Generalized Telegraphist's Equations for Waveguides

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In this paper Maxwell's partial differential equations and the boundary conditions for waveguides filled with a heterogeneous and non-isotropic medium are converted into an infinite system of ordinary differential equations. This system represents a generalization of "telegraphist's equations" for a single mode transmission to the case of multiple mode transmission. A similar set of equations is obtained for spherical waves. Although such generalized telegraphist's equations are very complicated, it is very likely that useful results can be obtained by an appropriate modal analysis.

From a purely mathematical point of view the problem of electromagnetic wave propagation inside a metal waveguide reduces to obtaining that solution of Maxwell's equations which satisfies certain boundary conditions *along* the waveguide and certain terminal conditions at the ends of the waveguide. If the medium inside the waveguide is homogeneous and isotropic and if the cross-section of the waveguide is either rectangular or circular or elliptic, the desired solution is obtained by the method of separating the variables. The method works for some other special cross-sections. It works also if the medium inside a rectangular waveguide consists of homogeneous, isotropic strata parallel to one of its faces. Similarly, it works if the medium inside a circular waveguide consists of coaxial, homogeneous, isotropic layers. But in general if the medium is either nonhomogeneous or non-isotropic or both, the method does not work. The mathematical reason for this is that the solution is of a more complicated form than a simple production of functions, each depending on a single coordinate. Any function that one usually encounters in physical problems, and therefore a solution of Maxwell's equations, may be expanded in a series of orthogonal functions. Sets of such functions are provided by the solutions for waveguides filled with homogeneous media. Such functions already satisfy the proper boundary conditions and the problem is to obtain series which also satisfy

Maxwell's equations. From the physical point of view this method represents a conversion of Maxwell's equations into generalized "telegraphist's equations."

Thus it is already known that Maxwell's partial differential equations *and* the boundary conditions *along* a waveguide are convertible into a set of independent ordinary differential equations, each resembling telegraphist's equations for electric transmission lines.[1] Each equation describes a "mode of propagation" in terms of concepts well known in electric circuit theory. A waveguide can be considered as an infinite system of transmission lines. If the medium inside the waveguide is homogeneous and isotropic and if the surface impedance of the boundary is zero, the method of separating the variables enables us to obtain a set of "normal", that is, *uncoupled* modes of propagation. Any irregularity or "discontinuity" in the waveguide provides a coupling between some, or all, modes of propagation. The irregularity may be in a boundary of the waveguide or in the dielectric within it. A heterogeneous dielectric may be considered as a homogeneous dielectric with distributed irregularities.[2] Similarly a heterogeneous non-isotropic dielectric may be considered as a homogeneous isotropic dielectric with distributed irregularities. Such irregularities provide a distributed coupling between the various modes appropriate to homogeneous isotropic waveguides. Our problem is to calculate the coupling coefficients. The generalized telegraphist's equations, obtained in this manner, are very complicated in that they represent an infinite number of coupled transmission modes. They are useful, however, in suggesting a physical picture of wave propagation under complicated conditions, and can be used in approximate analysis when we can ignore all but the most tightly coupled modes. For example, this picture was successfully employed by Albersheim[3] in studying the effect of gentle bending of a waveguide on propagation of circular electric waves. In this case it was important to consider the coupling between only two modes, TE_{01} and TM_{11}, which have the same cutoff frequency in a straight waveguide. More recently, Stevenson obtained exact equations for waves in horns of arbitrary shape.[4] His equations express the propagation of the axial components of E and H. The various modes are coupled through the boundary of the horn. In

[1] S. A. Schelkunoff, "Transmission Theory of Plane Electromagnetic Waves," *Proc. Inst. Radio Engrs.*, Nov. 1937, pp. 1457–1492.

[2] S. A. Schelkunoff, "Electromagnetic Waves," D. van Nostrand Co., (1943), pp. 92–93.

[3] W. J. Albersheim, "Propagation of TE_{01} Waves in Curved Waveguides," *Bell System Tech. J.*, Jan. 1949, pp. 1–32.

[4] A. F. Stevenson, "General Theory of Electromagnetic Horns," *J. Appl. Phys.*, Dec. 1951, pp. 1447–1460.

the present paper we shall consider waveguides of *constant* cross-section and *conical* horns of arbitrary shape filled with a heterogeneous and non-isotropic dielectric and derive the equations for propagation of the generalized voltages and currents representing the *transverse* field components. The various modes are coupled through the medium. It is very likely that our equations can be generalized to include the coupling through the boundary.

To understand the mechanism of coupling between the various modes through the medium consider Maxwell's equations

$$\text{curl } E = -j\omega B, \qquad \text{curl } H = {}^{c}J + j\omega D, \tag{1}$$

where ${}^{c}J$ is the density of conduction current while the other letter symbols have the usual meanings. In the most general linear case the components of B and D are linear functions of the components of H and E respectively, with the coefficients depending on the coordinates. These equations can always be rewritten as follows

$$\text{curl } E = -j\omega\mu H - M, \qquad \text{curl } H = j\omega\epsilon E + J, \tag{2}$$

where M and J are the densities of magnetic and electric polarization currents.[5]

$$M = j\omega(B - \mu H), \qquad J = {}^{c}J + j\omega(D - \epsilon E), \tag{3}$$

and μ, ϵ are constants (not necessarily those of vacuum). If M and J were given, they would act as sources exciting various modes of propagation in a homogeneous, isotropic waveguide. If M and J are functions of H and E, they can still be considered as the sources, acting on power borrowed from the wave, of the various modes. Thus M and J will provide the coupling between the modes existing in a homogeneous, isotropic waveguide.

Thus in order to derive the generalized telegraphist's equations we shall first consider the various modes of propagation in a homogeneous isotropic wave guide. Each mode is described by a transverse field distribution pattern[6] $T(u, v)$, where u and v are orthogonal coordinates of a point in a typical cross-section. This function is a solution of the following two-dimensional partial differential equation

$$\Delta T = \frac{1}{e_1 e_2}\left[\frac{\partial}{\partial u}\left(\frac{e_2}{e_1}\frac{\partial T}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{e_1}{e_2}\frac{\partial T}{\partial v}\right)\right] = -\chi^2 T, \tag{4}$$

[5] See Reference 2.

[6] S. A. Schelkunoff, "Electromagnetic Waves," D. van Nostrand Co. (1943), Chapter 10.

where χ is the separation constant and e_1, e_2 are the scale factors in the expression for the elementary distance

$$ds^2 = e_1^2\, du^2 + e_2^2\, dv^2. \tag{5}$$

In the case of TM waves the T-function must vanish on the boundary of zero impedance. This boundary condition restricts χ to a doubly infinite set of values χ_{mn} with the corresponding functions T_{mn}. In the case of TE waves the normal derivative of the T-function must vanish on the boundary of zero impedance. Since we have to consider both types of waves simultaneously, we shall distinguish between them by enclosing the *subscripts in parentheses* for TM waves and *in brackets* for TE waves. The double subscript designation of various modes has been standardized only for rectangular and circular waveguides. For waveguides of other shapes the standard is to use a single subscript by arranging the modes in the order of their cutoff frequencies. For convenience, we shall use this convention in the general case and denote TM modes by $T_{(n)}(u, v)$, and TE modes by $T_{[n]}(u, v)$. The corresponding cutoff constants will be $\chi_{(n)}$ and $\chi_{[n]}$. In what follows it is understood that whenever the T-functions should be designated by double subscripts, our single letter subscripts should be considered as symbols for ordered double subscripts.

The transverse field components may be derived from the potential and stream functions,[7] V and Π for TM waves and U and Ψ for TE waves. Thus

$$E_t = -\operatorname{grad} V - \operatorname{flux} \Psi, \qquad H_t = \operatorname{flux} \Pi - \operatorname{grad} U, \tag{6}$$

where the components of the gradient and flux of a scalar function W are

$$\begin{aligned} \operatorname{grad}_u W &= \frac{\partial W}{e_1\, \partial u}, & \operatorname{grad}_v W &= \frac{\partial W}{e_2\, \partial v}, \\ \operatorname{flux}_u W &= \frac{\partial W}{e_2\, \partial v}, & \operatorname{flux}_v W &= -\frac{\partial W}{e_1\, \partial u}. \end{aligned} \tag{7}$$

The T-functions corresponding to the various modes *of the same variety* are orthogonal; that is, the following integrals over the cross-section vanish,

$$\iint T_{(n)} T_{(m)}\, dS = 0, \qquad \iint T_{[n]} T_{[m]}\, dS = 0, \qquad \text{if } m \neq n. \tag{8}$$

It should be stressed that $T_{(n)}$ and $T_{[m]}$ are not, in general, orthogonal.

[7] See Reference 6.

Similarly the gradients of the T-functions of the same variety as well as the fluxes, are orthogonal,

$$\iint (\text{grad } T_{(n)}) \cdot (\text{grad } T_{(m)})\, dS = \iint (\text{flux } T_{(n)}) \cdot (\text{flux } T_{(m)})\, dS$$
$$= \iint (\text{grad } T_{[n]}) \cdot (\text{grad } T_{[m]})\, dS = \iint (\text{flux } T_{[n]}) \cdot (\text{flux } T_{[m]})\, dS = 0, \tag{9}$$

if $m \neq n$. The following gradients and fluxes of the T-functions are orthogonal for all m and n,

$$\iint (\text{grad } T_{(n)}) \cdot (\text{flux } T_{[m]})\, dS = \iint (\text{grad } T_{[n]}) \cdot (\text{flux } T_{(m)})\, dS$$
$$= \iint (\text{grad } T_{(n)}) \cdot (\text{flux } T_{(m)})\, dS = 0. \tag{10}$$

On the other hand, grad $T_{[m]}$ and flux $T_{[n]}$ are not, in general, orthogonal.

If all modes are present, the potential and stream functions are

$$V = -V_{(n)}(z)T_{(n)}(u, v), \qquad \Pi = -I_{(n)}(z)T_{(n)}(u, v),$$
$$\Psi = -V_{[n]}(z)T_{[n]}(u, v), \qquad U = -I_{[n]}(z)T_{[n]}(u, v), \tag{11}$$

where the tensor summation convention is used: whenever the same letter subscript is used in a product, it should receive all values in a given set and the resulting products should be added. The negative signs have been inserted in order to avoid a preponderance of negative signs in later equations. Substituting in (9), we have

$$E_t = V_{(n)} \text{ grad } T_{(n)} + V_{[n]} \text{ flux } T_{[n]}\,,$$
$$H_t = -I_{(n)} \text{ flux } T_{(n)} + I_{[n]} \text{ grad } T_{[n]}\,. \tag{12}$$

The T-functions for the various modes are determined by equation (4) and the boundary conditions except for arbitrary factors related to the power levels of the modes. If we choose these constants in such a way that

$$\iint (\text{grad } T) \cdot (\text{grad } T)\, dS \equiv \chi^2 \iint T^2\, dS = 1, \tag{13}$$

then the complex power carried by the wave is given by an expression similar to that in an ordinary transmission line,

$$P = \tfrac{1}{2} V_{(n)} I^*_{(n)} + \tfrac{1}{2} V_{[n]} I^*_{[n]}\,. \tag{14}$$

Hence, the V's and I's correspond to the voltages and currents in coupled transmission lines.

In an expanded form equations (12) are

$$E_u = V_{(n)} \frac{\partial T_{(n)}}{e_1 \, \partial u} + V_{[n]} \frac{\partial T_{[n]}}{e_2 \, \partial v}, \qquad E_v = V_{(n)} \frac{\partial T_{(n)}}{e_2 \, \partial v} - V_{[n]} \frac{\partial T_{[n]}}{e_1 \, \partial u},$$
$$H_v = I_{(n)} \frac{\partial T_{(n)}}{e_1 \, \partial u} + I_{[n]} \frac{\partial T_{[n]}}{e_2 \, \partial v}, \qquad H_u = -I_{(n)} \frac{\partial T_{(n)}}{e_2 \, \partial v} + I_{[n]} \frac{\partial T_{[n]}}{e_1 \, \partial u}. \tag{15}$$

To these we add the following expansions for the longitudinal components of E and H

$$E_z = \chi_{(n)} V_{z,(n)}(z) T_{(n)}(u, v), \qquad H_z = \chi_{[n]} I_{z,[n]}(z) T_{[n]}(u, v). \tag{16}$$

Equations of this form satisfy automatically the boundary conditions on E_z and H_z. The multipliers χ_n have been inserted arbitrarily in order to make the physical dimensions of the second factors to correspond to those of voltage and current.

Let us now write Maxwell's equations in an expanded form

$$\frac{\partial E_z}{e_2 \, \partial v} - \frac{\partial E_v}{\partial z} = -j\omega B_u, \qquad \frac{\partial H_z}{e_2 \, \partial v} - \frac{\partial H_v}{\partial z} = j\omega D_u,$$
$$\frac{\partial E_u}{\partial z} - \frac{\partial E_z}{e_1 \, \partial u} = -j\omega B_v, \qquad \frac{\partial H_u}{\partial z} - \frac{\partial H_z}{e_1 \, \partial u} = j\omega D_v, \tag{17}$$
$$\frac{\partial(e_2 E_v)}{\partial u} - \frac{\partial(e_1 E_u)}{\partial v} = -j\omega e_1 e_2 B_z, \qquad \frac{\partial(e_2 H_v)}{\partial u} - \frac{\partial(e_1 H_u)}{\partial v} = j\omega e_1 e_2 D_z.$$

Substituting from (15) and (16) in the left column of (17), we find

$$\chi_{(n)} V_{z,(n)} \frac{\partial T_{(n)}}{e_2 \, \partial v} - \frac{dV_{(n)}}{dz} \frac{\partial T_{(n)}}{e_2 \, \partial v} + \frac{dV_{[n]}}{dz} \frac{\partial T_{[n]}}{e_1 \, \partial u} = -j\omega B_u, \tag{18}$$

$$-\chi_{(n)} V_{z,(n)} \frac{\partial T_{(n)}}{e_1 \, \partial u} + \frac{dV_{(n)}}{dz} \frac{\partial T_{(n)}}{e_1 \, \partial u} + \frac{dV_{[n]}}{dz} \frac{\partial T_{[n]}}{e_2 \, \partial v} = -j\omega B_v, \tag{19}$$

$$V_{(n)} \frac{\partial^2 T_{(n)}}{\partial u \, \partial v} - V_{[n]} \frac{\partial}{\partial u} \left(\frac{e_2}{e_1} \frac{\partial T_{[n]}}{\partial u} \right) - V_{(n)} \frac{\partial^2 T_{(n)}}{\partial v \, \partial u} - V_{[n]} \frac{\partial}{\partial v} \left(\frac{e_1}{e_2} \frac{\partial T_{[n]}}{\partial v} \right) = -j\omega e_1 e_2 B_z. \tag{20}$$

In view of (4) the last equation reduces to

$$\chi_{[n]}^2 V_{[n]} T_{[n]} = -j\omega B_z. \tag{21}$$

Multiplying (18) by $[-\partial T_{(m)}/e_2\, \partial v]\, dS$, (19) by $[\partial T_{(m)}/e_1\, \partial u]\, dS$, adding, and integrating over the cross-section, we obtain

$$-\chi_{(m)} V_{z,(m)} + \frac{dV_{(m)}}{dz} = j\omega \iint \left(B_u \frac{\partial T_{(m)}}{e_2\, \partial v} - B_v \frac{\partial T_{(m)}}{e_1\, \partial u} \right) dS. \quad (22)$$

In the first term the summation convention should be ignored. Multiplying (18) by $[\partial T_{[m]}/e_1\, \partial u]\, dS$, (19) by $[\partial T_{[m]}/e_2\, \partial v]\, dS$, adding, and integrating we find

$$\frac{\partial V_{[m]}}{dz} = -j\omega \iint \left(B_u \frac{\partial T_{[m]}}{e_1\, \partial u} + B_v \frac{\partial T_{[m]}}{e_2\, \partial v} \right) dS. \quad (23)$$

Multiplying (21) by $T_{[m]}\, dS$ and integrating, we have

$$V_{[m]} = -j\omega \iint B_z T_{[m]}\, dS. \quad (24)$$

Subjecting the right column of (17) to a similar treatment, we obtain three additional equations. Summarizing, we have

$$\frac{\partial V_{(m)}}{dz} = j\omega \iint \left(B_u \frac{\partial T_{(m)}}{e_2\, \partial v} - B_v \frac{\partial T_{(m)}}{e_1\, \partial u} \right) dS + \chi_{(m)} V_{z,(m)}, \quad (25)$$

$$\frac{\partial I_{(m)}}{dz} = -j\omega \iint \left(D_u \frac{\partial T_{(m)}}{e_1\, \partial u} + D_v \frac{\partial T_{(m)}}{e_2\, \partial v} \right) dS, \quad (26)$$

$$\frac{dV_{[m]}}{dz} = -j\omega \iint \left(B_u \frac{\partial T_{[m]}}{e_1\, \partial u} + B_v \frac{\partial T_{[m]}}{e_2\, \partial v} \right) dS, \quad (27)$$

$$\frac{dI_{[m]}}{dz} = j\omega \iint \left(-D_u \frac{\partial T_{[m]}}{e_2\, \partial v} + D_v \frac{\partial T_{[m]}}{e_1\, \partial u} \right) dS + \chi_{[m]} I_{z,[m]}, \quad (28)$$

$$V_{[m]} = -j\omega \iint B_z T_{[m]}\, dS, \qquad I_{(m)} = -j\omega \iint D_z T_{(m)}\, dS. \quad (29)$$

In the last terms of equations (25) and (28) the summation convention should be ignored.

If the components of B and D are linear functions of the components of H and E respectively, then with the aid of (15) and (16) they can be expressed as linear functions of $V_{(n)}$, $V_{[n]}$, $I_{(n)}$, $I_{[n]}$, $V_{z,(n)}$, $I_{z,[n]}$. Solving (29) for $V_{z,(n)}$ and $I_{z,[n]}$ and making the appropriate substitutions in (25), (26), (27), (28), we obtain the generalized telegraphist's equations in the following form

$$\begin{aligned}
\frac{dV_{(m)}}{dz} &= -Z_{(m)(n)}I_{(n)} - Z_{(m)[n]}I_{[n]} - {}^{V}T_{(m)(n)}V_{(n)} - {}^{V}T_{(m)[n]}V_{[n]}\,, \\
\frac{dI_{(m)}}{dz} &= -Y_{(m)(n)}V_{(n)} - Y_{(m)[n]}V_{[n]} - {}^{I}T_{(m)(n)}I_{(n)} - {}^{I}T_{(m)[n]}I_{[n]}\,, \\
\frac{dV_{[m]}}{dz} &= -Z_{[m][n]}I_{[n]} - Z_{[m](n)}I_{(n)} - {}^{V}T_{[m][n]}V_{[n]} - {}^{V}T_{[m](n)}V_{(n)}\,, \\
\frac{dI_{[m]}}{dz} &= -Y_{[m][n]}V_{[n]} - Y_{[m](n)}V_{(n)} - {}^{I}T_{[m][n]}I_{[n]} - {}^{I}T_{[m](n)}I_{(n)}\,.
\end{aligned} \tag{30}$$

The transfer impedances Z, the transfer admittances Y, the voltage transfer coefficients ${}^{V}T$, and the current transfer coefficients ${}^{I}T$ between various modes are in general functions of z. They are constants if the properties of the waveguide are independent of the distance along it; in this case the problem of solving the generalized telegraphist's equations reduces to solving an infinite system of linear algebraic equations and the corresponding characteristic equation.

Similar equations may be derived for spherical waves either in an unlimited medium or in a medium bounded by a perfectly conducting conical surface of arbitrary cross-section. If the latter is circular and if the flare angle is 180°, we have a plane boundary. Hence, the case of spherical waves in a non-homogeneous medium is included. In the spherical case we shall use the general orthogonal system of coordinates (r, u, v) where r is the distance from the center and (u, v) are orthogonal angular coordinates. In this system the elements of length ds and area dS are given by

$$ds^2 = dr^2 + r^2(e_1^2\,du^2 + e_2^2\,dv^2), \qquad dS = r^2 d\Omega, \qquad d\Omega = e_1 e_2\,du\,dv. \tag{31}$$

The transverse field components may be expressed in a form similar to that for waveguides

$$rE_t = -\operatorname{grad} V - \operatorname{flux}\Pi, \qquad rH_t = \operatorname{flux}\Pi - \operatorname{grad} U, \tag{32}$$

where grad and flux of a typical scalar function are defined by equations (10). Instead of (11) we have

$$\begin{aligned}
V &= -V_{(n)}(r)T_{(n)}(u, v), \qquad & \Pi &= -I_{(n)}(r)T_{(n)}(u, v), \\
\Psi &= -V_{[n]}(r)T_{[n]}(u, v), \qquad & U &= -I_{[n]}(r)T_{[n]}(u, v),
\end{aligned} \tag{33}$$

where the T-functions satisfy equation (4) and appropriate boundary conditions. These functions, their gradients and fluxes are orthogonal.

They are assumed to be normalized as follows

$$\iint (\text{grad } T)\cdot(\text{grad } T)\, d\Omega = \chi^2 \iint T^2\, d\Omega = 1\,, \tag{34}$$

where $d\Omega$ is an elementary solid angle. Hence, equation (14) will again represent the complex power flow in the direction of propagation.

The various field components may then be expressed as follows

$$\begin{aligned} rE_u &= V_{(n)} \frac{\partial T_{(n)}}{e_1\, \partial u} + V_{[n]} \frac{\partial T_{[n]}}{e_2\, \partial v}, \qquad rE_v = V_{(n)} \frac{\partial T_{(n)}}{e_2\, \partial v} - V_{[n]} \frac{\partial T_{[n]}}{e_1\, \partial u}, \\ rH_v &= I_{(n)} \frac{\partial T_{(n)}}{e_1\, \partial u} + I_{[n]} \frac{\partial T_{[n]}}{e_2\, \partial v}, \qquad rH_u = -I_{(n)} \frac{\partial T_{(n)}}{e_2\, \partial v} + I_{[n]} \frac{\partial T_{[n]}}{e_1\, \partial u}, \\ & r^2 E_r = \chi_{(n)}\, V_{r,(n)}\, T_{(n)}\,, \qquad r^2 H_r = \chi_{[n]}\, I_{r,[n]}\, T_{[n]}\,. \end{aligned} \tag{35}$$

It should be noted that the physical dimensions of $V_{r,(n)}$ and $I_{r,[n]}$ are not those of voltage and current. Substituting in Maxwell's equations and using transformations similar to those in the case of plane waves, we find

$$\begin{aligned} \frac{dV_{(m)}}{dr} &= j\omega \iint \left(rB_u \frac{\partial T_{(m)}}{e_2\, \partial v} - rB_v \frac{\partial T_{(m)}}{e_1\, \partial u} \right) d\Omega + \chi_{(m)} r^{-2}\, V_{r,(m)}\,, \\ \frac{dI_{(m)}}{dr} &= -j\omega \iint \left(rD_u \frac{\partial T_{(m)}}{e_1\, \partial u} + rD_v \frac{\partial T_{(m)}}{e_2\, \partial v} \right) d\Omega, \\ \frac{dV_{[m]}}{dr} &= -j\omega \iint \left(rB_u \frac{\partial T_{[m]}}{e_1\, \partial u} + rB_v \frac{\partial T_{[m]}}{e_2\, \partial v} \right) d\Omega, \\ \frac{dI_{[m]}}{dr} &= j\omega \iint \left(-\, rD_u \frac{\partial T_{[m]}}{e_2\, \partial v} + rD_v \frac{\partial T_{[m]}}{e_1\, \partial u} \right) d\Omega + \chi_{[m]} r^{-2} I_{r,[m]}\,, \\ V_{[m]} &= -j\omega \iint (r^2 B_r) T_{[m]}\, d\Omega, \qquad I_{(m)} = -j\omega \iint (r^2 D_r) T_{(m)}\, d\Omega. \end{aligned} \tag{36}$$

Returning to the plane wave case and assuming the following general linear relations

$$\begin{aligned} B_u &= \mu_{uu} H_u + \mu_{uv} H_v + \mu_{uz} H_z\,, \qquad D_u = \epsilon_{uu} E_u + \epsilon_{uv} E_v + \epsilon_{uz} E_z\,, \\ B_v &= \mu_{vu} H_u + \mu_{vv} H_v + \mu_{vz} H_z\,, \qquad D_v = \epsilon_{vu} E_u + \epsilon_{vv} E_v + \epsilon_{vz} E_z\,, \\ B_z &= \mu_{zu} H_u + \mu_{zv} H_v + \mu_{zz} H_z\,, \qquad D_z = \epsilon_{zu} E_u + \epsilon_{zv} E_v + \epsilon_{zz} E_z\,, \end{aligned} \tag{37}$$

we find

$$B_u = I_{(n)}\left[-\mu_{uu}\frac{\partial T_{(n)}}{e_2\,\partial v} + \mu_{uv}\frac{\partial T_{(n)}}{e_1\,\partial u}\right] + I_{[n]}\left[\mu_{uu}\frac{\partial T_{[n]}}{e_1\,\partial u} + u_{uv}\frac{\partial T_{[n]}}{e_2\,\partial v}\right] + I_{z,[n]}\mu_{uz}\chi_{[n]}T_{[n]},$$

$$B_v = I_{(n)}\left[-\mu_{vu}\frac{\partial T_{(n)}}{e_2\,\partial v} + \mu_{vv}\frac{\partial T_{(n)}}{e_1\,\partial u}\right] + I_{[n]}\left[\mu_{vu}\frac{\partial T_{[n]}}{e_1\,\partial u} + \mu_{vv}\frac{\partial T_{[n]}}{e_2\,\partial v}\right] + I_{z,[n]}\mu_{vz}\chi_{[n]}T_{[n]},$$

$$B_z = I_{(n)}\left[-\mu_{zu}\frac{\partial T_{(n)}}{e_2\,\partial v} + \mu_{zv}\frac{\partial T_{(n)}}{e_1\,\partial u}\right] + I_{[n]}\left[\mu_{zu}\frac{\partial T_{[n]}}{e_1\,\partial u} + \mu_{zv}\frac{\partial T_{[n]}}{e_2\,\partial v}\right] + I_{z,[n]}\mu_{zz}\chi_{[n]}T_{[n]}, \tag{38}$$

$$D_u = V_{(n)}\left[\epsilon_{uu}\frac{\partial T_{(n)}}{e_1\,\partial u} + \epsilon_{uv}\frac{\partial T_{(n)}}{e_2\,\partial v}\right] + V_{[n]}\left[\epsilon_{uu}\frac{\partial T_{[n]}}{e_2\,\partial v} - \epsilon_{uv}\frac{\partial T_{[n]}}{e_1\,\partial u}\right] + V_{z,(n)}\epsilon_{uz}\chi_{(n)}T_{(n)},$$

$$D_v = V_{(n)}\left[\epsilon_{vu}\frac{\partial T_{(n)}}{e_1\,\partial u} + \epsilon_{vv}\frac{\partial T_{(n)}}{e_2\,\partial v}\right] + V_{[n]}\left[\epsilon_{vu}\frac{\partial T_{[n]}}{e_2\,\partial v} - \epsilon_{vv}\frac{\partial T_{[n]}}{e_1\,\partial u}\right] + V_{z,(n)}\epsilon_{vz}\chi_{(n)}T_{(n)},$$

$$D_z = V_{(n)}\left[\epsilon_{zu}\frac{\partial T_{(n)}}{e_1\,\partial u} + \epsilon_{zv}\frac{\partial T_{(n)}}{e_2\,\partial v}\right] + V_{[n]}\left[\epsilon_{zu}\frac{\partial T_{[n]}}{e_2\,\partial v} - \epsilon_{zv}\frac{\partial T_{[n]}}{e_1\,\partial u}\right] + V_{z,(n)}\epsilon_{zz}\chi_{(n)}T_{(n)}.$$

Substituting from equations (38) into equations (25) to (29) we obtain

$$\begin{aligned}\frac{dV_{(m)}}{dz} = {} & -j\omega I_{(n)}\iint\left[\mu_{uu}\frac{\partial T_{(n)}}{e_2\,\partial v}\frac{\partial T_{(m)}}{e_2\,\partial v} + \mu_{vv}\frac{\partial T_{(n)}}{e_1\,\partial u}\frac{\partial T_{(m)}}{e_1\,\partial u}\right.\\ & \left. - \mu_{uv}\frac{\partial T_{(n)}}{e_1\,\partial u}\frac{\partial T_{(m)}}{e_2\,\partial v} - \mu_{vu}\frac{\partial T_{(n)}}{e_2\,\partial v}\frac{\partial T_{(m)}}{e_1\,\partial u}\right]dS\\ & + j\omega I_{[n]}\iint\left[\mu_{uu}\frac{\partial T_{[n]}}{e_1\,\partial u}\frac{\partial T_{(m)}}{e_2\,\partial v} - \mu_{vv}\frac{\partial T_{[n]}}{e_2\,\partial v}\frac{\partial T_{(m)}}{e_1\,\partial u}\right.\\ & \left. + \mu_{uv}\frac{\partial T_{[n]}}{e_2\,\partial v}\frac{\partial T_{(m)}}{e_2\,\partial v} - \mu_{vu}\frac{\partial T_{[n]}}{e_1\,\partial u}\frac{\partial T_{(m)}}{e_1\,\partial u}\right]dS\\ & + j\omega I_{z,[n]}\iint\left[\mu_{uz}\frac{\partial T_{(m)}}{e_2\,\partial v} - \mu_{vz}\frac{\partial T_{(m)}}{e_1\,\partial u}\right]\chi_{[n]}T_{[n]}\,dS + \chi_{(m)}V_{z,(m)},\end{aligned} \tag{39}$$

$$\begin{aligned}\frac{dI_{(m)}}{dz} = -j\omega V_{(n)} \iint \Big[&\epsilon_{uu} \frac{\partial T_{(n)}}{e_1\,\partial u} \frac{\partial T_{(m)}}{e_1\,\partial u} + \epsilon_{vv} \frac{\partial T_{(n)}}{e_2\,\partial v} \frac{\partial T_{(m)}}{e_2\,\partial v} \\ &+ \epsilon_{uv} \frac{\partial T_{(n)}}{e_2\,\partial v} \frac{\partial T_{(m)}}{e_1\,\partial u} + \epsilon_{vu} \frac{\partial T_{(n)}}{e_1\,\partial u} \frac{\partial T_{(m)}}{e_2\,\partial v} \Big] dS \\ + j\omega V_{[n]} \iint \Big[&-\epsilon_{uu} \frac{\partial T_{[n]}}{e_2\,\partial v} \frac{\partial T_{(m)}}{e_1\,\partial u} + \epsilon_{vv} \frac{\partial T_{[n]}}{e_1\,\partial u} \frac{\partial T_{(m)}}{e_2\,\partial v} \\ &+ \epsilon_{uv} \frac{\partial T_{[n]}}{e_1\,\partial u} \frac{\partial T_{(m)}}{e_1\,\partial u} - \epsilon_{vu} \frac{\partial T_{[n]}}{e_2\,\partial v} \frac{\partial T_{(m)}}{e_2\,\partial v} \Big] dS \\ - j\omega V_{z,(n)} \iint \Big[&\epsilon_{uz} \frac{\partial T_{(m)}}{e_1\,\partial u} + \epsilon_{vz} \frac{\partial T_{(m)}}{e_2\,dv} \Big] \chi_{(n)} T_{(n)}\, dS,\end{aligned} \tag{40}$$

$$\begin{aligned}\frac{dV_{[m]}}{dz} = j\omega I_{(n)} \iint \Big[&\mu_{uu} \frac{\partial T_{(n)}}{e_2\,dv} \frac{\partial T_{[m]}}{e_1\,\partial u} - \mu_{vv} \frac{\partial T_{(n)}}{e_1\,\partial u} \frac{\partial T_{[m]}}{e_2\,\partial v} \\ &- \mu_{uv} \frac{\partial T_{(n)}}{e_1\,\partial u} \frac{\partial T_{[m]}}{e_1\,\partial u} + \mu_{vu} \frac{\partial T_{(n)}}{e_2\,\partial v} \frac{\partial T_{[m]}}{e_2\,\partial v} \Big] dS \\ - j\omega I_{[n]} \iint \Big[&\mu_{uu} \frac{\partial T_{[n]}}{e_1\,\partial u} \frac{\partial T_{[m]}}{e_1\,\partial u} + \mu_{vv} \frac{\partial T_{[n]}}{e_2\,\partial v} \frac{\partial T_{[m]}}{e_2\,\partial v} \\ &+ \mu_{uv} \frac{\partial T_{[n]}}{e_2\,\partial v} \frac{\partial T_{[m]}}{e_1\,\partial u} + \mu_{vu} \frac{\partial T_{[n]}}{e_1\,\partial u} \frac{\partial T_{[m]}}{e_2\,\partial v} \Big] dS \\ - j\omega I_{z,[n]} \iint \Big[&\mu_{uz} \frac{\partial T_{[m]}}{e_1\,\partial u} + \mu_{vz} \frac{\partial T_{[m]}}{e_2\,\partial v} \Big] \chi_{[n]} T_{[n]}\, dS,\end{aligned} \tag{41}$$

$$\begin{aligned}\frac{dI_{[m]}}{dz} = j\omega V_{(n)} \iint \Big[&-\epsilon_{uu} \frac{\partial T_{(n)}}{e_1\,\partial u} \frac{\partial T_{[m]}}{e_2\,\partial v} + \epsilon_{vv} \frac{\partial T_{(n)}}{e_2\,\partial v} \frac{\partial T_{[m]}}{e_1\,\partial u} \\ &- \epsilon_{uv} \frac{\partial T_{(n)}}{e_2\,\partial v} \frac{\partial T_{[m]}}{e_2\,\partial v} + \epsilon_{vu} \frac{\partial T_{(n)}}{e_1\,\partial u} \frac{\partial T_{[m]}}{e_1\,\partial u} \Big] dS \\ - j\omega V_{[n]} \iint \Big[&\epsilon_{uu} \frac{\partial T_{[n]}}{e_2\,\partial v} \frac{\partial T_{[m]}}{e_2\,\partial v} + \epsilon_{vv} \frac{\partial T_{[n]}}{e_1\,\partial u} \frac{\partial T_{[m]}}{e_1\,\partial u} \\ &- \epsilon_{uv} \frac{\partial T_{[n]}}{e_1\,\partial u} \frac{\partial T_{[m]}}{e_2\,\partial v} - \epsilon_{vu} \frac{\partial T_{[n]}}{e_2\,\partial v} \frac{\partial T_{[m]}}{e_1\,\partial u} \Big] dS \\ + j\omega V_{z,(n)} \iint \Big[&-\epsilon_{uz} \frac{\partial T_{[m]}}{e_2\,\partial v} + \epsilon_{vz} \frac{\partial T_{[m]}}{e_1\,\partial u} \Big] \chi_{(n)} T_{(n)}\, dS + \chi_{[m]} I_{z,[m]},\end{aligned} \tag{42}$$

$$\begin{aligned}I_{z,[n]} \iint j\omega\mu_{zz}\chi_{[n]} T_{[n]} T_{[m]}\, dS = &-V_{[m]} \\ &+ I_{(p)} \iint j\omega \Big[\mu_{zu} \frac{\partial T_{(p)}}{e_2\,\partial v} - \mu_{zv} \frac{\partial T_{(p)}}{e_1\,\partial u} \Big] T_{[m]}\, dS \\ &- I_{[p]} \iint j\omega \Big[\mu_{zu} \frac{\partial T_{[p]}}{e_1\,\partial u} + \mu_{zv} \frac{\partial T_{[p]}}{e_2\,\partial v} \Big] T_{[m]}\, dS,\end{aligned} \tag{43}$$

$$V_{z,(n)} \iint j\omega\epsilon_{zz}\chi_{(n)}T_{(n)}T_{(m)}\,dS = -I_{(m)}$$
$$- V_{(p)} \iint j\omega \left[\epsilon_{zu} \frac{\partial T_{(p)}}{e_1\,\partial u} + \epsilon_{zv} \frac{\partial T_{(p)}}{e_2\,\partial v} \right] T_{(m)}\,dS \qquad (44)$$
$$+ V_{[p]} \iint j\omega \left[-\epsilon_{zu} \frac{\partial T_{[p]}}{e_2\,\partial v} + \epsilon_{zv} \frac{\partial T_{[p]}}{e_1\,\partial u} \right] T_{(m)}\,dS.$$

If we solve the last two equations for $I_{z,[n]}$ and $V_{z,(n)}$ and substitute in the preceding four equations, we shall obtain the telegraphist's equations in their final form (30). Thus, let

$$\begin{aligned} {}^zZ_{[m][n]} &= \iint j\omega\mu_{zz}\chi_{[n]}T_{[n]}T_{[m]}\,dS, \\ {}^zY_{(m)(n)} &= \iint j\omega\epsilon_{zz}\chi_{(n)}T_{(n)}T_{(m)}\,dS. \end{aligned} \qquad (45)$$

From these coefficients we obtain another set

$$\begin{aligned} {}^zZ_{[n][m]} &= \text{normalized co-factor of } {}^zZ_{[m][n]}\,, \\ {}^zZ_{(n)(m)} &= \text{normalized co-factor of } {}^zY_{(m)(n)}\,. \end{aligned} \qquad (46)$$

Then,

$$\begin{aligned} I_{z,[n]} = &-V_{[m]}{}^zY_{[n][m]} \\ &+ I_{(p)}{}^zY_{[n][m]} \iint j\omega \left[\mu_{zu} \frac{\partial T_{(p)}}{e_2\,\partial v} - \mu_{zv} \frac{\partial T_{(p)}}{e_1\,\partial u} \right] T_{[m]}\,dS \\ &- I_{[p]}{}^zY_{[n][m]} \iint j\omega \left[\mu_{zu} \frac{\partial T_{[p]}}{e_1\,\partial u} + \mu_{zv} \frac{\partial T_{[p]}}{e_2\,\partial v} \right] T_{[m]}\,dS, \\ V_{z,(n)} = &-I_{(m)}{}^zZ_{(n)(m)} \\ &- V_{(p)}{}^zZ_{(n)(m)} \iint j\omega \left[\epsilon_{zu} \frac{\partial T_{(p)}}{e_1\,\partial u} + \epsilon_{zv} \frac{\partial T_{(p)}}{e_2\,\partial v} \right] T_{(m)}\,dS \\ &+ V_{[p]}{}^zZ_{(n)(m)} \iint j\omega \left[-\epsilon_{zu} \frac{\partial T_{[p]}}{e_2\,\partial v} + \epsilon_{zv} \frac{\partial T_{[p]}}{e_1\,\partial u} \right] T_{(m)}\,dS. \end{aligned} \qquad (47)$$

Before substituting in equations (39) to (42), the summation index m in (47) should be changed to avoid conflict with m in the former equations. It does not seem necessary to make these final substitutions in their most general form. The results are very complicated and in practice the various coefficients are not independent. Some coefficients may vanish; others may be small. In isotropic media, $\mu_{uu} = \mu_{vv} = \mu_{zz} = \mu$, $\epsilon_{uu} = \epsilon_{vv} = \epsilon_{zz} = \epsilon$ and the mutual coefficients vanish. In gyromagnetic media subjected to a strong magnetic field in the z-direction, the permeability coefficients of superposed *ac* fields are[8]

$$\mu_{uu} = \mu_{vv} = \mu, \qquad \mu_{vu} = -\mu_{uv}, \qquad \mu_{zu} = \mu_{zv} = \mu_{uz} = \mu_{vz} = 0. \tag{48}$$

If the entire waveguide is filled with such a medium, assumed to be homogeneous, equations (43) and (44) become

$$\begin{aligned} I_{z,[n]}\, j\omega\mu_{zz}\chi_{[n]} \iint T_{[n]}T_{[m]}\, dS &= -V_{[m]}\,, \\ V_{z,(n)}\, j\omega\epsilon\chi_{(n)} \iint T_{(n)}T_{(m)}\, dS &= -I_{(m)}\,. \end{aligned} \tag{49}$$

In view of the orthogonality of the T-functions and the normalization conditions (13), we have

$$I_{z,[m]} = -\frac{\chi_{[m]}}{j\omega\mu_{zz}}\, V_{[m]}\,, \qquad V_{z,(m)} = -\frac{\chi_{(m)}}{j\omega\epsilon}\, I_{(m)}\,, \tag{50}$$

where the summation convention is waived. In this case all the transfer coefficients in equations (30) vanish,

$$\begin{aligned} {}^{V}T_{(m)(n)} = {}^{V}T_{(m)[n]} = {}^{V}T_{[m][n]} = {}^{V}T_{[m](n)} = {}^{I}T_{(m)(n)} = {}^{I}T_{(m)[n]} \\ = {}^{I}T_{[m](n)} = {}^{I}T_{[m][n]} = 0. \end{aligned} \tag{51}$$

The transfer impedances and admittances are

$$\begin{aligned} Z_{(m)(n)} &= 0, \quad \text{if } n \neq m, \\ &= j\omega\mu + \frac{\chi_{(m)}^{2}}{j\omega\epsilon}, \quad \text{if } n = m; \\ Z_{(m)[n]} &= -j\omega\mu_{uv} \iint \left[\frac{\partial T_{[n]}}{e_1\,\partial u}\,\frac{\partial T_{(m)}}{e_1\,\partial u} + \frac{\partial T_{[n]}}{e_2\,\partial v}\,\frac{\partial T_{(m)}}{e_2\,\partial v}\right] e_1e_2\, du\, dv; \\ Y_{(m)(n)} &= 0, \quad \text{if } n \neq m, \\ &= j\omega\epsilon, \quad \text{if } n = m; \\ Y_{(m)[n]} &= 0, \quad \text{all } m, n; \\ Z_{[m][n]} &= j\omega\mu_{uv} \iint \left[\frac{\partial T_{[n]}}{\partial v}\,\frac{\partial T_{[m]}}{\partial u} - \frac{\partial T_{[n]}}{\partial u}\,\frac{\partial T_{[m]}}{\partial v}\right] du\, dv, \quad \text{if } n \neq m, \\ &= j\omega\mu, \quad \text{if } n = m; \end{aligned} \tag{52}$$

[8] C. L. Hogan, "The Ferromagnetic Faraday Effect at Microwave Frequencies and Its Applications—The Microwave Gyrator, *Bell System Tech. J.*, Jan. 1952, p. 9.

$$Z_{[m](n)} = j\omega\mu_{uv} \iint \left[\frac{\partial T_{(n)}}{e_1\,\partial u}\frac{\partial T_{[m]}}{e_1\,\partial u} + \frac{\partial T_{(n)}}{e_2\,\partial v}\frac{\partial T_{[m]}}{e_2\,\partial v}\right] e_1 e_2\,du\,dv;$$

$$Y_{[m][n]} = 0, \quad \text{if } n \neq m,$$

$$= j\omega\epsilon + \frac{\chi^2_{[m]}}{j\omega\mu_{zz}}, \quad \text{if } n = m;$$

$$Y_{[m](n)} = 0, \quad \text{all } m, n.$$

We note that $Z_{(m)[n]} = -Z_{[n](m)}$; $Z_{[m][n]} = -Z_{[n][m]}$, $(n \neq m)$.

In rectangular waveguides we choose cartesian coordinates; then $e_1 = e_2 = 1$, $u = x$, $v = y$ and

$$T_{(pq)} = 1_{pq}\chi^{-1}_{(pq)}(ab)^{-1/2} \sin\frac{p\pi x}{a} \sin\frac{q\pi y}{b},$$

$$T_{[st]} = 1_{st}\chi^{-1}_{[st]}(ab)^{-1/2} \cos\frac{s\pi x}{a} \cos\frac{t\pi y}{b}, \tag{53}$$

$$\chi^2_{(pq)} = \chi^2_{[pq]} = \frac{p^2\pi^2}{a^2} + \frac{q^2\pi^2}{b^2} \equiv \chi^2_{pq},$$

where $1_{pq} = 2$ if neither p nor q is equal to zero and $1_{0q} = 1_{p0} = \sqrt{2}$. Hence

$$Z_{(pq)[st]} = j\omega\mu_{xy}\frac{1_{pq}1_{st}\pi^2}{a^2b^2\chi_{pq}\chi_{st}}$$

$$\times \iint \left[(b/a)sp \sin\frac{s\pi x}{a}\cos\frac{p\pi x}{a}\cos\frac{t\pi y}{b}\sin\frac{q\pi y}{b}\right.$$

$$\left. + (a/b)tq\cos\frac{s\pi x}{a}\sin\frac{p\pi x}{a}\sin\frac{t\pi y}{b}\cos\frac{q\pi y}{b}\right] dx\,dy$$

$$= j\omega\mu_{xy}\frac{1_{pq}1_{st}pq[(s/a)^2 + (t/b)^2][1-(-)^{s+p}][1-(-)^{q+t}]}{\chi_{pq}\chi_{st}(s^2-p^2)(q^2-t^2)},$$

$$\text{if } s \neq p,\ q \neq t, \tag{54}$$

$$= 0, \quad \text{if } s = p \text{ or } q = t;$$

$$Z_{[pq][st]} = j\omega\mu_{xy}\frac{1_{pq}1_{st}(p^2t^2 - q^2s^2)[1-(-)^{s+p}][1-(-)^{q+t}]}{\chi_{pq}\chi_{st}(s^2-p^2)(q^2-t^2)ab},$$

$$\text{if } s \neq p,\ q \neq t,$$

$$= 0, \quad \text{if } s = p \text{ or } q = t, \text{ but not if } s = p \text{ and } q = t,$$

$$= j\omega\mu, \quad \text{if } s = p \text{ and } q = t.$$

Some of the mutual impedances vanish; thus

$$Z_{(pq)[st]} = 0, \tag{55}$$

if either $p + s$ *or* $q + t$ is *even*. If $p + s$, as well as $q + t$, is odd,

$$Z_{(pq)[st]} = \frac{4 \cdot 1_{pq} 1_{st} j\omega\mu_{xy} pq[(s/a)^2 + (t/b)^2]}{\chi_{pq}\chi_{st}(s^2 - p^2)(q^2 - t^2)}. \tag{56}$$

Similarly

$$Z_{[pq][st]} = 0, \tag{57}$$

if either $p + s$ *or* $q + t$ is *even*, provided $p \neq s$ and $q \neq t$. If $p + s$, as well as $q + t$, is odd,

$$Z_{[pq][st]} = \frac{4 \cdot 1_{pq} 1_{st} (p^2 t^2 - q^2 s^2) j\omega\mu_{xy}}{\chi_{pq}\chi_{st}(s^2 - p^2)(q^2 - t^2)ab}. \tag{58}$$

Consider now the set of modes which includes $TE_{[10]}$. This set includes $TE_{[01]}$ modes and all the other modes which are coupled to either of these modes. Noting that there are no $TM_{(p0)}$ and $TM_{(0q)}$ modes, we obtain the following table in which those modes which do not belong to the set are marked with a bar:

$$\begin{aligned}
&TE_{[10]}, TE_{[01]},\\
&\overline{TE}_{[20]}, \overline{TE}_{[11]}, \overline{TE}_{[02]}, \overline{TM}_{(11)},\\
&TE_{[30]}, TE_{[21]}, TE_{[12]}, TE_{[03]}, TM_{(21)}, TM_{(12)},\\
&\overline{TE}_{[40]}, \overline{TE}_{[31]}, \overline{TE}_{[22]}, \overline{TE}_{[13]}, \overline{TE}_{[04]}, \overline{TM}_{(31)}, \overline{TM}_{(22)}, \overline{TM}_{(13)},
\end{aligned} \tag{59}$$

From the preceding equations we obtain the coupling impedances,

$$\begin{aligned}
Z_{[10][01]} &= \frac{8}{\pi^2} j\omega\mu_{xy}, \quad Z_{[01][10]} = -\frac{8}{\pi^2} j\omega\mu_{xy},\\
Z_{[10][30]} &= Z_{[30][10]} = Z_{[01][03]} = Z_{[03][01]} = 0,\\
Z_{[30][01]} &= -Z_{[01][30]} = Z_{[10][03]} = -Z_{[03][10]} = \frac{8}{3\pi^2} j\omega\mu_{xy},\\
Z_{[21][10]} &= -Z_{[10][21]} = \frac{8\sqrt{2}}{3\pi^2} j\omega\mu_{xy}[1 + 4(b/a)^2]^{-1/2},\\
Z_{[01][12]} &= -Z_{[12][01]} = \frac{8\sqrt{2}}{3\pi^2} j\omega\mu_{xy}[1 + 4(a/b)^2]^{-1/2},\\
Z_{[10][12]} &= Z_{[12][10]} = Z_{[01][21]} = Z_{[21][01]} = 0,\\
Z_{[10](21)} &= -Z_{(21)[10]} = \frac{16\sqrt{2}}{3\pi^2} j\omega\mu_{xy}[4 + (a/b)^2]^{-1/2},
\end{aligned} \tag{60}$$

$$Z_{[01](12)} = -Z_{(12)[01]} = \frac{16\sqrt{2}}{3\pi^2} j\omega\mu_{xy}[4 + (b/a)^2]^{-1/2},$$

$$Z_{(12)[10]} = Z_{[10](12)} = Z_{(21)[01]} = Z_{[01](21)} = 0.$$

The principal effect of the gyromagnetic medium on the $TE_{[10]}$ and $TE_{[01]}$ modes may be understood by taking into account their mutual coupling but ignoring their coupling to other modes. The equations of propagation become

$$\begin{aligned}
\frac{dV_{[10]}}{dz} &= -j\omega\mu I_{[10]} - j\omega\mu_{xy}(8/\pi^2) I_{[01]}, \\
\frac{dI_{[10]}}{dz} &= -\left(j\omega\epsilon + \frac{\pi^2}{j\omega\mu_{zz}a^2}\right) V_{[10]}, \\
\frac{dV_{[01]}}{dz} &= j\omega\mu_{xy}(8/\pi^2) I_{[10]} - j\omega\mu I_{[01]}, \\
\frac{dI_{[01]}}{dz} &= -\left(j\omega\epsilon + \frac{\pi^2}{j\omega\mu_{zz}b^2}\right) V_{[01]}.
\end{aligned} \tag{61}$$

For exponentially propagated waves we have

$$\begin{aligned}
V_{[10]} &= \hat{V}_{[10]} e^{-j\beta z}, & V_{[01]} &= \hat{V}_{[01]} e^{-j\beta z}, \\
I_{[10]} &= \hat{I}_{[10]} e^{-j\beta z}, & I_{[01]} &= \hat{I}_{[01]} e^{-j\beta z}.
\end{aligned} \tag{62}$$

When the mutual permeability is zero, we have two independent modes whose phase constants are

$$\beta_{10} = \left(\omega^2\mu\epsilon - \frac{\mu\pi^2}{\mu_{zz}a^2}\right)^{1/2}, \qquad \beta_{01} = \left(\omega^2\mu\epsilon - \frac{\mu\pi^2}{\mu_{zz}b^2}\right)^{1/2}. \tag{63}$$

The phase constants of the perturbed modes may be expressed in terms of the unperturbed constants and the coefficient of coupling. When the losses are neglected, the mutual permeability is a pure imaginary. In this case it is convenient to define a *real* coupling coefficient

$$k = \frac{j8\mu_{xy}}{\pi^2\mu}. \tag{64}$$

Substituting from (62) in (61) and using (64), we find

$$\begin{aligned}
\beta\hat{V}_{[10]} &= \omega\mu\hat{I}_{[10]} - j\omega\mu k\hat{I}_{[01]}, & \beta\hat{I}_{[10]} &= \left(\omega\epsilon - \frac{\pi^2}{\omega\mu_{zz}a^2}\right)\hat{V}_{[10]}, \\
\beta\hat{V}_{[01]} &= j\omega\mu k\hat{I}_{[10]} + \omega\mu\hat{I}_{[01]}, & \beta\hat{I}_{[01]} &= \left(\omega\epsilon - \frac{\pi^2}{\omega\mu_{zz}b^2}\right)\hat{V}_{[01]}.
\end{aligned} \tag{65}$$

Eliminating $\hat{V}_{[10]}$ and $\hat{V}_{[01]}$, we have

$$(\beta^2 - \beta_{10}^2)\hat{I}_{[10]} = -jk\beta_{10}^2\hat{I}_{[01]}\,, \qquad (\beta^2 - \beta_{01}^2)\hat{I}_{[01]} = jk\beta_{01}^2\hat{I}_{[10]}\,. \tag{66}$$

Multiplying term by term, we obtain the characteristic equation

$$\beta^4 - (\beta_{10}^2 + \beta_{01}^2)\beta^2 + (1 - k^2)\beta_{10}^2\beta_{01}^2 = 0. \tag{67}$$

Solving, we have

$$\beta^2 = \tfrac{1}{2}(\beta_{10}^2 + \beta_{01}^2) \pm \tfrac{1}{2}[(\beta_{10}^2 - \beta_{01}^2)^2 + 4k^2\beta_{10}^2\beta_{01}^2]^{1/2}. \tag{68}$$

The effect of coupling is to increase the larger phase constant and decrease the smaller one; that is, to make the slower wave slower, and the faster wave faster.

Let us assume $a > b$; then $\beta_{10} > \beta_{01}$. Taking the upper sign in (68) and substituting in the second equation of the set (66), we have

$$\frac{\hat{I}_{[01]}}{\hat{I}_{[10]}} = \frac{jk(\beta_{01}/\beta_{10})}{p + (p^2 + k^2)^{1/2}}, \qquad p = \frac{1}{2}\left(\frac{\beta_{10}}{\beta_{01}} - \frac{\beta_{01}}{\beta_{10}}\right). \tag{69}$$

From (65) and (69) we find

$$\frac{\hat{V}_{[01]}}{\hat{V}_{[10]}} = \frac{\beta_{10}^2}{\beta_{01}^2}\,\frac{\hat{I}_{[01]}}{\hat{I}_{[10]}} = \frac{jk(\beta_{10}/\beta_{01})}{p + (p^2 + k^2)^{1/2}}\,. \tag{70}$$

Hence, the ratio of the power carried in the $TE_{[01]}$ mode to that in the $TE_{[10]}$ mode is

$$\frac{P_{01}}{P_{10}} = \frac{\hat{V}_{[01]}\hat{I}_{[01]}^*}{\hat{V}_{[10]}\hat{I}_{[10]}^*} = \frac{k^2}{[p + (p^2 + k^2)^{1/2}]^2}\,. \tag{71}$$

This ratio increases as k increases and p decreases.

If the phase constants of the uncoupled modes are equal, then $p = 0$ and $P_{01} = P_{10}$ for all values of the coupling coefficient. In this case (68) becomes

$$\beta^2 = \beta_{10}^2(1 \pm k) \quad \text{or} \quad \beta = \beta_{10}(1 \pm k)^{1/2}. \tag{72}$$

In terms of the original constants,

$$\beta = \left[\left(\mu \pm \frac{8}{\pi^2}\,j\mu_{xy}\right)\left(\omega^2\epsilon - \frac{\pi^2}{\mu_{zz}a^2}\right)\right]^{1/2}. \tag{73}$$

The cutoff frequencies of both normal modes are seen to be independent of either the transverse permeability or the mutual permeability. Since

μ_{xy} is a pure imaginary, it effectively increases the transverse permeability for one mode and decreases it for the other.

To evaluate the effect of higher order TE and TM modes on wave propagation we may substitute from (68) in all terms of the characteristic equation for telegraphist's equations except the first two diagonal terms and recalculate the β's. Alternatively we may replace $TE_{[10]}$ and $TE_{[01]}$ modes by the normal modes just obtained, recalculate the coupling coefficients, and evaluate the effect of the mode with the greatest coupling to the modes under consideration.