm\bar{g}Z_0(\gamma_0)H_0e^{-\gamma z}. \tag{124}$$

The total current per unit width carried by the stack is just $\bar{J}_z s$, where s is the thickness of the stack; there will also be small currents in the sheaths unless we assume the sheath impedance to be infinite. The potential difference between any two points y_1 and y_2 in the same transverse plane may easily be found from

$$V(y_2) - V(y_1) = -\int_{y_1}^{y_2} E_y\, dy. \tag{125}$$

For a Clogston 1 line of the proportions which we have been considering, the potential difference across the stacks will be small compared to the potential difference across the main dielectric.

In a coaxial Clogston 1 with infinitesimally thin laminae, the fields in the main dielectric are given to a good approximation by equations (46) of Section II, namely

$$\begin{aligned} H_\phi &\approx \frac{I}{2\pi\rho}\, e^{-\gamma z}, \\ E_\rho &\approx \sqrt{\frac{\mu_0}{\epsilon_0}}\, \frac{I}{2\pi\rho}\, e^{-\gamma z}, \\ E_z &\approx \frac{I}{2\pi \log(\rho_2/\rho_1)} \left[\frac{Z_1(\gamma_0)}{\rho_1} \log\frac{\rho_2}{\rho} + \frac{Z_2(\gamma_0)}{\rho_2} \log\frac{\rho_1}{\rho}\right] e^{-\gamma z}, \end{aligned} \tag{126}$$

where I is an arbitrary amplitude factor and $Z_1(\gamma_0)$ and $Z_2(\gamma_0)$ are expressed by (113). In the inner stack we have

$$\begin{aligned} H_\phi &\approx \frac{Z_1(\gamma_0)I}{2\pi\rho_1} \left[\frac{\bar{g}(\rho^2 - a^2)}{2\rho} + \frac{a}{\rho Z_a(\gamma_0)}\right] e^{-\gamma z}, \\ \bar{E}_\rho &\approx \sqrt{\frac{\bar{\mu}}{\bar{\epsilon}}}\, \frac{Z_1(\gamma_0)I}{2\pi\rho_1} \left[\frac{\bar{g}(\rho^2 - a^2)}{2\rho} + \frac{a}{\rho Z_a(\gamma_0)}\right] e^{-\gamma z}, \\ E_z &\approx \frac{Z_1(\gamma_0)I}{2\pi\rho_1}\, e^{-\gamma z}, \end{aligned} \tag{127}$$

while in the outer stack,

$$
\begin{aligned}
H_\phi &\approx \frac{Z_2(\gamma_0)I}{2\pi\rho_2}\left[\frac{\bar{g}(b^2-\rho^2)}{2\rho} + \frac{b}{\rho Z_b(\gamma_0)}\right]e^{-\gamma z}, \\
\bar{E}_\rho &\approx \sqrt{\frac{\bar{\mu}}{\bar{\epsilon}}}\,\frac{Z_2(\gamma_0)I}{2\pi\rho_2}\left[\frac{\bar{g}(b^2-\rho^2)}{2\rho} + \frac{b}{\rho Z_b(\gamma_0)}\right]e^{-\gamma z}, \qquad (128) \\
E_z &\approx -\frac{Z_2(\gamma_0)I}{2\pi\rho_2}\,e^{-\gamma z}.
\end{aligned}
$$

The average current density in either stack is uniform and is given by

$$\bar{J}_z = \bar{g}E_z, \qquad (129)$$

though in general the current density will not be the same in the two stacks because of the difference in cross-sectional areas. The potential difference between the surface of the inner core and any other point in the same transverse plane is

$$V(\rho) - V(a) = -\int_a^\rho E_\rho \, d\rho. \qquad (130)$$

If the stacks are thin compared to the thickness of the main dielectric, as we are assuming throughout Part I, then the potential difference across the stacks will be small compared to the potential difference across the main dielectric, and the characteristic impedance Z_k of the Clogston 1 cable will be approximately the same as the characteristic impedance of an ideal coaxial cable with perfect conductors of radii ρ_1 and ρ_2 and the same main dielectric, namely

$$Z_k = 60\sqrt{\frac{\mu_{0r}}{\epsilon_{0r}}}\log\frac{\rho_2}{\rho_1} \text{ ohms}. \qquad (131)$$

We shall defer making any field plots for Clogston-type transmission lines until Section IX of Part II, when we shall discuss the transition from Clogston 1 to Clogston 2 as the space originally occupied by the main dielectric is gradually filled with laminations. Our present results will then appear as the limiting case in which the thickness of the stacks is small compared to the thickness of the main dielectric.

In conclusion we shall mention briefly the question of how to dispose a given amount of laminated material in a Clogston 1 coaxial cable so as to achieve the minimum attenuation constant. The whole problem of optimum proportions for Clogston cables is a complicated one of which an adequate treatment would require a separate paper in itself, with the results depending to a great extent on engineering considerations which limit the ranges of the parameters that we can vary in any practical case. Here we shall discuss only the following rather highly idealized problem:

Given a coaxial Clogston 1 with infinitesimally thin laminae, having a high-impedance core and a high-impedance sheath of fixed radius b, and in which the total thickness $s_1 + s_2$ of both stacks is a fixed constant $2s$. Assuming that $2s$ is small compared to b, what should be the radius a of the core, and how should the total stack thickness be divided between the outer and inner stacks so as to minimize the attenuation constant of the line? Finally, what should be the fraction θ of conducting material in the stacks to minimize the attenuation constant, if the electrical constants of the conducting and insulating layers are fixed, but the properties of the main dielectric are at our disposal?

If the two inequalities

$$s_1 \ll a, \qquad s_2 \ll b, \tag{132}$$

are satisfied (these restrictions will be removed in Section IX, when we discuss Clogston cables having an arbitrary fraction of their total volume filled with laminations), then equation (118) for the attenuation constant of a Clogston 1 with infinitesimally thin laminae and high-impedance boundaries becomes, approximately,

$$\alpha \approx \frac{1}{2\eta_0 \bar{g} \log (b/a)} \left[\frac{1}{as_1} + \frac{1}{bs_2} \right]. \tag{133}$$

If we write

$$s_2 = 2s - s_1 , \tag{134}$$

and vary s_1 and s_2 in accordance with this relation while holding a and b constant, it is easy to show that the expression on the right side of (133) is a minimum when

$$s_1 = \frac{2s\sqrt{b}}{\sqrt{a} + \sqrt{b}}, \qquad s_2 = \frac{2s\sqrt{a}}{\sqrt{a} + \sqrt{b}}. \tag{135}$$

These equations tell us the most efficient way to divide the stacks in a Clogston 1 when the radii of the core and the outer sheath are a and b respectively, still assuming of course that the thickness of each stack is small compared to its mean radius.

If we introduce the optimum values of s_1 and s_2 into (133), we get

$$\alpha \approx \frac{1}{2\eta_0 \bar{g}(s_1 + s_2) \log (b/a)} \left[\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \right]^2. \tag{136}$$

If b is fixed, the last expression is a minimum, considered as a function of a, when

$$\log (b/a) = 1 + \sqrt{a/b}. \tag{137}$$

The root of this transcendental equation is

$$b/a = 4.3827, \qquad a = 0.22817b. \tag{138}$$

Substituting this value of b/a into (135), we find

$$\begin{aligned} s_1 &= 1.3535s, \\ s_2 &= 0.6465s, \\ s_1/s_2 &= 2.0935; \end{aligned} \tag{139}$$

while from (136) and (138) the minimum value of the attenuation constant is

$$\alpha \approx \frac{3.238}{\eta_0 \bar{g}(s_1 + s_2)b}. \tag{140}$$

To find the optimum value of θ, we observe that equation (118) for the attenuation constant of a Clogston 1 cable with infinitesimally thin laminae and high-impedance boundaries may be written in the form

$$\alpha = \frac{(\epsilon_0/\mu_0)^{\frac{1}{2}}}{\theta g_1} f(a, b, s_1, s_2), \tag{141}$$

where the first factor depends on the electrical constants of the components of the cable, while $f(a, b, s_1, s_2)$ is a function only of the geometry. By (110) the attenuation constant of a plane Clogston 1 has the same form, only with a different dependence on the geometrical factors. Now assume that the geometrical proportions of the line are fixed, and that the electrical constants μ_1, g_1, μ_2, and ϵ_2 of the conducting and insulating layers are given, but that the constants μ_0, ϵ_0 of the main dielectric and the fraction of space θ occupied by conducting layers in the stacks are at our disposal. The $\mu_0\epsilon_0$ product of the main dielectric is to be codetermined with θ so that Clogston's condition (102) is always satisfied. Solving (102) for θ gives

$$\theta = \frac{\mu_0\epsilon_0 - \mu_2\epsilon_2}{\mu_0\epsilon_0 + (\mu_1 - \mu_2)\epsilon_2}. \tag{142}$$

Hence the first factor in the expression (141) for α may be written

$$\frac{(\epsilon_0/\mu_0)^{\frac{1}{2}}}{\theta g_1} = \frac{\epsilon_0^{\frac{1}{2}}[\mu_0\epsilon_0 + (\mu_1 - \mu_2)\epsilon_2]}{g_1\mu_0^{\frac{1}{2}}[\mu_0\epsilon_0 - \mu_2\epsilon_2]}. \tag{143}$$

If we minimize the right side of (143) with respect to ϵ_0, all other quantities being held constant, by equating to zero the derivative with respect to ϵ_0 and then solving for ϵ_0, we get

$$\mu_0\epsilon_0 = \tfrac{1}{2}[(\mu_1 + 2\mu_2) + (\mu_1^2 + 8\mu_1\mu_2)^{\frac{1}{2}}]\epsilon_2 . \tag{144}$$

From (142) the value of θ is

$$\theta = \frac{\mu_1 + (\mu_1^2 + 8\mu_1\mu_2)^{\frac{1}{2}}}{3\mu_1 + (\mu_1^2 + 8\mu_1\mu_2)^{\frac{1}{2}}}, \tag{145}$$

and the corresponding attenuation constant is proportional to

$$\frac{(\epsilon_0/\mu_0)^{\frac{1}{2}}}{\theta g_1} = \frac{(\epsilon_0/\mu_0)^{\frac{1}{2}}}{g_1} \frac{3\mu_1 + (\mu_1^2 + 8\mu_1\mu_2)^{\frac{1}{2}}}{\mu_1 + (\mu_1^2 + 8\mu_1\mu_2)^{\frac{1}{2}}} . \tag{146}$$

It will be observed that so far we have determined only the optimum value of the product $\mu_0\epsilon_0$, and so we are still free to alter the ratio of μ_0 to ϵ_0 while holding the product of these two quantities constant. For given values of μ_1 and μ_2, we obtain the lowest attenuation constant by making ϵ_0 as small as possible and μ_0 as large as possible, subject of course to the practical restriction that ϵ_0 cannot be lower than the dielectric constant of free space. However if we permit μ_2 and μ_0 to be simultaneously increased, as by magnetic loading of both the insulating layers and the main dielectric, we find from (146) that on paper it is possible to decrease the attenuation constant without any definite limit. This observation is in accord with the fact that the attenuation constant of an ordinary coaxial cable may be decreased indefinitely, with a corresponding decrease in the velocity of propagation along the cable, if we are willing to assume an unlimited amount of lossless magnetic loading.

If $\mu_1 = \mu_2$, (144) and (145) take the form

$$\mu_0\epsilon_0 = 3\mu_2\epsilon_2 , \qquad \theta = 2/3, \tag{147}$$

from which we have the result given by Clogston:[14] If the conducting and insulating layers are infinitesimally thin and have equal permeabilities, then minimum attenuation is achieved when *the thickness of the conducting layers is twice the thickness of the insulating layers*. In this case, from (146) and (147) the attenuation is proportional to

$$\frac{(\epsilon_0/\mu_0)^{\frac{1}{2}}}{\theta g_1} = \frac{3(\epsilon_0/\mu_0)^{\frac{1}{2}}}{2g_1} . \tag{148}$$

When $\mu_0 = \mu_2$, corresponding to no magnetic loading, we must take $\epsilon_0 = 3\epsilon_2$, and (148) reduces to

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[14] Reference 1, pp. 513–514.

$$\frac{(\epsilon_0/\mu_0)^{\frac{1}{2}}}{\theta g_1} = \frac{3\sqrt{3}\,(\epsilon_2/\mu_2)^{\frac{1}{2}}}{2g_1}, \tag{149}$$

while if we load the main dielectric so that $\mu_0 = 3\mu_2$ and we can take $\epsilon_0 = \epsilon_2$, we have

$$\frac{(\epsilon_0/\mu_0)^{\frac{1}{2}}}{\theta g_1} = \frac{\sqrt{3}\,(\epsilon_2/\mu_2)^{\frac{1}{2}}}{2g_1}, \tag{150}$$

which is just one-third of the value with no magnetic loading.

As Clogston has pointed out, if the limitation is on the total thickness of conducting material in the stacks rather than on the stack thicknesses themselves, we shall find it advantageous to use a small value of θ (a high "dilution" of conducting material) so as to make the average dielectric constant $\epsilon_2/(1 - \theta)$ of the stacks, which has to be matched by the main dielectric, as small as possible. We shall see later that the effect of finite lamina thickness is in fact to limit the total thickness of conducting material which it is useful to employ in a single stack at high frequencies, so that for physical stacks of non-magnetic layers at high frequencies the optimum value of θ is less than 2/3. Quantitative results which take into account the finite thickness of the layers will be obtained in Section XI.

To illustrate the use of some of the equations derived above by means of a numerical example, we shall compare the attenuation constant of a conventional coaxial cable with that of a Clogston 1 cable of the same size. If a and b denote the radii of the inner and outer conductors of a conventional coaxial cable, and we take $b/a = 3.5911$ to minimize the attenuation constant, then we have from equation (48) of Section II, on setting $\rho_1 = a$ and $\rho_2 = b$,

$$\alpha = \frac{1.796}{\eta_0 g_1 \delta_1 b}, \tag{151}$$

where η_0 is the intrinsic impedance of the main dielectric, which may be air. For a Clogston 1 coaxial cable with infinitesimally thin laminae, no magnetic material in the stacks ($\mu_1 = \mu_2 = \mu_v$), and the optimum proportions given by (139) and (147), we have

$$\alpha \approx \frac{4.857}{\eta_0 g_1 (s_1 + s_2) b}, \tag{152}$$

where b is the outside radius of the outer stack and η_0 is the intrinsic impedance of the main dielectric, which cannot be air in a Clogston cable. The ratio of the attenuation constant α_c of this Clogston cable to the attenuation constant α_s of an *air-filled* standard coaxial of the same size, made of the same conducting material, is

$$\frac{\alpha_c}{\alpha_s} \approx \frac{2.704\ \delta_1}{(\mu_{0r}/\epsilon_{0r})^{\frac{1}{2}}(s_1 + s_2)}, \tag{153}$$

where μ_{0r} and ϵ_{0r} refer to the main dielectric of the Clogston cable.

Since the attenuation constant of a standard coaxial cable is proportional to the square root of frequency in the range we are considering, while the attenuation constant of the ideal Clogston cable is independent of frequency in this range, there will be a crossover frequency above which the Clogston cable has a lower attenuation constant than a conventional coaxial cable of the same size. If we are dealing with copper conductors and if frequencies are measured in $\text{Mc}\cdot\text{sec}^{-1}$ and linear dimensions in mils, then from equations (78) and (153) we find that the crossover frequency is given approximately by

$$f_{\text{Mc}} \approx \frac{49.50\ (\epsilon_{0r}/\mu_{0r})}{(s_1 + s_2)^2_{\text{mils}}}. \tag{154}$$

For example, let us take an ideal Clogston 1 cable of outer diameter 0.375 inches, excluding the sheath, with no magnetic loading, and assume the following values:

$$\begin{aligned} a &= 42.8 \text{ mils} & \theta &= 2/3 \\ b &= 187.5 \text{ mils} & \epsilon_{2r} &= 2.26 \text{ (polyethylene)} \\ s_1 &= 12.69 \text{ mils} & \epsilon_{0r} &= 3\epsilon_{2r} = 6.78 \\ s_2 &= 6.06 \text{ mils} & \mu_{0r} &= \mu_{1r} = \mu_{2r} = 1 \\ s_1 + s_2 &= 18.75 \text{ mils} \end{aligned} \tag{155}$$

This cable has a lower attenuation constant than a standard air-filled coaxial of the same size at frequencies above about 1 $\text{Mc}\cdot\text{sec}^{-1}$, the approximate formula (154) yielding 0.955 $\text{Mc}\cdot\text{sec}^{-1}$ for the crossover frequency and the exact equation (118), taken in conjunction with (151), yielding 1.251 $\text{Mc}\cdot\text{sec}^{-1}$.

The reader is cautioned that the comparison given by (153) between Clogston and conventional cables is based upon certain highly idealized assumptions. In the first place we have neglected the finite thickness of the laminae, which will in fact cause the attenuation constant of a physical Clogston cable to increase with increasing frequency, and ultimately to cross over again and become higher than the attenuation constant of a conventional air-filled coaxial. We have also neglected dielectric and magnetic losses, which are likely to be directly proportional to frequency and by no means negligible at the upper end of the frequency band. In practice, too, the $\mu_0\epsilon_0$ product of the main dielectric must be held very close to the Clogston value or the benefit of the large effective skin depth is lost; and the stacks must be extremely uniform or again the depth of penetration is greatly reduced. We shall take up all these matters in later sections, and shall see that while the results just given represent ultimate limits of performance, the practical improvements which can be achieved over conventional cables depend upon the degree to which one can solve the manufacturing problems that tend to make every actual Clogston cable differ more or less from the ideal structure considered above.

V. EFFECT OF FINITE LAMINA THICKNESS. FREQUENCY DEPENDENCE OF ATTENUATION IN CLOGSTON 1 LINES

The principal effect of finite lamina thickness in a Clogston cable is to introduce a frequency dependence into the propagation constant, and in particular to cause the attenuation constant to increase, with increasing frequency, above the value which we have found for infinitesimally thin laminae (or for finite laminae at low frequencies). The increased losses are associated with the fact that the penetration depth in a laminated stack decreases with increasing frequency, even when Clogston's condition is exactly satisfied, if the laminae are of finite thickness. We shall now obtain expressions for the surface impedance of a plane laminated stack of n double layers, such as is shown in Fig. 3, when Clogston's condition is satisfied but the individual layers are of finite thickness.

We first observe that Clogston's condition (102) implies

$$\begin{aligned} \eta_{2y}\kappa_2 t_2 &= \eta_2\sigma_2(1 - \gamma_0^2/\sigma_2^2)t_2 \\ &= i\omega\mu_2\left[1 - \frac{\theta\mu_1 + (1-\theta)\mu_2}{(1-\theta)\mu_2}\right]\frac{(1-\theta)t_1}{\theta} \\ &= -i\omega\mu_1 t_1 = -\eta_1\sigma_1 t_1 \\ &\approx -\eta_{1y}\kappa_1 t_1 , \end{aligned} \tag{156}$$

where in the last step we have used the fact that in the conducting layers η_{1y} is equal to η_1 and κ_1 is equal to σ_1 to a very good approximation. We now introduce the dimensionless parameter

$$\Theta = \sigma_1 t_1 = (1 + i)t_1/\delta_1 \approx \kappa_1 t_1 , \tag{157}$$

which may be regarded as a measure of the electrical thickness of the individual conducting layers. From (86) and (156) we have, for the propagation constant per double layer,

$$\text{ch}\,\Gamma = \text{ch}\,\Theta - \tfrac{1}{2}\Theta\,\text{sh}\,\Theta, \tag{158}$$

and from (87), for the iterative impedances,

$$\begin{aligned} K_1 &= \frac{\Theta}{g_1 t_1}\,[+\tfrac{1}{2}\Theta + (\tfrac{1}{4}\Theta^2 - \Theta\,\text{coth}\,\Theta + 1)^{\frac{1}{2}}], \\ K_2 &= \frac{\Theta}{g_1 t_1}\,[-\tfrac{1}{2}\Theta + (\tfrac{1}{4}\Theta^2 - \Theta\,\text{coth}\,\Theta + 1)^{\frac{1}{2}}], \end{aligned} \tag{159}$$

since $\eta_{1y} = \kappa_1/g_1 = \Theta/g_1 t_1$.

If the thickness t_1 of each conducting layer is moderately small compared to the skin depth δ_1 at the highest frequency of interest, the quantities Γ, K_1, and K_2 may conveniently be expanded in powers of Θ. The identity

$$\text{ch}\,x - 1 = 2\,\text{sh}^2\,\tfrac{1}{2}x \tag{160}$$

enables us to transform (158) into

$$\begin{aligned} \text{sh}^2\,\tfrac{1}{2}\Gamma &= \tfrac{1}{2}(\text{ch}\,\Theta - 1) - \tfrac{1}{4}\Theta\,\text{sh}\,\Theta \\ &= -\frac{\Theta^4}{48}\left[1 + \frac{\Theta^2}{15} + \frac{\Theta^4}{560} + \cdots\right], \end{aligned} \tag{161}$$

after we expand sh Θ and ch Θ by Dwight 657.1 and 657.2 and collect terms. Taking the square root by the binomial theorem gives

$$\text{sh}\,\tfrac{1}{2}\Gamma = -\frac{i}{4\sqrt{3}}\left[\Theta^2 + \frac{\Theta^4}{30} + \frac{17\Theta^6}{50400} + \cdots\right], \tag{162}$$

the negative sign being introduced because from (157) Θ^2 is a positive imaginary number and we want Re $\Gamma > 0$. Then

$$\begin{aligned} \Gamma &= 2\,\text{sh}^{-1}\left[-\frac{i}{4\sqrt{3}}\left(\Theta^2 + \frac{\Theta^4}{30} + \frac{17\Theta^6}{50400} + \cdots\right)\right] \\ &= -\frac{i}{\sqrt{3}}\left[\frac{\Theta^2}{2} + \frac{\Theta^4}{60} + \frac{\Theta^6}{525} + \cdots\right], \end{aligned} \tag{163}$$

provided that we expand the sh^{-1} function by Dwight 706. From (159) we get

$$\begin{aligned} K_1 &= \frac{1}{g_1 t_1}\left[\frac{(3 - i\sqrt{3})}{6}\,\Theta^2 + \frac{i\sqrt{3}}{45}\,\Theta^4 - \frac{i\sqrt{3}}{1575}\,\Theta^6 + \cdots\right], \\ K_2 &= \frac{1}{g_1 t_1}\left[-\frac{(3 + i\sqrt{3})}{6}\,\Theta^2 + \frac{i\sqrt{3}}{45}\,\Theta^4 - \frac{i\sqrt{3}}{1575}\,\Theta^6 + \cdots\right], \end{aligned} \tag{164}$$

where we have expanded coth Θ by Dwight 657.5 and chosen the sign of the square root to make Re K_1 and Re K_2 both positive.

Our first observation is that when the lamina thickness is finite the effective skin depth of the stack is also finite. We have, from (157) and (163),

$$\Gamma = \frac{1}{\sqrt{3}}\left[\frac{t_1^2}{\delta_1^2} + \frac{it_1^4}{15\delta_1^4} - \frac{8t_1^6}{525\delta_1^6} - \cdots\right], \tag{165}$$

and the average propagation constant per unit distance into the stack is

$$\Gamma_\ell = \frac{\Gamma}{(t_1 + t_2)} = \frac{1}{\sqrt{3}(t_1 + t_2)}\left[\frac{t_1^2}{\delta_1^2} + \frac{it_1^4}{15\delta_1^4} - \frac{8t_1^6}{525\delta_1^6} - \cdots\right]. \tag{166}$$

If as usual we define the effective skin depth Δ to be the distance, measured into an infinitely deep stack, at which the current density has fallen to $1/e$ of its value at the surface, then keeping only the first term in (166) we have

$$\Delta = \frac{1}{\text{Re}\,\Gamma_\ell} = \frac{\sqrt{3}(t_1 + t_2)\delta_1^2}{t_1^2} = \frac{\sqrt{3}(t_1 + t_2)}{\pi\mu_1 g_1 f t_1^2}, \tag{167}$$

a result also given by Clogston.[15] The number N of double layers in one effective skin depth is

$$N = \frac{\Delta}{(t_1 + t_2)} = \frac{\sqrt{3}\delta_1^2}{t_1^2} = \frac{\sqrt{3}}{\pi\mu_1 g_1 f t_1^2}, \tag{168}$$

while the total thickness T_Δ of conducting material in these layers is

$$T_\Delta = Nt_1 = \frac{\sqrt{3}\delta_1^2}{t_1} = \frac{\sqrt{3}}{\pi\mu_1 g_1 f t_1}. \tag{169}$$

T_Δ is essentially the thickness of conducting material in each stack which is effectively carrying current; it is evident that for small values of t_1/δ_1 this effective thickness is inversely proportional to the frequency f and to the thickness t_1 of the individual conducting layers, but independent of the thckness t_2 of the insulating layers, provided that t_2 is very small compared to the length of a free wave in the insulating material.

In the general case, still assuming of course that Clogston's condition is satisfied, the surface impedance $Z_0(\gamma_0)$ of a plane Clogston stack is given by equation (65) of Section III, namely

$$Z_0(\gamma_0) = \frac{\frac{1}{2}Z_n(\gamma_0)(K_1e^{n\Gamma} + K_2e^{-n\Gamma}) + K_1K_2 \operatorname{sh} n\Gamma}{Z_n(\gamma_0) \operatorname{sh} n\Gamma + \frac{1}{2}(K_1e^{-n\Gamma} + K_2e^{n\Gamma})}, \tag{170}$$

[15] Reference 1, equation (III-44).

where $Z_n(\gamma_0)$ is the impedance of the surface behind the stack. If $\Theta = 0$, (170) reduces to (105) of Section IV, that is,

$$Z_0(\gamma_0) = \frac{1}{\bar{g}s + 1/Z_n(\gamma_0)} = \frac{1}{g_1 T_1 + 1/Z_n(\gamma_0)}, \tag{171}$$

where T_1 is the total thickness of conducting material in the stack. If $Z_n(\gamma_0)$ is infinite, then for all values of Θ and n we have

$$Z_0(\gamma_0) = \frac{\Theta}{g_1 t_1} [\tfrac{1}{2}\Theta + (\tfrac{1}{4}\Theta^2 - \Theta \coth \Theta + 1)^{\frac{1}{2}} \coth n\Gamma]; \tag{172}$$

and if Re $n\Gamma$ is large, corresponding to a stack many effective skin depths thick, then for any $Z_n(\gamma_0)$ we have

$$Z_0(\gamma_0) = K_1 . \tag{173}$$

Once $Z_0(\gamma_0)$ has been computed for a particular frequency, the attenuation and phase constants of the plane Clogston 1 line at that frequency are given, as in Section II, by

$$\alpha = \text{Re } Z_0(\gamma_0)/\eta_0 b, \tag{174}$$

$$\beta = \omega\sqrt{\mu_0 \epsilon_0} + \text{Im } Z_0(\gamma_0)/\eta_0 b. \tag{175}$$

Explicit expressions for the surface impedance of a coaxial stack of finite layers have not been derived. However, if in a coaxial Clogston 1 the thickness of each stack is small compared to its mean radius, or if the depth of penetration given by (167) is small compared to the radius of the surface near which the currents flow, then the parallel-plane formula (170) may be used for the stack impedances $Z_1(\gamma_0)$ and $Z_2(\gamma_0)$ which are to be substituted into the equations of Section II for the attenuation and phase constants, namely

$$\alpha = \text{Re } \frac{Z_1(\gamma_0)/\rho_1 + Z_2(\gamma_0)/\rho_2}{2\eta_0 \log (\rho_2/\rho_1)}, \tag{176}$$

$$\beta = \omega\sqrt{\mu_0 \epsilon_0} + \text{Im } \frac{Z_1(\gamma_0)/\rho_1 + Z_2(\gamma_0)/\rho_2}{2\eta_0 \log (\rho_2/\rho_1)}. \tag{177}$$

If the plane approximations are regarded as insufficiently accurate, one can compute the surface impedance of a cylindrical stack by repeated multiplication of matrices similar to the one given by equations (88) of Section III. This procedure would obviously involve considerable numerical computation, but we can hardly expect that it would reveal anything qualitatively new for Clogston cables of the proportions considered in Part I.

It will be instructive to compare the impedance of a laminated plane stack with the impedance of a solid metal plate over the full frequency range from zero to very high frequencies.[16] If the stack contains n conducting layers, each of thickness t_1, and the metal plate is of thickness $T_1 = nt_1$, the impedances of the plate and of the stack will be equal at zero frequency, and also at very high frequencies where the first layer of the stack is already many skin depths thick. For simplicity we assume that both the plate and the stack are backed by infinite-impedance surfaces at all frequencies.

To orient ourselves we shall define three critical frequencies, for which respectively the thickness of the solid plate is equal to one skin depth in the metal, the thickness of the stack is equal to one "effective skin depth", and the thickness of a single conducting layer is equal to $\sqrt{3}$ skin depths in the metal. These frequencies are

$$\begin{aligned} f_1 &= 1/(\pi\mu_1 g_1 T_1^2) & &(T_1 = \delta_1), \\ f_2 &= \sqrt{3}/(\pi\mu_1 g_1 t_1 T_1) = \sqrt{3} n f_1 & &(T_1 = T_\Delta), \\ f_3 &= 3/(\pi\mu_1 g_1 t_1^2) = 3n^2 f_1 & &(t_1 = \sqrt{3}\delta_1). \end{aligned} \tag{178}$$

The approximate forms of the surface impedance functions of the plate and the stack in the various frequency ranges are then quite simple.

In the range $0 \leqq f \leqq f_1$, the surface impedance of the solid plate is approximately constant and given by

$$Z_0(\gamma_0) \approx 1/g_1 T_1, \tag{179}$$

while in the range $f \geqq f_1$ we see approximately the surface impedance of an infinite plate,

$$Z_0(\gamma_0) \approx (1 + i)/g_1\delta_1 = (1 + i)\sqrt{\pi\mu_1 f/g_1}, \tag{180}$$

which is proportional to $\sqrt{f}$. The surface impedance of the stack is approximately constant in the range $0 \leqq f \leqq f_2$, where

$$Z_0(\gamma_0) \approx 1/g_1 T_1, \tag{181}$$

while in the range $f_2 \leqq f \leqq f_3$ it is approximately equal to the impedance K_1 of an infinitely deep stack of moderately thin layers as given by the first of equations (164), namely

$$Z_0(\gamma_0) \approx (1/\sqrt{3} + i)\pi\mu_1 t_1 f, \tag{182}$$

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[16] In this connection see also Reference 1, Fig. 2, p. 494. Clogston compares a laminated stack with a solid plate of the same total thickness as the stack, hence a plate which contains more conducting material than the stack.

which is directly proportional to frequency (and independent of conductivity). For $f \geqq f_3$ the stack acts much like an infinitely thick solid plate, for which

$$Z_0(\gamma_0) \approx (1 + i)/g_1\delta_1 = (1 + i) \sqrt{\pi\mu_1 f/g_1}, \tag{183}$$

an impedance again proportional to $\sqrt{f}$.

The real parts of the approximate expressions for surface impedance may be plotted on log-log paper, where power-law relationships are represented by straight lines, to give quite a good idea of the way in which the stack resistance varies over the entire frequency range. To show how the exact resistance departs from the approximate formulas in the transition regions, we have calculated the resistance of a particular stack over the full frequency range from equation (172), and also the resistance of the corresponding solid plate from the formula

$$Z_0(\gamma_0) = (1 + i)\sqrt{\pi\mu_1 f/g_1} \coth [(1 + i)\sqrt{\pi\mu_1 g_1 f}T_1], \tag{184}$$

and plotted the results, together with those for an infinite plate and an infinite stack, in Fig. 7. The actual numerical values were chosen solely for ease in plotting, and are of no particular significance. It should be noted that the exact curves oscillate slightly around the asymptotic lines in the transition regions. For example, the resistance of the laminated stack is actually higher than the resistance of the solid plate at certain frequencies slightly above f_3. These oscillations appear clearly in the numerical results, but are scarcely visible on the plots because of the logarithmic compression of the upper ends of the frequency and resistance scales.

We shall next obtain an expression for the rate at which the surface impedance of a laminated stack begins to depart from its dc value as the frequency is increased. For this purpose we must expand the various factors appearing in equation (170) for $Z_0(\gamma_0)$ in powers of Θ. Using the expansions (163) and (164) which have already been derived for Γ, K_1, and K_2, it is a matter of straightforward if tedious algebra to show that:

$$e^{n\Gamma} = 1 - \frac{i\sqrt{3}n}{6}\Theta^2 - \frac{(15n^2 + i2\sqrt{3}n)}{360}\Theta^4 + \cdots, \tag{185}$$

$$e^{-n\Gamma} = 1 + \frac{i\sqrt{3}n}{6}\Theta^2 - \frac{(15n^2 - i2\sqrt{3}n)}{360}\Theta^4 + \cdots, \tag{186}$$

$$\text{sh}\, n\Gamma = -\frac{in}{2\sqrt{3}}\left[\Theta^2 + \frac{\Theta^4}{30} - \frac{(175n^2 - 48)}{12600}\Theta^6 + \cdots\right], \tag{187}$$

$$K_1 e^{n\Gamma} + K_2 e^{-n\Gamma} \tag{188}$$
$$= -\frac{i\Theta^2}{\sqrt{3}g_1t_1}\left[1 + \frac{(15n - 4)}{30}\Theta^2 - \frac{(175n^2 - 70n - 16)}{4200}\Theta^4 + \cdots\right],$$

$$K_1 e^{-n\Gamma} + K_2 e^{n\Gamma} \tag{189}$$
$$= -\frac{i\Theta^2}{\sqrt{3}g_1t_1}\left[1 - \frac{(15n + 4)}{30}\Theta^2 - \frac{(175n^2 + 70n - 16)}{4200}\Theta^4 + \cdots\right],$$

$$K_1K_2 \operatorname{sh} n\Gamma = \frac{1}{(g_1t_1)^2}\frac{in}{6\sqrt{3}}\Theta^6 + \cdots. \tag{190}$$

By substituting the above series into equation (170), we can obtain the variation of the stack impedance with frequency so long as t_1/δ_1 is sufficiently small. Although in principle there would be no difficulty in taking into account an arbitrary sheath impedance $Z_n(\gamma_0)$, for brevity we shall restrict ourselves here to the case in which the sheath impedance is so high that at all frequencies of interest the current in the sheath may be neglected. Then we have equation (191) (see next page).

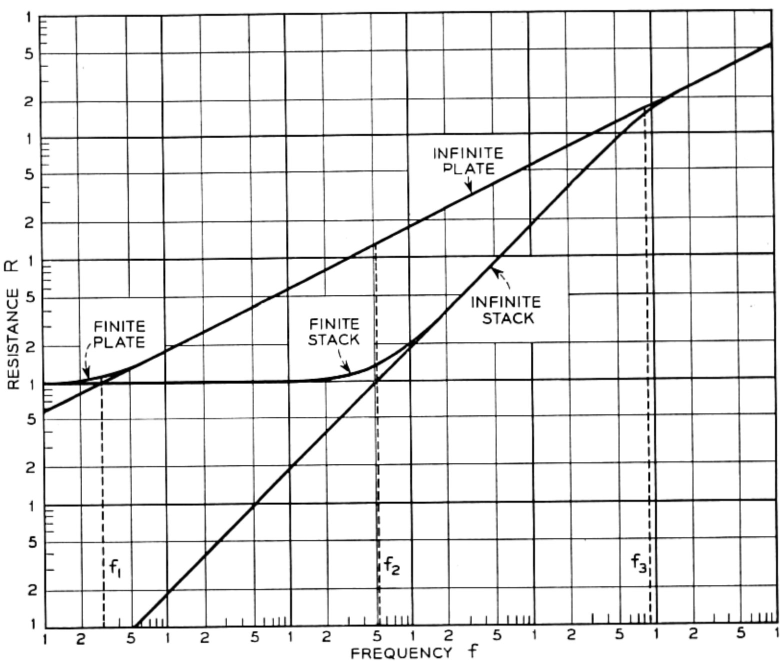

Fig. 7—Surface resistance R of solid plates and laminated stacks versus frequency f on log-log scale.

$$Z_0(\gamma_0) = \frac{K_1 e^{n\Gamma} + K_2 e^{-n\Gamma}}{2 \operatorname{sh} n\Gamma}, \tag{191}$$

which can be reduced to

$$\begin{aligned} Z_0(\gamma_0) &= \frac{1}{n g_1 t_1}\left[1 + \frac{(3n-1)}{6}\,\Theta^2 - \frac{(5n^2-1)}{180}\,\Theta^4 + \cdots\right] \\ &\approx \frac{1}{g_1 T_1}\left[1 + \frac{i T_1 t_1}{\delta_1^2} + \frac{T_1^2 t_1^2}{9\delta_1^4} + \cdots\right], \end{aligned} \tag{192}$$

the last expression being valid if the number of double layers is not too small ($n \geqq 5$, say). To this approximation the fractional changes in the resistance and reactance of the stack are

$$\frac{\Delta R}{R_0} = \frac{T_1^2 t_1^2}{9\delta_1^4} = \frac{T_1^2 t_1^2 \pi^2 \mu_1^2 g_1^2 f^2}{9}, \tag{193}$$

$$\frac{\Delta X}{R_0} = \frac{T_1 t_1}{\delta_1^2} = T_1 t_1 \pi \mu_1 g_1 f, \tag{194}$$

where

$$R_0 = 1/g_1 T_1 \tag{195}$$

is the dc resistance. From the exact calculations described above it appears that (193) and (194) are valid up to the neighborhood of the critical frequency

$$f_2 = \sqrt{3}/(\pi \mu_1 g_1 t_1 T_1), \tag{196}$$

at which frequency the approximate formulas yield

$$\Delta R/R_0 = 1/3, \qquad \Delta X/R_0 = \sqrt{3}. \tag{197}$$

For $f > f_2$, however, these approximations rapidly break down.

We may now answer the question: What must be the thickness t_1 of the individual conducting layers in a plane stack which contains a given total thickness T_1 of conducting material, if at a specified top frequency f_m the resistance of the stack is not to have increased by more than a specified small fraction of its dc value? We find that the permissible value of t_1 is

$$t_1 = \frac{3}{\pi \mu_1 g_1 T_1 f_m} \sqrt{\frac{\Delta R}{R_0}}, \tag{198}$$

and we note that this value of t_1 is inversely proportional both to f_m and to T_1. If we measure t_1 and T_1 in mils and f_m in $\text{Mc}\cdot\text{sec}^{-1}$, then on putting in the numerical values of μ_1 and g_1 for copper, we have

$$(t_1)_{\text{mils}} = \frac{20.31}{(f_m)_{\text{Mc}}(T_1)_{\text{mils}}} \sqrt{\frac{\Delta R}{R_0}}. \tag{199}$$

For a plane Clogston 1 with stacks of equal thickness, the attenuation constant is given by (174), and the fractional change in attenuation with frequency is equal to the fractional change in resistance of either stack, as calculated from (193). For a coaxial Clogston 1 with stacks thin enough so that the plane approximation is valid we may also use (193), but the fractional changes in resistance will be different for the two stacks if these are of different thicknesses, and the fractional change in the attenuation constant must be calculated from equation (176). If R_{10} and R_{20} are the dc resistances "per square" of the two stacks, and ΔR_1 and ΔR_2 their increments as obtained from (193), then the fractional increase in attenuation is given approximately by

$$\frac{\Delta\alpha}{\alpha_0} \approx \frac{\Delta R_1/\rho_1 + \Delta R_2/\rho_2}{R_{10}/\rho_1 + R_{20}/\rho_2}. \tag{200}$$

For either plane or cylindrical geometry we find that if we scale up a particular Clogston line by multiplying the thicknesses of the stacks and the main dielectric by the same factor, then the low-frequency attenuation constant will be divided by the square of the scale factor. However, the permissible thickness of the individual conducting layers, if we are to have the attenuation flat to a specified degree up to a fixed frequency, is inversely proportional to the scale factor. Thus if we double the overall dimensions of the line and double the amount of conducting material in the stacks, we shall divide the low-frequency attenuation constant by four, but we shall have to make the individual layers half as thick in order to maintain the same relative increase in attenuation constant at the same top frequency f_m. In addition it is clear that if we double the top frequency while maintaining the same requirement on $\Delta\alpha/\alpha_0$ for a line of given dimensions, we shall also have to cut the thickness of the individual layers in half.

As a numerical example, let us return to the cable whose specifications were given by (155) at the end of Section IV. For this cable we have:

$$\begin{aligned} \rho_1 &= 55.49 \text{ mils} & \theta s_1 &= 8.46 \text{ mils} \\ \rho_2 &= 181.44 \text{ mils} & \theta s_2 &= 4.04 \text{ mils} \\ \rho_2/\rho_1 &= 3.270 & R_{20}/R_{10} &\approx s_1/s_2 = 2.094 \end{aligned} \tag{201}$$

If the conducting layers are copper, we find that equation (200) for the fractional increase in attenuation becomes, numerically,

$$\Delta\alpha/\alpha_0 \approx 0.121(t_1)^2_{\text{mils}} f^2_{\text{Mc}} . \tag{202}$$

If for example the copper layers are 0.1 mil thick and the polyethylene layers 0.05 mil thick, since we are assuming $\theta = 2/3$, then the attenuation constant has increased by 10 per cent of its "flat" value at a frequency of about 9.1 $\text{Mc}\cdot\text{sec}^{-1}$.

We may also ask for the upper crossover frequency, above which the Clogston cable will have a higher attenuation constant than a standard air-filled coaxial of the same size. Such a crossover frequency must exist because the dielectric loading of the Clogston cable (in our case $\epsilon_{0r} = 6.78$) introduces a factor $\sqrt{\epsilon_{0r}}$ into the asymptotic expression for the attenuation constant at extremely high frequencies when the stacks look like solid metal walls; in addition there will be slight differences due to the fact that the geometric proportions of the conventional and Clogston cables are not exactly the same.

We assume, subject to a posteriori verification, that the upper crossover frequency lies between the critical frequencies f_2 and f_3, defined by (178), for each stack. Then we have in effect infinitely deep stacks of moderately thin laminae, whose surface resistances are equal and are given by (182) to be

$$R_1 = R_2 \approx \pi\mu_1 t_1 f/\sqrt{3} = 5.79 \times 10^{-5}(t_1)_{\text{mils}} f_{\text{Mc}} \text{ ohms.} \tag{203}$$

The attenuation constants of the conventional and Clogston cables are obtained from (151) and (176) respectively, where for the conventional coaxial we set $\eta_0 = \eta_v$. After a little arithmetic we find for the upper crossover frequency in this particular case,

$$f_{\text{Mc}} \approx 2.79/(t_1)^2_{\text{mils}} . \tag{204}$$

Thus if the copper layers are 0.1 mil thick, the upper crossover frequency is about 280 $\text{Mc}\cdot\text{sec}^{-1}$, which turns out to lie well inside the interval between the critical frequencies f_2 and f_3 for both stacks.

Comparing this result with the result at the end of Section IV, we see that a 0.375-inch Clogston 1 cable with 0.1-mil copper conductors and the other specifications given by (155) is nominally better than a conventional air-filled coaxial cable of the same size in the frequency range from about 1 $\text{Mc}\cdot\text{sec}^{-1}$ to 280 $\text{Mc}\cdot\text{sec}^{-1}$. We are still neglecting the effect of failure to satisfy Clogston's condition exactly, the effect of stack non-uniformity, and dielectric losses. All of these factors will be present to a greater or less degree in any physical embodiment of a Clogston cable, and will reduce, or in extreme cases even eliminate, the frequency range over which the Clogston cable exhibits lower loss than a conventional coaxial cable.

VI. EFFECT OF DIELECTRIC MISMATCH

We may think of Clogston's relation (102) as a condition imposed on the phase velocity in a laminated transmission line to maximize the depth of eddy current penetration into the stacks. If this condition is not exactly satisfied, that is, if the $\mu_0\epsilon_0$ product of the main dielectric is not equal to the $\bar{\mu}\bar{\epsilon}$ product of the stacks, then the effective skin depth of the stacks is finite at finite frequencies and decreases with increasing frequency even in the ideal case of infinitesimally thin layers, while if the layers are of finite thickness the effective skin depth is even less than it would be with a perfectly matched main dielectric. The losses in the stacks at moderate frequencies where Clogston's penetration effect is of importance are correspondingly increased by the presence of dielectric mismatch.

For a quantitative discussion we define the amount of dielectric mismatch $\Delta(\mu_0\epsilon_0)$ by

$$\Delta(\mu_0\epsilon_0) = \mu_0\epsilon_0 - \bar{\mu}\bar{\epsilon}, \tag{205}$$

and also the dielectric mismatch parameter k by

$$k = \frac{\Delta(\mu_0\epsilon_0)}{\bar{\mu}\bar{\epsilon} - \mu_2\epsilon_2} = \frac{(1-\theta)}{\theta}\frac{\Delta(\mu_0\epsilon_0)}{\mu_1\epsilon_2}. \tag{206}$$

In terms of k, the general expressions for Γ, K_1, and K_2 in a plane stack of finite layers take a relatively simple form. We have

$$\begin{aligned}\eta_{2y}\kappa_2 t_2 &= \eta_2\sigma_2(1 - \gamma_0^2/\sigma_2^2)t_2\\ &= \frac{i\omega\mu_1}{\mu_1\epsilon_2}[\mu_2\epsilon_2 - \mu_0\epsilon_0]\frac{(1-\theta)t_1}{\theta}\\ &= -i\omega\mu_1(1+k)t_1 = -(1+k)\eta_1\sigma_1 t_1\\ &\approx -(1+k)\eta_{1y}\kappa_1 t_1,\end{aligned} \tag{207}$$

after a little rearrangement, where the only approximation that has been made so far is to set $\eta_{1y} \approx \eta_1$ and $\kappa_1 \approx \sigma_1$. Substituting (207) into (86) and (87) gives

$$\text{ch}\,\Gamma = \text{ch}\,\Theta - \tfrac{1}{2}(1+k)\Theta\,\text{sh}\,\Theta, \tag{208}$$

and

$$K_1 = \frac{\Theta}{g_1 t_1}\left[\tfrac{1}{2}(1+k)\Theta + \sqrt{\tfrac{1}{4}(1+k)^2\Theta^2 - (1+k)\Theta\coth\Theta + 1}\right],$$
$$K_2 = \frac{\Theta}{g_1 t_1}\left[-\tfrac{1}{2}(1+k)\Theta + \sqrt{\tfrac{1}{4}(1+k)^2\Theta^2 - (1+k)\Theta\coth\Theta + 1}\right], \qquad (209)$$

where as usual

$$\Theta = \sigma_1 t_1 = (1+i)t_1/\delta_1 \approx \kappa_1 t_1 . \qquad (210)$$

If $k = 0$, equations (208) and (209) evidently reduce to (158) and (159) of the preceding section. For a stack of infinitesimally thin layers, the constants Γ_ℓ and K are given by equations (93) and (94) of Section III, namely

$$\Gamma_\ell = \left[\frac{i\bar{g}}{\omega\bar{\epsilon}}\,(\omega^2\bar{\mu}\bar{\epsilon} - \omega^2\mu_0\epsilon_0)\right]^{\frac{1}{2}} = (-2ik)^{\frac{1}{2}}\theta/\delta_1 , \qquad (211)$$

$$K = \Gamma_\ell/\bar{g} = (-2ik)^{\frac{1}{2}}/g_1\delta_1 . \qquad (212)$$

Up to this point we have set no restrictions on the magnitude of k, and we have not even assumed that k is necessarily real. Throughout the rest of this section, however, we shall assume that k is a positive or negative real number, as it must be if there is no dielectric or magnetic dissipation.

In practice both the lamina thickness and the amount of dielectric mismatch will be as small as it is feasible to make them. It will be useful, therefore, to obtain approximate expressions for Γ, K_1, and K_2 under the assumptions

$$|\Theta| \ll 1, \qquad |k| \ll 1. \qquad (213)$$

Then equation (208) yields

$$\operatorname{sh}^2 \tfrac{1}{2}\Gamma = \tfrac{1}{2}(\operatorname{ch}\Theta - 1) - \tfrac{1}{4}(1+k)\Theta \operatorname{sh}\Theta$$
$$= -\frac{k}{4}\,\Theta^2 - \frac{(1+2k)}{48}\,\Theta^4 - \cdots . \qquad (214)$$

If $|k| \ll 1$ we can neglect $2k$ compared to unity in the coefficient of Θ^4, but since we have made no assumptions as to the relative magnitudes of $|\Theta|$ and $|k|$, we cannot drop either the term in $k\Theta^2$ or the term in Θ^4. If we replace $\operatorname{sh}\tfrac{1}{2}\Gamma$ by $\tfrac{1}{2}\Gamma$ in (214), we get

$$\begin{aligned}\Gamma &\approx [-k\Theta^2 - \Theta^4/12]^{\frac{1}{2}} \\ &= \frac{t_1}{\sqrt{3}\delta_1}[(t_1/\delta_1)^2 - 6ik]^{\frac{1}{2}} \\ &= \frac{t_1}{\sqrt{3}\delta_1}\{[\sqrt{\tfrac{1}{4}(t_1/\delta_1)^4 + 9k^2} + \tfrac{1}{2}(t_1/\delta_1)^2]^{\frac{1}{2}} \\ &\qquad -i(\operatorname{sgn} k)[\sqrt{\tfrac{1}{4}(t_1/\delta_1)^4 + 9k^2} - \tfrac{1}{2}(t_1/\delta_1)^2]^{\frac{1}{2}}\},\end{aligned} \tag{215}$$

where we have taken the square root of the complex quantity by Dwight 58.2, and

$$\operatorname{sgn} k = \begin{matrix} +1 \text{ if } k > 0, \\ -1 \text{ if } k < 0. \end{matrix} \tag{216}$$

Similarly, from (209),

$$\begin{aligned}K_1 &= \frac{1}{g_1 t_1}\left[\frac{(1+k)}{2}\Theta^2 + \Theta\sqrt{-k - \frac{(1-2k-3k^2)}{12}\Theta^2 - \cdots}\right] \\ &\approx \frac{1}{g_1 t_1}[\tfrac{1}{2}\Theta^2 + \Theta\sqrt{-k - \Theta^2/12}] \\ &= \frac{it_1}{g_1\delta_1^2} + \frac{1}{\sqrt{3}g_1\delta_1}[(t_1/\delta_1)^2 - 6ik]^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{3}g_1\delta_1}\{[\sqrt{\tfrac{1}{4}(t_1/\delta_1)^4 + 9k^2} + \tfrac{1}{2}(t_1/\delta_1)^2]^{\frac{1}{2}} + i\sqrt{3}t_1/\delta_1 \\ &\qquad - i(\operatorname{sgn} k)[\sqrt{\tfrac{1}{4}(t_1/\delta_1)^4 + 9k^2} - \tfrac{1}{2}(t_1/\delta_1)^2]^{\frac{1}{2}}\}, \\ K_2 &\approx \frac{1}{\sqrt{3}g_1\delta_1}\{[\sqrt{\tfrac{1}{4}(t_1/\delta_1)^4 + 9k^2} + \tfrac{1}{2}(t_1/\delta_1)^2]^{\frac{1}{2}} - i\sqrt{3}t_1/\delta_1 \\ &\qquad - i(\operatorname{sgn} k)[\sqrt{\tfrac{1}{4}(t_1/\delta_1)^4 + 9k^2} - \tfrac{1}{2}(t_1/\delta_1)^2]^{\frac{1}{2}}\}.\end{aligned} \tag{217}$$

The effective skin depth of a stack of moderately thin layers in the presence of slight dielectric mismatch is, from (215),

$$\Delta = \frac{(t_1 + t_2)}{\operatorname{Re}\Gamma} = \frac{\sqrt{3}(t_1 + t_2)\delta_1/t_1}{[\sqrt{\tfrac{1}{4}(t_1/\delta_1)^4 + 9k^2} + \tfrac{1}{2}(t_1/\delta_1)^2]^{\frac{1}{2}}}. \tag{218}$$

An equation essentially equivalent to this was given by Clogston, in somewhat different notation.[17] It is clear from (211) or (218) that if the layers are infinitesimally thin, we have

$$\Delta = \delta_1/\theta \mid k \mid^{\frac{1}{2}}, \tag{219}$$

and the effective skin depth in the stack is proportional to the skin

[17] Reference 1, equation (III-42).

depth δ_1 in the conducting material at the operating frequency, although if the mismatch parameter k is small, the proportionality constant multiplying δ_1 will be large. In the general case, the number of double layers in one effective skin depth is

$$N = \frac{\Delta}{t_1 + t_2} = \frac{\sqrt{3}\delta_1/t_1}{[\sqrt{\frac{1}{4}(t_1/\delta_1)^4 + 9k^2} + \frac{1}{2}(t_1/\delta_1)^2]^{\frac{1}{2}}}, \tag{220}$$

and the total thickness of conducting material in these layers is

$$T_\Delta = Nt_1 = \frac{\sqrt{3}\delta_1}{[\sqrt{\frac{1}{4}(t_1/\delta_1)^4 + 9k^2} + \frac{1}{2}(t_1/\delta_1)^2]^{\frac{1}{2}}}. \tag{221}$$

It is instructive to plot the effective skin depth of a given stack at a fixed frequency as a function of dielectric mismatch. If

$$\Delta_0 = \sqrt{3}(t_1 + t_2)\delta_1^2/t_1^2 \tag{222}$$

denotes the effective skin thickness when there is no mismatch, then the relative skin thickness when the mismatch parameter is k is just

$$\frac{\Delta}{\Delta_0} = \frac{\sqrt{2}}{[\sqrt{1 + 36k^2\delta_1^4/t_1^4} + 1]^{\frac{1}{2}}}. \tag{223}$$

This ratio is plotted against k in Fig. 8, a universal curve being obtained by measuring k in units of $(t_1/\delta_1)^2$. It is worth noting that when $k = (t_1/\delta_1)^2$, the effective skin thickness is only 53 per cent of the skin thickness with perfect dielectric match.

The surface impedance $Z_0(\gamma_0)$ of a laminated plane stack at any frequency and with any amount of dielectric mismatch is given by equation (65),

$$Z_0(\gamma_0) = \frac{\frac{1}{2}Z_n(\gamma_0)(K_1e^{n\Gamma} + K_2e^{-n\Gamma}) + K_1K_2 \text{ sh } n\Gamma}{Z_n(\gamma_0) \text{ sh } n\Gamma + \frac{1}{2}(K_1e^{-n\Gamma} + K_2e^{n\Gamma})}. \tag{224}$$

For a stack with infinitesimally thin layers and total thickness s, the equation becomes

$$Z_0(\gamma_0) = K\frac{Z_n(\gamma_0) \text{ ch } \Gamma_\ell s + K \text{ sh } \Gamma_\ell s}{Z_n(\gamma_0) \text{ sh } \Gamma_\ell s + K \text{ ch } \Gamma_\ell s}, \tag{225}$$

where Γ_ℓ and K are given by (211) and (212). At zero frequency,

$$Z_0(\gamma_0) = \frac{1}{\bar{g}s + 1/Z_n(\gamma_0)} = \frac{1}{g_1T_1 + 1/Z_n(\gamma_0)}, \tag{226}$$

while if $Z_n(\gamma_0)$ is infinite, in general

$$Z_0(\gamma_0) = \frac{\Theta}{g_1 t_1} \{\tfrac{1}{2}(1 + k)\Theta + [\tfrac{1}{4}(1 + k)^2\Theta^2 - (1 + k)\Theta \coth \Theta + 1]^{\frac{1}{2}} \coth n\Gamma\}, \tag{227}$$

which simplifies, for infinitesimally thin layers, to

$$Z_0(\gamma_0) = K \coth \Gamma_\ell s. \tag{228}$$

If the stack is many effective skin depths thick, we have

$$Z_0(\gamma_0) = K_1 , \tag{229}$$

while if the individual layers are infinitesimally thin,

$$Z_0(\gamma_0) = K , \tag{230}$$

where K_1 and K are given by (209) and (211), respectively.

When $Z_0(\gamma_0)$ is known, the attenuation and phase constants of the parallel-plane Clogston 1 are given as usual by

$$\alpha = \operatorname{Re} Z_0(\gamma_0)/\eta_0 b, \tag{231}$$

$$\beta = \omega\sqrt{\mu_0\epsilon_0} + \operatorname{Im} Z_0(\gamma_0)/\eta_0 b. \tag{232}$$

For the coaxial cable we use

$$\alpha = \operatorname{Re} \frac{Z_1(\gamma_0)/\rho_1 + Z_2(\gamma_0)/\rho_2}{2\eta_0 \log (\rho_2/\rho_1)}, \tag{233}$$

$$\beta = \omega\sqrt{\mu_0\epsilon_0} + \operatorname{Im} \frac{Z_1(\gamma_0)/\rho_1 + Z_2(\gamma_0)/\rho_2}{2\eta_0 \log (\rho_2/\rho_1)}, \tag{234}$$

but the impedances of the cylindrical stacks are easy to compute only if we can employ the parallel-plane approximation for each stack. To take

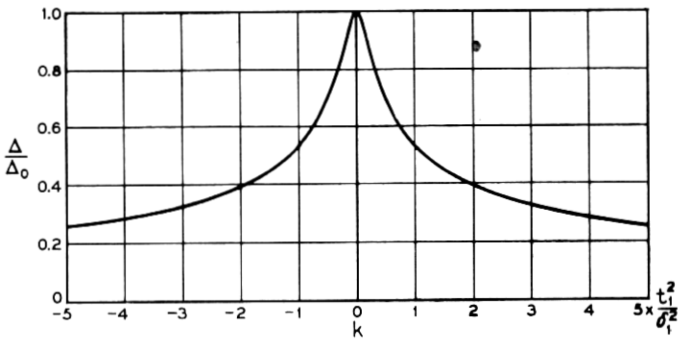

Fig. 8—Relative skin depth Δ/Δ_0 in a stack of finite layers versus dielectric mismatch parameter k, measured in units of $(t_1/\delta_1)^2$.

curvature effects into account would require a considerable amount of numerical calculation. Equation (98) of Section III provides an explicit expression for the surface impedance of a cylindrical stack of infinitesimally thin layers in the presence of dielectric mismatch, in terms of Bessel functions of complex argument; but if the layers are of finite thickness we can at present do nothing better than multiply out the matrices of the individual layers step by step.

The variation of the surface impedance of a laminated stack with frequency over the full frequency range is not quite so simple in the presence of dielectric mismatch as when Clogston's condition is exactly satisfied, but a somewhat analogous discussion may be given. As in the preceding section, we consider a plane stack of n conducting layers each of thickness t_1, where $nt_1 = T_1$, and backed by an infinite-impedance surface. When the mismatch parameter is k, the three critical frequencies are:

$$\begin{aligned} f_1 &= 1/(\pi\mu_1 g_1 T_1^2) && (T_1 = \delta_1), \\ f_2 &= \sqrt{3}/(\pi\mu_1 g_1 t_1 T_1 \sqrt{1 + 3n^2k^2}) && \\ &= \sqrt{3}nf_1/\sqrt{1 + 3n^2k^2} && (T_1 = T_\Delta), \\ f_3 &= 3/(\pi\mu_1 g_1 t_1^2) = 3n^2 f_1 && (t_1 = \sqrt{3}\delta_1). \end{aligned} \tag{235}$$

In the range $0 \leqq f \leqq f_2$, the surface impedance of the stack is approximately constant, being given by

$$Z_0(\gamma_0) \approx 1/g_1 T_1 . \tag{236}$$

In the range $f_2 \leqq f \leqq f_3$, we have

$$Z_0(\gamma_0) \approx K_1 , \tag{237}$$

where K_1 is given by (217) provided that k is small compared to unity. For infinitesimally thin layers the upper critical frequency f_3 is infinite, and we have for $f \geqq f_2$,

$$\begin{aligned} Z_0(\gamma_0) &\approx |k|^{\frac{1}{2}}(1 - i \operatorname{sgn} k)/g_1\delta_1 \\ &= (1 - i \operatorname{sgn} k)\sqrt{\pi\mu_1 |k| f/g_1} , \end{aligned} \tag{238}$$

which is proportional to $\sqrt{f}$. If the layers are of finite thickness but $k = 0$, we have the result obtained in the preceding section,

$$Z_0(\gamma_0) \approx (1/\sqrt{3} + i)\pi\mu_1 t_1 f, \tag{239}$$

which is proportional to f up to the critical frequency f_2. If neither the mismatch parameter k nor the layer thickness t_1 is zero, then the surface impedance $Z_0(\gamma_0)$ cannot be represented by a simple power of f in the range $f_2 \leqq f \leqq f_3$. At frequencies above f_3, if the layer thickness is finite, the impedance is approximately that of a solid conductor, namely

$$Z_0(\gamma_0) \approx (1 + i)/g_1\delta_1 = (1 + i)\sqrt{\pi\mu_1 f/g_1}\,, \tag{240}$$

which is proportional to $\sqrt{f}$.

Since in general the surface resistance depends upon the two parameters t_1/δ_1 and k, it is not possible to plot a single curve which shows the variation of resistance with frequency under all possible conditions of dielectric mismatch. However if we compare a matched stack of finite layers with a similar mismatched stack, we see that the asymptotic behavior of $Z_0(\gamma_0)$ is the same for both stacks at very low and very high frequencies. A numerical study of the exact equation for $Z_0(\gamma_0)$ shows that in the neighborhood of the critical frequency f_2, the resistance of the mismatched stack is higher than the resistance of the matched stack. (The critical frequency f_2 as defined in (235) is a function of the mismatch parameter k, but will be of the same order of magnitude for a slightly mismatched stack as for a perfectly matched stack.) The resistance of the mismatched stack exhibits relatively small fluctuations above and below the resistance of the matched stack in the neighborhood of the upper critical frequency f_3, but this region is not of as much practical interest as the region near f_2, where the stack resistance is definitely increased by the effect of dielectric mismatch.

An explicit expression for the rate at which the surface impedance of a mismatched stack begins to depart from its dc value as the frequency is increased has been worked out only for the ideal case of infinitesimally thin layers. For a plane stack of infinitesimal layers backed by an infinite-impedance surface, equation (228) gives, at moderately low frequencies,

$$\begin{aligned} Z_0(\gamma_0) &= \frac{K}{\Gamma_t s}\left[1 + \frac{(\Gamma_t s)^2}{3} - \frac{(\Gamma_t s)^4}{45} + \cdots\right] \\ &= \frac{1}{g_1 T_1}\left[1 - \frac{2ikT_1^2}{3\delta_1^2} + \frac{4k^2T_1^4}{45\delta_1^4} + \cdots\right], \end{aligned} \tag{241}$$

from which the fractional changes in resistance and reactance are

$$\frac{\Delta R}{R_0} = \frac{4k^2T_1^4}{45\delta_1^4} = \frac{4k^2\pi^2\mu_1^2 g_1^2 T_1^4 f^2}{45}\,, \tag{242}$$

$$\frac{\Delta X}{R_0} = -\frac{2kT_1^2}{3\delta_1^2} = -\frac{2k\pi\mu_1 g_1 T_1^2 f}{3}\,. \tag{243}$$

The admissible value of $|k|$, if the fractional change in resistance is not to exceed a specified value $\Delta R/R_0$ at a given top frequency f_m, is

$$|k| = \frac{3\sqrt{5}\delta_1^2}{2T_1^2}\sqrt{\frac{\Delta R}{R_0}} = \frac{3\sqrt{5}}{2\pi\mu_1 g_1 f_m T_1^2}\sqrt{\frac{\Delta R}{R_0}}, \qquad (244)$$

which is inversely proportional both to f_m and to the square of the total thickness of conducting material in the stack. If we express T_1 in mils, f_m in $\text{Mc}\cdot\text{sec}^{-1}$, and assume the conducting layers to be copper, we get

$$|k| = \frac{22.71}{(f_m)_{\text{Mc}}(T_1)^2_{\text{mils}}}\sqrt{\frac{\Delta R}{R_0}}. \qquad (245)$$

The variation with frequency of the surface impedance of a matched stack of finite layers at moderate frequencies (say $f \leqq f_2$) is given by equation (192) of Section V; but no simple formula has yet been derived for the surface impedance of a mismatched stack of finite layers in this frequency range. The derivation of such a formula would appear to involve nothing more than some rather formidable algebra, the difficulties centering around the fact that in the general case we can make no a priori assumptions as to the relative magnitudes of k and $(t_1/\delta_1)^2$. It is reasonable to suppose, however, that if both dielectric mismatch and finite lamina thickness contribute appreciably to $\Delta R/R_0$, the permissible values of $|k|$ and t_1 individually will be less, if we are to achieve a given flatness of the attenuation versus frequency curve, then the permissible value of either if the other factor were unimportant.

To exhibit the effect of dielectric mismatch from a slightly different point of view, we may plot the surface resistance of an infinitely deep plane stack of moderately thin layers (a finite stack several effective skin depths thick would show essentially the same behavior) at a fixed frequency, as a function of the mismatch parameter k. The surface resistance is just Re K_1, which may be obtained from (217) if k and t_1/δ_1 are assumed small compared to unity. Fig. 9 shows the dimensionless quantity

$$\text{Re } g_1\delta_1K_1 = \frac{1}{\sqrt{3}}\left[\sqrt{\tfrac{1}{4}(t_1/\delta_1)^4 + 9k^2} + \tfrac{1}{2}(t_1/\delta_1)^2\right]^{\frac{1}{2}}, \qquad (246)$$

for the three values $t_1/\delta_1 = 0$, $t_1/\delta_1 = 0.1$, and $t_1/\delta_1 = 0.2$. For an electrically thick solid conductor we have simply

$$\text{Re } g_1\delta_1K_1 = 1; \qquad (247)$$

hence to get any benefit from the laminated stack we must have Re $g_1\delta_1K_1$ smaller than unity. Actually, if we meet Clogston's condition by

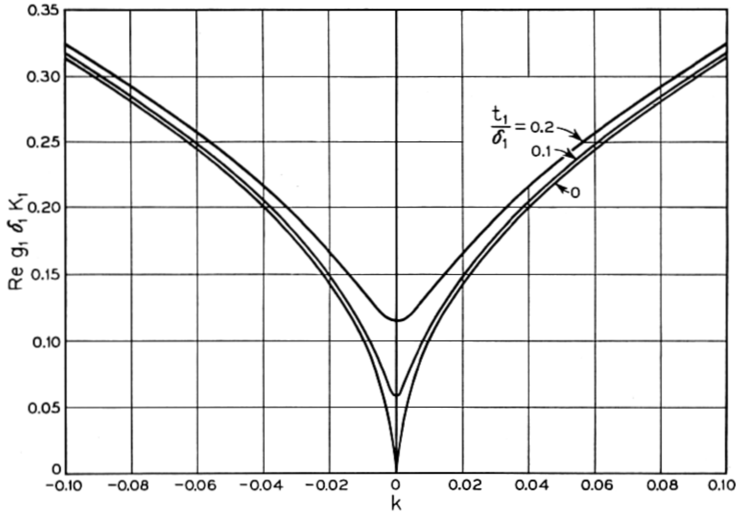

Fig. 9—Normalized stack resistance Re $g_1\delta_1K_1$ versus dielectric mismatch parameter k, for different values of t_1/δ_1.

raising the dielectric constant and thus lowering the impedance of the main dielectric, then since the attenuation constant of the line is proportional to the ratio of stack resistance to dielectric impedance, we must have Re $g_1\delta_1K_1$ considerably smaller than unity to obtain a lower attenuation with the Clogston line than with an ordinary air-filled line having solid metal walls.

For a plane Clogston 1 line with stacks of equal thickness, the fractional change in the attenuation constant with frequency is equal to the fractional change in the resistance of either stack, whether this change arises from the effects of finite lamina thickness or from dielectric mismatch or both. The fractional change in the attenuation constant of a coaxial Clogston 1 depends not only on the change in resistance of each stack, but also on the geometric proportions of the cable, in the manner expressed by equation (200) of Section V.

The effect of dielectric mismatch on the overall attenuation versus frequency characteristic of a Clogston cable is in general to reduce the total frequency range (in $\text{Mc}\cdot\text{sec}^{-1}$) over which the Clogston cable has a smaller attenuation constant than a conventional air-filled coaxial cable of the same size. To calculate the lower crossover frequency we may ordinarily neglect finite lamina thickness effects and use equation (241) for the stack impedances, while at the upper crossover frequency the stack impedances are very nearly equal to K_1, as given by (217).

It should be remembered that mismatch of the $\mu_0\epsilon_0$ product of the main dielectric will usually be accompanied by a change in the dielectric impedance $\sqrt{\mu_0/\epsilon_0}$. Thus under certain conditions the lower crossover frequency may even be reduced by choosing ϵ_0 slightly below the Clogston value, inasmuch as the increase in dielectric impedance may more than compensate for the increase in stack resistance at low frequencies; but it appears that this will be paid for in a steeper slope of the attenuation versus frequency curve and a consequent greater reduction of the upper crossover frequency.

It would be very useful to make a numerical study of the effects of dielectric mismatch in Clogston cables having a variety of different proportions; but in the present paper space limitations restrict us to a few observations concerning orders of magnitude. For the cable which we considered at the end of the preceding section, it turns out than an increase or decrease of 1 per cent in the value of ϵ_0 makes a change of at most a very few per cent in either crossover frequency; with a matched dielectric, we recall, these crossover frequencies were about 1 $\text{Mc}\cdot\text{sec}^{-1}$ and about 280 $\text{Mc}\cdot\text{sec}^{-1}$ respectively. However if we had designed a laminated cable with thicker stacks or thinner laminae or both, so as to increase the theoretical factor of improvement over a conventional cable in the working frequency range, we should have found that the tolerable deviation of ϵ_0 from Clogston's value, instead of being of the order of 1 per cent, was more nearly of the order of 0.1 per cent or even smaller; and the greater the improvement striven for, the more stringent the requirement of accurate dielectric match.

VII. DIELECTRIC AND MAGNETIC LOSSES IN CLOGSTON 1 LINES

Dielectric and magnetic dissipation in the main dielectric and in the stacks can be taken into account by introducing complex dielectric constants and permeabilities for the lossy materials. Thus we may write

$$\begin{aligned}
\epsilon_0 &= \epsilon_0' - i\epsilon_0'' = \epsilon_0' (1 - i \tan \phi_0), \\
\epsilon_2 &= \epsilon_2' - i\epsilon_2'' = \epsilon_2' (1 - i \tan \phi_2), \\
\mu_0 &= \mu_0' - i\mu_0'' = \mu_0' (1 - i \tan \zeta_0), \\
\mu_1 &= \mu_1' - i\mu_1'' = \mu_1' (1 - i \tan \zeta_1), \\
\mu_2 &= \mu_2' - i\mu_2'' = \mu_2' (1 - i \tan \zeta_2),
\end{aligned} \tag{248}$$

where in the most general case the loss tangents may all be different, though it will be assumed that they are all small compared to unity, so that the problem may be treated by first-order perturbation methods.

The average rate of energy dissipation per unit volume in a lossy dielectric by a harmonically varying electric field of maximum amplitude E is just $\frac{1}{2}\omega\epsilon''E^2$, since the imaginary part ϵ'' of the complex dielectric constant corresponds to a conductivity $g = \omega\epsilon''$. Similarly the average rate of energy dissipation per unit volume in a lossy magnetic material by a harmonically varying magnetic field of maximum amplitude H is $\frac{1}{2}\omega\mu''H^2$. The part of the attenuation constant which arises from dielectric and magnetic dissipation is one-half the ratio of power dissipated per unit length of line to total transmitted power, provided of course that the attenuation per wavelength is small. Since the loss tangents of the various materials are assumed small, we can use the fields found for the lossless case to calculate the transmitted and dissipated power.

If the volume occupied by currents in the stacks is small compared to the volume of the main dielectric, so that we can neglect the power flow in the stacks in the direction of wave propagation compared to the power flow in the main dielectric, then the part of the attenuation constant which is due to dielectric and magnetic dissipation is given by equation (51) of Section II, namely

$$\alpha_d = \tfrac{1}{2}\omega\sqrt{\mu_0'\epsilon_0'}(\tan\phi_0 + \tan\zeta_0) = \frac{\pi\sqrt{\mu_{0r}'\epsilon_{0r}'}}{\lambda_v}(\tan\phi_0 + \tan\zeta_0), \tag{249}$$

where λ_v is the vacuum wavelength and μ_{0r}', ϵ_{0r}' are the real parts of the relative permeability and relative dielectric constant of the main dielectric. This equation will be derived from energy considerations presently. It should be noted that the part of the attenuation constant given by (249) is directly proportional to frequency, provided that the loss tangents are independent of frequency; but it is the same for both plane and coaxial geometry and is independent of all the geometrical factors which describe the size and the relative proportions of the line.

Equation (249) will probably be sufficiently accurate for all Clogston 1 lines having the proportions (stacks thin compared to main dielectric) which we have considered in Part I. As an example wherein we also take into account the power flow in the stacks, however, we shall treat a parallel-plane line with infinitesimally thin laminae backed by high-impedance walls. Then, according to equations (120) and (121) of Section IV, the principal field components in the main dielectric are

$$H_x \approx H_0,$$
$$E_y \approx -\sqrt{\mu_0'/\epsilon_0'}H_0, \tag{250}$$

and in the stacks,

$$H_x \approx H_0(\tfrac{1}{2}a \mp y)/s,$$
$$\bar{E}_y \approx -\sqrt{\bar{\mu}'/\bar{\epsilon}'}H_0(\tfrac{1}{2}a \mp y)/s, \tag{251}$$

the propagation factor $e^{-\gamma z+i\omega t}$ being understood throughout. To take account of dielectric and magnetic dissipation in the stacks, we write

$$\bar{\epsilon} = \bar{\epsilon}' - i\bar{\epsilon}'' = [\epsilon_2'/(1-\theta)] - i[\epsilon_2''/(1-\theta)],$$
$$\bar{\mu} = \bar{\mu}' - i\bar{\mu}'' = [\theta\mu_1' + (1-\theta)\mu_2'] - i[\theta\mu_1'' + (1-\theta)\mu_2'']. \tag{252}$$

The average power P_0 transmitted through the main dielectric is obtained by integrating the real part of the z-component of the complex Poynting vector $\tfrac{1}{2}\mathbf{E} \times \mathbf{H}^*$ over unit width of the line; thus

$$P_0 = \tfrac{1}{2}\int_{-\frac{1}{2}b}^{\frac{1}{2}b} \sqrt{\mu_0'/\epsilon_0'}H_0H_0^*\,dy = \tfrac{1}{2}\sqrt{\mu_0'/\epsilon_0'}H_0H_0^*b. \tag{253}$$

Similarly, the average power P_1 transmitted per unit width of either stack is

$$P_1 = \tfrac{1}{2}\int_{\frac{1}{2}b}^{\frac{1}{2}a} [\sqrt{\bar{\mu}'/\bar{\epsilon}'}\; H_0H_0^*(\tfrac{1}{2}a - y)^2/s^2]\,dy$$
$$= \tfrac{1}{6}\sqrt{\bar{\mu}'/\bar{\epsilon}'}H_0H_0^*s. \tag{254}$$

The average power ΔP_0 dissipated in the main dielectric per unit length and width of the line is

$$\Delta P_0 = \tfrac{1}{2}\omega\int_{-\frac{1}{2}b}^{\frac{1}{2}b} [\epsilon_0''E_yE_y^* + \mu_0''H_xH_x^*]\,dy$$
$$= \tfrac{1}{2}\omega[\epsilon_0''(\mu_0'/\epsilon_0') + \mu_0'']H_0H_0^*b$$
$$= \tfrac{1}{2}\omega\mu_0'H_0H_0^*b(\tan\phi_0 + \tan\zeta_0), \tag{255}$$

while the average power ΔP_1 dissipated per unit length and width of either stack is

$$\Delta P_1 = \tfrac{1}{2}\omega\int_{\frac{1}{2}b}^{\frac{1}{2}a} [\bar{\epsilon}''\bar{E}_y\bar{E}_y^* + \bar{\mu}''H_xH_x^*]\,dy$$
$$= \tfrac{1}{6}\omega\bar{\mu}'H_0H_0^*s\,(\tan\phi_2 + \tan\bar{\zeta}), \tag{256}$$

where

$$\tan\bar{\zeta} = \frac{\bar{\mu}''}{\bar{\mu}'} = \frac{\theta\mu_1'' + (1-\theta)\mu_2''}{\theta\mu_1' + (1-\theta)\mu_2'}. \tag{257}$$

The attenuation constant due to dielectric and magnetic dissipation is

$$\begin{aligned} \alpha_d &= \frac{\Delta P_0 + 2\Delta P_1}{2(P_0 + 2P_1)} \\ &= \tfrac{1}{2}\omega\sqrt{\mu_0'\epsilon_0'} \; \frac{(\tan\phi_0 + \tan\zeta_0) + (2\bar{\mu}'s/3\mu_0'b)(\tan\phi_2 + \tan\bar{\zeta})}{1 + (2s/3b)\sqrt{\bar{\mu}'\epsilon_0'/\mu_0'\bar{\epsilon}'}}, \end{aligned} \tag{258}$$

which reduces to (249) if we neglect the terms in s/b. The total attenuation is the sum of the metal losses, given by equation (110), and the dielectric and magnetic losses.

For a coaxial Clogston 1 cable with infinitesimally thin laminae and high-impedance boundaries, the principal field components are given by equations (126)–(128) of Section IV. In the main dielectric we have

$$\begin{aligned} H_\phi &\approx \frac{I}{2\pi\rho}, \\ E_\rho &\approx \sqrt{\frac{\mu_0'}{\epsilon_0'}}\,\frac{I}{2\pi\rho}, \end{aligned} \tag{259}$$

while in the inner stack,

$$\begin{aligned} H_\phi &\approx \frac{I(\rho^2 - a^2)}{2\pi\rho(\rho_1^2 - a^2)}, \\ \bar{E}_\rho &\approx \sqrt{\frac{\bar{\mu}'}{\bar{\epsilon}'}}\,\frac{I(\rho^2 - a^2)}{2\pi\rho(\rho_1^2 - a^2)}, \end{aligned} \tag{260}$$

and in the outer stack,

$$\begin{aligned} H_\phi &\approx \frac{I(b^2 - \rho^2)}{2\pi\rho(b^2 - \rho_2^2)}, \\ \bar{E}_\rho &\approx \sqrt{\frac{\bar{\mu}'}{\bar{\epsilon}'}}\,\frac{I(b^2 - \rho^2)}{2\pi\rho(b^2 - \rho_2^2)}. \end{aligned} \tag{261}$$

The average power transmitted through the main dielectric is

$$P_0 = \frac{1}{2}\sqrt{\frac{\mu_0'}{\epsilon_0'}}\,\frac{II^*}{2\pi}\log\frac{\rho_2}{\rho_1}, \tag{262}$$

while for the average power transmitted through the inner and outer stacks it will be sufficient to replace the exact expressions by the following simple approximations,

$$P_1 \approx \frac{1}{2}\sqrt{\frac{\bar{\mu}'}{\bar{\epsilon}'}}\,\frac{II^*}{2\pi}\,\frac{s_1}{3\rho_1}, \tag{263}$$

$$P_2 \approx \frac{1}{2}\sqrt{\frac{\bar{\mu}'}{\bar{\epsilon}'}}\,\frac{II^*}{2\pi}\,\frac{s_2}{3\rho_2}. \tag{264}$$

For the average power dissipated per unit length of line in the main dielectric and the inner and outer stacks we have, respectively,

$$\Delta P_0 = \tfrac{1}{2}\omega\mu_0' \frac{II^*}{2\pi} \log \frac{\rho_2}{\rho_1} (\tan \phi_0 + \tan \zeta_0), \tag{265}$$

$$\begin{aligned} \Delta P_1 &\approx \tfrac{1}{2}\omega\bar{\mu}' \frac{II^*}{2\pi} \frac{s_1}{3\rho_1} (\tan \phi_2 + \tan \bar{\zeta}), \\ \Delta P_2 &\approx \tfrac{1}{2}\omega\bar{\mu}' \frac{II^*}{2\pi} \frac{s_2}{3\rho_2} (\tan \phi_2 + \tan \bar{\zeta}). \end{aligned} \tag{266}$$

The part of the attenuation constant which is due to dielectric and magnetic dissipation is therefore

$$\alpha_d = \frac{\Delta P_0 + \Delta P_1 + \Delta P_2}{2(P_0 + P_1 + P_2)}$$

$$= \tfrac{1}{2}\omega\sqrt{\mu_0'\epsilon_0'} \, \frac{\log \dfrac{\rho_2}{\rho_1} (\tan \phi_0 + \tan \zeta_0) + \dfrac{1}{3}\dfrac{\bar{\mu}'}{\mu_0'}\left(\dfrac{s_1}{\rho_1} + \dfrac{s_2}{\rho_2}\right)(\tan \phi_2 + \tan \bar{\zeta})}{\log \dfrac{\rho_2}{\rho_1} + \dfrac{1}{3}\sqrt{\dfrac{\bar{\mu}'\epsilon_0'}{\mu_0'\bar{\epsilon}'}}\left(\dfrac{s_1}{\rho_1} + \dfrac{s_2}{\rho_2}\right)}.$$

We need scarcely point out that if the loss tangents are not small compared to unity, it may be impossible to satisfy Clogston's condition (102) very closely with a real value of θ, and the resulting mismatch may reduce the depth of penetration and increase the metal losses in the stacks. In practice, however, the loss tangents will be of the order of 0.001 or even 0.0001, and matching the imaginary parts of $\mu_0\epsilon_0$ and $\bar{\mu}\bar{\epsilon}$ will be much less of a practical problem than matching the real parts.

Appendix I

BESSEL FUNCTION EXPANSIONS

Let ρ_1 and ρ_2 be the inner and outer radii of a cylindrical shell and let the thickness t, given by

$$t = \rho_2 - \rho_1, \tag{A1}$$

be less than ρ_1. Then, following Schelkunoff,[1] we may replace the Bessel functions appearing in equation (68) of Section III by their Taylor expansions, namely

[1] S. A. Schelkunoff, *Bell System Tech. J.*, **13,** pp. 561–562 (1934).

$$
\begin{aligned}
I_0(\kappa\rho_2) &= I_0(\kappa\rho_1 + \kappa t) = \sum_{n=0}^{\infty} \frac{(\kappa t)^n}{n!} I_0^{(n)}(\kappa\rho_1), \\
K_0(\kappa\rho_2) &= K_0(\kappa\rho_1 + \kappa t) = \sum_{n=0}^{\infty} \frac{(\kappa t)^n}{n!} K_0^{(n)}(\kappa\rho_1), \\
I_1(\kappa\rho_2) &= I_0'(\kappa\rho_2) = \sum_{n=0}^{\infty} \frac{(\kappa t)^n}{n!} I_0^{(n+1)}(\kappa\rho_1), \\
K_1(\kappa\rho_2) &= -K_0'(\kappa\rho_2) = -\sum_{n=0}^{\infty} \frac{(\kappa t)^n}{n!} K_0^{(n+1)}(\kappa\rho_1).
\end{aligned} \tag{A2}
$$

It follows that

$$
\begin{aligned}
K_0(\kappa\rho_1)I_1(\kappa\rho_2) + K_1(\kappa\rho_2)I_0(\kappa\rho_1) &= -\sum_{n=0}^{\infty} \frac{(\kappa t)^n}{n!} B_{n+1}(\kappa\rho_1), \\
K_0(\kappa\rho_1)I_0(\kappa\rho_2) - K_0(\kappa\rho_2)I_0(\kappa\rho_1) &= -\sum_{n=0}^{\infty} \frac{(\kappa t)^n}{n!} B_n(\kappa\rho_1), \\
K_1(\kappa\rho_1)I_1(\kappa\rho_2) - K_1(\kappa\rho_2)I_1(\kappa\rho_1) &= \sum_{n=0}^{\infty} \frac{(\kappa t)^n}{n!} A_{n+1}(\kappa\rho_1), \\
K_1(\kappa\rho_1)I_0(\kappa\rho_2) + K_0(\kappa\rho_2)I_1(\kappa\rho_1) &= \sum_{n=0}^{\infty} \frac{(\kappa t)^n}{n!} A_n(\kappa\rho_1),
\end{aligned} \tag{A3}
$$

where

$$
\begin{aligned}
A_n(x) &= I_0'(x)K_0^{(n)}(x) - K_0'(x)I_0^{(n)}(x), \\
B_n(x) &= I_0(x)K_0^{(n)}(x) - K_0(x)I_0^{(n)}(x).
\end{aligned} \tag{A4}
$$

The quantities $A_n(x)$ and $B_n(x)$ turn out to be finite polynomials in $1/x$, the general expressions for the coefficients having been derived in a rather inaccessible monograph by Pleijel.[2] When x is large, however, the leading terms are quite simple. From Pleijel's analysis, or directly by substituting the asymptotic series for $I_0(x)$ and $K_0(x)$ into (A4), we find

$$
\begin{aligned}
A_{2m}(x) &= 1/x + O(1/x^3), \\
A_{2m+1}(x) &= -m/x^2 + O(1/x^4), \\
B_{2m}(x) &= m/x^2 + O(1/x^4), \\
B_{2m+1}(x) &= -1/x + O(1/x^3),
\end{aligned} \tag{A5}
$$

where m is a positive integer or zero.

If we substitute these approximations into the first of equations (A3), we obtain

[2] H. Pleijel, *Beräkning af Motstånd och Självinduktion*, K. L. Beckmans Boktryckeri, Stockholm, 1906.

$$
\begin{aligned}
&K_0(\kappa\rho_1)I_1(\kappa\rho_2) + K_1(\kappa\rho_2)I_0(\kappa\rho_1) \\
&\approx \frac{1}{\kappa\rho_1}\sum_{m=0}^{\infty}\frac{(\kappa t)^{2m}}{(2m)!} - \frac{1}{(\kappa\rho_1)^2}\sum_{m=0}^{\infty}\frac{(m+1)(\kappa t)^{2m+1}}{(2m+1)!} \\
&= \frac{1}{\kappa\rho_2}\,\text{ch}\,\kappa t - \frac{1}{(\kappa\rho_1)^2}\left[\frac{1}{2}\frac{d}{dx}(x\,\text{sh}\,x)\right]_{x=\kappa t} \\
&= \left[\frac{1}{\kappa\rho_1} - \frac{t}{2\kappa\rho_1^2}\right]\text{ch}\,\kappa t - \frac{1}{2(\kappa\rho_1)^2}\,\text{sh}\,\kappa t.
\end{aligned}
\tag{A6}
$$

The other three equations may be treated similarly. Doing so, and remembering that

$$
\rho_2/\rho_1 = 1 + t/\rho_1 , \tag{A7}
$$

we obtain the results which were quoted in Section III, namely

$$
\begin{aligned}
\kappa\rho_2(K_{01}I_{12} + K_{12}I_{01}) &\approx \left[1 + \frac{t}{2\rho_1}\right]\text{ch}\,\kappa t - \frac{1}{2\kappa\rho_1}\,\text{sh}\,\kappa t, \\
\kappa\rho_2(K_{01}I_{02} - K_{02}I_{01}) &\approx \left[1 + \frac{t}{2\rho_1}\right]\text{sh}\,\kappa t, \\
\kappa\rho_2(K_{11}I_{12} - K_{12}I_{11}) &\approx \left[1 + \frac{t}{2\rho_1}\right]\text{sh}\,\kappa t, \\
\kappa\rho_2(K_{11}I_{02} + K_{02}I_{11}) &\approx \left[1 + \frac{t}{2\rho_1}\right]\text{ch}\,\kappa t + \frac{1}{2\kappa\rho_1}\,\text{sh}\,\kappa t,
\end{aligned}
\tag{A8}
$$

up to first order in t/ρ_1 .

Table of Symbols

Note: Rationalized MKS units are employed throughout. The subscripts 0, 1, 2 applied to symbols representing material constants, such as ϵ, μ, g, σ, and η, have the significance that 0 refers to the main dielectric in a Clogston line, while 1 refers to the conducting layers and 2 refers to the insulating layers in the stacks. Bars denote average values. Subscripts not included in the present table are explained in the context where they are used.

$\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$: Elements of the general circuit parameter matrix (Section III).

a: Distance between outer sheaths of plane Clogston line. Radius of inner core of coaxial Clogston line.

b: Thickness of main dielectric in plane Clogston line. Inner radius of outer sheath of coaxial Clogston line.

C: A parameter related to the degree of nonuniformity in a laminated medium (Section XII).

E: Electric field intensity; coordinate subscripts indicate components.

f: Frequency.

g: Electrical conductivity.

$\bar{g}$: θg_1; average conductivity parallel to laminated stack.

H: Magnetic field intensity; coordinate subscripts indicate components.

h: $-i\kappa_0$; a transverse separation constant (Section X).

I: Electric current.

i: $\sqrt{-1}$.

J: Electric current density; coordinate subscripts indicate components.

K: Characteristic impedance of stack of infinitesimally thin laminae.

K_1, K_2: Characteristic or iterative impedances of laminated stack (introduced in Section III).

k: A parameter related to dielectric mismatch in a Clogston 1 line (Section VI).

$\mathbf{M}$: The general circuit parameter matrix ($\mathfrak{ABCD}$-matrix).

m: A mode number.

n: Number of double layers in a laminated stack.

p: A mode number.

q: A parameter related to the propagation constant in a Clogston 2 line (Section XI).

R: A-c resistance of a laminated stack.

r: Ratio of attenuation constants of Clogston and conventional lines (Section XII).

s: Thickness of a laminated stack.

s_1, s_2: Thicknesses of inner and outer stacks in a coaxial Clogston 1.

T: Total thickness of conducting material in a laminated stack (subscripts explained in context).

T_Δ: Total thickness of conducting material in one effective skin depth.

t: Thickness of an electrically homogeneous layer. Time.

t_1: Thickness of a single conducting layer.

t_2: Thickness of a single insulating layer.

V: Electric potential.

w: An abbreviation for H_y in Section XII.

X: AC reactance of a laminated stack.

x: Rectangular coordinate in the direction of magnetic field in a plane Clogston line.

y: Rectangular coordinate in the direction normal to the stacks in a plane Clogston line.

Z: Surface impedance of a plane or cylindrical boundary; ratio of tangential components of the electric and magnetic fields (subscripts explained in context).

Z_k: Characteristic impedance of a transmission line.

z: Rectangular coordinate in the direction of wave propagation.

α: Re γ; attenuation constant.

β: Im γ; phase constant.

Γ: Propagation constant per double layer normal to laminated stack.

Γ_ℓ: $\Gamma/(t_1 + t_2)$; average propagation constant per unit distance normal to laminated stack.

γ: Propagation constant in longitudinal direction.

Δ: Effective skin depth; the depth at which the current density in an infinite plane stack has fallen to $1/e$ of its value at the surface. A small change in a quantity.

δ: $\sqrt{2/\omega\mu g}$; skin thickness in a solid conductor.

ϵ: Dielectric constant (capacitivity or permittivity).

$\bar{\epsilon}$: $\epsilon_2/(1 - \theta)$; average dielectric constant measured normal to laminated stack.

ϵ_r: ϵ/ϵ_v; relative dielectric constant.

ϵ_v: Dielectric constant of vacuum; 8.854×10^{-12} farads·meter^{-1}.

ϵ', ϵ'': Real and (negative) imaginary parts of complex dielectric constant.

ζ: $\tan^{-1}(\mu''/\mu')$; phase angle of complex permeability.

η: $\sqrt{i\omega\mu/(g + i\omega\epsilon)}$; intrinsic impedance of medium.

η_v: Intrinsic impedance of vacuum; 376.7 ohms.

η_y, η_ρ: $\eta(1 - \gamma^2/\sigma^2)^{\frac{1}{2}}$; characteristic impedance looking in the y- or ρ-direction in a homogeneous medium.

Θ: $(1 + i)t_1/\delta_1$; a parameter related to the electrical thickness of a conducting layer.

θ: $t_1/(t_1 + t_2)$; fraction of stack volume filled by conducting layers.

κ: $(\sigma^2 - \gamma^2)^{\frac{1}{2}}$; transverse propagation constant in the y- or ρ-direction in a homogeneous medium.

Λ:	A parameter related to the propagation constant in a Clogston 2 (Section XII).
λ:	Wavelength.
λ_v :	Free-space wavelength.
μ:	Permeability.
$\bar{\mu}$:	$\theta\mu_1 + (1 - \theta)\mu_2$; average permeability measured parallel to laminated stack.
μ_r :	μ/μ_v ; relative permeability.
μ_v :	Permeability of vacuum; $4\pi \times 10^{-7}$ henrys·meter^{-1}.
μ', μ'':	Real and (negative) imaginary parts of complex permeability.
ξ:	$y/a + \frac{1}{2}$; normalized coordinate transverse to a plane Clogston 2 line (Section XII).
ρ:	Radial coordinate in cylindrical system.
ρ_1 , ρ_2 :	Inner and outer radii of main dielectric in coaxial Clogston line.
σ:	$\sqrt{i\omega\mu(g + i\omega\epsilon)}$; intrinsic propagation constant of medium.
ϕ:	Angular coordinate in cylindrical system. Phase angle, $\tan^{-1}(\epsilon''/\epsilon')$, of complex dielectric constant.
χ:	$-i\Gamma_t$; a transverse separation constant.
ω:	Angular frequency in radians·second^{-1}.

FUNCTION SYMBOLS

Re:	Real part.
Im:	Imaginary part.
log:	Natural logarithm.
sh:	Hyperbolic sine.
ch:	Hyperbolic cosine.
J_0 , J_1 :	Bessel functions of the first kind.
N_0 , N_1 :	Bessel (Neumann) functions of the second kind.
I_0 , I_1 :	Modified Bessel functions of the first kind.
K_0 , K_1 :	Modified Bessel functions of the second kind.