# Principle Strains in Cable Sheaths and Other Buckled Surfaces

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(Manuscript Received February 25, 1952)

*Equations are developed for rigorous determination of magnitudes and directions of principal strains in plastic deformation, by means of measurements of rectangular strain rosettes. Application to the study of telephone cable sheath is described.*

In the course of certain studies of the polyethylene used in the sheath of telephone cable, it was necessary to calculate the magnitudes and directions of the principal strains from data obtained by measurements of the distortion of a square grid which had previously been printed on the surface of the cable. The strains were large, rendering useless the usual expressions for analysis of strain rosette data[1]. Such large strains are characteristically sustained for a wide variety of high polymeric materials of increasing importance for wire and cable sheathing as well as other structural uses. In this article the requisite formulas are developed.

The basic assumptions are:

(1) The strains may be large.

(2) The strains are uniform over any square of the grid (equivalent to the condition that a square transforms into a parallelogram).

(3) The square may be regarded as plane.

(4) Two of the principal strains are parallel to the surface.

We shall first consider only the two principal strains in the plane of the surface of the cable. Suppose these two strains to be parallel with the $x$ and $y$ coordinate axes, respectively, and that one side of the square is aligned, before straining, at the angle $\phi$ with the $x$ axis. This is illustrated in Fig. 1.

Let $e_x$ = maximum principal strain

$e_y$ = minimum principal strain

$$\lambda_x = 1 + e_x$$
$$\lambda_y = 1 + e_y$$

[1] Cf. for example, Max Frocht, "Photoelasticity," **1**, p. 37, 1941.

If primes are used to refer to the strained state,

$$\lambda_x = \frac{x_b' - x_a'}{x_b - x_a} = \frac{x_d' - x_c'}{x_d - x_c}$$

$$\lambda_y = \frac{y_b' - y_a'}{y_b - y_a} = \frac{y_d' - y_c'}{y_d - y_c}$$

If $L_1$ and $L_2$ are the lengths of the sides of the unstrained square, and $L_3$ and $L_4$ the diagonals,

$$(L_1 + \Delta L_1)^2 = \lambda_x^2(x_b - x_a)^2 + \lambda_y^2(y_b - y_a)^2 \tag{1a}$$

$$(L_2 + \Delta L_2)^2 = \lambda_x^2(x_d - x_c)^2 + \lambda_y^2(y_d - y_c)^2 \tag{1b}$$

$$(x_b - x_a)^2 = (y_d - y_c)^2 = L_1^2 \cos^2\phi_1 = L_2^2 \cos^2\phi_1 \tag{2a}$$

$$(y_b - y_a)^2 = (x_d - x_c)^2 = L_1^2 \sin^2\phi_1 = L_2^2 \sin^2\phi_1 \tag{2b}$$

whence, if

$$\frac{L_1 + \Delta L_1}{L_1} = L_1', \ \frac{L_2 + \Delta L_2}{L_2} = L_2', \text{ etc.} \tag{2c}$$

$$L_1'^2 = \lambda_x^2 \cos^2 \phi_1 + \lambda_y^2 \sin^2 \phi_1 \tag{3a}$$

$$L_2'^2 = \lambda_x^2 \sin^2 \phi_1 + \lambda_y^2 \cos^2 \phi_1 \tag{3b}$$

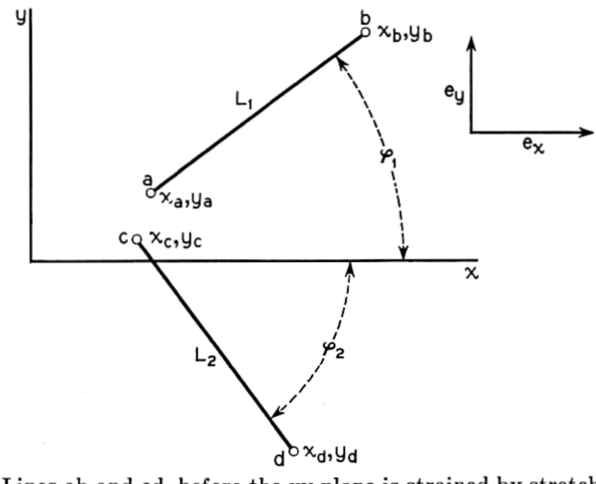


Fig. 1—Lines ab and cd, before the xy plane is strained by stretching (or compressing) in the x and y directions.

Henceforth, for clarity, suppose the subscript "1" to refer to the longer side of the parallelogram, "2" to the shorter side, "3" to the longer diagonal, and "4" to the shorter.

$$S_1 = \text{original slope of } L_1 = \tan \phi_1 = \frac{y_b - y_a}{x_b - x_a}$$

$$S_2 = \text{original slope of } L_2 = \tan \phi_2 = \frac{y_d - y_c}{x_d - x_c}$$

$$S_1' = \tan \phi_1' = \frac{\lambda_y}{\lambda_x} S_1 \tag{4a}$$

$$S_2' = \tan \phi_2' = \frac{\lambda_y}{\lambda_x} S_2 \tag{4b}$$

Since $\phi_1 - \phi_2 = 90°$,

$$S_2 = -\frac{1}{S_1} \tag{5a}$$

$$S_2' = -\lambda_y/\lambda_x S_1 \tag{5a}$$

By expansion and substitution from Equations (4) and (5),

$$\tan (\phi_1' - \phi_2') = \frac{\frac{\lambda_y}{\lambda_x}\left(S_1 + \frac{1}{S_1}\right)}{1 - \left(\frac{\lambda_y}{\lambda_x}\right)^2} \tag{6}$$

Let $$\psi = 90° - (\phi_1' - \phi_2')$$

then

$$\tan (90° - (\phi_1' - \phi_2')) = \tan \psi = \frac{1 - \left(\frac{\lambda_y}{\lambda_x}\right)^2}{\frac{\lambda_y}{\lambda_x}\left(S_1 + \frac{1}{S_1}\right)} \tag{7}$$

which is the shear between $L_1'$ and $L_2'$.

$$(S_1 + 1/S_1) = \tan \phi_1 + \cot \phi_1 = \frac{2}{\sin 2\phi_1} \tag{8}$$

and substituting this in equation (7),

$$\sin 2\phi_1 = \frac{2\lambda_x\lambda_y \tan \psi}{(\lambda_x^2 - \lambda_y^2)} \tag{9}$$

whence

$$\cos 2\phi_1 = \sqrt{1 - \frac{4\lambda_x^2\lambda_y^2 \tan^2\psi}{(\lambda_x^2 - \lambda_y^2)^2}} \tag{10}$$

Remembering that

$$\cos^2\phi_1 = \frac{1 + \cos 2\phi_1}{2} \tag{11a}$$

and

$$\sin^2 \phi_1 = \frac{1 - \cos 2\phi_1}{2} \tag{11b}$$

and substituting Equation (10) in Equation (11), Equation (11) in Equation (3), and then solving the quadratic equation thus formed for $\lambda_x$ and $\lambda_y$, we have

$$\lambda_x^2, \lambda_y^2 = \frac{(L_1'^2 + L_2'^2) \pm \sqrt{(L_1'^2 + L_2'^2)^2 - 4L_1'^2 L_2'^2 \cos^2 \psi}}{2} \tag{12}$$

Referring to Fig. 2, and using the law of cosines, and remembering that $L_3'$ is the ratio of the strained to the unstrained length of the diagonal,

$$-\cos\theta = \sin\psi = \frac{2L_3'^2 - (L_1'^2 + L_2'^2)}{2L_1'L_2'} \tag{13a}$$

whence

$$\cos^2\psi = \frac{4L_3'^2 (L_1'^2 + L_2'^2 - L_3'^2) - (L_1'^2 - L_2'^2)^2}{4L_1'^2L_2'^2} \tag{13b}$$

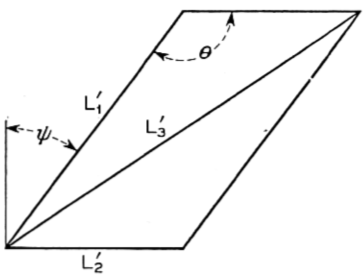


Fig. 2—A parallelogram formed by straining a square. $L_1'$, $L_2'$ and $L_3'$ are the ratios of the lengths of the indicated lines to their original lengths.

This expression, substituted in Equation (12) and reduced, gives

$$\lambda_x^2, \lambda_y^2 = \frac{(L_1'^2 + L_2'^2) \pm \sqrt{2(L_1'^2 - L_3'^2)^2 + 2(L_2'^2 - L_3'^2)^2}}{2} \tag{14}$$

It may be noted here that a property of the parallelogram, namely, in the notation used here,

$$L_1'^2 + L_2'^2 = L_3'^2 + L_4'^2 \tag{15}$$

makes it immaterial which diagonal is used. This may be readily seen by substituting

$$L_3'^2 = L_1'^2 + L_2'^2 - L_4'^2$$

in Equation (14). The effect is merely that of substituting $L_4$ for $L_3$. In Equation (13a), however, the result is a change in the sign of $\psi$.

As an example of the application of these equations, the measurements of one specimen were:

$$L_1' = 2.1$$
$$L_2' = 1.2$$
$$L_3' = 2.0$$

From Equation (14),

$$\lambda_x^2 = 4.758, \qquad \lambda_x = 2.181, \qquad e_x = 1.181$$
$$\lambda_y^2 = 1.092 \qquad \lambda_y = 1.045 \qquad e_y = 0.045$$

From Equation (13a),

$$\sin\psi = 0.4266, \text{ whence}$$
$$\psi = 25.3°$$
$$\tan\psi = 0.472$$

From Equation (9),

$$\sin 2\phi_1 = 0.587$$
$$\phi_1 = 18.0°$$
$$\tan\phi_1 = 0.324$$

From Equation (4a),

$$\tan\phi_1' = 0.1554$$
$$\phi_1' = 8.83°$$

FromEquation (9), it is obvious that the maximum value of $\tan\psi$ occurs at $\phi_1 = 45°$, and is in this case equal to 0.804.

This example is illustrated in Fig. 3.

The question of direction of the $x$ and $y$ axes is simply settled by drawing a line through either of the acute angles of the parallelogram, crossing the parallelogram at an angle $\phi_1'$ with the longer side. This line will be parallel to the $x$ direction, which is, according to the convention, that of greatest strain.

So far no mention has been made of strain in the third dimension; that is, a change in thickness of the sheath. In plastic deformation, the volume change is generally negligible. This requires that

$$\lambda_x \lambda_y \lambda_z = 1$$

whence

$$\lambda_z = \frac{1}{\lambda_x \lambda_y}$$

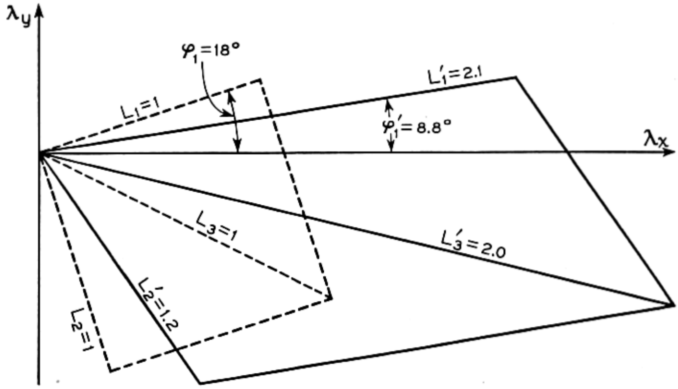


Fig. 3—A square and the parallelogram resulting from stretching to length ratios $\lambda_x = 2.181$ in the $x$-direction and $\lambda_y = 1.045$ in the $y$-direction.

TABLE I

| Degrees of twist in 3 feet | Principal Strains, per cent | | |
|---|---|---|---|
| | Parallel to Surface | | Perpendicular to Surface |
| | Max. | Min. | |
| 180 | 16 | 06 | −19 |
| 270 | 26 | 09 | −27 |
| 360 | 33 | 14 | −34 |
| 450 | 36 | 20 | −39 |
| 540 | 42 | 19 | −41 |
| 630 | 43 | 22 | −43 |
| 720 | 46 | 24 | −45 |

In the example given,

$$\lambda_z = 0.439, \qquad e_z = -0.561$$

Polyethylene sheaths of cable specimens 3 feet long buckled severely over their entire length when the cables were twisted 720° and showed the strains given in Table I at steps up to the final twist[2].

The ratio of maximum to minimum strain parallel to the surface is about 2:1. Tests with a 1:1 ratio[3], a more severe condition, have shown that the principal strains at rupture will be of the order of 300 per cent. Therefore it is evident that the strains incidental to the most severe types of handling will not, of themselves, cause rupture of the sheaths.

[2] Unpublished memorandum by A. G. Hall.

[3] I. L. Hopkins, W. O. Baker and J. B. Howard, *J. Appl. Phys.*, **21**, No. 3, pp. 206–213, March, 1950.