Network Synthesis Using Tchebycheff Polynomial Series†

By SIDNEY DARLINGTON

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A general method is developed for finding functions of frequency which approximate assigned gain or phase characteristics, within the special class of functions which can be realized exactly as the gain or phase of finite networks of linear lumped elements. The method is based upon manipulations of two Tchebycheff polynomial series, one of which represents the assigned characteristic, and the other the approximating network function. The wide range of applicability is illustrated with a number of examples.

1. INTRODUCTION

Network synthesis is the opposite of network analysis—namely, the design of a network to have assigned characteristics, as opposed to the evaluation of the characteristics of an assigned network. In general, there are specifications on the internal constitution of the network, as well as requirements relating to its external performance. A common form of the general problem is the design of a finite network of linear lumped elements, to produce an assigned gain or phase characteristic over a prescribed interval of useful frequencies. The present paper relates to this particular form.

In general, the restrictions on the network are such that the assigned performance cannot be matched exactly. This gives rise to an approximation or interpolation problem. For present purposes, the problem is: to choose a function of frequency which matches the assigned gain or phase to a satisfactory accuracy, from that special class of functions which can be realized exactly with physical finite networks of linear lumped elements. The function of frequency may be defined in terms of network singularities (natural modes and infinite loss points). The

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interpolation problem may then be regarded as solved when a suitable set of network singularities has been obtained; for quite different techniques are used to design actual networks with these singularities.

The interpolation problem may be attacked in a number of different ways; and a variety of different techniques are, in fact, needed to cover the wide diversity of practical applications. The present topic is a fairly general way of attacking the problem, based upon manipulations of two series of Tchebycheff polynomials. The two series represent expansions of two functions of frequency—one, the ideal assigned gain or phase, the other, the network approximation to the ideal. The interpolation problem may be solved in this way because it is feasible, as will be shown, to determine network singularities from arbitrarily assigned values of coefficients in the corresponding Tchebycheff polynomial series.

The techniques to be described were derived originally from studies of the so-called potential analogy; but they can now be developed most easily without reference thereto.† In a sense they may be regarded as extensions of familiar filter theory, using Tchebycheff polynomials, to more general gain and phase functions. The extensions, however, depend upon a number of new principles. Extensions of the filter theory applied to more general problems have been noted in published papers; but those noted have not used the specific general approach employed here.‡ The wide applicability of this general approach will be illustrated by specific examples.

2. NETWORK AND TRANSMISSION FUNCTION

It will be sufficient for our present purposes to limit the discussion to the general 4-pole shown in Fig. 1. The 4-pole may be active or passive, but it must be a stable finite network of linear lumped elements. E and V are steady state ac voltages, E the driving voltage and V the response. The gain α and phase β are here defined as the real and imaginary parts of $\log V/E$.

For a finite network of lumped elements, $\alpha + i\beta$ always has the following form:

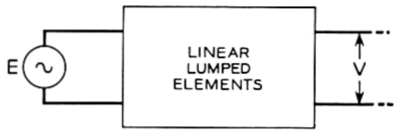
$$\alpha + i\beta = \log K \frac{(p - p_1')(p - p_2') \cdots}{(p - p_1'')(p - p_2'') \cdots} \tag{1}$$

† Tchebycheff polynomials are related to potential analogue charges on ellipses, as described in the author's paper "The Potential Analogue Method of Network Synthesis"[1], Section 15.

‡ For the most part, they have used the potential analogy, in such a way that Tchebycheff polynomials do not appear at all in general applications. For examples, see methods of Matthaei[2], Bashkow[3], and Kuh[4].

The "frequency variable" p represents, of course, $i\omega$. The zeros p'_σ of the rational fraction are those values of p at which there is infinite loss. The poles p''_σ are the so-called natural modes, or values of p at which response V can exist in the absence of driving voltage E. The scale factor K determines the average level of transmission. The zeros, poles, and scale factor together determine the gain and phase completely.

For a physical stable network, the zeros and poles must meet certain well known restrictions, which are commonly stated in terms of locations in the complex plane for frequency variable p. Within these restrictions, the zeros and poles can be subject to arbitrary choice, say for purposes of network synthesis.



$\alpha + i\beta = \log V/E$

Fig. 1—A general 4-pole.

The symmetries required by the physical restrictions permit α and β to be represented separately as follows:†

$$\begin{aligned} 2\alpha &= \log K^2 \frac{(p_1'^2 - p^2)(p_2'^2 - p^2)\cdots}{(p_1''^2 - p^2)(p_2''^2 - p^2)\cdots} \\ i2\beta &= \log \frac{(p_1' - p)\cdots(p_1'' + p)\cdots}{(p_1' + p)\cdots(p_1'' - p)\cdots} \end{aligned} \qquad (2)$$

These expressions hold at all real frequencies, but only at real frequencies.

3. TCHEBYCHEFF POLYNOMIALS

It is functions of these special types which we are to synthesize with the help of Tchebycheff polynomials. More generally, Tchebycheff polynomials come in various forms, and may be analyzed in various ways. For our special purposes, however, they take somewhat special forms (a little different from textbook definitions); and they are best analyzed in quite special ways.‡ It will be simplest to start with arbitrary definitions, to be justified later on by demonstrations of usefulness.

† The phase equation omits a possible 180° phase reversal, which is trivial for present purposes.

‡ For discussions of Tchebycheff polynomials from other viewpoints, see Courant and Hilbert[5], and also a paper by Lanczos[6] on trigonometric interpolation.

Actually, the definitions must vary with the nature of the useful frequency interval. For the present, however, it will be assumed that the useful interval extends from $\omega = 0$ to ω_c; or more precisely, from $\omega = -\omega_c$ to $+\omega_c$ (in accordance with the symmetries of gain and phase functions). Useful intervals which do not include $\omega = 0$ require changes in the definitions, which will be taken up in Section 28.

For our present purposes, Tchebycheff polynomials T_k may be defined as follows:

$$
\begin{aligned}
p &= i\omega = i\omega_c \sin \phi \\
T_k &= \cos k\phi, \quad k \text{ even} \\
T_k &= i \sin k\phi, \quad k \text{ odd}
\end{aligned}
\tag{3}
$$

The first equation defines an auxiliary angle variable, ϕ, in terms of which T_k is especially simple. The imaginary scale factor i, associated with polynomials of odd order, simplifies later analysis. In addition, it

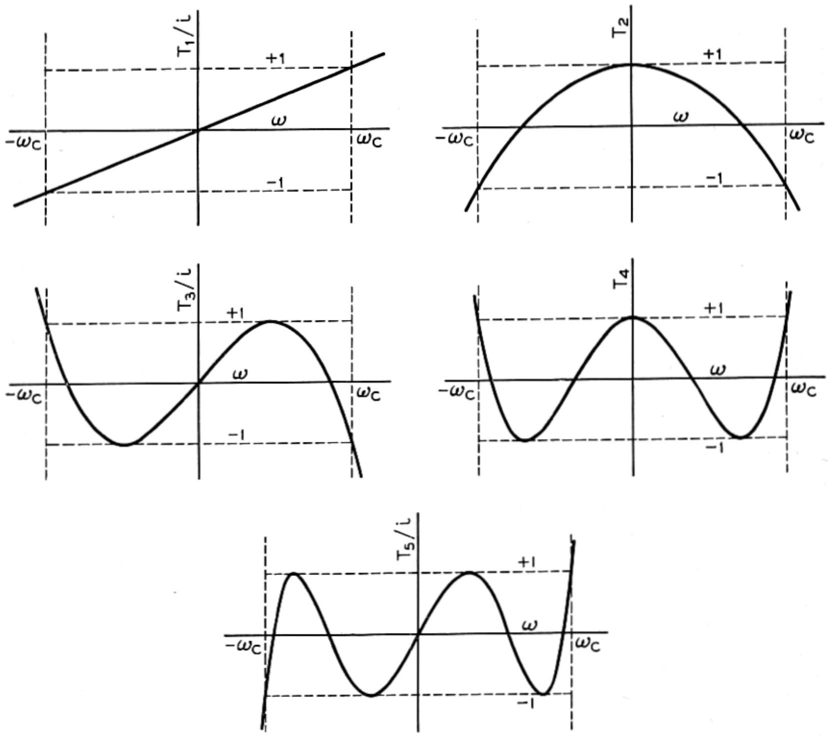


Fig. 2—Tchebycheff polynomials.

is especially appropriate for general network applications, because the odd ordered polynomials contribute to the imaginary parts of complex network functions—such as $i\beta$ in $\alpha + ib$.†

It is apparent from (3) that the Tchebycheff polynomials become simply Fourier harmonics, if they are plotted against a *distorted* frequency scale—that is, against ϕ. This means that they must be orthogonal, over that particular range of frequencies which corresponds to real values of ϕ. From the relation between ϕ and ω, it is clear that real values of ϕ cover the frequency interval between $-\omega_c$ and $+\omega_c$, which is our useful interval. In other words, the interval of orthogonality coincides with the useful frequency interval. The corresponding interval of p is of course $p = -i\omega_c$ to $+i\omega_c$.

If a given function is plotted against ϕ, instead of ω, it may be expanded in a Fourier series. Each term in the series may be replaced by a Tchebycheff polynomial, to obtain an expansion of a given function in terms of polynomials, for the specific useful interval $\omega = -\omega_c$ to $+\omega_c$. Established techniques are available for expanding experimental, or other numerical data, in Fourier series, as well as actual analytic functions.

In Fig. 2, some of the Tchebycheff polynomials are plotted against ω. The frequencies $-\omega_c$ and $+\omega_c$ are also indicated. Frequencies between these limits correspond to real values of the angle variable ϕ. If this part of the frequency scale is stretched, in the proper non-uniform way, the various "loops" not only have the same maximum values, but also the same shapes. In other words, they become periodic. More specifically, a stretch which changes the frequency scale into a ϕ scale changes the plots into $\sin k\phi$ or $\cos k\phi$.

4. TRANSFORMATION OF VARIABLE

An alternate to (3) may be obtained by relating a new variable, z, to ϕ by

$$z = e^{i\phi} \tag{4}$$

Substituting z in the exponential equivalent of $\sin \phi$, in the first equation of (3), gives an alternative definition of z directly in terms of p, namely:

$$p = \frac{\omega_c}{2}\left(z - \frac{1}{z}\right) \tag{5}$$

† A small change in the definition of ϕ would bring the definitions closer to convention, by replacing both sines by cosines (without altering T_k as a function of p). This however, would complicate our later analysis.

Substitutions in the exponential equivalents of the other sine and cosine in (3) give:

$$
\begin{aligned}
T_k &= \frac{1}{2}\left(z^k + \frac{1}{z^k}\right), \quad k \text{ even} \\
T_k &= \frac{1}{2}\left(z^k - \frac{1}{z^k}\right), \quad k \text{ odd}
\end{aligned}
\tag{6}
$$

Network applications depend upon the nature of the relationship between the variable p, and the variable z. The relationship is illustrated in Fig. 3, which indicates corresponding contours in the p and z planes.

Since angle ϕ is real in the useful interval, z, as defined by (4), has unit magnitude. In equivalent conformal mapping terms, the unit circle in the z plane maps onto a segment of the axis of real frequencies in the p plane—namely the segment extending from $p = -i\omega_c$ to $+i\omega_c$. Hereafter, we shall say merely that the useful interval in the z plane is the unit circle, or $|z| = 1$. The real frequency intervals outside the useful interval map onto the imaginary axis in the z-plane.

z-plane circles with radii other than unity map onto p-plane ellipses, all with foci at $p = \pm i\omega_c$. This is reminiscent of filter theory using Tchebycheff polynomials, and in fact such a filter may be obtained by spacing z-plane mappings of natural modes uniformly around such a circle.†

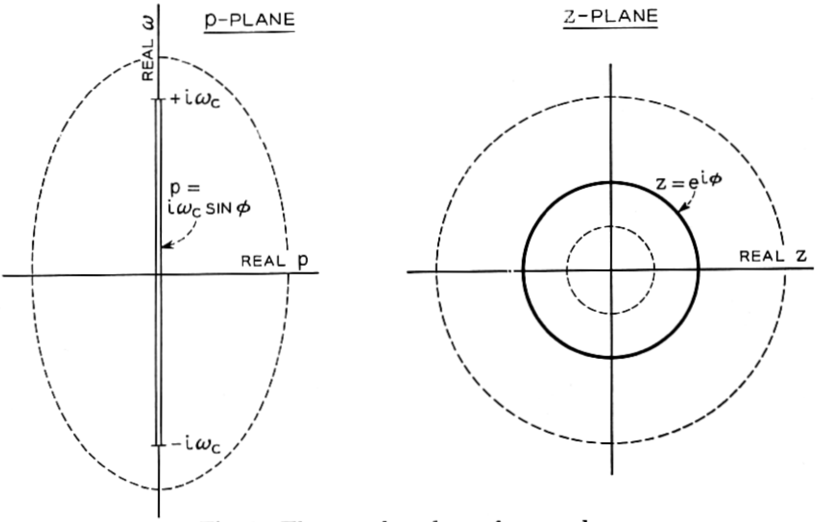


Fig. 3—The complex planes for p and z.

† The filter theory is developed in detail in a monograph by Wheeler[7], which also includes an extensive bibliography.

5. Z-PLANE MAPPINGS OF NETWORK SINGULARITIES

z-plane mappings of network singularities are also an essential part of synthesis applications. The mapping z_σ of a typical zero or pole p_σ is illustrated in Fig. 4. From (5), the analytic relation must be:

$$p_\sigma = \frac{\omega_c}{2}\left(z_\sigma - \frac{1}{z_\sigma}\right) \tag{7}$$

By its quadratic nature, there must be exactly *two* values of z_σ, corresponding to one p_σ. The relation is such that replacing z_σ by $-1/z_\sigma$ leaves p_σ unchanged; and hence the two values of z_σ must be negative reciprocals, each of the other. Thus, one mapping of p_σ falls outside the unit z-plane circle, and the other inside.

A unique definition of z_σ may be obtained by requiring that z_σ must be the mapping *outside* the unit circle. Then $|z_\sigma| > 1$ by definition, and the complete definition of z_σ may be:

$$p_\sigma = \frac{\omega_c}{2}\left(z_\sigma - \frac{1}{z_\sigma}\right) \qquad |z_\sigma| > 1 \tag{8}$$

This definition is unique provided network singularities p_σ are excluded from that very special line segment of the real frequency axis which corresponds to the useful frequency interval, $-\omega_c < \omega < +\omega_c$ (where $|z_\sigma|$ would be exactly 1).

We are going to solve the interpolation problem by choosing the z_σ first, instead of the p-plane singularities p_σ, after formulating the interpolation problem in suitable z-plane terms. For this, however, we must

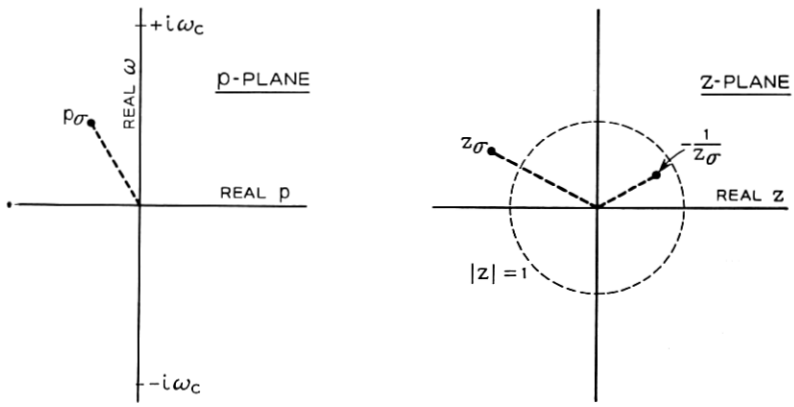


Fig. 4—Mappings of a network singularity.

know what further conditions must be imposed upon the z_σ, so that the corresponding p_σ will meet the special conditions necessary for physical networks. A simple analysis of the definition (8) of z_σ, and of the well known restrictions on the p_σ, leads to the following assertion;

The physical restrictions on z_σ are exactly the same as those on p_σ.

It is obvious, for example, that conjugate complex z_σ are necessary for conjugate complex p_σ. Also, because $|z_\sigma| > 1$, z_σ dominates $-1/z_\sigma$. Then the sign of Re p_σ is the same as that of the Re z_σ, and p_σ with negative real parts require z_σ with negative real parts, and so on.

Thus the direct choice of z_σ is restricted in exactly the same way as the choice of p_σ, except for the additional general requirement $|z_\sigma| > 1$. The last condition imposes no important restriction on the corresponding p_σ. Initially, it was adopted to make z_σ unique for any p_σ (not at a useful real frequency); but this condition does also play an essential role in the z-plane formulation of the interpolation problem.

6. NETWORK GAIN AND PHASE IN TERMS OF z

A first step in the z-plane formulation of the interpolation problem is the formulation of the network gain and phase functions, (1) and (2), in terms of z. This is most usefully examined as a transformation of functional form, rather than as a conformal mapping.

The gain and phase function (1) transforms as follows: The analytic relation between p and z is regular in the neighborhood of the singularities p_σ of the network function. Therefore, there will be similar singularities of the transformed function at the z-plane mappings of p_σ, which are z_σ and $-1/z_\sigma$. These singularities, and also suitable behavior at infinity, are exhibited by the following expression for $\alpha + i\beta$ as a function of z.

$$\alpha + i\beta = \log K_z' \frac{\prod\left(1 - \frac{z}{z_\sigma'}\right)\left(1 + \frac{1}{z_\sigma' z}\right)}{\prod\left(1 - \frac{z}{z_\sigma''}\right)\left(1 + \frac{1}{z_\sigma'' z}\right)} \quad (9)$$

$\prod$ is used here to designate a product of factors of the type following it.†

† The expression is readily confirmed in the following very elementary manner: For every factor $\left(1 - \frac{z}{z_\sigma}\right)$, in (9), there is also a factor $\left(1 + \frac{1}{z_\sigma z}\right)$. The product of the two may be expanded as follows:

$$\left(1 - \frac{z}{z_\sigma}\right)\left(1 + \frac{1}{z_\sigma z}\right) = \frac{1}{z_\sigma}\left[\left(z_\sigma - \frac{1}{z_\sigma}\right) - \left(z - \frac{1}{z}\right)\right] \quad (10)$$

If we define a new scale factor K_z by $K'_z = K_z^2$, we may write (9) as follows:

$$\alpha + i\beta = \log \left\{ K_z \frac{\prod \left(1 - \frac{z}{z'_\sigma}\right)}{\prod \left(1 - \frac{z}{z''_\sigma}\right)} \right\} \left\{ \begin{matrix} \text{Same Rational} \\ \text{Function in } -1/z \end{matrix} \right\} \tag{13}$$

Similar expressions for the separate gain and phase functions may be derived from (2):

$$\begin{aligned} 2\alpha &= \log \left\{ K_z^2 \frac{\prod \left(1 - \frac{z^2}{z'^2_\sigma}\right)}{\prod \left(1 - \frac{z^2}{z''^2_\sigma}\right)} \right\} \left\{ \begin{matrix} \text{Same Rational} \\ \text{Function in } -1/z \end{matrix} \right\} \\ i2\beta &= \log \left\{ \prod \frac{1 - \frac{z}{z'_\sigma}}{1 + \frac{z}{z'_\sigma}} \prod \frac{1 + \frac{z}{z'_\sigma}}{1 - \frac{z}{z''_\sigma}} \right\} \left\{ \begin{matrix} \text{Same Rational} \\ \text{Function in } -1/z \end{matrix} \right\} \end{aligned} \tag{14}$$

Equation (13) holds at all values of p and z, while (14) holds at all real frequencies. Simplifications of (14) should be noted, good for the useful interval only. When $|z| = 1$, $1/z = z^*$. Recalling also that $\log |x|^2$ is $2 \log |x|$, and similar elementary relations, one obtains from (14):

When $|z| = 1$,

$$\begin{aligned} \alpha &= \log \left| K_z^2 \frac{\prod \left(1 - \frac{z^2}{z'^2_\sigma}\right)}{\prod \left(1 - \frac{z^2}{z''^2_\sigma}\right)} \right| \\ \beta &= \text{Phase} \prod \frac{1 - \frac{z}{z'_\sigma}}{1 + \frac{z}{z'_\sigma}} \prod \frac{1 + \frac{z}{z''_\sigma}}{1 - \frac{z}{z''_\sigma}} \end{aligned} \tag{15}$$

Comparison with (5) and (7) gives:

$$\left(1 - \frac{z}{z_\sigma}\right)\left(1 + \frac{1}{z_\sigma z}\right) = -\frac{2}{\omega_c z_\sigma}(p - p_\sigma) \tag{11}$$

Thus (9) is equivalent to (1) provided

$$K'_z = K \frac{\prod \left(-\frac{z'_\sigma \omega_c}{2}\right)}{\prod \left(-\frac{z''_\sigma \omega_c}{2}\right)} \tag{12}$$

7. THE POWER SERIES IN z

Our applications to network synthesis depend upon a correspondence which may be shown to exist between certain functions of z and certain power series in z. The functions of z may be formulated in terms of network singularities. The power series in z may be derived from the Tchebycheff polynomial series in p representing the corresponding gain and phase.

The Tchebycheff polynomial expansion of a gain and phase function may be written:

$$\alpha + i\beta = \sum C_k T_k \tag{16}$$

If $\alpha + i\beta$ corresponds to a finite network, it may be represented by the function of z in (13). At the same time, T_k may be represented by the function of z in (6). With these changes, (16) becomes:

$$\log \left\{ K_z \frac{\prod\left(1 - \frac{z}{z_\sigma'}\right)}{\prod\left(1 - \frac{z}{z_\sigma''}\right)} \right\} \left\{ \begin{matrix} \text{Same Rational} \\ \text{Function in } -1/z \end{matrix} \right\} = \sum C_k \frac{1}{2} \left[z^k + \left(\frac{-1}{z} \right)^k \right] \tag{17}$$

The logarithm of the product of the two rational functions, in z and $-1/z$ respectively, may be written as the sum of two logarithms. The series in sums of z^k and $(-1/z)^k$ may be written as the sum of two series. Then

$$\log \left\{ K_z \frac{\prod\left(1 - \frac{z}{z_\sigma'}\right)}{\prod\left(1 - \frac{z}{z_\sigma''}\right)} \right\} + \log \left\{ \begin{matrix} \text{Same Rational} \\ \text{Function in } -1/z \end{matrix} \right\} = \sum \frac{C_k}{2} z^k + \sum \frac{C_k}{2} \left(\frac{-1}{z} \right)^k \tag{18}$$

The above expression equates the sum of two similar functions, in z and $-1/z$ respectively, to the sum of two power series, also respectively in z and $-1/z$. The theorem on which the synthesis methods are based asserts that the functions and power series in z and $-1/z$ may be equated separately, throughout the useful interval. That is:

When $|z| = 1$,

$$\log\left\{K_z \frac{\prod\left(1 - \frac{z}{z'_\sigma}\right)}{\prod\left(1 - \frac{z}{z''_\sigma}\right)}\right\} = \tfrac{1}{2} \sum C_k z^k \tag{19}$$

$$\log\left\{\begin{matrix}\text{Same Rational} \\ \text{Function in } -1/z\end{matrix}\right\} = \tfrac{1}{2} \sum C_k \left(\frac{-1}{z}\right)^k$$

The relation (18) does not, by itself, *require* (19) to be true. (19) follows from (18) if and only if the function of z has a power series expansion involving only positive powers of z, and the function in $-1/z$ has a power series expansion in $-1/z$, with the same coefficients. This added condition, however, is readily established for the useful interval.†

Combining (19) and (16) yields a most useful relationship connecting the z-plane mappings z_σ, of the network singularities p_σ, and the coefficients C_k, of the Tchebycheff polynomial expansion of $\alpha + i\beta$:

$$\alpha + i\beta = \sum C_k T_k$$

$$\sum \tfrac{1}{2} C_k z^k = \log K_z \frac{\prod\left(1 - \frac{z}{z'_\sigma}\right)}{\prod\left(1 - \frac{z}{z''_\sigma}\right)} \tag{20}$$

In more qualitative terms:

The transformation from variable p to variable z converts an expansion in Tchebycheff polynomials in p into an expansion in a power series in z.

Thus, by working with the z_σ, in place of the p_σ, one may use a power series sort of analysis in calculating, or in choosing, the coefficients C_k in the Tchebycheff polynomial series.

The relations (20) refer to the combined gain and phase function. Exactly similar relations can readily be obtained, however, for gain and

† As defined in (8), $|z_\sigma| > 1$. In the useful interval, $|z| = 1$. Hence $|z/z_\sigma| < 1$. It follows that $\log(1 - z/z_\sigma)$ has a power (MacClauren) series expansion in positive powers of z, convergent on and within the circle $|z| = 1$. Finally the first logarithm in (19) may be expressed as a sum of logarithms of this simple type, each of which may be expanded separately. Substituting $-1/z$ for z maps the unit circle onto itself. It follows that the second logarithm in (19) has an expansion in positive powers of $-1/z$, in the useful interval, provided the first logarithm has an expansion in positive powers of z; and the coefficients in the two series will be the same.

phase separately. These may be derived from (14), and take the form:

$$\left.\begin{aligned} \alpha &= \sum C_k T_k \\ \sum C_k z^k &= \log K_z^2 \frac{\prod \left(1 - \frac{z^2}{z_\sigma'^2}\right)}{\prod \left(1 - \frac{z^2}{z_\sigma''^2}\right)} \end{aligned}\right\} k, \quad \text{even}$$

$$\left.\begin{aligned} i\beta &= \sum C_k T_k \\ \sum C_k z^k &= \log \prod \frac{1 - \frac{z}{z_\sigma'}}{1 + \frac{z}{z_\sigma'}} \prod \frac{1 + \frac{z}{z_\sigma''}}{1 - \frac{z}{z_\sigma''}} \end{aligned}\right\} k, \quad \text{odd} \qquad (21)$$

(The absence of factors $\frac{1}{2}$ in $\sum C_k z^k$, as compared with (20), reflects the factors 2 associated with α and β in (14).)

8. REPRESENTATION OF ASSIGNED GAIN AND PHASE

In synthesis problems, the network gain or phase, α or β, is to approximate an assigned (ideal) gain or phase, say $\bar{\alpha}$ or $\bar{\beta}$. To make effective use of the z-plane analysis, in network synthesis, we need to describe $\bar{\alpha}$ and $\bar{\beta}$ by relations analogous to (20) and (21), which express α and β in z-plane terms. These relations, while similar to (20) and (21), must take a more general form (since $\bar{\alpha}$ or $\bar{\beta}$ need only be approximately the gain or phase of a finite network). For our present purposes, the appropriate relations are those noted below.

Let $\bar{\alpha} + i\bar{\beta}$ be any function of p which has the following properties: It must be analytic throughout the useful interval. Further, there are to be no singularities within a (p-plane) distance ϵ of the useful interval, where ϵ is *finite* (but may be small). Finally, at real frequencies, $\bar{\alpha}$ and $i\bar{\beta}$ are to equal respectively the even and odd parts of $\bar{\alpha} + i\bar{\beta}$.

Under the conditions stated, $\bar{\alpha} + i\bar{\beta}$ may always be expanded in terms of our Tchebycheff polynomials T_k. Let $\sum \bar{C}_k T_k$ be the expansion. To obtain a parallel to (20), we may form (arbitrarily) a power series $\sum \frac{1}{2}\bar{C}_k z^k$. Then we may *define* a function $\bar{R}(z)$ by identifying $\log \bar{R}(z)$ with the power series. All this adds up to the following, comparable to (20):

$$\begin{aligned} \bar{\alpha} + i\bar{\beta} &= \sum \bar{C}_k T_k \\ \sum \tfrac{1}{2}\bar{C}_k z^k &= \log \bar{R}(z) \end{aligned} \qquad (22)$$

The functions of z have the following properties: Because of the mild restrictions, which we have imposed on the singularities of $\bar{\alpha} + i\bar{\beta}$, the series $\sum \bar{C}_k z^k$ defines a function which is analytic within, and on the circle $|z| = 1$. Then $\bar{R}(z)$, also, is analytic within, and on the circle. Further, $\bar{R}(z)$ has no zeros anywhere in the same region. ($\bar{R}(z)$, however, may be more general than the rational fraction in (20).) Finally, because of the even and odd symmetries, required of $\bar{\alpha}$ and $i\bar{\beta}$, (22) may be broken into the following parallels of the equations (21):

$$\left.\begin{aligned} \bar{\alpha} &= \sum \bar{C}_k T_k \\ \sum \bar{C}_k z^k &= \log [\bar{R}(z)\bar{R}(-z)] \end{aligned}\right\} k, \quad \text{even}$$
$$\left.\begin{aligned} i\bar{\beta} &= \sum \bar{C}_k T_k \\ \sum \bar{C}_k z^k &= \log \left[\frac{\bar{R}(z)}{\bar{R}(-z)}\right] \end{aligned}\right\} k, \quad \text{odd} \tag{23}$$

In some applications, it is possible to express $\bar{R}(z)$ in closed form. In all applications, it is possible to expand $\bar{R}(z)$ as a power series, convergent in the region $|z| \leqq 1$. The same is true of $1/\bar{R}(z)$, since there are no zeros in the region. Coefficients of either series ($\bar{R}(z)$ or $1/\bar{R}(z)$) may readily be calculated by means which we shall examine a little later. For the present we shall say merely that $\bar{R}(z)$ is a *known* function, corresponding to an assigned $\bar{\alpha} + i\bar{\beta}$.

9. A DESIGN CRITERION

When the gain and phase function, $\alpha + i\beta$, is to approximate $\bar{\alpha} + i\bar{\beta}$, the error in the approximation is $(\alpha - \bar{\alpha}) + i(\beta - \bar{\beta})$. The error may be expressed in terms of z by taking the difference of corresponding equations in (20), (22). The difference of the logarithms may be expressed as a single logarithm of a ratio. Alternatively, and also more conveniently for our later purposes, it may be expressed as the negative of the logarithm of the reciprocal ratio. When this is done,

$$\begin{aligned} (\alpha - \bar{\alpha}) + i(\beta - \bar{\beta}) &= \sum (C_k - \bar{C}_k) T_k \\ \sum \tfrac{1}{2}(C_k - \bar{C}_k) z^k &= -\log \left\{ \frac{1}{K_z} \frac{\prod \left(1 - \frac{z}{z_\sigma''}\right)}{\prod \left(1 - \frac{z}{z_\sigma'}\right)} \cdot \bar{R}(z) \right\} \end{aligned} \tag{24}$$

Consider the following arbitrary requirement, as a design criterion: The series $\sum C_k T_k$ is to match exactly the series $\sum \bar{C}_k T_k$, through terms of order m. If both series have converged to small remainders when $k = m$, this criterion will surely make $\alpha + i\beta$ a good approximation to $\bar{\alpha} + i\bar{\beta}$.† In terms of the coefficients, the criterion requires:

$$C_k = \bar{C}_k, \qquad k \leqq m \tag{25}$$

If (25) is applied to the second equation of (24), the power series is zero through terms of order m. In other words, the logarithm, equated to the series, will approximate zero in the power series, or "maximally flat" manner, to order m. The logarithm is zero when the expression in brackets is unity. Further, the logarithm will approximate zero in the maximally flat manner when, and only when the bracket approximates unity in the maximally flat manner. Thus a condition which is equivalent to (25) is the following:

$$\frac{1}{K_z} \frac{\Pi\left(1 - \frac{z}{z_\sigma''}\right)}{\Pi\left(1 - \frac{z}{z_\sigma'}\right)} \cdot \bar{R}(z) = 1 + \epsilon_{m+1} z^{m+1} + \epsilon_{m+2} z^{m+2} \cdots \tag{26}$$

This may be represented symbolically by

$$\frac{1}{K_z} \frac{\Pi\left(1 - \frac{z}{z_\sigma''}\right)}{\Pi\left(1 - \frac{z}{z_\sigma'}\right)} \cdot \bar{R}(z) \overset{m}{=} 1 \tag{27}$$

where $\overset{m}{=}$ is used to indicate equality through power series terms of order m.

When (27) is applied to network synthesis, the singularities z_σ, and scale factor K_z are the unknowns, while $\bar{R}(z)$ is known. If m is equal to the total number of z_σ, (27) will determine the network function completely. When m is smaller, (27) will furnish $m + 1$ relations between the network parameters (including the zero order condition), which may be combined with specifications of other sorts. Since (27) is equivalent to (25), this procedure amounts to the determination of network parameters which will yield assigned values of the coefficients, $C_k = \bar{C}_k$, $k \leqq m$, in the Tchebycheff polynomial expansion of $\alpha + i\beta$.

Equation (27) applies when both gain and phase are to be approximated. For approximation to gain only, or to phase only, similar relations may be derived from (21) and (23). Only even ordered Tchebycheff

† When both residues are relatively large, the approximation *may* still be good, for the remainders may be quite similar, and the error will be their difference. In practical applications, this is a not uncommon situation.

polynomials contribute to gain. The following condition turns out to be the equivalent of $C_{2k} = \bar{C}_{2k}$, $k \leqq m$:

$$\frac{1}{K_z^2} \frac{\prod \left(1 - \dfrac{z^2}{z_\sigma''^2}\right)}{\prod \left(1 - \dfrac{z^2}{z_\sigma'^2}\right)} \cdot \bar{R}(z)\bar{R}(-z) \overset{me}{=} 1 \tag{28}$$

where $\overset{me}{=}$ means approximation in accordance with a power series of even ordered terms, through order $2m$. Correspondingly, only odd ordered Tchebycheff polynomials contribute to phase. The following condition is equivalent to $C_{2k-1} = \bar{C}_{2k-1}$, $k = 1$ to m:

$$\prod \frac{1 - \dfrac{z}{z_\sigma''}}{1 + \dfrac{z}{z_\sigma''}} \prod \frac{1 + \dfrac{z}{z_\sigma'}}{1 - \dfrac{z}{z_\sigma'}} \cdot \frac{\bar{R}(z)}{\bar{R}(-z)} \overset{2m}{=} 1 \tag{29}$$

The remaining sections (except the last) develop in more detail the application of z-plane techniques to more specific synthesis problems, of various sorts. Most of these (but not quite all) are based directly on (27), (28), or (29). The exceptions use a modification of (28), in which the function of z on the left is retained, but with the zeros and poles adjusted for a different kind of approximation to unity, $\cong$ but not $\overset{me}{=}$.

In all cases, unity is approximated with one of the functions appearing in (27), (28), (29). It will be convenient to use $H(z)$ to represent the error in the approximation, or departure from unity. When gain only is of interest, the function in (28) is used, and $H(z)$ is defined by:

$$\frac{1}{K_z^2} \frac{\prod \left(1 - \dfrac{z^2}{z_\sigma''^2}\right)}{\prod \left(1 - \dfrac{z^2}{z_\sigma'^2}\right)} \cdot \bar{R}(z)\bar{R}(-z) = 1 + H(z) \tag{30}$$

In developing the specific techniques, we shall start with a very definite, rather special example, in order to illustrate the techniques with specific operations. This will be discussed in considerable detail in Sections 10 through 14. Thereafter we shall examine how these specific operations may be generalized, in a number of different respects.

10. AN INTRODUCTORY EXAMPLE

The example which has been chosen for detailed discussion is the equalization of the gain distortion produced by two resistance-capacity type cut-offs. The equalization is to be accomplished with a network which has n natural modes, but no finite frequencies of infinite loss. (This is simply one of the arbitrary specifications which define this problem.)

The two cut-offs may be due to circuits or devices at two different points in a communication system, which may be represented schematically as in Fig. 5. Their effect can be described in terms of two assigned natural modes. Two assigned modes are assumed, instead of only one, because a single mode would make the problem too simple. Our present purposes will be served well, however, if we simplify the problem by requiring the two assigned modes to be *identical*, say at $p = \bar{p}_0$.

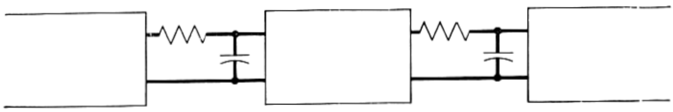


Fig. 5—A system with two resistance-capacity type cut-offs.

The two natural modes would be cancelled completely by two infinite loss points at the same location in the p plane. A network with two infinite loss points, however, is not physically possible unless it has also at least two natural modes; and the natural modes will have to introduce distortion of their own. Thus no finite network will give *perfect* equalization of unwanted natural modes. Sometimes it is desirable, in practice, to use an equalizer configuration which produces *no* finite frequencies of infinite loss, the entire equalization being accomplished by a suitable choice of its n modes. Configurations of this sort are illustrated in Fig. 6. Thus, our simple illustrative problem, though chosen to introduce principles, is also of some practical interest.

The exclusion of finite frequencies of infinite loss simplifies the repre-

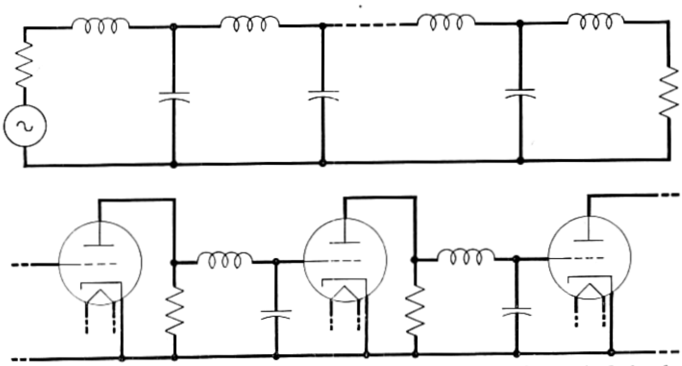


Fig. 6—Configurations which produce no finite frequencies of infinite loss.

sentation of the network gain α. In (21), the z'_σ correspond to finite frequencies of infinite loss, and are to be omitted when there are to be natural modes only. What is left is the logarithm of the reciprocal of a polynomial, which is of course the negative of the logarithm of the polynomial itself. Thus α may be described as follows, for this particular application:

$$\alpha = \sum C_{2k} T_{2k}$$
$$\sum C_{2k} z^{2k} = -\log K_z^2 \prod \left(1 - \frac{z^2}{z_\sigma^2}\right), \qquad \sigma = 1, \cdots, n \tag{31}$$

(For convenience, z_σ has been written for z''_σ, and K_z has been redefined to avoid the $1/K_z$ required if it is defined as in (21).)

The assigned gain $\bar{\alpha}$ is even more special. In this particular problem, $\bar{\alpha}$ has most of the properties of a network gain α. Specifically, it is the negative of the gain to be equalized, which in fact corresponds to a finite network. As a result, $\bar{R}(z)$ of (22) may be expressed in closed (rational) form. (Later on, we shall modify the methods appropriate for this very special situation, so that $\bar{R}(z)$ need be representable only by series.)

The specific representation of our present $\bar{\alpha}$ may be very similar to the representation of α in (31), as follows:

$$\bar{\alpha} = \sum \bar{C}_{2k} T_{2k}$$
$$\sum \bar{C}_{2k} z^{2k} = +\log \bar{K}_z^2 \left(1 - \frac{z^2}{\bar{z}_0^2}\right)^2 \tag{32}$$

(Both (31) and (32) apply only to the useful interval, $|z| = 1$.)

The constant $\bar{z}_0$ is the z-plane mapping of the assigned unwanted natural modes at $p = \bar{p}_0$, and may be calculated therefrom by (8). In (32), $\bar{z}_0$ determines the $\bar{C}_{2k}$, which in turn determine $\bar{\alpha}$. The constants z_σ, in (31), are the z-plane mappings of the arbitrary natural modes of the equalizer. They are to be adjusted to make α approximate $\bar{\alpha}$. Then the network natural modes p_σ may be calculated from them, by means of (8).

Taking the difference of corresponding equations in (31) and (32) gives the following, analogous to (24):

$$\alpha - \bar{\alpha} = \sum (C_{2k} - \bar{C}_{2k}) T_{2k}$$
$$\sum (C_{2k} - \bar{C}_{2k}) z^{2k} = -\log \left\{ K_z^2 \bar{K}_z^2 \left(1 - \frac{z^2}{\bar{z}_0^2}\right)^2 \prod \left(1 - \frac{z^2}{z_\sigma^2}\right) \right\} \tag{33}$$

This differs from (24) in two regards. It relates to the gain error, $(\alpha - \bar{\alpha})$, without regard to phase. It reflects the more specific functional forms of our present α and $\bar{\alpha}$.

The formulas show that the coefficients in the Tchebycheff polynomial expansion of our present $\alpha - \bar{\alpha}$ are fixed by the logarithm of a polynomial in z^2, of degree $n + 2$. Since the Tchebycheff polynomial series is simply one representation of the function $\alpha - \bar{\alpha}$, this means that $\alpha - \bar{\alpha}$ itself is determined by the polynomial in z^2. Out of the $n + 2$ zeros, in terms of z^2, n are subject to arbitrary choice, but the other two are required to be at $z^2 = \bar{z}_0^2$.

To arrive at a useful choice of the zeros, one may start with the expanded form of the polynomial, which replaces the second equation of (33) by:

$$\sum (C_{2k} - \bar{C}_{2k})z^{2k} = -\log \{\hat{K}_0 + \hat{K}_1 z^2 + \cdots \hat{K}_{n+2} z^{2n+4}\} \qquad (34)$$

All but two of the coefficients $\hat{K}_k$ may be assigned arbitrary values, provided the remaining two are then adjusted to give the required two zeros at $z^2 = \bar{z}_0^2$. The corresponding zeros z_σ^2 may then be found by ordinary root extraction methods.

The coefficients may be chosen in such a way that the complex polynomial approximates unity, when $|z| = 1$. Then the logarithm approximates zero, the coefficients in the power series (34) are small, and since these are also the coefficients in the Tchebycheff polynomial series in (33), $\alpha - \bar{\alpha}$ is small.

11. TCHEBYCHEFF POLYNOMIAL SERIES MATCHED THROUGH n TERMS

A special choice of coefficients, which meets these requirements fairly well, is the choice determined by (28), with $m = n$. The function on the left side of (28) is here the polynomial in (34). For our present purposes, therefore, (28) becomes:

$$\{\hat{K}_0 + \hat{K}_1 z^2 + \cdots \hat{K}_{n+2} z^{2n+4}\} \overset{ne}{=} 1 \qquad (35)$$

This requires $\hat{K}_0 = 1$, and $\hat{K}_k = 0$ for $k = 1$ to n. Then $\hat{K}_{n+1}$ and $\hat{K}_{n+2}$ must be adjusted to give the two required zeros at $z^2 = \bar{z}_0^2$. This gives:

$$\hat{K}_0 + \cdots \hat{K}_{n+2} z^{2n+4} = 1 - (n + 2)\left(\frac{z}{\bar{z}_0}\right)^{2n+2} + (n + 1)\left(\frac{z}{\bar{z}_0}\right)^{2n+4} \qquad (36)$$

In accordance with Section 9, this special choice of coefficients corresponds to a match of Tchebycheff polynomial series, α to $\bar{\alpha}$, through terms of order $2n$:

$$C_{2k} = \bar{C}_{2k}, \qquad k \leqq n \qquad (37)$$

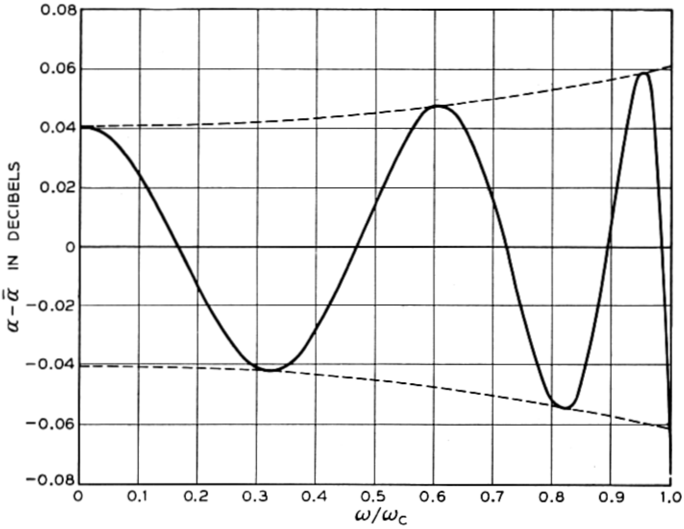


Fig. 7—Error when four natural modes equalize two natural modes at $\bar{p}_0 = -0.75\omega_c$.

The actual accuracy may be calculated from the final zeros and poles, from non-zero terms in the expansion of $\alpha - \bar{\alpha}$, or by an analysis in terms of z which will be described later.

A sample plot of $\alpha - \bar{\alpha}$ is shown in Fig. 7. This corresponds to an equalizer of four natural modes, compensating for an initial loss which rises to about 8 db at the top useful frequency, or a distortion of ± 4 db about the median loss. Residual errors are order of ± 0.06 db.

A little later we shall return to the question of accuracy, to take up methods of estimating what can be done with other numbers of arbitrary natural modes, and other values of the assigned modes. First, however, we should investigate whether the network singularities z_σ determined by (36) meet the other necessary conditions.

12. PROPERTIES OF z_σ

In the first place, $|z_\sigma|$ must be >1. Otherwise, the function of z in (31) will have no power series expansion over the useful interval $|z| = 1$; and (31) will not, in fact, determine the gain α over the useful interval. It turns out, however, that the condition does not give trouble in the synthesis of natural modes, when there are no arbitrary frequencies of infinite loss. This may be demonstrated by the argument outlined below.

The z_σ are zeros of the polynomial in (34), which we have given the special form (36), by applying (35). A function theoretic test for $|z_\sigma| < 1$

makes use of the contour in the complex plane for the polynomial, corresponding to the z-plane circle $|z| = 1$. (This is like a Nyquist diagram except that the contour for the variable, z, is different.) There will be $|z_\sigma| < 1$ if and only if the contour for the polynomial encloses the origin.

Now the polynomial in (34), and (35), is merely a special case of the function on the left in (28), and (30). For this special case (30) becomes

$$\sum \hat{K}_k z^{2k} = 1 + H(z) \tag{38}$$

The polynomial cannot enclose the origin without passing through some negative real value. But this requires an $|H(z)| > 1$, at some point on the contour in question, $|z| = 1$, which happens to be also our useful interval. On the other hand, $\alpha - \bar{\alpha} = 0$ when $\sum \hat{K}_k z^{2k} = 1$, and $H(z)$ is in the nature of a correction term, which is small in the useful interval when $\alpha - \bar{\alpha}$ is small.

The conclusion is: There will be no $|z_\sigma| < 1$ unless the approximation, α to $\bar{\alpha}$, is so poor that $\alpha - \bar{\alpha}$ exceeds several db in the useful interval.

Besides the requirement $|z_\sigma| > 1$, the z_σ must meet physical restrictions, which we found to be the same as those limiting the natural modes p_σ. The z_σ may be calculated as follows: The z_σ^2 are roots of the polynomial in (35), in terms of z^2. All the roots in terms of z^2 are z_σ^2, except the two required roots at $\bar{z}_0^2$, which correspond to assigned gain $\bar{\alpha}$. Each z_σ is a square root of a z_σ^2. There are two possible square roots, however, differing only as to sign. As far as gain α is concerned, either choice of sign is permissible; for α depends only on z_σ^2. For a physical network, however, the choice must be such that Re $z_\sigma < 0$. This choice is possible if, and only if $\sqrt{z_\sigma^2}$ has a finite real part. A pure imaginary z_σ corresponds to a negative real z_σ^2, and thus negative real roots in terms of z^2 are excluded by physical considerations.

Table I lists both z_σ^2 and z_σ for a number of values of n. When n is even, all roots are physical. On the other hand, when n is odd, one root is always non-physical. In a sense, an odd n is not really appropriate for the present illustrative problem, with any physical design. An odd n must necessarily bring in a real natural mode, which merely increases the sort of distortion we are trying to equalize—that is the distortion due to unwanted real modes.

The following argument substantiates the suggestion, and also illustrates manipulations of a sort which are frequently useful in more general applications: The highest order coefficient in (34), $\hat{K}_{n+2}$, may be set aside for adjustments to satisfy physical requirements. The rest of

TABLE I—*Z-Plane Natural Modes for Equalization of Two Identical Unwanted Modes*

n	$z_\sigma^2/\bar{z}_0^2$	$\sqrt{z_\sigma^2/\bar{z}_0^2}$	$z_\sigma/\lvert \bar{z}_0 \rvert$
1	$-.5000$	$0 \pm i\ .7071$	Non Physical
2	$-.3333 \pm i\ .4714$	$\pm(.3492 \pm i\ .6747)$	$-.3492 \pm i\ .6747$
3	$-.6059$	$0 \pm i\ .7784$	Non Physical
	$-.0720 \pm i\ .6384$	$\pm(.5340 \pm i\ .5977)$	$-.5340 \pm i\ .5977$
4	$+.1378 \pm i\ .6782$	$\pm(.6441 \pm i\ .5264)$	$-.6441 \pm i\ .5264$
	$-.5378 \pm i\ .3582$	$\pm(.2328 \pm i\ .7695)$	$-.2328 \pm i\ .7695$
5	$-.6703$	$0 \pm i\ .8187$	Non Physical
	$+.2942 \pm i\ .6684$	$\pm(.7157 \pm i\ .4670)$	$-.7157 \pm i\ .4670$
	$-.3757 \pm i\ .5701$	$\pm(.3918 \pm i\ .7275)$	$-.3918 \pm i\ .7275$

the coefficients may then be chosen to eliminate terms from the series $\sum (C_{2k} - \bar{C}_{2k})T_{2k}$, representing $\alpha - \bar{\alpha}$, subject to the previous condition that two zeros must be $z^2 = \bar{z}_0^2$. This replaces $\overset{ne}{=}$ by $\overset{(n-1)e}{=}$, in (35), and changes (36) to:

$$\sum \hat{K}_k z^{2k} = 1 - (n+1)\left(\frac{z}{\bar{z}_0}\right)^{2n} + n\left(\frac{z}{\bar{z}_0}\right)^{2n+2} + \hat{K}_{n+2} z^{2n}(\bar{z}_0^2 - z^2)^2 \tag{39}$$

If n is odd, all the roots z_σ can be physical only if $\hat{K}_{n+2}$ is negative. On the other hand, any finite negative $\hat{K}_{n+2}$ leads to a larger error, $\alpha - \bar{\alpha}$, than $\hat{K}_{n+2} = 0$. Reducing $\hat{K}_{n+2}$ to zero is the same as reducing the degree of the polynomial by one, which amounts to reducing n by 1, from an odd to the next smaller even integer. In other words, a *physical* design with an odd number of natural modes is less effective, for the present application, than a simpler network, with the next smaller even number of modes.

Note that the z_σ in Table I are proportional to $\bar{z}_0$. This means that root extraction methods need be used only once for each value of n, after which the roots may be quickly adjusted for any value of $\bar{z}_0$, corresponding to any assigned value of the two identical modes, $\bar{p}_0$.

13. ACCURACY

The accuracy of a completed design can be checked by calculating α from the natural modes p_σ, and comparing α with $\bar{\alpha}$. It is important, however, to have at least some information about accuracy in advance of the detailed calculation of the p_σ. Otherwise, it may be necessary to carry out several designs, in all detail, in order to obtain one satisfactory design.

The needed information about accuracy can in fact be obtained from the error function $H(z)$, which we formulated for general gain applications in (30), and for the present application in (38). The analysis which yields (15) may be used to obtain a very similar expression for $\alpha - \bar{\alpha}$, in which $\bar{R}(z)\bar{R}(-z)$ appears in combination with the rational function of z from (15). It may be expressed in terms of the error function $H(z)$ of (30), as follows:

$$\alpha - \bar{\alpha} = -\log |\, 1 + H(z) \,| \tag{40}$$

When $H(z)$ is zero, $\alpha - \bar{\alpha}$ is zero. When $H(z)$ is small, $\alpha - \bar{\alpha}$ depends on phase $H(z)$ as much as on $|\, H(z) \,|$. When $H(z)$ is a positive real, $\alpha - \bar{\alpha}$ is negative. When $H(z)$ is imaginary, $\alpha - \bar{\alpha}$ is very small. When $H(z)$ is a negative real, $\alpha - \bar{\alpha}$ is positive. When $H(z)$ is complex, $|\, \alpha - \bar{\alpha} \,|$ is always smaller than with a real $H(z)$ of the same magnitude. The last statement may be expressed as follows:

$$-\log \{1 + |\, H(z) \,|\} \leqq \alpha - \bar{\alpha} \leqq -\log \{1 - |\, H(z) \,|\} \tag{41}$$

The left hand relation is an equality when phase $H(z)$ is an even number of π radians; the right hand side, when it is an odd number of π radians.

In the useful interval, where $z = e^{i\phi}$, the $H(z)$ corresponding to (36) is as follows:

$$\begin{aligned} H(z) &= -(n + 2)\left(\frac{z}{\bar{z}_0}\right)^{2n+2} + (n + 1)\left(\frac{z}{\bar{z}_0}\right)^{2n+4} \\ |H(z)| &= \frac{n + 2}{\bar{z}_0^{2n+2}}\left|\, 1 - \frac{n + 1}{(n + 2)\bar{z}_0^2}\, e^{i\phi} \,\right| \\ \text{phase } H(z) &= \pi + (2n + 2)\phi + \text{phase}\left\{1 - \frac{n + 1}{(n + 2)\bar{z}_0^2}\, e^{2i\phi}\right\} \end{aligned} \tag{42}$$

As ω varies from 0 to ω_c, ϕ varies by $\frac{\pi}{2}$ radians. The corresponding phase of $H(z)$ varies by $(n + 1)\pi$ radians, which means that $H(z)$ is successively positive real, imaginary, negative real, imaginary, through $n + 1$ half cycles. This accounts for the oscillatory nature of the $\alpha - \bar{\alpha}$ curve, illustrated in Fig. 7.

The amplitudes of the oscillations are fixed by $|\, H(z) \,|$, which varies relatively slowly. Specifically, the two logarithms in (41) determine envelopes, between which the actual error curve oscillates. These are the dashed lines in Fig. 7.

The maximum error, in the useful interval, is determined by the maximum value of the envelopes, i.e.,

$$(\alpha - \bar{\alpha})_{\text{max.}} \cong \frac{n+2}{\bar{z}_0^{2n+2}}\left[1 + \frac{n+1}{n+2}\frac{1}{\bar{z}_0^2}\right] \tag{43}$$

This function is plotted in Fig. 8, for various values of n. The abscissae "distortion before equalization" represent distortion relative to the median loss in the useful interval, or one half the total variation in the interval. (This is a function of the top useful frequency ω_c, relative to the assigned natural mode $\bar{p}_0$; and (7) makes $\omega_c/\bar{p}_0$ a simple function of $\bar{z}_0$.) The figure is convenient for estimating the values of n needed for specific applications.

The various ripples in $\alpha - \bar{\alpha}$ do not all have the same amplitude, (43). For some values of n and $\bar{z}_0$, the amplitudes are almost uniform; for others they are quite variable. A measure of the variability in ripple

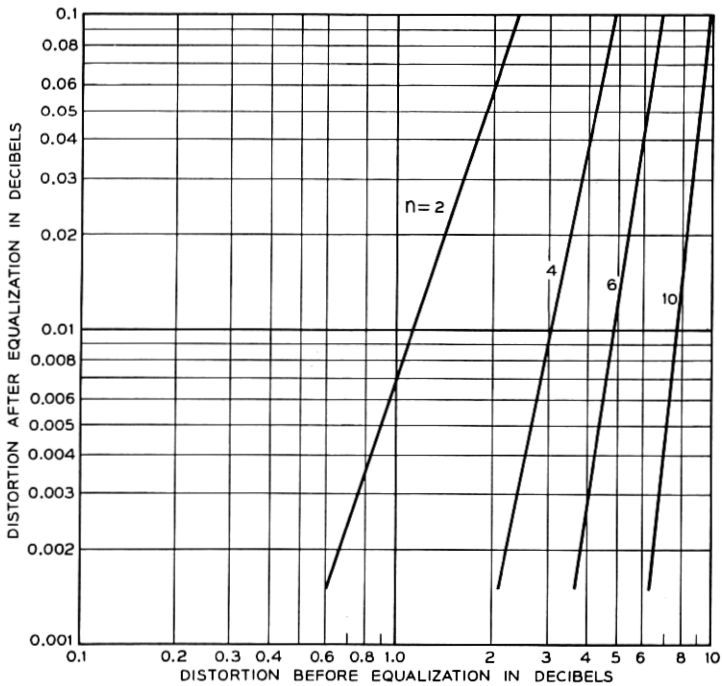


Fig. 8—Distortion before and after equalization—n natural modes equalizing 2 identical natural modes.

amplitude, across the useful interval, is:

$$\frac{|H(z)|_{\max.}}{|H(z)|_{\min.}} = \frac{(n+2)\bar{z}_0^2 + (n+1)}{(n+2)\bar{z}_0^2 - (n+1)} \tag{44}$$

14. APPROXIMATION IN THE TCHEBYCHEFF SENSE

The above analysis suggests a way of improving the design determined by (37) (or the equivalent, (36)). An *optimum* $\alpha - \bar{\alpha}$ is commonly one which has the following properties, in the useful interval:

A maximum number of "ripples,"
all maxima of $|\alpha - \bar{\alpha}|$ *equal.*

(This usually minimizes the largest departure in the useful interval, thereby yielding an "approximation in the Tchebycheff Sense.") Since the variation in phase $H(z)$ determines the number of ripples, while $|H(z)|$ determines the amplitudes of the ripples, the above conditions will be met if $H(z)$ has the following properties, in the useful interval:

Phase $H(z)$ *as variable as possible,* (45)
$|H(z)|$ *constant.*

These conditions may be regarded as alternative design criteria, replacing $C_{2k} = \bar{C}_{2k}$. They can in fact be applied to our special example, and also to certain other special problems which will be noted later. For more general applications, a suitable $H(z)$ can be defined, but no reasonably simple procedure has yet been found for calculating the required constants. (The difficulties will be particularized in a later section.)

For the present example, (38) may be used to replace (34), and hence also the second equation of (33), by:

$$\sum (C_{2k} - \bar{C}_{2k})z^{2k} = -\log[1 + H(z)] \tag{46}$$

(33) requires $H(z)$ to be a polynomial in z^2, of degree $n + 2$, with two zeros of $[1 + H(z)]$ at $z^2 = \bar{z}_0^2$. The object is to find an $H(z)$ of this sort, which also satisfies (45), at least to a good approximation.

The following $H(z)$ does in fact exhibit the required properties:

$$H(z) = Gz^{2n+2}\frac{[1 - Jz^2][1 - J^{n+2}/z^{2n+4}]}{[1 - J/z^2]} \tag{47}$$

The function is a polynomial because the factor $[1 - J^{n+2}/z^{2n+4}]$ is divisible by $(1 - J/z^2)$. The constants J and G are to be chosen to give the required double zero of $[1 + H(z)]$ at $z^2 = \bar{z}_0^2$. One value of J, so determined, is real and of order $1/\bar{z}_0^2$. This is the appropriate solution. Then $| J^{n+2}/z^{2n+4} |$ is of order $1/\bar{z}_0^{2n+4}$, when $| z | \geqq 1$. This suggests the following approximation in place of (47):

$$H(z) \cong Gz^{2n+2} \frac{1 - Jz^2}{1 - J/z^2} \tag{48}$$

The approximation is at least as good as $1/\bar{z}_0^{2n+4}$, compared with unity, both in the useful interval and in the neighborhood of the singularities $\bar{z}_0$, and z_σ. This means that the approximation can be used: in estimating the error $\alpha - \bar{\alpha}$ (in the useful interval), in calculating J and G, and in finding the roots z_σ.

In the useful interval, $| z | = 1$, and therefore $1/z = z^*$. Then $(1 - J/z^2)$ is $(1 - Jz^2)^*$; and their ratio has magnitude unity. Thus $| H(z) | = | G |$, in the useful interval, to order of $1/\bar{z}_0^{2n+4}$ compared with unity†. With $| J | < 1$, phase $H(z)$ varies over the useful interval to the same extent as the phase of z^{2n+2}.‡ Fig. 9 illustrates the difference in $\alpha - \bar{\alpha}$, as determined by (42) and (48). These curves, however, are for single values of n and $\bar{z}_0$; and the improvement obtained by using (48) would be different with different values of n or $\bar{z}_0$.

The values of J and G, determined from (48), and the requirement that $[1 + H(z)]$ must have two zeros at $z^2 = \bar{z}_0^2$, turn out to be:

$$J = \frac{n+1}{n+2} \frac{1}{\bar{z}_0^2} \frac{2}{1 + \frac{1}{\bar{z}_0^4} + \sqrt{\left(1 - \frac{1}{\bar{z}_0^4}\right)^2 + \frac{4}{(n+2)^2 \bar{z}_0^4}}}$$
$$G = -\frac{1}{\bar{z}_0^{2n+2}} \frac{1 - J/\bar{z}_0^2}{1 - J\bar{z}_0^2} \tag{49}$$

Note that this J is in fact smaller than $1/\bar{z}_0^2$.

15. GENERALIZATION

The several sections preceding describe a quite specific example, as an introduction to synthesis applications. The next several sections describe how the specific methods of the example may be generalized, in several respects.

First, the ideal gain, $\bar{\alpha}$, is generalized, so that it need not even have the sort of functional form associated with finite networks. Then, the

† The $| H(z) |$ determined by (42) is constant only to order $1/\bar{z}_0^2$.
‡ This is the most we can expect, when we have n singularities, which can prevent the dominance of only lower order terms, through z^{2n}.

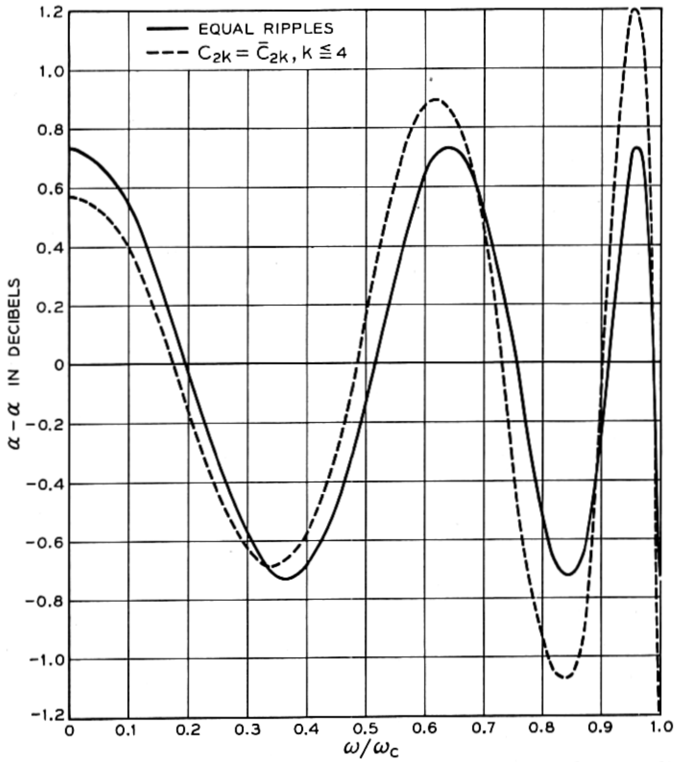


Fig. 9—Comparison of design procedures—four natural modes equalizing two natural modes at $\bar{p}_0 = -\frac{5}{12}\,\omega_c$.

approximating network gain is generalized, by introducing arbitrary frequencies of infinite loss, in addition to the arbitrary natural modes. The methods are also modified for approximation to an assigned phase, instead of gain, or to phase and gain simultaneously. Finally, the analysis is modified to permit useful intervals of the "high-pass" type, or (in the case of gain simulation) of the "band-pass" type.

16. APPROXIMATION TO A GENERAL ASSIGNED GAIN $\bar{\alpha}$

If we now permit the assigned gain $\bar{\alpha}$ to be general, in the sense of Section 8, we must return to the formulation:

$$\begin{aligned} \bar{\alpha} &= \sum \bar{C}_{2k} T_{2k} \\ \sum \bar{C}_{2k} z^{2k} &= \log\,[\bar{R}(z)\bar{R}(-z)] \end{aligned} \qquad (50)$$

If we retain simulation with a network which has n natural modes, and no frequencies of infinite loss, we must retain the formulation:

$$\begin{aligned} \alpha &= \sum C_{2k} T_{2k} \\ \sum C_{2k} z^{2k} &= -\log K_z^2 \prod \left(1 - \frac{z^2}{z_\sigma^2}\right), \qquad \sigma = 1, \cdots, n \end{aligned} \tag{51}$$

The corresponding formulation of the error is (in place of (33)):

$$\begin{aligned} \alpha - \bar{\alpha} &= \sum (C_{2k} - \bar{C}_{2k}) T_{2k} \\ \sum (C_{2k} - \bar{C}_{2k}) z^{2k} &= -\log \left[K_z^2 \prod \left(1 - \frac{z^2}{z_\sigma^2}\right) \cdot \bar{R}(z) \bar{R}(-z) \right] \end{aligned} \tag{52}$$

For $C_{2k} = \bar{C}_{2k}$, $k \leqq m$, the following special case of (28) is now required:

$$K_z^2 \prod \left(1 - \frac{z^2}{z_\sigma^2}\right) \cdot \bar{R}(z) \bar{R}(-z) \overset{me}{=} 1 \tag{53}$$

Now the reciprocal of $\bar{R}(z)\bar{R}(-z)$ has a power series expansion, in the region of interest. (Recall Section 8.)

It follows that (53) may be multiplied by this quantity, without damaging the equality of power series coefficients. In other words (53) is equivalent to:

$$K_z^2 \prod \left(1 - \frac{z^2}{z_\sigma^2}\right) \overset{me}{=} \frac{1}{\bar{R}(z)\bar{R}(-z)} \tag{54}$$

Let K_k be the coefficient of z^{2k} in the polynomial expansion of the left hand side; and let $\bar{K}_k$ be the coefficient of z^{2k} in the infinite series expansion of the right hand side. Then,

$$\begin{aligned} K_z^2 \prod \left(1 - \frac{z^2}{z_\sigma^2}\right) &= K_0 + K_1 z^2 + \cdots K_n z^{2n} \\ \frac{1}{\bar{R}(z)\bar{R}(-z)} &= \sum \bar{K}_k z^{2k}, \qquad k = 0, ---, \infty \end{aligned} \tag{55}$$

Substitution in (54) gives

$$K_0 + K_1 z^2 + \cdots K_n z^{2n} \overset{me}{=} \sum \bar{K}_k z^{2k} \tag{56}$$

In other words,

$$K_k = \bar{K}_k, \qquad k \leqq m \tag{57}$$

These relations are directly applicable to network synthesis, provided the coefficients $\bar{K}_k$ can be calculated. Formulae for their calculation may be derived in the following way:

If (55) is substituted in (50), the result is:†

$$\bar{\alpha} = \sum \bar{C}_{2k} T_{2k} \qquad (58)$$
$$\sum \bar{C}_{2k} z^{2k} = -\log \sum \bar{K}_k z^{2k}$$

When $z = 0$, the second equation reduces to:

$$\bar{K}_0 = e^{-\bar{C}_0} \qquad (59)$$

If the functions of z are differentiated, a simple rearrangement gives:

$$\sum k\bar{K}_k z^{2k-2} = [-\sum k\bar{C}_{2k} z^{2k-2}][\sum \bar{K}_k z^{2k}] \qquad (60)$$

The right hand side may be expanded as a single power series, and then like powers on the two sides may be equated separately. The result is:

$$\begin{aligned} \bar{K}_1 &= -\bar{C}_2\bar{K}_0 \\ 2\bar{K}_2 &= -\bar{C}_2\bar{K}_1 - 2\bar{C}_4\bar{K}_0 \\ 3\bar{K}_3 &= -\bar{C}_2\bar{K}_2 - 2\bar{C}_4\bar{K}_1 - 3\bar{C}_6\bar{K}_0 \end{aligned} \qquad (61)$$

Synthesis calculations may now be carried out in the following stages. The assigned gain $\bar{\alpha}$ is expanded as a Tchebycheff polynomial series, to determine coefficients $\bar{C}_{2k}$, say through order $k = n$. The equations (61) are then used to calculate coefficients $\bar{K}_k$, also through order n. Each successive coefficient is computed in terms of those previously determined. Note that the $\bar{K}_k$, $k \leqq n$, are fixed by the same number of $\bar{C}_{2k}$—that is, orders $k \leqq n$.

Equation (57) is now applied to identify K_k with $\bar{K}_k$, $k \leqq m$. If all the network degrees of freedom are to be used to get $C_{2k} = \bar{C}_{2k}$, index $m = n$, and (57) determines the polynomial in (55) completely. Otherwise, $m < n$, and coefficients K_{m+1} to K_n are to be adjusted in accordance with specifications of other kinds. When all the K_k have been determined, the singularities z_σ are found by root extraction methods, applied to the right hand side of the first equation of (55).

The previous example might have been carried out in these terms, but happened to be simpler in the terms used. If (32) is regarded as a special case of (50), and if (32) is simplified (for purposes illustration) by using $\bar{K}_z = 1$, the corresponding $\bar{R}(z)\bar{R}(-z)$ becomes simply $\left(1 - \frac{z^2}{z_0^2}\right)^2$. Then $\sum \bar{K}_k z^{2k}$ is

† We may think of these equations as defining an *infinite* network, with natural modes only, which would match the assigned gain $\bar{\alpha}$ exactly.

$$\sum \bar{K}_k z^{2k} = \frac{1}{(1 - z^2/\bar{z}_0^2)^2} = \sum (k + 1) \left(\frac{z^2}{\bar{z}_0^2}\right)^k \tag{62}$$

thus $\bar{K}_k$ and K_k become

$$\begin{aligned} \bar{K}_k &= \frac{k+1}{\bar{z}_0^{2k}}, \qquad k = 0 \text{ to } \infty \\ K_k &= \frac{k+1}{\bar{z}_0^{2k}}, \qquad k = 0 \text{ to } n \end{aligned} \tag{63}$$

If these K_k are used to evaluate the polynomial on the left hand side of (53), in accordance with (55), and if the polynomial is then multiplied by the above special $\bar{R}(z)\bar{R}(-z)$, the result is exactly (36).

The error function $H(z)$, of (30) and (42), may now be defined as follows:

$$K_s^2 \prod \left(1 - \frac{z^2}{z_\sigma^2}\right) R(z)R(-z) = 1 + H(z) \tag{64}$$

The error $\alpha - \bar{\alpha}$ is again:

$$\alpha - \bar{\alpha} = -\log |1 + H(z)| \tag{65}$$

If (64) is used to express (53) in terms of $H(z)$, a "1" may be subtracted from each side of the relation, to get

$$H(z) \overset{me}{=} 0 \tag{66}$$

When $m = n$, this requires an $H(z)$ of the following form, in terms of the coefficients $\bar{K}_k$ derived from $\bar{R}(z)\bar{R}(-z)$:

$$H(z) = -\frac{\bar{K}_{n+1}z^{2n+2} + \bar{K}_{n+2}z^{2n+4} + \cdots}{\sum \bar{K}_k z^{2k}} \tag{67}$$

17. CHARACTERISTICS OF z_σ

As in the previous example, $|z_\sigma| > 1$ when the approximation, α to $\bar{\alpha}$, is at all reasonable. The z_σ are again zeros of $1 + H(z)$; but now $H(z)$ is defined by (64). If $|H(z)| < 1$, when $|z| = 1$, there will be the same number of zeros of $1 + H(z)$ as poles, in the region $|z| < 1$. Any poles would have to be poles of $\bar{R}(z)\bar{R}(-z)$. In Section 8, we noted that this function is regular in the region $|z| \leqq 1$. Hence, there will be no poles, and there will be no $|z_\sigma| < 1$, under ordinary accuracy conditions.

As before, the z_σ can be chosen in accordance with the physical conditions, provided none are pure imaginaries. Since an imaginary z_σ is a negative real z_σ^2:

There must be no negative real z_σ^2.

There will be no negative real z_σ^2 if the polynomial $\sum K_k z^{2k}$ in (55) is non-zero at all negative real z^2. If the requirement is violated, initially, one or more K_k, of the highest orders, must be modified. Graphical methods are likely to be useful for this, combining plots of the original polynomial, and proposed changes. An approximation of the form $C_{2k} = \bar{C}_{2k}$, $k \leqq m$, will still be realized, but with $m < n$.

The error function corresponding to $m < n$ is as follows, in place of (67):

$$H(z) = \frac{(K_{m+1} - \bar{K}_{m+1})z^{2m+2} \cdots - \bar{K}_{n+1}z^{2n+2} \cdots}{\sum \bar{K}_k z^{2k}} \tag{68}$$

18. ACCURACY

The accuracy of match again may be estimated by the means of (41), using $H(z)$ of (67) or (68). $H(z)$, however, may not be so easily calculated as for the previous example.

A simpler but less reliable estimate of accuracy is furnished by the error in the first unmatched coefficient in the Tchebycheff polynomial series. If $K_k = \bar{K}_k$ through $k = m$, the leading terms in various series are as follows. First, from (64) and (68),

$$K_z^2 \prod \left(1 - \frac{z^2}{z_\sigma^2}\right) R(z)R(-z) = 1 + \frac{K_{m+1} - \bar{K}_{m+1}}{K_0} z^{2m+2} \cdots \tag{69}$$

using this in (52) gives:

$$\sum (C_{2k} - \bar{C}_{2k})z^{2k} = -\log\left[1 + \left(\frac{K_{m+1} - \bar{K}_{m+1}}{\bar{K}_0}\right) z^{2m+2} \cdots\right] \tag{70}$$

Then, from the properties of logarithms,

$$\sum (C_{2k} - \bar{C}_{2k})z^{2k} = -\frac{K_{m+1} - \bar{K}_{m+1}}{\bar{K}_0} z^{2m+2} \cdots \tag{71}$$

Consequently (also from (52)):

$$\alpha - \bar{\alpha} = \frac{\bar{K}_{m+1} - K_{m+1}}{\bar{K}_0} T_{2m+2} \cdots \tag{72}$$

This is the same as the leading term of $H(z)$, except that z^{2m+2} is replaced by $-T_{2m+2}$. If $m = n$, the same equation holds with $K_{m+1} = 0$.

The coefficient $\frac{\bar{K}_{m+1} - K_{m+1}}{K_0}$ in (72) is a sort of average of the envelopes of the ripples in $\alpha - \bar{\alpha}$. The variability of the envelopes, across the useful interval, depends upon higher order coefficients, in comparison with the leading term. Calculation of higher order coefficients is relatively complicated.

19. APPROXIMATION IN THE TCHEBYCHEFF SENSE

The criteria (45) carry over to general assigned gains, as conditions on $H(z)$ which, if realized, are usually sufficient to establish approximation in the Tchebycheff sense. For this purpose we must use the $H(z)$ of (64), rather than (67) or (68) (which correspond explicitly to $C_{2k} = \bar{C}_{2k}$, $k \leqq m$). In terms of the polynomial and series representations of (55), the $H(z)$ of (64) becomes:

$$H(z) = \frac{K_0 + K_1 z^2 \cdots K_n z^{2n}}{\sum \bar{K}_k z^{2k}} - 1 \tag{73}$$

The following somewhat special problem is easily solved, in these terms, and has a direct bearing on various quite different synthesis techniques: A network is to be designed which combines the functions of an equalizer or simulator, with those of a filter, or selective network. In the useful interval, an assigned gain variation $\bar{\alpha}$ is to be approximated in the Tchebycheff sense. At higher frequencies, there is to be a rapidly increasing loss, or "sharp filter cut-off." The number of natural modes, n, is to be more than sufficient to match $\bar{\alpha}$ to the required accuracy, in the absence of a selectivity requirement, the latitude being used to produce the required sharp cut-off. In particular, n is to be large enough so that an n term match of Tchebycheff coefficients produces errors that are negligible compared with those accepted as a price of the sharp cut-off.

On the above assumption of an ample n, the infinite series $\sum \bar{K}_k z^{2k}$ in (73) may be truncated after the term of order n, *and the errors due to the truncation may be neglected in calculating the design error* $\alpha - \bar{\alpha}$.† Then (73) becomes

$$H(z) = \frac{K_0 + K_1 z^2 \cdots K_n z^{2n}}{\bar{K}_0 + \bar{K}_1 z^2 \cdots \bar{K}_n z^{2n}} - 1 \tag{74}$$

† The truncated series is merely the polynomial on the left side of (56) which *would* be obtained if the filter selectivity were ignored, and m were given the maximum value, n.

If a sharp cut-off were not required, this approximate $H(z)$ could be made exactly zero, by using $K_k = \bar{K}_k$ for all coefficients. Then the actual design error would be determined by the approximation inherent in the use of (74) in place of (73). For high selectivity, however, K_n should be much larger than $\bar{K}_n$, as large as possible within assigned limits on $\alpha - \bar{\alpha}$ in the useful range. (It is readily established that $K_n z^n$ will determine α at asymptotically high frequencies.) The other K_k are then to be adjusted so that $\alpha - \bar{\alpha}$ exhibits the desired "equal ripples."

The following $H(z)$ has the functional form (74), and also meets conditions (45):

$$H(z) = Gz^{2n} \frac{\bar{K}_0 + \dfrac{\bar{K}_1}{z^2} \cdots \dfrac{\bar{K}_n}{z^{2n}}}{\bar{K}_0 + \bar{K}_1 z^2 \cdots \bar{K}_n z^{2n}} \tag{75}$$

Multiplying Gz^{2n} into the numerator gives a rational fraction which is obviously consistent with (74). The coefficients K_k of (74) which correspond to (75) are:

$$K_k = \bar{K}_k + G\bar{K}_{n-k} \tag{76}$$

In the useful interval $[\sum \bar{K}_k/z^{2k}]$ is $[\sum \bar{K}_k z^{2k}]^*$. Hence the polynomials in z and $1/z$ have identical magnitudes, in the useful interval; and, since also $|z| = 1$, $|H(z)| = |G|$ in (75). The phase variation, over the useful interval, is the same for $H(z)$ as for z^{2n}, which yields the same number of ripples in $\alpha - \bar{\alpha}$ as an ordinary Tchebycheff filter of like degree.†

The constant G is arbitrary, except that its sign must be properly chosen to avoid non-physical natural modes. Increasing G increases the filter selectivity, but also increases $\alpha - \bar{\alpha}$ in the useful interval. G and n are to be chosen together, to realize an assigned selectivity within an assigned limit on distortion.

The above analysis may be related to the following filter problem: Required to design a filter which has flat gain, in the useful interval, but which has m *assigned* frequencies of infinite loss, in addition to n arbitrary natural modes ($m \leqq n$). The n arbitrary natural modes may be regarded as compensating for gain variations due to the assigned frequencies of infinite loss, in the useful interval, while reinforcing their effects at other frequencies. *Compensation* of effects of the infinite loss points is the same as *simulation* of the effects of natural modes at the same (assigned) frequencies. The approximation in the useful interval

† This assumes an $\bar{\alpha}$ with the general characteristics described in Section 8, which are such that the numerator and denominator of the fraction in (75) will each have a net phase shift of *zero*, across the useful interval.

is to be no better than necessary, so that there may be a maximum reinforcing of losses at other frequencies. In these terms, (58), and $\sum \bar{K}_k z^{2k}$ in (73), correspond to the assigned natural modes (at the same locations as the assigned frequencies of infinite loss). Then the ideal $\sum \bar{K}_k z^{2k}$ is itself a polynomial, of degree $m \leqq n$, and (74) is exact, rather than an approximation to (73). Then (76) determines the n arbitrary modes in such a way that the net filter gain approximates zero in the Tchebycheff sense, over the useful interval.

A different procedure for obtaining the same result is described in the author's paper "Synthesis of Reactance 4-Poles".[8] The above analysis of the filter problem is of interest in relating the more general synthesis techniques, in terms of Tchebycheff polynomial series, to previous filter theory.

Similar filters have also been obtained by Matthaei[2], on a potential analogy basis. He includes, however, somewhat more general filter characteristics, for which he obtains only approximately equal ripple errors. Analysis of the sort described above may be used to clarify Matthaei's analysis of the conditions under which he obtains exactly equal-ripples.

Equation (75) may be related to work of Bashkow[3]. The (arbitrary) amplitude of the (equal) maxima of $|\alpha - \bar{\alpha}|$, computed from $H(z)$ of (75), depends only on $|G|$. The frequencies at which the maxima occur correspond to phase $H(z) = s\pi$, which is independent of $|G|$. Thus, the locations of the maxima are invariant to the arbitrary amplitude, *within the range where* (75) *applies*. (75) applies only when (74) may be used in place of (73). Generally, (74) only approximates (73), and the approximation introduces small variations in the maxima of $|\alpha - \bar{\alpha}|$ (when α corresponds to (76)). If the maxima themselves are sufficiently small, the small variations will be large *percentage* variations; and the adjustments to compensate for the variations will yield significant shifts in the location of the maxima. In other words, the locations of the maxima of $|a - \bar{\alpha}|$, required for *equal* amplitudes, are largely invariant to the magnitude of the equal amplitudes, but only to an approximation which becomes worse as the amplitudes are decreased.

Bashkow states the invariance of the frequencies of maximum $|\alpha - \bar{\alpha}|$, as a more or less empirical conclusion, based on a quite different approach to the same synthesis problem.

Equation (75) may be related also to work of Kuh.[4] The natural modes z_σ are zeros of $1 + H(z)$. In other words, $H(z_\sigma) = -1$. Using the $H(z)$ of (75) gives the following:

$$K_0 + K_1 z_\sigma^2 + \cdots K_n z_\sigma^{2n} = -G z_\sigma^{2n} \left\{ \bar{K}_0 + \frac{\bar{K}_1}{z_\sigma^2} + \cdots \frac{\bar{K}_n}{z_\sigma^{2n}} \right\} \quad (77)$$

It must be remembered, however, that this formulation is permissible only if the approximations, inherent in (75), are justified when $z = z_\sigma$ (as well as in the useful interval). Taking the logarithm of each side gives:

$$\log \{\bar{K}_0 + \cdots \bar{K}_n z_\sigma^{2n}\}$$
$$= \log(-G) + 2n \log z_\sigma + \log\left\{\bar{K}_0 + \cdots \frac{\bar{K}_n}{z_\sigma^{2n}}\right\} \quad (78)$$

Equation (58) may now be applied, to replace the logarithms by power series, provided the truncation of the series is again justified, and provided the convergence of $\sum \bar{C}_{2k} z^{2k}$ is also proper, at both $z = z_\sigma$ and $z = 1/z_\sigma$. This gives

$$-\sum \bar{C}_{2k} z_\sigma^{2k} = -\sum \bar{C}_{2k}/z_\sigma^{2k} + 2n \log z_\sigma + \log(-G) \quad (79)$$

(Summations $\sum$ are all with respect to k; and there is one equation for each σ.) This is a suitable rule, for obtaining an $|\alpha - \bar{\alpha}|$ with equal maxima, whenever the approximations are in fact unimportant. It is not at all clear, however, just when the approximations become significant.

Kuh uses the potential analogy approach for the same sort of synthesis problem. He spaces the natural modes along a p-plane contour defined in fairly complicated potential analogy terms. It can be shown, however, that mapping his potentials from p plane to z plane leads to (79).

When the network is to approximate $\bar{\alpha}$ in the useful interval, but is *not* required to supply selectivity at other frequencies, the approximation (74) is usually untenable. It is generally necessary to retain the exact formulation (73).

When selectivity is not required, the phase excursion of $H(z)$, in the useful interval, can usually be increased to that of z^{2n+2} (as in (67), corresponding to $C_{2k} = \bar{C}_{2k}$, $k \leqq n$). As a step toward meeting the first condition of (45), one may then write (73) as follows:

$$H(z) = -\bar{K}_{n+1} z^{2n+2} \frac{\sum \frac{\bar{K}_{n+1+k}}{\bar{K}_{n+1}} z^{2k} + \sum_1^{n+1} Q_k \frac{1}{z^{2k}}}{\sum \bar{K}_k z^{2k}} \quad (80)$$

$$Q_k = \frac{\bar{K}_{n+1-k} - K_{n+1-k}}{\bar{K}_{n+1}}, \qquad k = 1, \cdots, n+1$$

The coefficients $\bar{K}_k$ and $\frac{\bar{K}_{n+1+k}}{\bar{K}_{n+1}}$ are fixed by $\bar{\alpha}$. The only arbitrary design constants are the Q_k. They are to be small enough so that they do not affect the total phase excursion $H(z)$, when $|z| = 1$. Their specific values are to make $|H(z)|$ approximately constant, when $|z| = 1$. In general the (required) series in z^2 in the numerator makes it extremely difficult to determine the required values for the Q_k. No reasonably simple general procedure has yet been found.

20. ARBITRARY RATIONAL FRACTIONS

The preceding sections were devoted to the approximation α to $\bar{\alpha}$, using n arbitrary natural modes, but no arbitrary frequencies of infinite loss. Similar techniques may be used when there are n'' arbitrary natural modes and n' arbitrary frequencies of infinite loss. As the applications become more involved, however, routine calculations must be supplemented increasingly with an element of art.

For simultaneous design of natural modes and frequencies of infinite loss, we must go back from (31) to the α formulation in (21). This we shall now write:

$$\begin{aligned} \alpha &= \sum C_{2k} T_{2k} \\ \sum C_{2k} z^{2k} &= -\log \frac{N}{D} \end{aligned} \tag{81}$$

The functions N and D are polynomials. The coefficients will be defined as follows:

$$\begin{aligned} N &= K_0'' + K_1'' z^2 \cdots K_{n''}'' z^{2n''} \\ D &= 1 + K_1' z^2 \cdots K_{n'}' z^{2n'} \end{aligned} \tag{82}$$

By comparison with (21), the zeros of N, in terms of z^2, are the $z_\sigma''^2$. (Note the minus sign in 81.) The zeros of D are then the $z_\sigma'^2$. For physical networks, $n'' \geqq n'$.

Equations (50), describing $\bar{\alpha}$, may be retained as they stand. Combining (50) and (81) gives, in place of (52):

$$\begin{aligned} \alpha - \bar{\alpha} &= \sum (C_{2k} - \bar{C}_{2k}) T_{2k} \\ \sum (C_{2k} - \bar{C}_{2k}) z^{2k} &= -\log \left[\frac{N}{D} \bar{R}(z) \bar{R}(-z) \right] \end{aligned} \tag{83}$$

The function $\bar{R}(z)\bar{R}(-z)$ is exactly the same as before. The reciprocal of the function will still be $\sum \bar{K}_k z^{2k}$, with the $\bar{K}_k$ related to $\bar{C}_{2k}$ by (58), (59), (61). The new rational fraction N/D will appear where previously we had the polynomial $K_0 + \cdots K_n z^{2n}$.

Accordingly, the rule for $C_{2k} = \bar{C}_{2k}$, $k \leqq m$, now becomes, in place of (56),

$$\frac{N}{D} \overset{me}{=} \sum \bar{K}_k z^{2k} \tag{84}$$

This condition may be used to determine the coefficients K_k'', K_k' of N and D (in combination with conditions of other sorts, when $m < n'' + n'$). When the coefficients have been calculated, the (z-plane) natural modes z_σ'' may be determined from the roots of N, exactly as the z_σ of previous sections. The infinite loss points z_σ' may be calculated from the roots of D, in exactly the same way except that Re z_σ' need not be negative. Signs of the z_σ' must be such that complex *and imaginary* z_σ' are in conjugate pairs. Note that there can be *conjugate* imaginary z_σ' only if D has a corresponding *double* negative real zero.

When $m = n'' + n'$, the following method may be used to calculate the K_k'' and K_k' determined by (84). The relation is first multiplied by D, to get:

$$N \overset{(n''+n')}{=} D \sum \bar{K}_k z^{2k} \tag{85}$$

Then algebraic manipulation is used to evaluate the power series equivalent of the right hand side, through terms of order $n'' + n'$, using the known values of the $\bar{K}_k$, but general values of the K_k' of D. Each coefficient is a linear function of the unknowns, K_k'.

Now the polynomial N, in (85), has no terms of order $k > n''$. Therefore (85) requires zero coefficients in the expansion of the right hand side, from order $n'' + 1$ to order $n'' + n'$. Equating these coefficients to zero gives n' linear equations in the n' unknown K_k'. Solving for the K_k' determines polynomial D. The values calculated for the K_k' may then be used in lower order coefficients of the expansion of the right hand side of (85), which are exactly the coefficients K_k'' of N.

When $n'' - n' = 0$ or 1, a continued fraction method is likely to be preferable. Various established techniques† may be used to convert the series $\sum \bar{K}_k z^2$ into a continued fraction of the form:

$$\sum \bar{K}_k z^{2k} = a_0 + \cfrac{1}{\cfrac{a_1}{z^2} + \cfrac{1}{a_2 + \cfrac{1}{\cfrac{a_3}{z^2} + \cfrac{1}{a_4 \cdots}}}} \tag{86}$$

† See, for example, Fry's applications of continued fractions to network design.[9]

If the continued fraction is truncated after the term of order m, and is rearranged as a rational fraction N/D, it will obey equation (84). The degrees of N and D will be such that $n'' + n' = m$, and $n'' - n' = 0$ or 1. The continued fraction may be associated with the hypothetical ladder network shown in Fig. 10, with variable impedance shunt branches proportional to z^2. The impedance of the (truncated) ladder is N/D.

21. ACCURACY

The accuracy of match, $\alpha - \bar{\alpha}$, may again be evaluated from the final network singularities; or by (41), with $H(z)$ as in (30), before the singularities have been determined from roots of N and D. A rougher estimate may again be obtained from the error in the first unmatched term in the Tchebycheff polynomial series. As before, (equation (72)), this is equal to the leading term in $H(z)$, with z^{2m+2} replaced by $-T_{2m+2}$.

The rational fraction in (30) is the same as our present N/D. In terms of N/D, (30) becomes:

$$\frac{N}{D}\bar{R}(z)\bar{R}(-z) = 1 + H(z) \tag{87}$$

If (86), or Fig. (10), is used to determine N and D, the leading term in $H(z)$ turns out to be:

$$H(z) = \frac{(-)^{m+1}z^{2m+2}}{(a_1a_2 \cdots a_m)^2 a_{m+1}\bar{K}_0} \cdots \tag{88}$$

The corresponding mismatch in Tchebycheff polynomial terms is:

$$C_{2m+2} - \bar{C}_{2m+2} = \frac{(-)^m}{(a_1a_2 \cdots a_m)^2 a_{m+1}\bar{K}_0} \tag{89}$$

22. ZEROS AND POLES

When frequencies of infinite loss are to be chosen, as well as natural modes, the situation in regard to $|z_\sigma| < 1$ is somewhat less favorable.

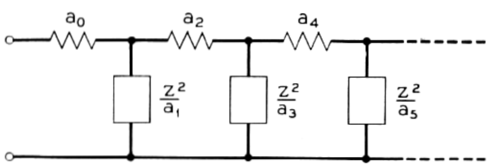


Fig. 10—A ladder network representation of the continued fraction (86).

It is still true that $1 + H(z)$ will have the same number of zeros as poles in the region $| z | < 1$, so long as $\alpha - \bar{\alpha}$ is reasonably small in the interval $| z | = 1$. In equation (87), however, the poles of $1 + H(z)$ include the zeros z'_σ of D (the arbitrary infinite loss points), as well as the poles of $\bar{R}(z)\bar{R}(-z)$. When the coefficients of D are to be chosen as in the previous section, the contour rule merely says that any z'_σ and z''_σ in the region $| z | < 1$ will occur in like numbers.

Fortunately, the frequent occurrence of $| z_\sigma | < 1$ is softened by the following curious circumstance. Almost always, any $| z'_\sigma |$ and $| z''_\sigma | < 1$ are so nearly identical that factors $(z - z'_\sigma)$ and $(z - z''_\sigma)$ may be cancelled out without any important effect on $H(z)$, or $\alpha - \bar{\alpha}$. Cancellations of this sort were encountered a number of times before an explanation was discovered. Actually the explanation is quite simple.

At any zero of $1 + H(z)$, $H(z) = -1$. On the other hand, $H(z)$ is small when $| z | = 1$. Generally, it is much *smaller* in most of the interval $| z | < 1$. For instance, when $C_{2k} = \bar{C}_{2k}$ through $k = m$, $H(z)$ is proportional to z^{2m+2}, in the neighborhood of $z = 0$. As a result, $| H(z) |$ rarely becomes as large as 1, in the region $| z | < 1$, except in the very close proximity of a pole. In other words, in the region $| z | < 1$, any zero z''_σ is usually very close to a pole z'_σ—usually so close that the corresponding factors $z - z_\sigma$ may be canceled out without significant effect on $\alpha - \bar{\alpha}$.

The occurrence of non-physical natural modes ($Re z''_\sigma = 0$) is the same as before; but adjustments to correct for these, in an efficient manner, are much more complicated. In addition, there may be non-physical infinite loss points, z'_σ. To correct for non-physical singularities, the *simplest* procedure would be to change one or both of the highest order coefficients in N and D of (82), that is $K''_{n''}$ and $K'_{n'}$. This would be inefficient, however, for it would spoil the match of $C_{n''}$ to $\bar{C}_{n''}$, or $C_{n'}$ to $\bar{C}_{n'}$. The unmodified design, defined by (84), can match terms through order $n'' + n'$, and it is desirable to change only the highest order terms in adjusting the design.

More efficient adjustments are in fact feasible. They sometimes require an increased element of art; but the art may be based on specific principles. Some particularly useful principles are described in the next section. These apply to various other corrections besides correction of non-physical zeros and poles. Examples are reduction in phase to make two-terminal realization possible, and increase in shunt capacity in two-terminal designs. In general, they offer a means of making $m < n' + n''$ in (84), and using the remaining degrees of freedom to meet other conditions.

23. MODIFICATION OF N/D

Suppose N_1/D_1 and N_2/D_2 are two rational fractions, representing special choices of N and D in (82), such that:

$$\frac{N_1}{D_1} \overset{m_1 e}{=} \sum \bar{K}_k z^{2k}$$
$$\frac{N_2}{D_2} \overset{m_2 e}{=} \sum \bar{K}_k z^{2k} \qquad (90)$$

Then suppose F is a function of z^2 such that

$$F \overset{\mu e}{=} 0 \qquad (91)$$

The following combination represents an approximation to $\sum \bar{K}_k z^{2k}$, of the order indicated:†

$$\frac{N_1 + FN_2}{D_1 + FD_2} \overset{me}{=} \sum \bar{K}_k z^{2k} \qquad (92)$$

$m = m_1$, or $m_2 + \mu$, whichever is the smaller.

The left hand side of (92) may be used as N/D in (84), to match C_{2k} to $\bar{C}_{2k}$ through $k = m$. Adjustment of the function F may be used to satisfy other requirements, in addition to accuracy specifications.

Frequently, N_1/D_1 may be the rational fraction corresponding to truncation of the continued fraction, (86), after the term in $a_{n''+n'}$. Then N_2/D_2 is likely to be a truncation of order $m < n'' + n'$. The corresponding F is likely to be proportional to $z^{2\mu}$, or at most a simple polynomial in z^2. When F is a constant, and $m = n'' + n' - 1$, the use of these particular rational functions in (92), to determine N/D, corresponds to matching C_{2k} to $\bar{C}_{2k}$ through $k = n'' + n' - 1$, but leaving $C_{2(n''+n')}$ subject to adjustment. Specificially, $C_{2(n''+n')}$ depends on the choice of F, which may hinge upon such special conditions as the elimination of non-physical singularities.

Problems which call for more complicated combinations are by no means uncommon. Skill may be needed in the choice of specific combinations which will solve specific problems. Computations may be

† The relationship is easily established by noting that:

$$\frac{N_1 + FN_2}{D_1 + FD_2} - \Sigma\, \bar{K}_k z^{2k} = \frac{\dfrac{N_1}{D_1} - \Sigma\, \bar{K}_k z^{2k}}{1 + F\,\dfrac{D_2}{D_1}} + F\,\frac{\dfrac{N_2}{D_2} - \Sigma\, \bar{K}_k z^{2k}}{\dfrac{D_1}{D_2} + F} \qquad (93)$$

systematized, to a considerable extent, by using the error formula (88), and other relations between the coefficients of the continued fraction, and the rational fraction truncations of various orders.†

24. APPROXIMATION TO BOTH GAIN AND PHASE

The applications described in previous sections relate to approximations to prescribed gain, $\bar{\alpha}$, without regard to the associated phase. Quite similar methods apply, however, to the simultaneous approximation of gain and phase.

The starting point is equation (20). Replacing products of factors by polynomials gives, in place of (81):

$$\begin{aligned} \alpha + i\beta &= \sum C_k T_k, \qquad k \text{ even and odd} \\ \sum \tfrac{1}{2} C_k z^k &= -\log \frac{N}{D} \end{aligned} \tag{94}$$

The polynomials are now as follows, in place of (82):

$$\begin{aligned} N &= K_0'' + K_1'' z \cdots K_{n''}'' z^{n''} \\ D &= 1 + K_1' z \cdots K_{n'}' z^{n'} \end{aligned} \tag{95}$$

(If only natural modes are to be used, the suitable replacement for the first equation or (55) is here obtained merely by using $D = 1$, and K_z, z, z_σ, in place of their squares.)

A comparable expression is needed for the assigned gain and phase $\bar{\alpha} + i\bar{\beta}$. In place of (50), we now repeat (22), and redefine the coefficients $\bar{K}_k$ in accordance with

$$\begin{aligned} \bar{\alpha} + i\bar{\beta} &= \sum \bar{C}_k T_k, \qquad k \text{ even and odd} \\ \sum \tfrac{1}{2} \bar{C}_k z^k &= \log \bar{R}(z) = -\log \sum \bar{K}_k z^k \end{aligned} \tag{96}$$

The definition of $\bar{K}_k$ has been changed in such a way that it is now related to $\bar{C}_k/2$ exactly as it was previously (in 58) related to $\bar{C}_{2k}$. Equations (59), (61) may be applied to calculating the $\bar{K}_k$ by merely substituting therein a $\bar{C}_k/2$ for every $\bar{C}_{2k}$.

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† For example, a simple recursion formula may be used to assemble the polynomials N and D which correspond to truncation of the continued fraction (86) at a number of different points. Specifically, $P_n = P_{n-1} + \dfrac{z^2}{a_n a_{n-1}} P_{n-2}$, where P is either N or D and P_n corresponds to truncation of the continued fraction after the term in a_n. The formula holds for $n \geqq 2$.

Equations (84), (85), (86) may now be modified, for the new N, D and $\bar{K}_k$, by merely using z in place of z^2 wherever it occurs (including z^k in place of z^{2k}).† The modifications of equations (84), (85), and the truncation of (86) after a_m now lead to $C_k = \bar{C}_k$, $k \leqq m$, instead of the previous $C_{2k} = \bar{C}_{2k}$. This means that m must be twice as great to match coefficients out to the same actual orders. This is to be expected since now one half of our design parameters are used to approximate phase $\bar{\beta}$, leaving only half for approximating gain $\bar{\alpha}$. Equation (89) must be changed not only in regard to the orders of C_k, $\bar{C}_k$, but also in regard to the factors $\frac{1}{2}$ in (94), (96). This gives

$$\frac{C_{m+1} - \bar{C}_{m+1}}{2} = \frac{(-)^m}{(a_1 a_2 \cdots a_m)^2 \, a_{m+1} \bar{K}_0} \tag{97}$$

The most important change is in regard to the zeros and poles z_σ. The polynomials N and D now determine z_σ'' and z_σ' directly, instead of their squares. There is no opportunity to adjust the sign of Re z_σ'' by choosing the correct sign of $\sqrt{z_\sigma''^2}$. When non-physical singularities appear, adjustments of high order coefficients may be tried. Section 23 applies provided z^2 is replaced by z. If the specification of the problem permits added delay, linear phase may be added to $\bar{\alpha} + i\bar{\beta}$ to increase the probability of physical singularities‡. (Addition of linear phase changes only $\bar{C}_1$, in $\sum \bar{C}_k T_k$. A *negative* change in $\bar{C}_1$ *increases* the delay.)

25. APPROXIMATION TO AN ASSIGNED PHASE $\bar{\beta}$§

Sometimes it is required to approximate an assigned phase, without regard to gain. More commonly, it is required to approximate an assigned phase, using an "all-pass" network, which has a theoretically zero gain. These two problems, however, are very nearly identical, due to circumstances explained at the end of this section.

For approximation to phase only, we go back to the β equation in (21). As before, products of factors $(z - z_\sigma)$ are replaced by polynomial

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† and $\overset{m}{=}$ in place of $\overset{me}{=}$.

‡ The well known relation between the gain and phase of any physical network (See for instance Bode[10]) may give some information regarding the reasonableness of $\bar{\beta}$. It must be remembered, however, that departures of network gain α, from the assigned gain $\bar{\alpha}$, outside the useful interval, may affect the permissible phase β, within the interval.

§ Up to the present, applications to phase problems have not been developed to the same extent as for gain. Techniques have been explored, however, to determine how such applications may in fact be carried out.

equivalents. Then, in place of (81) or (94), we have

$$\left.\begin{aligned} i\beta &= \sum C_k T_k \\ \sum C_k z^k &= -\log \frac{N}{D} \end{aligned}\right\} k \text{ odd} \tag{98}$$

Using n to represent $n'' + n'$, the total number of network singularities, we may write N and D as follows, in place of (82) or (95):

$$\begin{aligned} N &= 1 + K_1 z + K_2 z^2 + K_3 z^3 \cdots + K_n z^n \\ D &= 1 - K_1 z + K_2 z^2 - K_3 z^3 \cdots (-)^n K_n z^n \end{aligned} \tag{99}$$

Notice that N and D are related by

$$D(z) = N(-z), \tag{100}$$

which is required by the form of the β equation in (21).

To arrive at *a* design procedure most easily (but not the simplest design procedure), one may express the *assigned* gain $\bar{\beta}$ in the following way (comparable to (58) and (96)):

$$\begin{aligned} i\bar{\beta} &= \sum \bar{C}_k T_k, \qquad k \text{ odd} \\ \sum \bar{C}_k z^k &= -\log \sum \bar{K}_k z^k \end{aligned} \tag{101}$$

Coefficients $\bar{K}_k$ may again be calculated by a modification of (61). This time $\bar{C}_{2k}$ is replaced by $\bar{C}_k$, wherever it appears in (61), and then all even ordered $\bar{C}_k$ are made zero (since only odd terms appears in $\sum \bar{C}_k z^k$ of (101)). Note that even ordered $\bar{K}_k$ are *not* usually zero, even though even ordered $\bar{C}_k$ are.

The degrees of N and D, in (99), are such that we can make $C_k = \bar{C}_k$ through terms of order $k = 2n$. This requires merely:

$$\frac{N}{D} \overset{2n}{=} \sum \bar{K}_k z^k \tag{102}$$

As stated, the condition applies to C_k of both even and odd orders. Since even ordered $\bar{C}_k$ are zero, it means that at least n even ordered C_k will be zero, in addition to the match between n odd orders. (102) is sufficient to determine an N and a D without reference to (100). If the (equal) degrees n of (99) are assumed, however, the N and D determined by (102) will be found to obey (100) automatically (provided $\sum \bar{K}_k z^k$ corresponds to an *odd* series $\sum \bar{C}_k z^k$, as here assumed).†

A simpler method for computing the same N and D takes advantage

† This was discovered by Mrs. M. D. Stoughton.

of the known relation (100), connecting N and D. Let E and O be respectively the sums of even and odd terms in N. Then N is $E + O$ and (100) requires D to be $E - O$. The ratio O/E may be related to $\sum C_k z^k$ of (98) as follows:

$$\frac{O}{E} = -\tanh \tfrac{1}{2} \sum C_k z^k \tag{103}$$

Now let two convergent series, respectively even and odd, be such that:

$$\frac{\bar{O}}{\bar{E}} = -\tanh \tfrac{1}{2} \sum \bar{C}_k z^k \tag{104}$$

Let coefficients $\bar{K}'_k$ be defined by:

$$\bar{E} + \bar{O} = \sum \bar{K}'_k z^k \tag{105}$$

Then $\bar{E}$ and $\bar{O}$ are respectively the sums of the even and odd terms. The complete series may now be related to the (odd) series $\sum \bar{C}_k z^k$ as follows:

$$\sum \tfrac{1}{2}\bar{C}_k z = -\log \sum \bar{K}'_k z^k \tag{106}$$

This fixes the $\bar{K}'_k$ of $\bar{E} + \bar{O}$ in terms of the $\bar{C}_k$.

To make $C_k = \bar{C}_k$ through m odd orders, (102) is now replaced by

$$\frac{O}{E} \overset{mo}{=} \frac{\bar{O}}{\bar{E}} \tag{107}$$

The symbol $\overset{mo}{=}$ designates equality of power series through m *odd* orders. (*All* even terms are now zero on both sides.) The right hand side may be expressed as a continued fraction of the following form, comparable to (86):

$$\frac{\bar{O}}{\bar{E}} = \cfrac{1}{\cfrac{a_1}{z} + \cfrac{1}{\cfrac{a_2}{z} + \cfrac{1}{\cfrac{a_3}{z} \cdots}}} \tag{108}$$

Truncation after only the m^{th} term gives the O/E of (107).

The coefficients of $\bar{O}$ and $\bar{E}$ may be calculated by an appropriate modification of (61). (Calculate like $\bar{K}_k$ of (101), after dividing all $\bar{C}_k$ by 2). After O and E have been evaluated, by truncating the continued fraction (108), their sum gives polynomial N of (99).

The natural modes and frequencies of infinite loss are determined from the zeros of the polynomial N. By (21), each zero is either a (z-plane) natural mode, z_σ'', or the negative of an infinite loss point $-z_\sigma'$. If gain variations are inconsequential, there is likely to be some latitude in designating each zero as a z_σ'', or as a $-z_\sigma'$.

A zero of N with a *positive* real part would make a non-physical natural mode, and hence it *must* be a $-z_\sigma'$, corresponding to an infinite loss point. A zero of N with a *negative* real part *can* be a natural mode z_σ'', but this may not be *required*. It may be either a z_σ'' or a $-z_\sigma'$, provided conjugate zeros are assigned in the same way, and provided the total number of $-z_\sigma'$ does not exceed the total number of z_σ''. The latter condition requires:

At least half the zeros of N must have negative real parts.

The continued fraction (108) shows how many zeros will have negative real parts, before any zeros have been calculated. The following theorem makes this easy:

The number of zeros of N which have negative real parts is equal to the number of positive coefficients in the truncation of the continued fraction (108) *which determined N.*

When gain is not to be disregarded, but is to be exactly zero, the synthesis technique needs few changes. The phase of an "all-pass" network is related to the natural modes z_σ'' as follows:

$$i\beta = \sum C_k T_k, \qquad k \text{ odd}$$

$$\sum \frac{C_k z^k}{2} = -\log \frac{\prod \left(1 - \frac{z}{z_\sigma''}\right)}{\prod \left(1 + \frac{z}{z_\sigma''}\right)} \tag{109}$$

This may be regarded as a special case of (20) for $\alpha = 0$ (which makes $C_k = O$ for k even, and also happens to require $z_\sigma' = -z_\sigma''$). In functional form however, it is more like $i\beta$ of (21). It differs in only two regards. In the power series in z, each C_k is divided by two. In the rational fraction in z, all the zeros correspond to natural modes, and the poles correspond to frequencies of infinite loss; but the poles are also exactly the negatives of the zeros, as in the $i\beta$ equations of (21).

Accordingly, the phase synthesis technique which ignores gain variations may be applied to the zero gain form of the problem by cutting every $\bar{C}_k$ in two. All zeros of $N = E + O$ must be construed as natural modes z''_σ. Finally, the network must have as many infinite loss points as natural modes, such that $z'_\sigma = -z''_\sigma$. (Integer n is now the number of natural modes, rather than the total number of singularities.)

For physical networks, all the first n terms of the continued fraction (108) must now be positive. To meet this condition it may be necessary to add linear phase to the assigned phase (by adding a *negative* correction to $\bar{C}_1$). It appears that sufficient linear phase will always lead to a physical design, provided the number of modes n is increased to retain a reasonable accuracy.

26. LINEAR PHASE

When the assigned phase $\bar{\beta}$ is linear, the calculations are relatively simple.

If a delay D is to be approximated over a frequency interval extending to $\omega = \omega_c$,

$$i\bar{\beta} = -D\omega_c T_1 \tag{110}$$

If delay D is to be realized without regard to gain variations, the appropriate $\bar{O}/\bar{E}$ is

$$\frac{\bar{O}}{\bar{E}} = \tanh \frac{D\omega_c z}{2} \tag{111}$$

A known continued fraction expansion of tanh X may be applied to (111), to obtain the coefficients of (108) without bothering with (105).†
The result may be arranged as follows:

$$\frac{\bar{O}}{\bar{E}} = \cfrac{1}{\cfrac{2}{D\omega_c z} + \cfrac{1}{\cfrac{3\cdot 2}{D\omega_c z} + \cfrac{1}{\cfrac{5\cdot 2}{D\omega_c z} \cdots}}} \tag{112}$$

Truncation of the continued fraction gives O/E, and then $O + E$. The zeros z_σ turn out to be proportional to $\frac{1}{D}$, and therefore root extraction techniques are required only for one D, for each n. The zeros are tabulated for sample n's, in Table II.

† For the expansion of tanh X, reference may be made to a text on continued fractions by Wall[11], page 349, equation 91.6.

TABLE II—*Z-Plane Natural Modes for Linear Phase*

n	$D\omega_c z_\sigma$
1	-2
2	$-3 \pm i\sqrt{3}$
3	-4.64438 $-3.67782 \pm i\ 3.50876$
4	$-5.79242 \pm i\ 1.73446$ $-4.20758 \pm i\ 5.31484$
5	-7.29348 $-6.70392 \pm i\ 3.48532$ $-4.64934 \pm i\ 7.14204$
6	$-8.49668 \pm i\ 1.73510$ $-7.47142 \pm i\ 5.25256$ $-5.03190 \pm i\ 8.98532$

The error in the first mismatched Tchebycheff coefficient is a rough measure of accuracy. It may be shown to be

$$C_{2n+1} - \bar{C}_{2n+1} = \frac{(-)^n (D\omega_c)^{2n+1}}{4^n[1\cdot 3\cdot 5 \cdots (2n-1)]^2(2n+1)} \tag{113}$$

This measure of accuracy is plotted in Fig. 11, for various numbers of natural modes n. A sample detailed curve of $\beta - \bar{\beta}$ is shown in Fig. 12, with dotted lines corresponding to the estimated error (113).

If delay D is such that the error is reasonable, all the zeros may be natural modes. If these are combined with a like number of infinite loss points, such that $z_\sigma' = -z_\sigma''$, an all-pass network will be obtained, instead of one which approximates D without regard to gain. The all pass network will produce twice the delay, and twice the nonlinearity of phase. In other words, for an all pass network, both coördinates in Fig. 11 must be doubled.

27. SIMPLIFICATION OF SINGULARITY ARRAYS

In complex communication systems, a single equalizer may be required to correct for a number of effects. In a coaxial cable system, for instance, a single network in the standard repeater may be required to compensate for the following: Cable attenuation, characteristics of input and output networks, effects of interstages (significant because the feedback is limited), and distortion due to variable controls at mean settings. Tchebycheff polynomial methods may not be efficient when applied

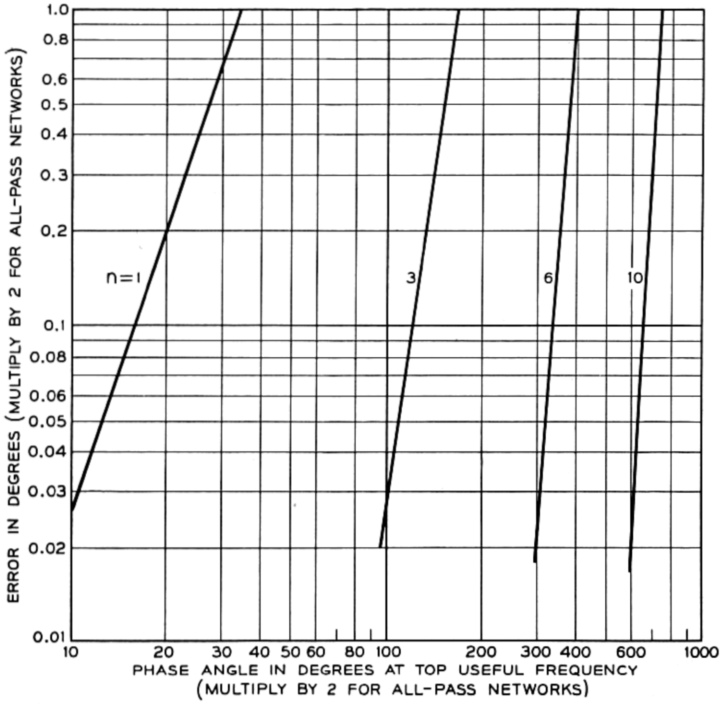

Fig. 11—Estimated error for n natural modes approximating linear phase.

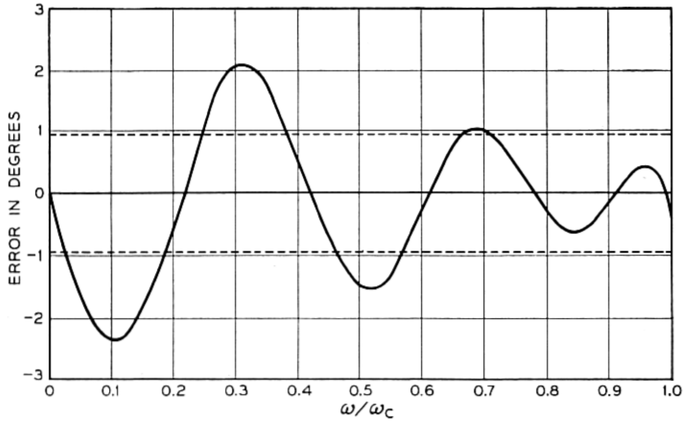

Fig. 12—Error when six natural modes approximate a linear phase, with a slope giving 402° at top useful frequency.

directly to all these effects. They may still be useful, however, when applied in the following way:

Separate arrays of singularities are determined, which match the separate effects to required accuracy, using any convenient methods. Minimum network complexity is not required at this point. Combining all the singularities in a single array gives an initial design which is sufficiently accurate, but may use many more singularities than are actually necessary. Tchebycheff polynomial methods are now used to obtain a simpler set of singularities, which approximates the initial set to sufficient accuracy. This has been designated "boiling down" the original set.

In a problem of this sort the assigned characteristic has the network form, as well as the network characteristic which is to approximate it. (The example discussed in Sections 10 to 14 is also a problem of this sort.) As a result, equations (20) and (21) apply to $\bar{\alpha}$ and $\bar{\beta}$, as well as to α and β. This makes it possible to replace $\sum \bar{K}_k z^{2k}$ and $\sum \bar{K}_k z^k$, of (55), (56), (96) etc., by a finite rational fraction $\bar{N}/\bar{D}$. If both $\bar{\alpha}$ and $\bar{\beta}$ are to be approximated, the following is derived from (20).

$$\sum \tfrac{1}{2}\bar{C}_k z^k = \log \bar{K}_z \frac{\prod\left(1 - \dfrac{z}{\bar{z}'_\sigma}\right)}{\prod\left(1 - \dfrac{z}{\bar{z}''_\sigma}\right)} = -\log \frac{\bar{N}}{\bar{D}} \tag{114}$$

The singularities $\bar{z}''_\sigma$, $\bar{z}'_\sigma$ correspond, of course, to the network singularities which are to be boiled down. If only $\bar{\alpha}$, or only $\bar{\beta}$, is to be approximated, suitable modifications are readily derived from (21).

The boiling down is accomplished by requiring

$$\frac{N}{D} \overset{m}{=} \frac{\bar{N}}{\bar{D}} \tag{115}$$

where N/D is of lower total degree than $\bar{N}/\bar{D}$. If $m = n'' + n'$, and $n'' - n' = 0$ or 1, the continued fraction method can again be used. This requires expansion of $\bar{N}/\bar{D}$ in continued fraction form, instead of $\sum \bar{K}_k z^k$.

An example of a boiled down set of singularities is illustrated in Fig. 13.

28. GENERALIZATION OF THE USEFUL INTERVAL

All the previous analysis applies to a useful frequency interval $-\omega_c < \omega < +\omega_c$. Its important characteristics are as follows: It is a single continuous interval, with $\omega = 0$ at its center. Useful intervals with other

other characteristics may be obtained, within limits, by changing the definitions of z and z_σ, in terms of p and p_σ (equations (5) and (8)).

In all cases, the definition of Tchebycheff polynomial T_k remains the same in terms of z. The interval of orthogonality remains $|z| = 1$; and the relation between p and z is always such that the useful frequency interval is the p-plane mapping of $|z| = 1$. The relation must also be such that rational functions of p may be expressed as products of rational functions, in z and $1/z$ respectively, corresponding to (13), (14). At the same time, the physical restrictions on network singularities p_σ must translate into manageable restrictions on the z-plane singularities z_σ. It is restrictions such as these that limit the manageable useful intervals.

It is easy to apply the "low-pass" techniques to "high-pass" intervals, extending from ω_c, through ∞, to $-\omega_c$. The appropriate trans-

SINGULARITIES

	ORIGINAL	REDUCED
ω_c/p'_σ	−2.2613	−1.0363
	−0.8493	−0.2798
	−0.3897	
	−0.2418	
	−0.02913 $\pm i$ 0.11384	
	−0.03856 $\pm i$ 0.06881	
ω_c/p''_σ	−2.2221	−0.9239
	−0.7549	−0.2265
	−0.3819	
	−0.2039	
	−0.02926 $\pm i$ 0.11370	
	−0.03817 $\pm i$ 0.06859	

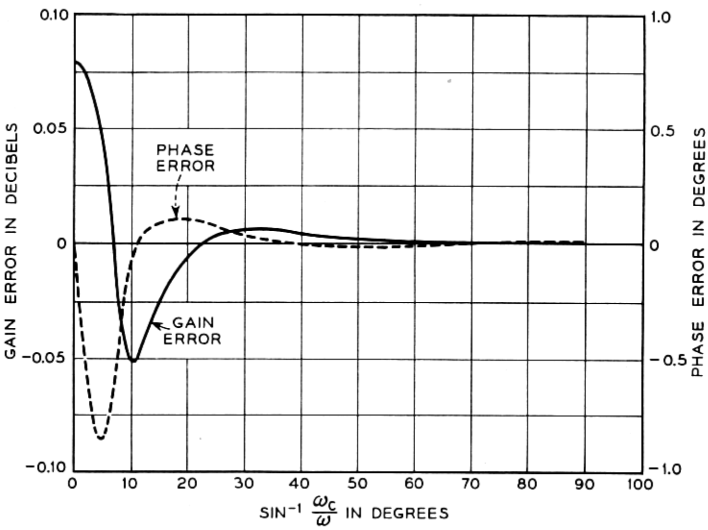

Fig. 13—An example of reduction in complexity.

formation, replacing (5), is:

$$p = \frac{2\omega_c}{z - \dfrac{1}{z}} \tag{116}$$

With this transformation, the whole of the previous z-plane analysis may be applied at once to high-pass useful intervals, except where linear phase is involved.

The situation is more complicated in regard to "band-pass" intervals. If the useful interval includes the frequencies between ω_{c1} and ω_{c2}, the *complete* useful interval (p-plane mapping of $|z| = 1$) must include also the "image" frequencies, between $-\omega_{c1}$ and $-\omega_{c2}$. Otherwise, conjugate complex z-plane singularities z_σ will not lead to conjugate network singularities p_σ. When there are two disjoint parts of the useful interval, the appropriate relation between p and z is relatively complicated. Up to the present, no corresponding technique has been discovered for approximating assigned phases over band-pass intervals, in Tchebycheff polynomial terms. Gain approximations can be handled, however, and for a quite simple reason. Gain functions are even functions, and behave in the p^2 plane much as gain-and-phase functions behave in the p plane. In the p^2 plane, $-\omega$ and $+\omega$ are identical, and a band-pass useful interval is a single segment of the ω^2 axis.

For gain approximations over a band-pass interval, (5) may be replaced by:

$$p^2 = \frac{a + b\left(z - \dfrac{1}{z}\right)^2}{1 + c\left(z - \dfrac{1}{z}\right)^2} \tag{117}$$

The three coefficients, a, b, c are subject to two conditions, stemming from the requirement that the interval $|z| = 1$ must map onto the interval $\omega_{c1}^2 < \omega^2 < \omega_{c2}^2$. This leaves one arbitrary degree of freedom. Its choice may be related to ordinary least squares approximations in the following way:

If $\alpha = \sum C_{2k}T_{2k}$, the first n terms approximate α in the least squares sense. In other words, the integrated square of the error is a minimum, relative to all possible choices of the first n coefficients C_{2k}, provided the integration extends over the useful frequency interval, and includes an appropriate "weight function". When (117) relates z to p, the arbitrary degree of freedom in the choice of the constants a, b, c permits selection of any one of a *family* of weight functions. Conventional least squares analysis may be applied to determine these functions.†

In applying least squares analysis, it must be borne in mind that the network gain α does not approximate the assigned gain $\bar{\alpha}$ in the simple least squares sense. When $C_{2k} = \bar{C}_{2k}$ for $k \leqq n$, $\alpha - \bar{\alpha}$ depends upon *two* least squares approximations. The first n terms of $\sum C_{2k}T_{2k}$ represent a least squares approximation to α, and are made identical with the first n terms of $\sum \bar{C}_{2k}T_{2k}$, which represent a least squares approximation to $\bar{\alpha}$.

When (117) relates p to z, z-plane singularities z_σ may be defined by:

$$p_\sigma^2 = \frac{a + b\left(z_\sigma - \dfrac{1}{z_\sigma}\right)^2}{1 + c\left(z_\sigma - \dfrac{1}{z_\sigma}\right)^2} \tag{118}$$

$$| z_\sigma | > 1,$$

$$\text{Re } z_\sigma \text{ to have same sign as Re } p_\sigma$$

An additional singularity, z_0^2, is also needed, corresponding to the finite poles of (117). It may be defined as follows:

$$1 + c\left(z_0 - \frac{1}{z_0}\right)^2 = 0 \qquad | z_0 | > 1 \tag{119}$$

When $p_\sigma^2 - p^2$, in α of (2), is expressed in terms of z and z_σ, (117) introduces denominator factors $(1 - z^2/z_0^2)$ and $(1 - 1/z_0^2z^2)$. As a result, α of (21) must be changed to the following, for band-pass intervals:

$$\alpha = \sum C_{2k}T_{2k}$$

$$\sum C_{2k}z^{2k} = \log K_z^2 \frac{\prod\left(1 - \dfrac{z^2}{z_\sigma'^2}\right)}{\prod\left(1 - \dfrac{z^2}{z_\sigma''^2}\right)}\left(1 - \frac{z^2}{z_0^2}\right)^{n''-n'} \tag{120}$$

When definite values have been chosen for a, b, c of (117) (in order that the $\bar{C}_k$ may be calculated), $(1 - z^2/z_0^2)$ in (120) is not subject to arbitrary adjustment. This situation can be handled by defining N/D as the rational fraction in the α equations of (21), as before, but re-

† For general discussions of orthogonal functions and least squares approximations, see Courant and Hilbert[5], and also a short text by Jackson.[12]

placing (84) by

$$\frac{N}{D} \overset{me}{=} \left(1 - \frac{z^2}{z_0^2}\right)^{n''-n'} \sum \bar{K}_k z^{2k} \tag{121}$$

Fig. 14 illustrates an application of the technique to the simulation of a coaxial cable attenuation (which is nearly proportional to $\sqrt{\omega}$).

29. RECAPITULATION

Tchebycheff polynomial series may be applied advantageously to a very wide range of network synthesis applications. The scope of their usefulness may depend upon the skill of the designer, as with any synthesis tools, but the underlying principles are reasonably simple. The most important principles are perhaps the following:

A Tchebycheff polynomial series in frequency may be related to a power series in a new variable z. When the Tchebycheff polynomial series corresponds to a finite network gain or phase, the power series corresponds to an analytic function of z, quite similar in form to the network function of p, with singularities at z-plane mappings of the

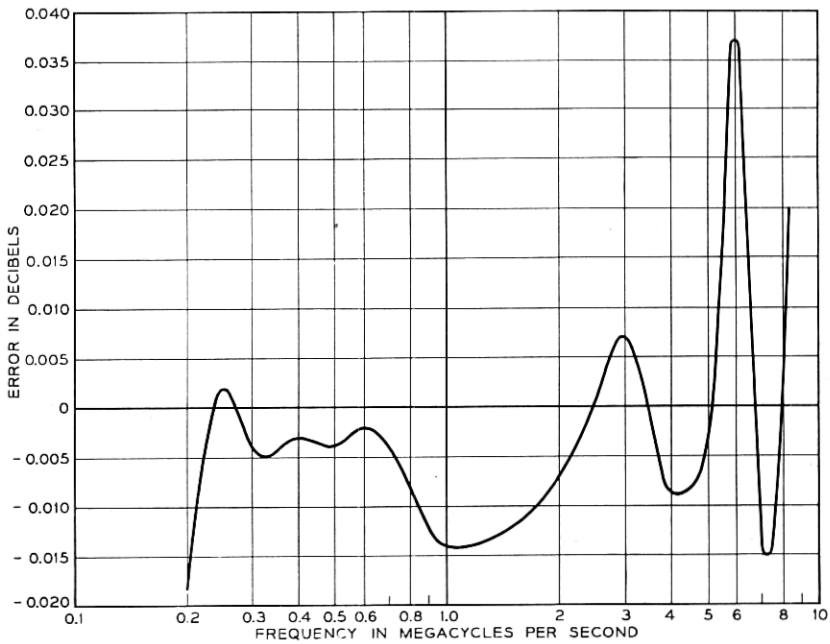

Fig. 14—Simulation of a coaxial cable attenuation—Attenuation at top useful frequency = 46 db; Network = four constant-resistance sections.

network singularities. This makes it possible to apply power series approximation methods, in terms of z, to obtain approximations based on Tchebycheff polynomial series, in terms of frequency.

"Maximally flat" approximations in terms of z may be used to match the first m terms in the Tchebycheff polynomial series representing network gain or phase to the corresponding terms in the series representing assigned gain or phase. In this way, a Tchebycheff polynomial type of least squares approximation to the network function is made identical to the corresponding least squares approximation to the ideal function. The overall error, network function minus ideal function, is then the difference between the two least squares errors.

The z-plane analysis may also be manipulated, in a quite different way, to approach an equal ripple type of approximation (which usually represents approximation in the Tchebycheff sense). The complications are such that applications have been limited to problems of certain quite special types. On the other hand, analysis of this sort has been found useful in clarifying various other ways of seeking equal ripple approximations.

REFERENCES

1. S. Darlington, "The Potential Analogue Method of Network Synthesis," Bell System Tech. J., **30**, pp. 315–365, Apr. 1951.
2. G. L. Matthaei, "A General Method for Synthesis of Filter Transfer Functions as Applied to L-C and R-C Filter Examples," *Stanford University Electronics Laboratory Technical Report No. 39*, Aug. 31, 1951 (for Office of Naval Research, NR-078-360).
3. T. R. Bashkow, "A Contribution to Network Synthesis by Potential Analogy," *Stanford University Electronics Laboratory Technical Report No. 25*, June 30, 1950 (for Office of Naval Research, NR-078-360).
4. E. S. Kuh, "A Study of the Network-Synthesis Approximation Problem for Arbitrary Loss Functions," *Stanford University Electronics Laboratory Technical Report No. 44*, Feb. 14, 1952 (for Office of Naval Research, NR-078-360).
5. R. Courant and D. Hilbert, *Methoden der Mathematischen Physik*, Vol. I, Chap. 2, Julius Springer, Berlin, 1931.
6. C. Lanczos, "Trigonometric Interpolation of Empirical and Analytic Functions," *J. of Math. and Phys.*, **17**, pp. 123–199, 1938–39.
7. H. A. Wheeler, "Potential Analog for Frequency Selectors with Oscillating Peaks," *Wheeler Monograph No. 15*, Wheeler Laboratories, Great Neck, N. Y., 1951.
8. S. Darlington, "Synthesis of Reactance 4-Poles," *J. of Math. and Phys.*, **18**, pp. 257–353, Sept., 1939.
9. T. C. Fry, "Use of Continued Fractions in Design of Electrical Networks," *American Math. Soc. Bulletin*, **35**, pp. 463–498, July-Aug., 1929.
10. H. W. Bode, *Network Analysis and Feedback Amplifier Design*, D. Van Nostrand Co., New York, 1945.
11. H. S. Wall, *Analytic Theory of Continued Fractions*, D. Van Nostrand Co., New York, 1948.
12. Dunham Jackson, *Fourier Series and Orthogonal Polynomials*, The Mathematical Association of America, Oberlin, Ohio, 1941.