THE BELL SYSTEM TECHNICAL JOURNAL

VOLUME XXXI MARCH 1952 NUMBER 2

Introduction to Formal Realizability Theory—I

By BROCKWAY McMILLAN

(Manuscript received October 15, 1951)

This paper offers a general approach to the realizability theory of networks with many accessible terminals. The methods developed are applied to give a complete characterization of all finite passive networks.

I. SUMMARY

1.0 A principal result of this paper is to characterize those matrices $Z(p)$, functions of the frequency parameter p, which can be realized as open-circuit impedance matrices of finite passive networks. This characterization is provided by the following theorem:

1.1 *Theorem*:* Let $Z(p)$ be an $n \times n$ matrix whose elements are $Z_{rs}(p)$, $1 \leq r, s \leq n$, where

(i) Each $Z_{rs}(p)$ is a rational function

(ii) $\overline{Z_{rs}(p)} = Z_{rs}(\bar{p})$ (the bar denotes complex conjugate)

(iii) $Z_{rs}(p) = Z_{sr}(p)$

(iv) For each set of real constants $k_1, \cdots, k_n$, the function

$$\varphi_Z(p) = \sum_{r,s=1}^{n} Z_{rs}(p)k_rk_s$$

has a non-negative real part whenever $\text{Re}(p) > 0$.

Then there exists a finite passive network, a $2n$-pole, which has the impedance matrix $Z(p)$.

* Presented to the American Mathematical Society, April 17, 1948. Abstract 260, *Bulletin of the A.M.S. No. 54*, July, 1948.

Conversely, if a finite passive $2n$-pole has an impedance matrix $Z(p)$, that matrix has the properties (i), (ii), (iii), (iv).

A formally identical dual theorem holds for open-circuit admittance matrices $Y(p)$.

1.2 A general realizability theorem, applicable to and characterizing completely all finite passive networks, whether having impedance matrices or not, is also proved.

1.3 An effort is made to lay a foundation adequate for the realizability theory of both active and passive multi-terminal devices. To this end, a large part of the paper is devoted to the scrutiny of fundamental properties of networks.

II. INTRODUCTION AND FOREWORD

2.0 Network theory provides direct means for associating with an electrical network a mathematical description which characterizes the behavior of that network. Typically, this results in shifting engineering attention from a detailed, possibly quite intricate, electrical structure to a mathematical entity which succinctly describes the relevant behavior of that structure. An essential feature of this shift in focus is emphasized by the word "relevant": only those terminals of the network which are directly relevant to the problem at hand are considered in the mathematical description. Design work can then be done in terms of constructs relating explicitly to these accessible terminals, the effect of the internal structure being felt only by implication.

The physical origins of these mathematical constructs, and the implications of the internal structure upon them, cannot however be entirely forgotten, for they have mathematical consequences which are not always immediately evident. Until he knows these limitations—imposed upon him by the physical nature or the necessary structural form of the networks he is designing—a design engineer cannot make free use of the mathematical tools that network theory has provided.

We give the name "realizability theory" to that part of network theory which aims at the isolation and understanding of those broad limitations upon network performance, i.e., upon the mathematical constructs which describe that performance—which are imposed by limitations on the network structure. One may also include in the province of realizability theory some of the converse questions: the study of those structural features common to all networks whose performance is limited in some specified way.

Realizability theory would have little content were it not that "performance" here must be construed to mean *performance as viewed from the accessible terminals only*. Were all branch currents and node potentials in a network available to observation, a mathematical statement of performance would be equivalent to stating the full system of differential equations governing these quantities, i.e., equivalent to giving the detailed network diagram.

2.1 With a few important exceptions, the converse kind of problem in realizability theory does not lead to a strict implication from functional limitations to structural features, because the field of equivalent structures for a specified performance is very broad. Typically, it is only by imposing some general *a priori* limitations on structure that further conclusions can be firmly drawn from a functional limitation. In studying this kind of problem one is rapidly led from those basic issues which are clearly part of realizability theory toward general, difficult, and usually unsolved problems of network synthesis. One cannot, and should not, draw a sharp boundary here, but Nature so far has provided a fairly definite one for us, in that most of these problems have proved too difficult of solution.

2.2 The direct realizability problems, the passage from structural properties to functional properties, have been somewhat more tractable. Here, again, there is no clear dividing line between general realizability theory and the sort of design theory in which, for example, one specifies a particular filter structure depending on a limited number of parameters and examines the performance of the structure as a function of these parameters. There is an extensive literature at or near this latter level of generality, most of it relating to filters or filter-like structures (e.g., interstage couplers in amplifiers).

At a more basic level, the limitations on a network's structure which are commonly met in practice are of the following kinds:

a. Limitations on the kind of elements appearing, e.g., to passive networks, networks without coupled coils, networks whose elements have specified parasitics, etc;

b. Limitations on the general form of the network diagram, e.g., to ladder or lattice structures, without limitation to a specified number of elements or parameters.

Here the problems are varied and difficult. We survey briefly the present status of some of them.

2.3 Networks with two accessible terminals, two-poles, are basic in network technology. Fortunately, also, two-poles are unique among networks in that there is always a simple way to describe their performance. Except for the trivial limiting case of an open circuit, every two-pole has a well-defined impedance, $Z(p)$, a function of the complex frequency parameter p, which describes its performance in a way which is by now well understood. Dually, except for the limiting case of a short circuit, every two-pole has a well-defined admittance function $Y(p)$. Even the limiting cases are tractable: every open circuit has the admittance function $Y(p) \equiv 0$ and every short circuit the impedance function $Z(p) \equiv 0$.

In other words, by exercising his option to speak in terms either of impedance or of admittance, one can always describe the performance of a two-pole by using a single function of frequency.

The descriptive simplicity and practical importance of two-poles led early to a fairly complete realizability theory for them. In 1924 R. M. Foster[7] gave a function-theoretic characterization of the impedance functions of finite passive two-poles containing only reactances. The corresponding problem for two-poles which are not at all limited as to structure, beyond being finite and passive, was solved by O. Brune[2] in 1931. The effects of various structural limitations have since been studied by several writers (cf. Darlington,[6] Bott and Duffin[13]).

2.4 Technology, and the promptings of conscience, have meanwhile urged the study of devices with more than two accessible terminals. Here, however, Nature has been less kind, in that no uniquely simple method is available for describing the performance of such devices as viewed from their terminals.

Indeed, basic network theory has been remiss here, in not even making available a mode of description which is generally applicable—whether simple or not.

W. Cauer[5] showed that, when one admits ideal transformers among his network components, it is sufficient to study networks which are natural and direct generalizations of two-poles, namely, $2n$-poles,* for arbitrary values of n. The corresponding natural generalization of the impedance function $Z(p)$ of a two-pole is the impedance *matrix* of a $2n$-pole: just as one multiplies a scalar current by a scalar impedance to get a scalar voltage, one multiplies a vector current by an impedance matrix to get a vector voltage.

2.41 Not all descriptive difficulties are resolved, however, by considering $2n$-poles and their impedance or admittance matrices. For the moment, a simple example will suffice to show this: the 2×2-pole which consists simply of one pair of short-circuited terminals and one pair of

* Defined in Cauer,[5] and also later here.

open-circuited terminals is a finite passive $2n$-pole ($n = 2$) which has neither an impedance matrix nor an admittance matrix.

2.42 When one eliminates this kind of descriptive difficulty by fixing his attention only upon $2n$-poles for which an impedance matrix (or, dually, an admittance matrix) is available, the general realizability problem for finite passive devices is solved. A partial solution, for the case $n = 2$, was published by C. M. Gewertz[8] in 1933. The solution (Theorem 1.1) of the problem for a general value of n has been announced recently by three authors, independently: Y. Oono,[10] in 1946,* the present author, in 1948,† and M. Bayard,[1] in 1949. The problem for reactive $2n$-poles is much simpler and was solved by Cauer,[3] in 1931.

2.5 Intermediate between the fairly specific problems of filter theory on the one hand and the general realizability theory of multi-terminal devices on the other, lies the study of four-poles as transducers. There is a considerable literature on the realization of transfer functions or transfer impedances under various structural limitations. The basic and extensive work of Bode[14] on active systems belongs also in this category since it is avowedly limited to single-loop structures.

2.6 Beyond the important result that, by sufficiently elaborate circumventions, one may avoid the use of transformers in the synthesis of any two-pole, (Bott and Duffin[13]) little in general is known about networks which do not have transformers.

2.7 The present paper is an essay in the realizability theory of devices with many accessible terminals. Ideal transformers are admitted as network elements; indeed, their use is essential. This fact is indicated by the adjective "formal" appearing in the title.

The availability of ideal transformers makes it possible to exploit the simplification noted by Cauer and to consider only networks which are $2n$-poles in his sense. The aim of the paper, therefore, is to set a foundation for realizability theory which is completely general within this framework.

2.71 The first problem is that of description. We observed above an example—entirely trivial—of a passive four-pole which had neither an impedance nor an admittance matrix. Unfortunately, opportunities

* Date of Japanese publication. This reference, and manuscript of Oono[10, 11], were sent by Professor Oono in a personal communication to R. L. Dietzold, who showed them to me in December, 1948, while an early draft of the present paper was in preparation. The recent (1950) American republication of Oono[10] unfortunately omits reference to the original.

† Cf. footnote to 1.1.

for this kind of degeneracy become manifold in multi-terminal devices, and some degree of degeneracy is the rule rather than the exception. Consider an entirely practical example: that of an amplifier chassis from which the tubes have been removed.* Here the degeneracy is essential and intrinsic; it would be highly artificial to regard it as the mere accident of a limiting case. True, given any *particular* degenerate network, there is usually evident a method for representing or describing its behavior. What is needed, however, is a mode of representation which is applicable generally to any $2n$-pole without *a priori* knowledge of its structure or peculiar degeneracies.

2.72 The mode of representation adopted in this paper, embodied in the notions of general $2n$-pole (Section 4) and linear correspondence (Section 6), is an obvious one, and so completely general that it solves no problems other than the elemental one for which it was introduced. It provides a definite mathematical construct whose properties one can discuss with mathematical precision. This is all that we ask of it.

Realizability theory begins and ends with the study of these properties. It would be more accurate to say that the notion of general $2n$-pole describes a particular, but still very large, class of mathematical entities; realizability theory consists in the study of certain subclasses of the whole class of these entities, the particular subclasses being distinguished by special, and to us interesting, properties.

2.73 Despite its avowed aim at generality, the paper is oriented toward the realizability theory of finite passive networks. It ultimately provides a proof of 1.1 and indeed a complete characterization of finite passive $2n$-poles, however degenerate. This characterization is accomplished in a sequence of postulates, each one delineating a property of general $2n$-poles, i.e., a subclass consisting of all $2n$-poles having this property. The class of $2n$-poles having all of these properties is then identified with the class of $2n$-poles obtained from finite passive networks.

2.74 If we have succeeded here in our hope to set an adequate foundation for the realizability theory of devices with many terminals, it will be because of the nature and organization of the postulates themselves. They describe what at present seem to be individually significant properties of $2n$-poles, of progressively greater specificity, which in the aggregate characterize finite passive devices. By eliminating them in various combinations one obtains larger classes of objects. Further re-

* It is exactly this example, and the practical need of an adequate theory for it, which led the author first to study the realizability theory of passive multi-terminal devices.

search alone will tell whether or not one obtains in this way the kinds of device which are significant. For example, one would like general realizability theorems for structures containing vacuum tubes with frequency-independent transconductances, vacuum tubes with non-vanishing transit times, unilateral devices with specified parasitics, etc.

2.75 Actually, the postulates as we have given them are certainly not adequate for such an ambitious program. Exigencies of the presentation have dictated a number of condensations and compromises. It is hoped that the basic ideas are still evident even if not isolated individually in separate and entirely independent postulates. In any event, it is the author's firm belief that the presentation as given is at least illustrative of the kind of approach, and the level of mathematical detail, which will be needed if one is ever to provide a truly adequate realizability theory: a theory which will cover, for example, the broad range of active linear systems which present-day technology allows us to consider.

2.8 Apart from the network theoretic concepts, which must be evaluated by their effectiveness in solving problems—an assessment which is by no means yet complete—this paper is strongly marked by an idiosyncracy of its author: a consistent and insistent use of geometric ideas and terminology. This is based on the personal experience that linear algebra achieves logical unity and a freedom from encumbering notation when viewed in this way. A general reference covering most of the linear algebra (geometry) required here is P. R. Halmos' elegant monograph[9].

2.9 For a proof solely of 1.1, which has already been three times proved in the literature,[1, 10, 11] this paper provides an apparatus which is too cumbersome. There is even a sense in which 1.1 alone provides a characterization of all finite passive devices, for it seems to be generally accepted that, by the use of ideal transformers, any finite passive network can be represented as a network which has an impedance matrix to which is adjoined suitable ideal transformers. Therefore we cannot claim that, in using this cumbrous apparatus to characterize all finite passive $2n$-poles (including the degenerate ones), we have offered anything not already provided by a simpler proof of 1.1.

Three things may be said in rebuttal. First, we have already emphasized that the apparatus here exhibited was designed for more problems than that to which it is here applied. It is presented in the belief that it will prove of further use.

Second, even in the study of passive networks, it has seemed to the author helpful to look at the manifold things which are *not* passive networks. One gets then a clearer view of the unique position occupied by passive devices among all linear systems.

Third, there is a kind of semantic issue here: the assertion that any finite passive *network* (sic) can be put in such a form that 1.1 applies seems to this author to give a kind of circular characterization of such devices. A characterization which did not itself involve the concept of a network seems more satisfying. Logically, there is no circle here, but this is a fact requiring proof. A careful reading of this paper will show that it provides a proof. This particular subtlety does not of itself justify the lengths to which we have gone. It is, however, no longer a subtlety if one wishes to consider devices which do not have a representation in terms of something non-degenerate to which ideal transformers have been added.

2.91 The present Part I of the paper is so organized that at the end of Section 8 the reader is in possession of all of its principal results and its basic ideas. The remaining Sections, 9 through 20, may then be regarded as an Appendix containing the details of proofs. Indeed, Part II will be largely devoted to further details of proof, though there will be there one important idea not mentioned, save casually, in Part I—the idea of degree for a matrix.

In Sections 4 through 11, technical paragraphs have been distinguished from explanatory or heuristic ones by starring the paragraph numeral.

Part II of the paper contains the bulk of the proof of 1.1. This proof is modelled after that of Brune[2] for the realizability of two-poles. One familiar with the Brune process will probably find Part II readable without extensive reference to Part I.

Let the reader be warned that the Brune process is not a practical one for realizing networks because of its critical dependence upon a difficult minimization and balancing operation. The same criticism applies to the generalized Brune process of Part II.

The Brune process is of theoretical importance because it does realize a network with the minimum number of reactive elements. These facts will be brought to light in Part II.

The proofs of Oono[10] and Bayard[1] are different from ours. That of Oono[11] again follows the Brune model.

III. INTRODUCTION TO PART I

3.0 We keep before us first the problem of finding a mathematical description applicable to and characterizing the behavior of all finite passive networks. Second, we seek to make mathematically precise those ideas which appear to form the basis of general realizability theory. Sections 4 through 7 introduce the immediate mathematical machinery for this. Section 8 states the fundamental realizability theorem and outlines its proof. At this point the reader has had an introduction to the results of the paper. The remainder of the paper is then devoted to the technical details of proof. Beginning with Section 12, the device of starring the technical passages will be dropped.

3.1 Cauer[5] distinguished precisely the class of networks called $2n$-poles from the class of all multi-terminal networks. He also showed that, by the use of ideal transformers, any multi-terminal network is equivalent to a network which is a $2n$-pole (for some n) in his sense. We shall in Section 4 define a class of objects to be called general $2n$-poles. This class includes all electrical networks which are $2n$-poles in Cauer's sense. Its definition abstracts the significant properties isolated by Cauer.

For the study, alone, of finite passive networks, this definition is unnecessary, since one can in fact so put the arguments as to deal only with $2n$-poles which are finite passive networks, and therefore to deal only with concepts already defined in Cauer[5]. The somewhat physical notion of a general $2n$-pole is a convenient backdrop against which to display the important physical properties of finite passive networks, and, indeed, of networks in general. Having it available, we use it throughout the realizability arguments.

IV. DEFINITION OF GENERAL $2n$-POLE

4.0* Network theory establishes a correspondence between oriented linear graphs and systems of differential equations. With each node of the graph is associated a potential $E_n = E_n(t)$ and with each oriented branch a current $I_b = I_b(t)$. These potentials and currents are constrained, first by Krichoff's laws, and second by differential equations which depend upon the nature of the branches but not upon the topology of the graph.

4.01* A finite passive network is one whose graph has the following properties:

(i) There are finitely many nodes, $1, 2, \cdots, N$.

(ii) There are finitely many branches, $1, 2, \cdots, B$.

* Technical paragraph as explained in Section 2.91.

(iii) Let the b-th branch begin at node n_b and end at n_b'. Let $v_b = E_{n_b} - E_{n_b'}$. Then for each b, one of

$$v_b = R_b I_b, \qquad R_b > 0 \tag{a}$$

$$I_b = C_b \frac{dv_b}{dt}, \qquad C_b > 0 \tag{b}$$

$$v_b = \sum_{b'} L_{bb'} \frac{dI_{b'}}{dt} \tag{c}$$

holds, where the matrix $L_{b,b'}$, is real, symmetric, and semi-definite.

Cauer has shown[5] how an ideal transformer can be defined as the limiting case of a finite passive network. It is indeed no more nor less ideal than an open circuit ($R_b = \infty$ or $C_b = 0$) or a short circuit ($R_b = 0$ or $C_b = \infty$).

4.02 We seldom deal with networks in the detail which is implicit in (iii) above. We are usually interested in the external characteristics, so to speak, of such networks as viewed from a relatively small number of terminals (nodes). These multi-terminal devices, however, we continue to incorporate into larger network diagrams. It is usually clear how Kirchoff's laws are to be applied in these cases, and what the differential equations of the resulting system are. We are obliged, however, to make these matters precise before we can deal intelligently with the most general physical properties of networks.

4.1 We have seen the two kinds of constaint that a multi-terminal device imposes on the branch currents and node voltages in a network in which it is incorporated: the topological ones contained in Kirchoff's laws and the dynamical ones described by differential equations. Correspondingly, there are two aspects to the concept of general $2n$-pole.

4.11* In its relation to Kirchoff's laws, a general $2n$-pole is indicated as an object with n pairs of terminals (T_r, T_r'), $1 \leq r \leq n$. Each terminal can be made a node in an arbitrary finite diagram constructed out of network elements and other general $2m$-poles, with arbitrary values of m. This diagram is not an oriented linear graph, so we have no basis for the use of Kirchoff's laws. From it, however, we construct an oriented linear graph, called the *ideal graph* of the diagram, by the following rule:

The nodes of the ideal graph are those of the original diagram. Every

* Technical paragraph as explained in Section 2.91.

oriented branch of the original diagram is repeated in the ideal graph, similarly situated and oriented. Between those nodes which, in the original diagram, were the (T_r, T'_r) of a $2n$-pole **N**, is drawn a branch β_r, called the r-th *ideal branch* of **N**, oriented from T_r to T'_r. This is done for each such terminal pair.

Kirchoff's laws now apply to this ideal graph.

4.12* Consider a particular $2n$-pole **N**. Let E_r be the potential of T_r, E'_r that of T'_r. Define

$$v_r(t) = E_r - E'_r .$$

Then $v_r(t)$ is the voltage across the ideal branch β_r so oriented that $v_r(t) \geq 0$ when T_r is positive relative to T'_r. Let $k_r(t)$ represent the current entering T_r. Then $k_r(t) = I_r(t)$, the current in β_r, so $k_r(t)$ is also the current leaving T'_r. This is the force of the notion of ideal branch and the fact which distinguishes a network which is a $2n$-pole from an arbitrary network with $2n$ terminals.

4.13 For example, the network at (a) of Fig. 1 is not a 2×2 pole because its currents are not constrained to meet the ideal branch requirement. The addition of ideal transformers in either of the ways shown in (b) or (c) of the figure converts it to a 2×2 pole. Of course in a circuit in which the currents are constrained externally—as they would be, for

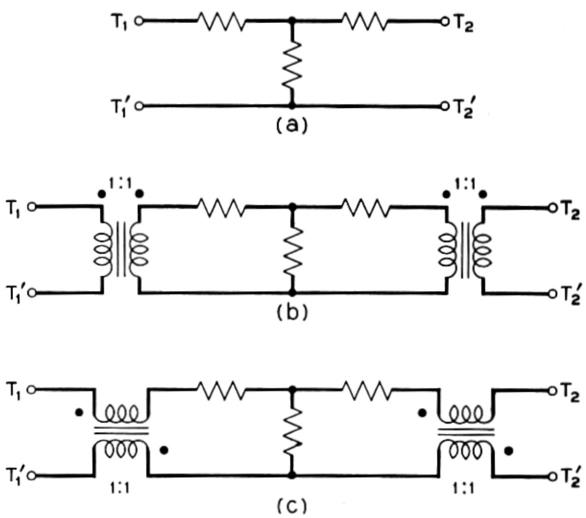

Fig. 1—Conversions of a four pole to a 2×2 pole.

* Technical paragraph as explained in Section 2.91.

example, when the 2×2 pole is driven by separate generators in the two external meshes—these transformers can be eliminated. The definition of $2n$-pole requires however that in every context the ideal branch concept is valid.

4.2* The second aspect of the concept of general $2n$-pole is that it imposes some kind of constraint—other than that implied by 4.11 and Kirchoff's laws—upon the voltages across and currents in its ideal branches. Define the symbols

$$\underline{v} = \underline{v}(t) = [v_1(t), v_2(t), \cdots, v_n(t)]$$

and

$$\underline{k} = \underline{k}(t) = [k_1(t), k_2(t), \cdots, k_n(t)]$$

as the n-tuples, respectively, of voltages across (T_r, T_r') and currents into T_r, $1 \leq r \leq n$. These are added and multiplied by scalars by the usual rules of vector algebra. If $\underline{v}$ and $\underline{k}$ represent *simultaneous* values of voltage and current in the $2n$-pole **N**—i.e., values satisfying all the constraints—then we say that **N** admits the pair $[\underline{v}, \underline{k}]$.

We say that **N** admits $\underline{v}$ if there is a $\underline{k}$ such that **N** admits the pair $[\underline{v}, \underline{k}]$. This $\underline{k}$ is said to correspond to $\underline{v}$. Dually, **N** admits $\underline{k}$ if there is a $\underline{v}$ (corresponding to $\underline{k}$) such that **N** admits $[\underline{v}, \underline{k}]$.

The constraints imposed by a general $2n$-pole **N** on voltages and currents are completely described by the totality of pairs $[\underline{v}, \underline{k}]$ which **N** admits. We shall *define* a general $2n$-pole, therefore, as

(i) a collection of n oriented ideal branches, as in 4.11, and

(ii) a list of pairs $[\underline{v}, \underline{k}]$ of voltages and currents admitted in these branches.

Hereafter we shall usually drop the adjective "general."

4.21 The definition we have just given is, in a way, a postulational form of an n-dimensional Thevenin's theorem. It postulates that a $2n$-pole is a thing† which, as far as the outside world is concerned, can be represented by a collection of two-poles β_r, $1 \leq r \leq n$, among which there exists a complicated agreement as to what currents and voltages will be admitted.

4.22 The passive networks (b) and (c) of Fig. 1 define 2×2 poles, because they satisfy 2.01 and clearly permit a complete specification of the admissible pairs $[\underline{v}, \underline{k}]$. Any equivalent network would also specify

* Technical paragraph as explained in Section 2.91.
† In fact, at this level of generality, *any* thing.

the same 2×2 pole, because—by its very equivalence—it would admit the same pairs. The closest association we can make between a $2n$-pole and a network, then, is to identify the $2n$-pole with an equivalence class of networks.

4.23 The completely symmetric role played by voltages and currents in this definition of general $2n$-pole will make it possible to take early advantage of the well-known duality principle of network theory. We shall do so freely.

4.3* We shall call a $2n$-pole physically realizable if its admissible pairs $[\underline{v}, \underline{k}]$ are the solutions of a system of differential equations obtained from a finite passive network, admitting the limiting elements: ideal transformers, open circuits, and short circuits.

V. PHYSICAL PROPERTIES OF NETWORKS

5.0 There are clearly a great many properties of finite passive networks which are not yet possessed by the general $2n$-poles now introduced. It is instructive to examine these properties physically.

5.1 In the first place, the dynamical constraints (a), (b), and (c) of 4.01 are expressed by linear, time invariant, differential equations. Accordingly, the $2n$-poles of network theory are:

5.11 Linear, in that the class of admissible pairs $[\underline{v}, \underline{k}]$ is a linear space;

5.12 Time invariant, admitting with each $[\underline{v}(t), \underline{k}(t)]$ also all $[\underline{v}(t + \tau), \underline{k}(t + \tau)]$ for aribtrary τ.

5.2 In the second place, a physical network **N** cannot predict the future, i.e., it cannot respond before it is excited. This can be formalized in terms of the pairs $[v, k]$ admitted by **N**, but to do so would require some digression. The reasons will be seen under 5.7 below.

5.3 We have already mentioned the constraints imposed on voltages and currents in a network by the topology of the network, through the medium of Kirchoff's laws. These constraints have three important properties:

5.31 They are workless, since they are imposed by resistanceless connections, leakless nodes, and, in the formal theory, by ideal transformers.

5.32 Though it seems scarcely necessary to say it, they are the only workless constraints. All other constraints are dynamical and have powers or energies associated with them.

---

* Technical paragraph as explained in Section 2.91.

5.33 They are frequency independent, that is, holonomic in the sense of dynamics.

5.4 The workless and the dynamical constraints in a physical network are all defined by relations with real coefficients. The space of admissible pairs is then a real linear space.

5.5 The positivities specified in 4.01 are characteristic of passive systems. They correspond to the fact that the power dissipation and the stored energies are all positive.

5.6 By definition, finite passive networks contain finitely many lumped elements. Correspondingly, their resonances and anti-resonances are finite in number.

5.7 We are accumstomed to dealing with networks which have, in addition to the properties listed above, a kind of non-degeneracy, in that the list of admissible pairs $[\underline{v}, \underline{k}]$ satisfies:

5.71 At least one of $\underline{v}$ or $\underline{k}$ can be specified arbitrarily—any real function is admitted;

5.72 When the free number of $[\underline{v}, \underline{k}]$ is specified, the other is uniquely determined.

For these non-degenerate networks, the property 5.2 above is easily formalized: if, say, $\underline{k}$ is determined by $\underline{v}$, then

$$\underline{v}^1(t) = \underline{v}^2(t) \quad \text{for} \quad t \leq t_0$$

implies

$$\underline{k}^1(t) = \underline{k}^2(t) \quad \text{for} \quad t \leq t_0 ,$$

where $[\underline{v}^i, \underline{k}^i]$ are admissible pairs, $i = 1, 2$. The general statement of 5.2 involves this condition and some discussion of the $\underline{v}$'s for which **N** admits $[\underline{v}, 0]$, and the dual notions.

5.8 The reason for speaking in terms of pairs $[\underline{v}, \underline{k}]$, instead of in terms of "cause" and "effect," or "impulse" and "response," is hinted at by 5.7 above. For the tacit implications of the cause and effect language completely obscure the fact that 5.71 and 5.72 are properties which are not automatically possessed by electrical networks. In fact, the simple four-pole of 2.41—a pair of unconnected terminals T_1 , T_1' , and a pair of shorted terminals T_2 , T_2'—has neither property, yet it is a perfectly good linear time invariant four pole. Its admissible pairs are

$$[(v_1 , 0), (0, k_2)],$$

where v_1 and k_2 are arbitrary real functions of the time

VI. LINEAR CORRESPONDENCES

6.0 In developing the formal properties of $2n$-poles which are equivalent to the physical ones just listed, it would be instructive to adjoin requirements piecemeal, much in the order given in Section 5. Space does not permit us full enjoyment of this luxury, but the reader will find a rough parallel between Section 5 and the developments of this Section and Section 7.

6.1 It is well known that linear time invariant systems are best studied by the tools of Fourier or Laplace analysis. We make this fact the basis of our first step in characterizing physically realizable $2n$-poles simply by phrasing our whole discussion in the frequency language. The content of the following paragraph will be obvious enough, but it does define terms to be used later.

6.11* Let v and k, without underscores, represent n-tuples of complex numbers:

$$v = [v_1, v_2, \cdots, v_n], \tag{1}$$

$$k = [k_1, k_2, \cdots, k_n]. \tag{2}$$

These are to be manipulated by the rules of vector algebra. Let p be a complex number. We shall say that a $2n$-pole **N** admits the pair $[v, k]$ at frequency p, if in the sense of 4.2 **N** admits the pair $[\underline{v}, \underline{k}]$ (*with* underscores) where $\underline{v}$ has components

$$v_r(t) = \mathrm{Re}(v_r e^{pt}), \qquad 1 \leq r \leq n, \tag{3}$$

and $\underline{k}$ has components

$$k_r(t) = \mathrm{Re}(k_r e^{pt}), \qquad 1 \leq r \leq n. \tag{4}$$

Also analogously to 4.2, we say that **N** admits v at frequency p if there is a k such that **N** admits $[v, k]$ at frequency p, and that this k corresponds to v (at frequency p). Similarly, **N** admits k at frequency p if there is a (corresponding) v such that **N** admits $[v, k]$ (at p).

6.12* Let **V** denote the aggregate of all n-tuples (1), and **K** the aggregate of all n-tuples (2). These are then complex linear spaces.

6.2* As our first step toward characterizing realizable $2n$-poles, let us consider a *linear correspondence* L between **V** and **K** described by the postulates:

P1. There is a set Γ_L of complex numbers and for each $p \epsilon \Gamma_L$ a list $L(p)$ of pairs $[v, k]$, $v \epsilon \mathbf{V}$, $k \epsilon \mathbf{K}$.

* Technical paragraph as explained in Section 2.91.

P2. If $[v^1, k^1] \epsilon L(p)$ and $[v^2, k^2] \epsilon L(p)$, then

$$[a_1 v^1 + a_2 v^2, a_1 k^1 + a_2 k^2] \epsilon L(p)$$

for any complex numbers a_1, a_2.

6.21* Given such a linear correspondence L, we can always describe a $2n$-pole $\mathbf{N}_L$ by:

$\mathbf{N}_L$ admits $[v, k]$ at frequency p if and only if $[v, k] \epsilon L(p)$.

That is, we can always interpret the pairs $[v, k]$ generated by (3) and (4) from the $[v, k] \epsilon L(p)$, for each $p \epsilon \Gamma_L$, as the voltages across and currents in a set of n ideal branches. We call $\mathbf{N}_L$ the $2n$-pole associated with L.

6.22* We call Γ_L the frequency domain of L (or of $\mathbf{N}_L$).

6.23 From here on, the words "$2n$-pole" can with some strain be regarded as suggestive but unnecessary. We in fact deal with linear correspondences—having properties as yet unspecified—and shall show how physical networks can be constructed which admit the pairs $[v, k] \epsilon L(p)$. Actually we use freely the concept of general $2n$-pole and thereby avoid some elaborate circumlocutions.

6.24* We identify two correspondences L_1 and L_2 as being the same if (i) their frequency domains differ only by a finite set, and (ii) for each p where both are defined the lists $L_1(p)$ and $L_2(p)$ are the same.

6.3 The simplest linear correspondences are those generated by matrices. For example, let $Z(p)$ be an $n \times n$ matrix with, say, elements $Z_{rs}(p)$ which are rational functions of p, $1 \leq r, s \leq n$. Let Γ_L consist of all the values of p at which $Z(p)$ is defined. For $p \epsilon \Gamma_L$, define $L(p)$ as the class of all pairs

$$[v, k] \tag{5}$$

obtained by letting k range over $\mathbf{K}$, where for each k, v is defined by the matrix equation

$$v = Z(p)k. \tag{6}$$

This kind of matrix equation will be used throughout to symbolize the n component equations

$$v_r = \sum_{s=1}^{n} Z_{rs}(p)k_s, \qquad 1 \leq r \leq n. \tag{7}$$

The list of pairs $L(p)$ defined by (5) clearly satisfies P1 and P2. It can therefore be used to define a $2n$-pole $\mathbf{N}_L$. It is easy to see that $\mathbf{N}_L$

* Technical paragraph as explained in Section 2.91.

in fact is non-degenerate in a sense similar to that of 5.7, for the current amplitudes k can be specified arbitrarily, and the resulting voltage amplitudes v are then fixed by k and p, by (6).

$Z(p)$ is called the impedance matrix of the $2n$-pole $\mathbf{N}_L$. It is also sometimes called the open-circuit impedance matrix, because each $Z_{rs}(p)$ is, by (7), the voltage amplitude across (T_r, T_r') when the current amplitudes at all terminals save (T_s, T_s') are zero—i.e., when all pairs save the s-th are on open circuit.

6.31 Dually, the pairs

$$[v, Y(p)v]$$

defined by an admittance matrix $Y(p)$ as v ranges over $\mathbf{V}$ define a linear time invariant $2n$-pole which is non-degenerate.

VII. WORK AND ENERGY

7.0* A linear correspondence satisfying P1 and P2 is something which abstracts the properties of linearity and time invariance. Most of the remaining properties of physical networks involve the mention of work or energy. These concepts enter our picture by way of the scalar product (v, k) between a voltage n-tuple (1) and a current n-tuple (2), of 6.11. This scalar product is defined by

$$(v, k) = \sum_{r=1}^{n} v_r \bar{k}_r. \tag{1}$$

7.01 If $p = i\omega$, one easily calculates from (3) and (4) of 6.11 that

$$2 \operatorname{Re}(v, k) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left[\sum_{r=1}^{n} v_r(t) k_r(t) \right] dt.$$

That is, when $p = i\omega$, the real part of $2(v, k)$ measures the average total power dissipated by the system of currents $k_r(t)$ against the driving voltages $v_r(t)$.

When p is not a pure imaginary, the interpretation of the scalar product (v, k) is not so clearly physical as this. The reader will ultimately observe that our significant statements about such products can all be reduced to statements applicable when $p = i\omega$, i.e., when the power interpretation is valid.

7.1* An important concept in what follows is that of the annihilator of a linear manifold (Halmos[9], par. 16). Let $\mathbf{V}_1 \subseteq \mathbf{V}$ be a linear manifold.

* Technical paragraph as explained in Section 2.91.

Then its annihilator $(\mathbf{V}_1)^0$ is the set of all k such that

$$v \epsilon \mathbf{V}_1 \quad \text{implies} \quad (v, k) = 0.$$

$(\mathbf{V}_1)^0$ is a linear manifold in $\mathbf{K}$.

Dually, given $\mathbf{K}_1 \subseteq \mathbf{K}$, $(\mathbf{K}_1)^0$ is the linear manifold of all $v \epsilon \mathbf{V}$ such that

$$k \epsilon \mathbf{K}_1 \quad \text{implies} \quad (v, k) = 0.$$

The annihilator concept is the analog in our general geometric framework of the idea of orthogonality. It clearly suggests a connection with workless constraints.

7.2* The complex conjugate of an n-tuple v (or k) is defined in the obvious way: if

$$v = [v_1, \cdots, v_n]$$

then

$$\bar{v} = [\bar{v}_1, \cdots, \bar{v}_n].$$

This conjugation operation clearly has the properties

$$\begin{aligned} \bar{\bar{\xi}} &= \xi \\ \overline{a\xi + b\eta} &= \bar{a}\bar{\xi} + \bar{b}\bar{\eta} \end{aligned} \tag{2}$$

where a and b are scalars and ξ and η are (consistently) elements of $\mathbf{V}$ or $\mathbf{K}$. Furthermore, at once from (1) of 7.0,

$$\overline{(v, k)} = (\bar{v}, \bar{k}). \tag{3}$$

7.21* A linear manifold will be called real if it contains, with any n-tuple also the conjugate of that n-tuple.

7.22* A real manifold is spanned by real n-tuples. This will be proved in the Appendix, Section 20.

7.23* The annihilator of a real manifold is real. For let $\mathbf{K}_1$ be real and $k^1, \cdots, k^r$ be real n-tuples which span $\mathbf{K}_1$. Then if $v \epsilon (\mathbf{K}_1)^0$ every

$$(v, k^s) = 0,$$

and conversely. But then also

$$(\bar{v}, k^s) = \overline{(v, k^s)} = \bar{0} = 0,$$

so $\bar{v} \epsilon (\mathbf{K}_1)^0$.

* Technical paragraph as explained in Section 2.91.

7.3* Given a linear correspondence L, we make several definitions:

$\mathbf{V}_L(p)$ is the set of all $v \epsilon \mathbf{V}$ such that there is a k with $[v, k] \epsilon L(p)$.

$\mathbf{K}_L(p)$ is the set of all $k \epsilon \mathbf{K}$ such that there is a v with $[v, k] \epsilon L(p)$.

$\mathbf{V}_{L0}(p)$ is the set of $v \epsilon \mathbf{V}_L(p)$ such that

$$[v, 0] \epsilon L(p).$$

$\mathbf{K}_{L0}(p)$ is the set of $k \epsilon \mathbf{K}_L(p)$ such that

$$[0, k] \epsilon L(p).$$

7.31* The postulate P2 implies that for each $p \epsilon \Gamma_L$, $\mathbf{V}_L(p)$, $\mathbf{K}_L(p)$, $\mathbf{V}_{L0}(p)$ and $\mathbf{K}_{L0}(p)$ are all linear manifolds.

7.32 $\mathbf{V}_L(p)$, for example, is the set of $v \epsilon \mathbf{V}$ such that $\mathbf{N}_L$ admits v at frequency p.

7.4* We now postulate

P3. There exist fixed linear manifolds $\mathbf{V}_L \subseteq \mathbf{V}$, $\mathbf{K}_L \subseteq \mathbf{K}$ such that

(A) For every $p \epsilon \Gamma_L$, $\mathbf{V}_L(p) = \mathbf{V}_L = (\mathbf{K}_{L0}(p))^0$

(I) For every $p \epsilon \Gamma_L$, $\mathbf{K}_L(p) = \mathbf{K}_L = (\mathbf{V}_{L0}(p))^0$.

7.41* We may henceforth write $\mathbf{V}_{L0}$, $\mathbf{K}_{L0}$, for $\mathbf{V}_{L0}(p)$, $\mathbf{K}_{L0}(p)$, knowing that, under P3

$$\mathbf{V}_{L0} = (\mathbf{K}_L)^0,$$
$$\mathbf{K}_{L0} = (\mathbf{V}_L)^0.$$

7.42 Linear correspondences satisfying P3 abstract the properties mentioned in 5.3. The equalities $\mathbf{V}_L(p) = \mathbf{V}_L$, $\mathbf{K}_L(p) = \mathbf{K}_L$ guarantee the frequency-independence of the workless constraints. The equalities $\mathbf{V}_L(p) = (\mathbf{K}_{L0}(p))^0$, $\mathbf{K}_L(p) = (\mathbf{V}_{L0}(p))^0$ in a sense guarantee that the only constraints imposed upon admissible currents and voltages (as opposed to constraints relating currents and voltages) are those which arise from open or short circuits, i.e., are workless.

7.43 An illustrative consequence of P3, for example, is that if L satisfies P3 and if $\mathbf{N}_L$ is such that all of the current amplitudes can be specified arbitrarily, then indeed the voltages are determined by the currents. This will appear as a consequence of 8.1. It is a very general theorem about networks of a kind that this author, at least, has not heretofore encountered.

---

* Technical paragraph as explained in Section 2.91.

7.5* Continuing toward realizability, we introduce

P4. If $p \epsilon \Gamma_L$, then $\bar{p} \epsilon \Gamma_L$. If $[v, k] \epsilon L(p)$, then $[\bar{v}, \bar{k}] \epsilon L(\bar{p})$.

This postulate embodies most of the reality properties of networks. It has as an immediate consequence the

7.51* *Lemma:* If L satisfies P1, P2, P3, and P4, then all of

$$\mathbf{V}_L, \mathbf{V}_{L0}, \mathbf{K}_L, \mathbf{K}_{L0}$$

are real.

Proof: By P4, $v \epsilon \mathbf{V}_L(p) = \mathbf{V}_L$ implies $\bar{v} \epsilon \mathbf{V}_L(\bar{p}) = \mathbf{V}_L$. Hence $\mathbf{V}_L$ is real. Then $\mathbf{K}_{L0} = (\mathbf{V}_L)^0$ is real, and dually.

7.6* The three remaining postulates on L refer to scalar products. They are concerned with the energy questions related to passivity, rather than with the workless constraint questions.

P5. If $[u, j] \epsilon L(p)$ and $[v, k] \epsilon L(p)$, and if

(A) u and v are real, or if

(I) j and k are real,

then

$$(u, k) = (v, j).$$

7.61 This is the property which provides the reciprocity law. In its presence, the relations in P3 may be weakened to

$$\mathbf{V}_L(p) = \mathbf{V}_L \supseteq (\mathbf{K}_{L0}(p))^0,$$
$$\mathbf{K}_L(p) = \mathbf{K}_L \supseteq (\mathbf{V}_{L0}(p))^0.$$

This fact will appear as a consequence of the lemma of Section 12.

7.7* *Lemma:* A consequence of P2 and P3(A) is that if

$$[v, k_r] \epsilon L(p), \qquad r = 1, 2,$$

then for any $u \epsilon \mathbf{V}_L$,

$$(u, k_1) = (u, k_2).$$

For by P2 we have that

$$[v - v, k_1 - k_2] = [0, k_1 - k_2] \epsilon L(p),$$

hence $k_1 - k_2 \epsilon \mathbf{K}_{L0}$. Then however, by P3(A), $u \epsilon \mathbf{V}_L$ implies $u \epsilon (\mathbf{K}_{L0})^0$, so

* Technical paragraph as explained in Section 2.91.

that

$$0 = (u, k_1 - k_2) = (u, k_1) - (u, k_2).$$

Q.E.D. A dual result follows from P3(I).

7.71* The result of 7.7 above means that the scalar product (v, k) is fixed by v alone when we know that $[v, k] \epsilon L(p)$. This means that, given $v \epsilon \mathbf{V}_L$, there is a unique function $F_v(p)$ defined for $p \epsilon \Gamma_L$ by

$$F_v(p) = \overline{(v, k)}$$

where $[v, k] \epsilon L(p)$. Dually,

$$J_k(p) = (v, k)$$

is defined for each fixed $k \epsilon \mathbf{K}_L$.

7.72* (P6.) The complement of Γ_L is finite and

(I) For each $v \epsilon \mathbf{V}_L$, $F_v(p)$ is rational

(A) For each $k \epsilon \mathbf{K}_L$, $J_k(p)$ is rational.

7.73* (P7.) (A) $\text{Re}(p) \geq 0$ implies $\text{Re}(F_v(p)) \geq 0$

(I) $\text{Re}(p) \geq 0$ implies $\text{Re}(J_k(p)) \geq 0$.

VIII. THE FUNDAMENTAL REALIZABILITY THEOREM

8.0* We can now state our fundamental realizability theorem: If a linear correspondence L satisfies P1, ⋯ , P7, the associated $2n$-pole $\mathbf{N}_L$ is physically realizable. Conversely, given a physically realizable $2n$-pole $\mathbf{N}$, the associated linear correspondence satisfies P1, ⋯ , P7.

8.01 Actually, the postulates P1, ⋯ , P7 are not unique nor even entirely independent. Many changes may be rung on them. We indicated one above. At the expense of apparent asymmetry, the (A) or (I) portions, in various combinations, can be deleted or weakened. We shall not pursue this subject further at this point, but must come back to it in Section 12.

8.02 We close this Section by outlining the proof of 8.0. The details are then contained in the remainder of the paper.

8.03 The proof that P1 through P7 are necessary for physical realizability will be a direct one: it will be shown that, considered individually, each network branch and each ideal transformer satisfies the postulates.

* Technical paragraph as explained in Section 2.91.

By an application of Kron's method (described by Synge[12]), it will then be shown that the imposition of Kirchoff's laws preserves the postulates. This work is most efficiently performed after the full machinery of the sufficiency proofs is available, and will be done in Section 19.

8.04 The sufficiency of P1 through P7 can be deduced—and we will do so—from the lemmas to be quoted below. Apart from Section 19 on necessity, the remainder of the paper is devoted to the proofs of these lemmas.

8.1* *Lemma:* If L is a linear correspondence satisfying P1, P2, P3, and P4, then there exists a fixed real nonsingular matrix W such that

8.11 The list $L_W(p)$ of all pairs†

$$[W^{-1}v, W'k],$$

where $[v, k] \epsilon L(p)$, describes a linear correspondence L_W satisfying P1, P2, P3, and P4.

8.12 The $2n$-pole $\mathbf{N}_W$ $(= \mathbf{N}_{L_W})$ associated with L_W consists of

(i) Some number r of open-circuited terminal pairs $(T_1, T_1'), \cdots, (T_r, T_r')$,

(ii) Some number s of short-circuited terminal pairs $(T_{n-s+1}, T'_{n-s+1}), \cdots, (T_n, T'_n)$,

(iii) A set of $m = n - r - s$ terminal pairs $(T_{r+1}, T'_{r+1}), \cdots, (T_{r+m}, T'_{r+m})$.

8.13 Either $m = 0$, or the terminal pairs in (iii) are those of a $2m$-pole $\mathbf{N}_1$ which has a nonsingular impedance matrix $Z_1(p)$.

This lemma, and the following, will be proved in 13.2.

8.2* *Lemma:* If L satisfies P5, P6, and P7, then $Z_1(p)$ is a positive real‡ matrix, that is, $Z_1(p)$ satisfies (i), $\cdots$, (iv) of 1.1.

8.3* *Lemma:* If a $2m$-pole $\mathbf{N}_1$ has a positive real impedance matrix, then $\mathbf{N}_1$ is physically realizable.

This is the sufficiency half of the matrix realizability theorem 1.1. Part II will be devoted to its proof.

8.4* *Lemma:* If $\mathbf{N}_W$ is physically realizable, then $\mathbf{N}$ can be constructed from it by the use of ideal transformers.

This is Cauer's Transformation Theorem[5] about which we shall say more in Section 9.

---

* Technical paragraph as explained in Section 2.91.
† W^{-1} and W' are respectively the reciprocal and the transpose of W.
‡ Gewertz's terminology[8], by now traditional.

8.5* The sufficiency half of 8.0 is now clear. By 8.2 and 8.3, $\mathbf{N}_1$ is physically realizable. Clearly then $\mathbf{N}_W$ is, simply by the adjunction of the necessary open and short circuits. Finally $\mathbf{N}$ is by Cauer's theorem, 8.4.

8.6* We can see now how to prove the necessity of positive reality for the realizability of a positive real matrix $Z(p)$. This is the necessity half of the matrix theorem 1.1. Let $Z(p)$ be the matrix of a realizable $\mathbf{N}$. Then $\mathbf{N}$ has an associated linear correspondence L satisfying P1, $\cdots$, P7, by the necessity half of 8.0. The pairs of L are the pairs

$$[Z(p)k,\ k]$$

generated as k ranges over all n-tuples. By definition, then, the pairs of L_W are

$$[W^{-1}Z(p)k,\ W'k].$$

As k ranges over all n-tuples, the nonsingularity of W implies that $W'k$ does also. Let $U = W^{-1}$. Then the pairs above are the same as

$$[UZ(p)U'k,\ k]$$

as k ranges over all n-tuples. Hence L_W has the impedance matrix $UZ(p)U'$, where $U = W^{-1}$ is real and nonsingular. Because L_W has an impedance matrix, $r = 0$ in 8.12.

Now by 8.1 and 8.2, $Z_1(p)$ is positive real and the matrix $UZ(p)U'$ of L_W is just $Z_1(p)$ bordered by s rows and columns of zeros. It is then easy to see that $UZ(p)U'$ is positive real, and finally also that $Z(p)$ is. These last two facts will be proved formally in Section 16.

IX. CAUER'S TRANSFORMATION THEOREM

9.0 Cauer's transformation theorem[5] is the cornerstone of formal realizability theory. In one form, the theorem reads:

9.1* Let $Z(p)$ be the impedance matrix of a physically realizable $2n$-pole $\mathbf{N}$. Let U be a real, constant, nonsingular matrix. Then

$$UZ(p)U' \tag{1}$$

is again the impedance matrix of a physically realizable $2n$-pole, $\mathbf{N}_U$. $\mathbf{N}_U$ can be constructed from $\mathbf{N}$ by the use of ideal transformers.

9.2* A superficial generalization of this theorem can be obtained at once from Cauer's proof. It asserts that if $\mathbf{N}$ is physically realizable and is described by the linear correspondence L, then there is a physically realizable $2n$-pole $\mathbf{N}_W$, obtainable from $\mathbf{N}$ by the use of ideal trans-

* Technical paragraph as explained in Section 2.91.

formers, which is described by the linear correspondence L_W whose pairs at each p are the pairs

$$[W^{-1}v,\ W'k], \tag{2}$$

where $[v, k] \epsilon L(p)$.

We refer to Cauer[5] for the proof. It is straightforward.

9.21 We shall use the second form (9.2) of Cauer's theorem in our realization process. Notice that it is in a sense a "physical" theorem, about the way one physical network is related to another. It is used in this way: we shall always solve a realizability problem by finding some network **N** which is easily realized, and then a W such that $\mathbf{N}_W$, which is now realizable, provides a solution to the given problem.

9.22* We shall call the $2n$-pole $\mathbf{N}_W$ a Cauer equivalent of **N**.

9.3 Although Cauer's theorem will be applied, in a sense, only *a posteriori*, its effect is fundamental. For it implies that formal physical realizability is a property of matrices which is invariant under the operation (1) or a property of correspondences which is invariant under (2). There is an extensive classical literature on the properties of matrices invariant under operations like that of (1), and the effect of Cauer's theorem is to make these results all available to formal realizability theory.

9.31* It is worth observing here that we are already well set up to use Cauer's theorem:

Lemma: If L is a linear correspondence satisfying P1, $\cdots$, P7, then the correspondence L_W of 9.2 also satisfies P1, $\cdots$, P7.

Proof: Let $M = L_W$. P1 and P2 for M are obvious, with $\Gamma_M = \Gamma_L$. By definition of M,

$$\mathbf{V}_M(p) = W^{-1}\mathbf{V}_L(p) = W^{-1}\mathbf{V}_L$$

$$\mathbf{K}_M(p) = W'\mathbf{K}_L(p) = W'\mathbf{K}_L$$

$$\mathbf{V}_{M0}(p) = W^{-1}\mathbf{V}_{L0}(p) = W^{-1}\mathbf{V}_{L0}$$

$$\mathbf{K}_{M0}(p) = W'\mathbf{K}_{L0}(p) = W'\mathbf{K}_{L0}$$

where $W^{-1}\mathbf{S}$ for a manifold **S** consists of all n-tuples $W^{-1}v$, where $v \epsilon \mathbf{S}$. Hence in P3,

$$\mathbf{V}_M(p) = \mathbf{V}_M = W^{-1}\mathbf{V}_L$$

$$\mathbf{K}_M(p) = \mathbf{K}_M = W'\mathbf{K}_L$$

for fixed manifolds $\mathbf{V}_M$, $\mathbf{K}_M$ as defined.

* Technical paragraph as explained in Section 2.91.

Now if $v \epsilon \mathbf{V}_{L0}$, then

$$(v, k) = 0$$

for every $k \epsilon \mathbf{K}_L = (\mathbf{V}_{L0})^0$. Then, however, by direct calculation from Section 7.0,

$$(W^{-1}v, W^*k) = 0,$$

where W^* is the adjoint, i.e. transposed conjugate matrix of W. But because W is real, $W^* = W'$. Hence if $v \epsilon \mathbf{V}_{L0}$, then

$$(W^{-1}v, k) = 0$$

for every $k \epsilon W'\mathbf{K}_L = \mathbf{K}_M$. Hence

$$\mathbf{K}_M = (W^{-1}\mathbf{V}_{L0})^0 = (\mathbf{V}_{M0}(p))^0.$$

By this and its dual, P3 is completed for M.

The reamining postulates for M follow from those for L by the simple equality

$$(v, k) = (W^{-1}v, W'k)$$

already established, combined with $\Gamma_M = \Gamma_L$.

9.32 For fixed $Z(p)$, the matrices (1), as U ranges over a group, form an equivalence class. Classical matrix theory treats of such equivalence classes. This author's predilection is to regard this theory from a geometrical point of view. In part this prejudice may be justified by the ease with which that slightly more general object, a linear correspondence, can be treated by geometrical methods. In any event we shall begin our program of proofs with a brief introduction to the geometrical approach.

X. GEOMETRICAL PRELIMINARIES

10.0* We now wish to consider $\mathbf{V}$ and $\mathbf{K}$ as complex n-dimensional linear spaces† respectively of voltage vectors v and current vectors k. The distinction here is in point of view. A vector v is regarded as an absolute geometrical object; an n-tuple $[v] = [a_1, \cdots, a_n]$ is regarded as a set of coordinates for the vector v, relative to some coordinate basis. Given a fixed coordinate basis, there is a one-to-one correspondence between vectors v and the n-tuples $[v]$ which represent them in that basis, a correspondence which preserves the operations of vector algebra.

---

* Technical paragraph as explained in Section 2.91.

† For a reference concerning the ideas in this section, see Halmos[9], Chapters I and II.

10.01 The effect of attaching a geometric identity to vectors, rather than to n-tuples, is to make it possible to choose coordinate bases freely and as convenient, without elaborate constructions or even interpretations. We can then discuss properties of n-tuples (and other objects, e.g. matrices) which are invariant under the kind of operations exemplified by (1) and (2) of Section 9 as *properties of a single geometric object*, rather than as properties shared by an extensive class of concrete objects which are converted into each other by the group of operations.

10.1 This change in point of view need not change formally anything we have said to date; it simply erects a conceptual superstructure, or provides a conceptual foundation, depending on the reader's personal attitude.

We shall support this statement by going through the important ideas of Sections 4, 6, and 7 and examining their geometrical meanings or counterparts. It is convenient to consider first and at some length the notions of scalar product and complex conjugate. The geometric structure will then be complete enough to permit a rapid survey of the remaining ideas.

10.11* The geometrical counterpart of the scalar product introduced in 7.0 is a numerically valued function $\sigma = \sigma(v, k)$ of two vector variables. Its first argument v ranges over **V** and its second argument k ranges over **K**. The function $\sigma(v, k)$ is linear in v and conjugate linear in k:

$$\begin{aligned} \sigma(au + bv, k) &= a\sigma(u, k) + b\sigma(v, k), \\ \sigma(v, ak + b\ell) &= \bar{a}\sigma(v, k) + \bar{b}\sigma(v, \ell). \end{aligned} \tag{1}$$

We denote this function $\sigma(v, k)$ by the simple bracket notation (v, k).

10.12 With this scalar product, the geometry of **V** and **K** is that of a space **K** and the space $\mathbf{K}^* = \mathbf{V}$ of conjugate linear functionals over **K**. This is analogous to the real geometry of space and conjugate space discussed at length in Halmos[9]. In fact, in the introduction to Chapter III of Halmos[9], the modifications introduced by the conjugate linearity of (v, k) over **K** are treated in detail.

10.13* Because of its importance, we quote here a paraphrase of the results covered in Halmos[9], par. 12.

(i) If $f(v)$ is any numerically valued homogeneous linear function of $v \epsilon \mathbf{V}$, then there is a unique vector $k_f \epsilon \mathbf{K}$ such that

$$f(v) = (v, k_f)$$

for all $v \epsilon \mathbf{V}$.

* Technical paragraph as explained in Section 2.91.

(ii) If $g(k)$ is any numerically values homogeneous conjugate-linear function of $k \epsilon \mathbf{K}$ (i.e., if $\overline{g(k)}$ is linear in k) then there is a unique $v_g \epsilon \mathbf{V}$ such that

$$g(k) = (v_g , k)$$

for all $k \epsilon \mathbf{K}$.

10.2* The annihilator $(\mathbf{V}_1)^0$ of a manifold $\mathbf{V}_1 \subseteq \mathbf{V}$ is, as in 7.1, the set of all $k \epsilon \mathbf{K}$ such that

$$v \epsilon \mathbf{V}_1 \quad \text{implies} \quad (v, k) = 0.$$

10.21* It is shown in Halmos[9] that to each basis $v^1, \cdots, v^n$ in $\mathbf{V}$ there exists a unique dual basis $k^1, \cdots, k^n$ in $\mathbf{K}$ such that

$$(v^r, k^s) = \delta_{rs} , \tag{2}$$

where δ_{rs} is the Kronecker symbol: $\delta_{rs} = 0$ if $r \neq s$, $\delta_{rr} = 1$, $1 \leq r, s \leq n$.

10.22 If

$$\begin{aligned} [v] &= [a_1 , \cdots , a_n] \\ [k] &= [b_1, \cdots , b_n] \end{aligned} \tag{3}$$

are the n-tuples representing v and k relative to a pair of dual bases, then it is easily computed from (1) and (2) that

$$(v, k) = \sum_{r=1}^{n} a_r \bar{b}_r. \tag{4}$$

Therefore the concrete scalar product of 7.0 is indeed the geometric scalar product here considered, when we restrict our pairs of bases in $\mathbf{V}$ and $\mathbf{K}$ always to be dual in the sense of (2).

10.23* We shall use the words "coordinate frame" or simply "frame" to denote a pair of dual bases in $\mathbf{V}$ and $\mathbf{K}$. Any basis in $\mathbf{V}$ (or $\mathbf{K}$) specifies a frame by the uniqueness result quoted above.

10.24 We shall henceforth deal always with coordinate frames, in fact, ultimately, real coordinate frames, rather than arbitrary pairs of bases. This means in classical language that we are considering as "geometrical properties" all properties which are preserved under the group of linear transformations which leave the bilinear form (4) invariant. The properties related to physical realizability will turn out to be invariant only under the subgroup of real linear transformations preserving (4).

* Technical paragraph as explained in Section 2.91.

10.3* Conjugation is an operation which to each $v \epsilon \mathbf{V}$ associates a vector $\bar{v}$ uniquely determined by v with the properties

$$\begin{aligned} \bar{\bar{v}} &= v, \\ \overline{(au + bv)} &= \bar{a}\bar{u} + \bar{b}\bar{v}, \end{aligned} \tag{5}$$

where a and b are any complex numbers and $\bar{a}$, $\bar{b}$ their conjugates.

10.31* Given any such conjugation operation in $\mathbf{V}$, and given any $k \epsilon \mathbf{K}$, define a function $g_k(v)$ by

$$g_k(v) = \overline{(\bar{v}, k)} \tag{6}$$

for $v \epsilon \mathbf{V}$. Then $g_k(v)$ is linear in v, by (5) above and (1) of 10.11. Therefore, by 10.13, there is a unique vector $\bar{k} \epsilon \mathbf{K}$ such that

$$g_k(v) = (v, \bar{k}). \tag{7}$$

10.32* Directly from (1) of 10.11 and (6) above, if $j = ak + b\ell$, then

$$g_j(v) = ag_k(v) + bg_l(v).$$

From (7), therefore

$$(v, \bar{j}) = a(v, \bar{k}) + b(v, \bar{\ell})$$

for all $v \epsilon \mathbf{V}$. Comparing this with (1) of 10.11, we see that

$$\bar{j} = \bar{a}\bar{k} + \bar{b}\bar{\ell}. \tag{8}$$

The second item of (5) above then holds for vectors $k \epsilon \mathbf{K}$.

That $\bar{\bar{k}} = k$ follows easily: We have from (6) and (7), written for the vector $\bar{k}$, that

$$\overline{(\bar{v}, \bar{k})} = (v, \bar{\bar{k}}). \tag{9}$$

We also have, by writing (6) and (7) for vectors $\bar{v}$ and k that

$$\overline{(\bar{\bar{v}}, k)} = (\bar{v}, \bar{k}).$$

Taking complex conjugates of these two numbers, and using $\bar{\bar{v}} = v$ from (5), we have

$$(v, k) = \overline{(\bar{v}, \bar{k})}. \tag{10}$$

Then (9) and (10), which hold for all $v \epsilon \mathbf{V}$, identify k and $\bar{\bar{k}}$ by 10.13.

10.34* We have now showed in (5), (8) and (10) that this complex conjugate satisfies the formal properties of the conjugate for n-tuples introduced in 7.2.

* Technical paragraph as explained in Section 2.91.

10.35. The abstract scalar product of 10.11 turned out in the end to be no more than the concrete one of 7.0 when we restrict our attention to n-tuples derived from vectors by the use of coordinate frames. In a similar way, it is not hard to show that there always exists a coordinate frame in which the abstract conjugation now introduced has the form of 7.2. This will be done in the Appendix (20.2).

10.36* Our need for writing out the components of vectors has now almost vanished. Henceforth we shall use subscripts to denote particular vectors, e.g. v_1, rather than components.

10.4* A vector will be called real if it is equal to its own conjugate. A manifold will be called real if it contains with each vector also the conjugate of that vector. **V** and **K** are then real. A basis will be called real if it is made up of real vectors, and a frame will be called real if its bases are real. Any frame in terms of which our conjugation operation takes the form of 7.2 is real by definition because its basis vectors *in that frame* have components which are 0 or 1. The vector 0 is real, similarly.

10.41* The basis dual to a real basis is real, for if

$$(v_r, k_s) = \delta_{rs},$$

then by (10) of 10.3 and the hypothesis that $v_r = \bar{v}_r$, we have

$$(v_r, \bar{k}_s) = \bar{\delta}_{rs} = \delta_{rs}$$

so the $\bar{k}_s$ satisfy the same equations as the k. The uniqueness of the basis dual to $v_1, \cdots, v_r$ then proves that $\bar{k}_s = k_s$, $1 \leq s \leq n$.

10.42* Any vector v can be written

$$v = v_1 + iv_2$$

where v_1 and v_2 are real. Namely

$$v_1 = \frac{1}{2}(v + \bar{v}),$$

$$v_2 = \frac{1}{2i}(v - \bar{v}).$$

10.5* It is shown in Halmos[9], par. 34, that if $v \epsilon \mathbf{V}$, $k \epsilon \mathbf{K}$ are represented by $[v]$, $[k]$ in some coordinate frame, and by $[v]_1$, $[k]_1$ in some other frame, then there is a nonsingular matrix $[W]$, which (a) depends only upon the

* Technical paragraph as explained in Section 2.91.

two frames, and (*b*) relates these n-tuples as follows:

$$[v]_1 = [W]^{-1}[v], \quad [k]_1 = [W]^*[k]. \tag{11}$$

It is easy to show that if $[W]$ has real elements, so that $[W]^* = [W]'$, then the two frames involved above are either both real, or else neither is real. Also, conversely, if both frames are real, then necessarily the $[W]$ of (11) has real elements and $[W]^* = [W]'$.

10.6* Some further important geometrical notions must be mentioned before we proceed.

If $\mathbf{V}_1$ and $\mathbf{V}_2$ are disjoint linear manifolds in $\mathbf{V}$—i.e. linear manifolds having in common only the single vector 0—we write

$$\mathbf{V}_1 \oplus \mathbf{V}_2$$

for the linear manifold consisting of all vectors $v = v_1 + v_2$, where $v_i \epsilon \mathbf{V}_i$, $i = 1, 2$. The circle around the plus sign is used to denote the disjointness of $\mathbf{V}_1$ and $\mathbf{V}_2$.

It is shown in Halmos[9], par. 19, that if

$$\mathbf{V} = \mathbf{V}_1 \oplus \mathbf{V}_2 \tag{12}$$

then

$$\mathbf{K} = \mathbf{K}_1 \oplus \mathbf{K}_2, \tag{13}$$

where $\mathbf{K}_1 = (\mathbf{V}_2)^0$, $\mathbf{K}_2 = (\mathbf{V}_1)^0$ and the dimension of $\mathbf{K}_i$ is equal to that of $\mathbf{V}_i$, $i = 1, 2$. We call (13) the decomposition dual to (12). We sometimes write $\mathbf{K}_i = \mathbf{V}_i^*$ to denote the $\mathbf{K}_i$ dual to $\mathbf{V}_i$ in the decomposition (13). It is shown in Halmos[9], loc. cit., that there exists a basis $v_1, \cdots, v_n$ in $\mathbf{V}$ and its dual $k_1, \cdots, k_n$ in $\mathbf{K}$ such that, if r is the dimension of $\mathbf{V}_1$,

$$\begin{aligned} v_1, \cdots, v_r &\quad \text{is a basis for } \mathbf{V}_1 \\ v_{r+1}, \cdots, v_n &\quad \text{is a basis for } \mathbf{V}_2 \\ k_1, \cdots, k_r &\quad \text{is a basis for } \mathbf{K}_1 \\ k_{r+1}, \cdots, k_n &\quad \text{is a basis for } \mathbf{K}_2. \end{aligned} \tag{14}$$

Furthermore, if $v_1, \cdots, v_n$ is any basis in $\mathbf{V}$ satisfying the first half of (14), its dual basis satisfies the second half, and dually.

We shall show in the Appendix that if any one of $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{K}_1$, or

* Technical paragraph as explained in Section 2.91.

$\mathbf{K}_2$ is real, then they all are, and that in this case the bases (14) can be chosen to be real.

Similar considerations apply to decompositions into more summands: if

$$\mathbf{V} = \mathbf{V}_1 \oplus \mathbf{V}_2 \oplus \cdots \oplus \mathbf{V}_m$$

then

$$\mathbf{K} = \mathbf{K}_1 \oplus \mathbf{K}_2 \oplus \cdots \oplus \mathbf{K}_m,$$

where

$$\mathbf{V}_i^* = \mathbf{K}_i = \bigcap_{j \neq i} \mathbf{V}_j^0 = \Big(\sum_{j \neq i} \mathbf{V}_j\Big)^0.$$

XI. GEOMETRICAL CORRESPONDENCES

11.0 With the geometry of $\mathbf{V}$ and $\mathbf{K}$ now in hand, we consider the geometric aspects of our network theoretic concepts.

The definition in Section 4 of general $2n$-pole describes a concrete thing and stands unaltered in our geometric view. The definitions in 6.11 of the terminology typified by "$\mathbf{N}$ admits $[v, k]$ at frequency p" are unchanged except that we should now explicitly indicate that we are discussing concrete n-tuples of complex numbers by placing brackets around the vector symbols, thus: $[v]$, $[k]$. In other words, a $2n$-pole is described by a concrete relation between n-tuples.

11.1* All of the postulates P1, $\cdots$, P7 are stated in a language which now has been given an absolute geometric meaning. In this meaning, P1 and P2 describe a *geometrical linear correspondence* between vectors $v \epsilon \mathbf{V}$ and $k \epsilon \mathbf{K}$. This is the geometric counterpart of the concrete notion of a linear correspondence between n-tuples.

11.11 An impedance matrix, as in 6.3, describes a particularly tightly knit linear correspondence, namely a linear function from $\mathbf{K}$ to $\mathbf{V}$. The geometrical counterpart is an *impedance operator* which for each p is by definition a linear homogeneous function which assigns to each vector $k \epsilon \mathbf{K}$ a unique $v = Z(p)k \epsilon \mathbf{V}$. That is: an operator is a functional relationship between vectors and as such has a geometric identity.

11.12 It is easy to prove† that, given an impedance operator $Z(p)$, and given any coordinate bases in $\mathbf{V}$ and $\mathbf{K}$ respectively, there is a matrix $[Z(p)]$, with elements $Z_{rs}(p)$, $1 \leq r, s \leq n$, such that relative to these bases the coordinates k_s of a vector k and the coordinates v_r of $v = Z(p)k$ are related by (7) of 6.3. We call $[Z(p)]$ the matrix of $Z(p)$

* Technical paragraph as explained in Section 2.91.
† Cf. Halmos[9], par. 26.

relative to the given pair of bases. A strong analog of this observation is contained in the following lemma.

11.13* *Lemma:* (i) Let L be a geometrical linear correspondence. Fix any real coordinate frame and let $[L]$ be the linear correspondence whose paired n-tuples are

$$[[v], [k]],$$

where

$$[v, k] \epsilon L(p).$$

(ii) Alternatively, let $[L]$ be a (concrete) linear correspondence between n-tuples. Interpret the n-tuples related by $[L]$ as representing vectors in some real coordinate frame. Let L be the geometrical correspondence whose pairs, expressed as n-tuples in this frame, are those of the concrete correspondence $[L]$.

In either case, (i) or (ii), the geometric correspondence L satisfies the geometric postulates P1 , $\cdots$, P7 if and only if the concrete correspondence $[L]$ satisfies the concrete forms of these postulates.

The proof of this lemma consists essentially in reading the postulates carefully. We shall not reproduce it.

11.2 Our position is now this: We have on the one hand geometrical objects, vectors v, k, operators $Z(p)$, $Y(p)$, and geometrical correspondences L. On the other hand, we have concrete n-tuples $[v]$, $[k]$, matrices $[Z(p)]$, $[Y(p)]$, and linear correspondences $[L]$. Given any pair of bases in **V** and **K**, in particular, given any coordinate frame, each geometric object generates a corresponding concrete object which represents it relative to those bases or that frame. Conversely, given a concrete object $[\xi]$, we can choose a frame in **V** and **K** and find that geometric object ξ whose coordinates in the chosen frame are given by $[\xi]$.

11.21* The concrete object, linear correspondence, defines a linear time-invariant $2n$-pole by 6.21. To complete the picture, we might say that a geometrical correspondence L defines a *Cauer class* of $2n$-poles.

11.22* This terminology is motivated by the following observation: if $[L]$ and $[L]_1$ are linear correspondences representing L in two distinct real frames, then there exists a real nonsingular matrix $[W]$ relating the

$$[[v], [k]] \epsilon [L](p)$$

and the

$$[[v]_1 , [k]_1] \epsilon [L]_1(p)$$

* Technical paragraph as explained in Section 2.91.

by the formulas of 10.5. This means that $[L]$ and $[L]_1$ are related like the $[L]$ and $[L_W]$ of 9.2. The $2n$-pole associated with $[L]_1$ therefore is a Cauer equivalent of that associated with $[L]$.

11.23 The observation of 11.22, combined with (ii) of 11.13, gives an alternative proof of 9.31. This proof is deceptively free of calculation, but of course the calculations are concealed in the extensive geometrical developments of Section 10, many of which are there offered on faith.

XII. THE FUNDAMENTAL LEMMA

12.0 This section is devoted to the statement, and the proof in part, of a lemma which, on the face of it, looks like an exercise in manipulating the postulates. In fact, the content of the lemma, and most of the details of its proof, are essential in what follows. To postpone them would force us into needless duplication of effort.

Lemma: Let L be a geometrical linear correspondence satisfying P1, P2, P4, P5(I), P6(I), P7(I) and the following weak form of P3(I):

P3′(I): If $p \epsilon \Gamma_L$, then $\mathbf{K}_L(p) = \mathbf{K}_L \supseteq (\mathbf{V}_{L0}(p))^0$.

Then there is a frequency domain $\Gamma'_L \subseteq \Gamma_L$, differing from Γ_L by a finite set, such that L satisfies all of the postulates for $p \epsilon \Gamma'_L$.

The statement of the dual result is evident and will be omitted.

The proof that L satisfies P3 will be given in this section. Verification of the remaining postulates will follow in paragraph 16.6.

We assume that the properties of positive real (PR) functions are known. They are summarized for later use in Section 15. We make occasional advance references thereto.

To the proof:

12.01 First, $\mathbf{K}_L$ is a real manifold and for $p \epsilon \Gamma_L$

$$\mathbf{K}_L \subseteq (\mathbf{V}_{L0}(p))^0. \tag{1}$$

This, with P3′(I), gives P3(I) for L.

Proof: $\mathbf{K}_L$ is real, as in 7.51. Consider now a $p \epsilon \Gamma_L$ and a $v \epsilon \mathbf{V}_{L0}(p)$; then $[v, 0] \epsilon L(p)$. Consider any real $j \epsilon \mathbf{K}_L$; then there is a $u \epsilon \mathbf{V}_L(p)$ such that $[u, j] \epsilon L(p)$. Now 0 and j are real. Hence by P5(I)

$$(v, j) = (u, 0) = 0.$$

Therefore any real $j \epsilon \mathbf{K}_L$ has a vanishing scalar product with every $v \epsilon \mathbf{V}_{L0}(p)$. Since $\mathbf{K}_L$ is real, it is spanned by real j and (1) follows.

12.1 By the dual of 7.7, if we know that

$$[v, k] \epsilon L(p),$$

then the value of (v, k) is determined by k. This makes it possible to state P6(I) and P7(I) for L (we take P6(I) to include the hypothesis that Γ_L has a finite complement).

12.11 If $k \epsilon \mathbf{K}_L$, then $J_k(p)$ is PR.

Proof: if k is real then

$$\overline{J_k(p)} = \overline{(v, k)} = (\bar{v}, k), \tag{2}$$

where, of course, $[v, k] \epsilon L(p)$. Then however $[\bar{v}, k] \epsilon L(\bar{p})$, by P4. Hence by 12.1, (2) gives us

$$\overline{J_k(p)} = J_k(\bar{p}).$$

From this and P6(I), P7(I) we conclude that $J_k(p)$ is PR for any real $k \epsilon \mathbf{K}_L$.

Now, given any $k \epsilon \mathbf{K}_L$, we have $\bar{k} \epsilon \mathbf{K}_L$ by 12.01. Then

$$k = k_1 + ik_2$$

where k_1 and k_2 are real and in $\mathbf{K}_L$, since $\mathbf{K}_L$ is a linear manifold (see 10.42). Let

$$[v_r, k_r] \epsilon L(p),$$

$r = 1, 2$. Then we have (P2)

$$[v_1 + iv_2, k] \epsilon L(p).$$

Then

$$J_k(p) = (v_1, k_1) + (v_2, k_2) + i(v_1, k_2) - i(v_2, k_1).$$

Now by P5(I), $(v_1, k_2) = (v_2, k_1)$. Hence

$$J_k(p) = (v_1, k_1) + (v_2, k_2) \tag{3}$$

for any $p \epsilon \Gamma_L$. Since each summand in (3) is a PR function, it follows that $J_k(p)$ is PR for any $k \epsilon \mathbf{K}$.

12.12 Let $\mathbf{K}_1$ be the set of all vectors $k \epsilon \mathbf{K}_L$ such that

$$J_k(p) = 0 \qquad \text{for every } p \epsilon \Gamma_L.$$

Notice that we do not assert that $\mathbf{K}_1$ is a linear manifold.

If $k \epsilon \mathbf{K}_1$ then $k \epsilon \mathbf{K}_L$ and (3) above applies. Then

$$(v_1, k_1) + (v_2, k_2) = 0$$

and, using this and the PR property of each summand, we conclude that k_1 and k_2 are in $\mathbf{K}_1$.

12.13 We wish now to show that $\mathbf{K}_1 \subseteq \mathbf{K}_{L0}(p)$. Consider a real $j \epsilon \mathbf{K}_L$ and a real $k \epsilon \mathbf{K}_1$. Let

$$\begin{aligned}&[u(p), j] \epsilon L(p),\\&[v(p), k] \epsilon L(p).\end{aligned} \tag{4}$$

Then, given any real λ, by P2

$$[u(p) + \lambda v(p), j + \lambda k] \epsilon L(p).$$

Then, because $k \epsilon \mathbf{K}_1$,

$$(u + \lambda v, j + \lambda k) = (u, j) + \lambda(v, j) + \lambda(u, k).$$

Since j and k are real, by P5(I) this can be written

$$(u + \lambda v, j + \lambda k) = (u, j) + 2\lambda(v, j). \tag{5}$$

Choose any p_1 such that $\text{Re}(p_1) \geq 0$. Then P7(I) implies that the left side of (5) has a non-negative real part at $p = p_1$. The right side, by suitable choice of λ, can have any chosen real part unless

$$\text{Re}(v(p_1), j) = 0. \tag{6}$$

Hence P7(I) implies (6). Now $(v(p), j)$ is a rational function, by P6(I) applied to the other members of (5). By (6), this rational function has a vanishing real part throughout the right half p-plane. Hence it is an imaginary constant:

$$(v(p), j) \equiv ia. \tag{7}$$

Then

$$\overline{(v(p), j)} = (\overline{v(p)}, j) = -ia. \tag{8}$$

But $[v(p), k] \epsilon L(p)$, so $[\overline{v(p)}, k] \epsilon L(\bar{p})$ by P4. Since also $[v(\bar{p}), k] \epsilon L(\bar{p})$, by 12.1, we have from (8) that

$$(v(\bar{p}), j) = -ia.$$

Comparing this with (7) written for $\bar{p}$, we have $a = 0$ and

$$(v(p), j) = 0 \quad \text{for} \quad p \epsilon \Gamma_L. \tag{9}$$

Now $v(p)$ was determined by (4) wherein k is real. For any $k \epsilon \mathbf{K}_1$, $k = k_1 + ik_2$, where k_1 and k_2 are real and in $\mathbf{K}_1$ (12.11). A corresponding $v(p)$ satisfying (4) can be written

$$v(p) = v_1(p) + iv_2(p), \tag{10}$$

by P2, where $[v_r(p),\ k_r] \epsilon L(p)$, $r = 1, 2$. Then (9) holds for each of $v_1(p)$, $v_2(p)$ and therefore also for the $v(p)$ of (10).

We have showed now that for any $p \epsilon \Gamma_L$ and any $k \epsilon \mathbf{K}_1$, the $v(p)$ of (4) has a vanishing scalar product with every real $j \epsilon \mathbf{K}_L$. Since $\mathbf{K}_L$ is spanned by real j,

$$v(p) \epsilon (\mathbf{K}_L)^0 = \mathbf{V}_{L0}. \tag{11}$$

12.14 By (11),

$$[v(p),\ 0] \epsilon L(p).$$

Comparing this with (4), and applying P2,

$$[v(p) - v(p),\ k - 0] = [0,\ k] \epsilon L(p).$$

Since k is now any vector in $\mathbf{K}_1$, we have

$$\mathbf{K}_1 \subseteq \mathbf{K}_{L0}(p) \subseteq \mathbf{K}_L \tag{12}$$

for every $p \epsilon \Gamma_L$.

12.15 We can now also show that $\mathbf{V}_L(p) \subseteq (\mathbf{K}_1)^0$. We return to 12.13 and read (9) thereof as originally derived for real j and k. Applying P5(I), we have from (9) that

$$(u(p),\ k) = 0 \qquad \text{for } p \epsilon \Gamma_L. \tag{13}$$

By the argument immediately following (9), (13) also holds for any $k \epsilon \mathbf{K}_1$, provided j is real. As in 12.11 any $j \epsilon \mathbf{K}_L$ can be written $j = j_1 + ij_2$, where j_1 and j_2 are real, and the corresponding

$$u(p) = u_1(p) + iu_2(p)$$

where $[u_r(p),\ j_r] \epsilon L(p)$. Therefore, finally, (13) holds for any $u(p)$ satisfying (4)—i.e., any $u(p) \epsilon \mathbf{V}_L(p)$—and any $k \epsilon \mathbf{K}_1$. Therefore

$$\mathbf{V}_L(p) \subseteq (\mathbf{K}_1)^0 \tag{14}$$

for any $p \epsilon \Gamma_L$.

12.2 We now fix our attention on a specific *real* $p_0 \epsilon \Gamma_L$

12.21 By P4, if

$$[v,\ k] \epsilon L(p_0)$$

we have also

$$[\bar{v},\ \bar{k}] \epsilon L(\bar{p}_0) = L(p_0).$$

In particular, $\mathbf{K}_{L0}(p_0)$ is real.

12.22 We can now show that $\mathbf{K}_1$ is a real linear manifold. Consider a real $k \epsilon \mathbf{K}_{L0}(p_0)$. Then $[0, k] \epsilon L(p_0)$ and by 12.1

$$J_k(p_0) = 0.$$

Then by 12.11 (and 15.12), $J_k(p) = 0$, so $k \epsilon \mathbf{K}_1$. Since $\mathbf{K}_{L0}(p_0)$ is spanned by real k (12.21), we have

$$\mathbf{K}_{L0}(p_0) \subseteq \mathbf{K}_1 .$$

Comparing this with (12) gives us

$$\mathbf{K}_{L0}(p_0) = \mathbf{K}_1 . \tag{15}$$

Since $\mathbf{K}_{L0}(p_0)$ is a real linear manifold by definition and 12.21, we see that $\mathbf{K}_1$ is.

12.3 Let us now write, by (12) and (15),

$$\mathbf{K}_L = \mathbf{K}_2 \oplus \mathbf{K}_1 \tag{16}$$

where $\mathbf{K}_2$ is an arbitrary fixed manifold disjoint from $\mathbf{K}_1$ and with it spanning $\mathbf{K}_L$. All three manifolds are real (12.21, (15), 10.6).

Choose a $\mathbf{K}_3$ disjoint from $\mathbf{K}_L$ such that

$$\mathbf{K} = \mathbf{K}_3 \oplus \mathbf{K}_2 \oplus \mathbf{K}_1 . \tag{17}$$

Let the decomposition of $\mathbf{V}$ dual to (17) be (10.6)

$$\mathbf{V} = \mathbf{V}_3 \oplus \mathbf{V}_2 \oplus \mathbf{V}_1 .$$

Then $\mathbf{V}_3 = (\mathbf{K}_2 \oplus \mathbf{K}_1)^0 = (\mathbf{K}_L)^0 = \mathbf{V}_{L0}$ by 12.01. Hence

$$\mathbf{V} = \mathbf{V}_{L0} \oplus \mathbf{V}_2 \oplus \mathbf{V}_1 . \tag{18}$$

By (14) and the definitions,

$$\mathbf{V}_{L0} \subseteq \mathbf{V}_L(p) \subseteq \mathbf{V}_{L0} \oplus \mathbf{V}_2 . \tag{19}$$

12.31 Consider a real p_0. Then by P3′(I), (15) and (16) we have

$$\mathbf{K}_{L0}(p_0) \subseteq \mathbf{K}_L(p_0) \subseteq \mathbf{K}_2 \oplus \mathbf{K}_{L0}(p_0). \tag{20}$$

This is now an expression dual to (19). We shall prove next that, given any $k \epsilon \mathbf{K}_L(p_0) \cap \mathbf{K}_2 (= \mathbf{K}_2)$, there is a unique $v_k \epsilon \mathbf{V}_L(p_0) \cap \mathbf{V}_2$ such that

$$[v_k , k] \epsilon L(p_0). \tag{21}$$

Dually, given any $v \epsilon \mathbf{V}_L(p_0) \cap \mathbf{V}_2$, there is a unique $k_v \epsilon \mathbf{K}_L(p_0) \cap \mathbf{K}_2$ such that

$$[v, k_v] \epsilon L(p_0).$$

The proof is a standard one in algebra and depends only upon P2, (19), and (20).

Proof: Given $k \epsilon \mathbf{K}_L(p_0) \cap \mathbf{K}_2$, there is some $v \epsilon \mathbf{V}_L(p_0)$ such that

$$[v, k] \epsilon L(p_0). \tag{22}$$

By (19), then,

$$v = v_0 + v_2$$

where $v_0 \epsilon \mathbf{V}_{L0}$, $v_2 \epsilon \mathbf{V}_2$. Then

$$[v_0, 0] \epsilon L(p_0)$$

so, applying P2 to this and (22),

$$[v - v_0, k - 0] = [v_2, k] \epsilon L(p_0). \tag{23}$$

Hence $v_2 \epsilon \mathbf{V}_L(p_0) \cap \mathbf{V}_2$ and $v_k = v_2$ satisfies (21). Suppose now $v_3 \epsilon \mathbf{V}_L(p_0) \cap \mathbf{V}_2$ and

$$[v_3, k] \epsilon L(p_0).$$

Then using this with (23) and P2

$$[v_2 - v_3, 0] \epsilon L(p_0).$$

Hence $(v_2 - v_3) \epsilon \mathbf{V}_{L0}$. Now $\mathbf{V}_L(p_0) \cap \mathbf{V}_2$ is a linear manifold and contains v_2, v_3. Hence

$$(v_2 - v_3) \epsilon \mathbf{V}_{L0} \cap \mathbf{V}_L(p_0) \cap \mathbf{V}_2 = 0.$$

Therefore $v_2 = v_3$.

The dual argument completes the proof.

12.32 The argument actually exhibited in 12.31 uses only P2 and (19), hence the v_k of (21) is unique whether or not p_0 is real. Indeed, this is true even when $k \epsilon \mathbf{K}_L$.

12.33 The result of 12.31 establishes a bi-unique linear mapping between $\mathbf{K}_2$ and $\mathbf{V}_L(p_0) \cap \mathbf{V}_2$. Hence these two manifolds are of the same dimension. Since $\mathbf{K}_2$ and $\mathbf{V}_2 = \mathbf{K}_2^*$ are of the same dimension by construction, it follows that

$$\mathbf{V}_L(p_0) \cap \mathbf{V}_2 = \mathbf{V}_2$$

and, by (19), that

$$\mathbf{V}_L(p_0) = \mathbf{V}_{L0} \oplus \mathbf{V}_2.$$

12.4 Let us now introduce a real frame in **V** and **K** which provides real bases in $\mathbf{K}_1$, $\mathbf{K}_2$, $\mathbf{K}_3$ and in $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_{L0}$ of (17) and (18). Let $k_1, \cdots, k_m$

be the basis vectors spanning $\mathbf{K}_2$. By 12.32, there are unique vectors $u_1(p), \cdots, u_m(p)$ in $\mathbf{V}_2$ such that

$$[u_r(p), k_r] \epsilon L(p).$$

Let $v_1, \cdots, v_m$ be the (real) basis vectors in $\mathbf{V}_2$ dual to the $k_1, \cdots, k_m$:

$$(v_r, k_s) = \delta_{rs} \qquad 1 \leq r \leq s. \tag{24}$$

Since the $u_r(p)$ are all in $\mathbf{V}_2$ we have for each $p \epsilon \Gamma_L$

$$u_s(p) = \sum_{r=1}^{m} a_{rs}(p) v_r \tag{25}$$

where the coefficients $a_{rs}(p)$ are calculated by (24) to be

$$a_{rs}(p) = (u_s(p), k_r). \tag{26}$$

12.41 Because the k_r are real, P5(I) implies that

$$a_{sr}(p) = (u_r(p), k_s) = (u_s(p), k_r) = a_{rs}(p). \tag{27}$$

By the reasoning just following (8) and by the uniqueness of the $u_s(p) \epsilon \mathbf{V}_2$, since $\mathbf{V}_2$ is real, we have $\overline{u_s(\bar{p})} = u_s(\bar{p})$. Then

$$\overline{a_{rs}(p)} = (\overline{u_s(p)}, k_r) = (u_s(\bar{p}), k_r) = a_{rs}(\bar{p}).$$

12.42 We have by P2 that

$$[u_r(p) + \lambda u_s(p), k_r + \lambda k_s] \epsilon L(p), \tag{28}$$

for any λ. The identity

$$\begin{aligned}(u_r + u_s, k_r + k_s) - (u_r - u_s, k_r - k_s)& \\ = 2(u_r, k_s) + 2(u_s, k_r)&\end{aligned} \tag{29}$$

holds in fact for any vectors u_r, u_s, k_r, k_s. Using (27), (28) and P6(I), it exhibits $a_{rs}(p)$ as a rational function.

12.5 Consider the $m \times m$ matrix $[Z_1(p)]$ whose elements are the $a_{rs}(p)$. the s-th column of this matrix consists of the components of $u_s(p)$. The rank of the matrix is by definition the dimension of the space spanned by these columns.

12.51 Now the rank of $[Z_1(p)]$ can be expressed in terms of the vanishing or not of its various minor determinants. There are finitely many such minors and each is a rational function. Each is either identically zero or else vanishes at only finitely many points. Hence the rank of $[Z_1(p)]$, except at these finitely many points, and at the p in the complement of Γ_L, is a constant. We call this constant the nominal rank of $[Z_1(p)]$.

12.52 Let Γ_L' consist of all $p \epsilon \Gamma_L$ where $[Z_1(p)]$ has its nominal rank. Then Γ_L' has a finite complement. By the reality result of 12.41, if $p \epsilon \Gamma_L'$ then $\bar{p} \epsilon \Gamma_L'$.

It is clear that at any $p \epsilon \Gamma_L$ the rank of $[Z_1(p)]$ does not exceed its nominal rank.

12.53 By construction, the vectors $u_1(p), \cdots, u_m(p)$ all lie in $\mathbf{V}_L(p) \cap \mathbf{V}_2$. By the reasoning of 12.33, at any real $p_0 \epsilon \Gamma_L$ they span $\mathbf{V}_2$. Hence the nominal rank of $[Z_1(p)]$ is m. Therefore, for any $p \epsilon \Gamma_L'$, $[Z_1(p)]$ has rank m and the $u_1(p), \cdots, u_m(p)$, lying in $\mathbf{V}_2$, still span $\mathbf{V}_2$. Therefore for all $p \epsilon \Gamma_L'$

$$\mathbf{V}_L(p) \cap \mathbf{V}_2 = \mathbf{V}_2 .$$

By (19), then,

$$\mathbf{V}_L(p) = \mathbf{V}_{L0} \oplus \mathbf{V}_2 = \mathbf{V}_L , \tag{30}$$

a fixed manifold, for all $p \epsilon \Gamma_L'$.

12.54 It is clear by its construction (cf. Halmos[1], par. 26) that $[Z_1(p)]$ describes the mapping of 12.32 from $\mathbf{K}_2$ to $\mathbf{V}_2 = \mathbf{V}_L(p) \cap \mathbf{V}_2$ by

$$[v_k] = [Z_1(p)][k].$$

Here the m-tuples $[v_k]$ and $[k]$ are the components of v_k and k relative to the bases now available in $\mathbf{V}_2$ and $\mathbf{K}_2$.

12.55 We repeat

$$\mathbf{K}_1 \subseteq \mathbf{K}_{L0}(p) \subseteq \mathbf{K}_L = \mathbf{K}_1 \oplus \mathbf{K}_2 . \tag{12}$$

Fix a $p \epsilon \Gamma_L'$ and a $k \epsilon \mathbf{K}_{L0}(p) \cap \mathbf{K}_2$. Then $[0, k] \epsilon L(p)$. Since $0 \epsilon \mathbf{V}_2$, it follows from 12.54 that $[Z_1(p)]$ annihilates k. Suppose $m \neq 0$. Since the rank of $[Z_1(p)]$ is m, it follows that $k = 0$. Hence for $p \epsilon \Gamma_L'$

$$\mathbf{K}_{L0}(p) \cap \mathbf{K}_2 = 0.$$

By (12), then, (31)

$$\mathbf{K}_{L0}(p) = \mathbf{K}_1 = \mathbf{K}_{L0} ,$$

a fixed manifold. This, with the result of 12.53, proves that L satisfies P3(A), when $m \neq 0$.

If $m = 0$ then $\mathbf{V}_2 = 0$, $\mathbf{K}_2 = 0$ and (31) follows from (12) and (16).

12.56 $[Z_1(p)]$ is of dimension m and rank m for any $p \epsilon \Gamma_L'$. Therefore the correspondence of 12.32 and 12.54 between $\mathbf{V}_2$ and $\mathbf{K}_2$ is bi-unique for any $p \epsilon \Gamma'_L$. This extends 12.31 to any $p \epsilon \Gamma'_L$.

12.57 If $m = 0$, i.e., if $\mathbf{V}_2 = \mathbf{K}_2 = 0$, then $\mathbf{V}_{L0} = (\mathbf{K}_{L0})^{\gamma}$ and the fact that L satisfies all the postulates is trivial because all scalar products (v, k) for $v \epsilon \mathbf{V}_L = \mathbf{V}_{L0}$ and $k \epsilon \mathbf{K}_L = \mathbf{K}_{L0}$ are zero. If $m \neq 0$, we have yet to show that L satisfies P5(A), P6(A), P7(A).

12.6 Since now L satisfies P3, 7.7 as given is applicable and we find (with 12.1) that if $p \epsilon \Gamma'_L$ and

$$[v, k] \epsilon L(p),$$

then (v, k) is fixed by either v or k. Furthermore,

$$(v, k) = (v + v_0, k + k_0)$$

for any $v_0 \epsilon \mathbf{V}_{L0}$, $k_0 \epsilon \mathbf{K}_{L0}$.

12.61 If $p \epsilon \Gamma'_L$ and $[v, k] \epsilon L(p)$, then $v \epsilon \mathbf{V}_L$, $k \epsilon \mathbf{K}_L$. By (30), (31), and (16), therefore, there exist $v_0 \epsilon \mathbf{V}_{L0}$, $k_0 \epsilon \mathbf{K}_{L0}$ such that $u = v - v_0 \epsilon \mathbf{V}_2$, $j = (k - k_0) \epsilon \mathbf{K}_2$. Then by P2

$$[u, j] \epsilon L(p). \tag{32}$$

By 12.6, then, any value assumed by a scalar product (v, k) with $[v, k] \epsilon L(p)$ is also assumed by a product (u, j), where (32) holds and $u \epsilon \mathbf{V}_2$, $j \epsilon \mathbf{K}_2$.

XIII. SUFFICIENCY OF THE POSTULATES

13.0 We suppose that L satisfies the postulates of 12.0. Then the results of Section 12 are applicable. The ones of first importance are contained in the facts from (15), (30) and (31), that

$$\mathbf{V}_L = \mathbf{V}_{L0} \oplus \mathbf{V}_2,$$

$$\mathbf{K}_L = \mathbf{K}_2 \oplus \mathbf{K}_{L0},$$

where the choice of $\mathbf{K}_2$ was governed only by the requirement that the second of these formulae hold.

13.01 Considering $\mathbf{K}_2$ and $\mathbf{V}_2$ as separate spaces, $\mathbf{V}_2 = \mathbf{K}_2^*$ by 10.6. Let M be the geometrical linear correspondence between them with frequency domain Γ'_L and pairs described by 12.31 and 12.56 (or 12.54). That is, as vectors in $\mathbf{V}_2$ and $\mathbf{K}_2$

$$[v, k] \epsilon M(p)$$

if and only if, as vectors in **V** and **K**,

$$[v, k] \epsilon L(p).$$

13.02 In the real frame of 12.4 let us renumber the basis vectors so that

$$v_1, \cdots, v_r \quad \text{span } \mathbf{V}_{L0},$$

$$v_{r+1}, \cdots, v_{r+m} \quad \text{span } \mathbf{V}_2,$$

$$v_{r+m+1}, \cdots, v_n \quad \text{span } \mathbf{V}_1.$$

Then

$$k_1, \cdots, k_{r+m} \quad \text{span } \mathbf{K}_2,$$

$$k_{r+m+1}, \cdots, k_n \quad \text{span } \mathbf{K}_{L0}.$$

We say that such a frame *reduces* L.

13.1 Let us now interpret the s-th components of $[v]$ and $[k]$ in this frame respectively as the voltage across and the current in an ideal branch β_s of a $2n$-pole **N**, $1 \leq s \leq n$.

By construction, the vectors $v \epsilon \mathbf{V}_L$ in this frame have components $a_{r+m+1} = \cdots = a_n = 0$, since $v_1, \cdots, v_{r+m}$ span $\mathbf{V}_L$. At the same time, the components $b_{r+m+1}, \cdots, b_n$ of $[k]$ may be chosen arbitrarily without altering the fact that $[[v], [k]] \epsilon [L](p)$ because of 12.06. Therefore, the ideal branches $\beta_{r+m+1}, \cdots, \beta_n$ can each be realized physically by a short circuit.

In a dual way, since $k_{r+1}, \cdots, k_n$ span $\mathbf{K}_L$, any $k \epsilon \mathbf{K}_L$ has components $b_1, \cdots, b_r$ all zero in our chosen frame. Furthermore, the components $a_1, \cdots, a_r$ of $[v]$ can be chosen at will. Hence the ideal branches $\beta_1, \cdots, \beta_r$ can each be realized physically by an open circuit.

Let $\mathbf{N}_1$ now be the $2m$-pole whose ideal branches are $\beta_{r+1}, \cdots, \beta_{r+m}$. Let the pairs $[[v], [k]]$ admitted by $\mathbf{N}_1$ at each $p \epsilon \Gamma_L'$ be the $[[v], [k]]$, where $[v, k] \epsilon M(p)$ (13.01). The representation just found for **N** shows that **N** is physically realizable if and only if $\mathbf{N}_1$ is.

13.11 The matrix $[Z_1(p)]$ of 12.54 is the impedance matrix of the $2m$-pole $\mathbf{N}_1$.

13.12 We now show that $[Z_1(p)]$ is a positive real matrix. The displayed formulae of 12.41 show (ii) and (iii) of 1.1, and 12.42 shows (i). Now suppose that $[v, k] \epsilon M(p)$. Then, as vectors in **V** and **K**, $[v, k] \epsilon L(p)$ by definition of $M(p)$. Then, however, if k is fixed

$$J_k(p) = (v, k)$$

is a PR function (12.11). Regarding v and k in $\mathbf{V}_2$ and $\mathbf{K}_2$ let

$$[b_{r+1}, \cdots, b_{r+m}] = [k].$$

Then by (1) of 7.0

$$(v, k) = \sum_{t=1}^{m} \sum_{s=1}^{m} a_{st}(p) b_{t+r} \bar{b}_{s+r}$$

and this has a non-negative real part if $\mathrm{Re}(p) > 0$. This is (iv) of 1.1·

13.2 We can now prove the lemmas 8.1 and 8.2. Given a linear correspondence $[L]$ which satisfies P1, $\cdots$, P7 by 11.13 we can interpret $[L]$ as the concrete correspondence representing a geometrical correspondence L in some chosen real frame, and L satisfies P1, $\cdots$, P7. Then by the results in 13.01–13.12 there exists a real frame in which the representative $[L]_1$ of L has the properties claimed in 8.1 and 8.2 for L_W. But we saw in 11.22 that $[L]$ and $[L]_1$ are related by a real matrix W like the L and L_W of Section 8. Q.E.D.

13.21 With the proofs of 8.1 and 8.2 we have reduced the sufficiency claimed for P1, $\cdots$, P7 in 8.0 to the sufficiency of positive reality of $[Z(p)]$ claimed in 1.1, by the argument outlined in 8.5.

XIV. OPERATOR-VALUED FUNCTIONS OF p

The next three sections are directed principally toward the proof of the matrix theorem of 1.1. They do however, contribute to 12.10 and to the necessity proof.

14.0 We continue to use the geometric language. The reader who regards this as unduly pedantic is free to place a concrete interpretation upon every argument, for all of the arguments are either frankly based on matrix representations or upon the three identities:

14.01 $(Zj, k) = \overline{(Z^*k, j)}$ for all $j, k \epsilon \mathbf{K}$.

1402 $\bar{Z}\bar{k} = \overline{(Zk)}$ for all $k \epsilon \mathbf{K}$.

14.03 $Z' = (\bar{Z})^* = \overline{(Z^*)}$

14.04 These identities are obvious for matrices using 7.0 and 7.2. Geometrically, the first and second define Z^* and $\bar{Z}$, and the third defines Z' in two ways. The equivalence of these two ways is a theorem based on (10) of 10.33.

14.05 The symbol Z will always denote an impedance (operator, matrix, scalar), and Y will always denote an admittance. An impedance operates from **K** to **V**, an admittance dually. The operators in Halmos[9] are physically dimensionless, in that they operate, e.g., from **V** to **V**. This difference is scarcely noticeable.

We shall regularly omit the duals to concepts or proofs given in terms of impedances. In doing so, we adopt the rule that the dual to an expression

$$(Zk, k)$$

is

$$\overline{(v, Yv)}.$$

14.1 An operator is called symmetric if $Z = Z'$. Such operators have three useful special properties:

14.11 If Z is symmetric and j and k are real, then

$$(Zj, k) = \overline{(\overline{Z}j, k)} = ((\overline{Z})^*k, j) = (Z'k, j) = (Zk, j)$$

by (10) of 10.33, 14.02, 14.01, 14.03, and hypothesis.

14.12 Let $k = k_1 + ik_2$, where k_1 and k_2 are real (10.42). If Z is symmetric then

$$(Zk, k) = (Zk_1, k_1) + (Zk_2, k_2),$$

for, by 14.11,

$$\begin{aligned}(Zk_1, ik_2) = -i(Zk_1, k_2) &= -i(Zk_2, k_1) \\ &= -(Z(ik_2), k_1).\end{aligned}$$

(Cf. the similar identity in 12.11.)

14.13 The symmetric operator Z is completely defined by the quadratic form

$$(Zk, k) \tag{1}$$

as a function of *real* $k \epsilon \mathbf{K}$. For 14.11 permits the formula (29) of 12.42 in any real frame, where $u_s = Zk_s$. The matrix elements of $[Z(p)]$ in that frame are then defined by that formula in terms of values of (1) for real k.

The form (1) specifies any Z (symmetric or not) if k is allowed to range over all of **K** (Halmos[9], par. 53).

14.2 Let $Z(p)$ now be an impedance operator depending on p. We say that $p_0 \neq \infty$ is a pole of order m of $Z(p)$ if

$$\ell(k) = \lim_{p \to p_0} (p - p_0)^m (Z(p)k, k) \tag{2}$$

exists for every $k \epsilon \mathbf{K}$ and is not identically zero. By 15.13, this limit $\ell(k)$ defines an operator R_0, the residue* of $Z(p)$ at p_0, by

$$(R_0 k, k) = \ell(k) \quad \text{for} \quad k \epsilon \mathbf{K}.$$

The changes in (2) required to define a pole at $p = \infty$ are obvious.

14.21 A pole p_0 of order m of $Z(p)$ is a pole of some matrix element of $[Z(p)]$, of order m, in any frame, and no element of $[Z(p)]$ has a pole at p_0 of order exceeding m. For the elements of $[Z(p)]$ are defined by the values of $(Z(p)k, k)$, by 14.11 and Halmos[9] loc. cit.

XV. POSITIVE REAL FUNCTIONS

15.0 Let $f(p)$ be a scalar function of the complex variable p. Following Brune[2] we define $f(p)$ to be positive real if

(i) $f(p)$ is a rational function of p,

(ii) $\overline{f(p)} = f(\bar{p})$,

(iii) $\text{Re}(p) > 0$ implies $\text{Re}(f(p)) \geq 0$.

The property (i) of being rational is of course on a quite different level of ideas from the other properties, but it saves words later to include it specifically in the meaning of positive real.

We abbreviate the words positive real to PR.

15.01 The open region of the complex plane consisting of all finite p such that $\text{Re}(p) > 0$—the right half plane—we denote by Γ_+.

15.1 Brune, loc. cit., established a number of properties of PR functions $f(p)$ which will be useful to us here:

15.11 $f(p)$ has no poles in Γ_+.

15.12 If $\text{Re}(f(p)) = 0$ for some $p \epsilon \Gamma_+$, then $f(p) \equiv 0$ for all p.

15.13 If it exists, $\dfrac{1}{f(p)}$ is PR.

15.14 If $f(p)$ has a pole at $p = p_0$, it has one at $p = \overline{p_0}$.

15.15 If $f(p)$ has a pole at $p = i\omega_0$, that pole is simple and

$$f(p) = \frac{2p}{p^2 + \omega_0^2} r + f_1(p),$$

where $r > 0$, and $f_1(p)$ is PR.

* Properly, R_0 is a residue only when $m = 1$. There is no convenient name available for general m.

15.16 If $f(p)$ has a pole at $p = \infty$, that pole is simple and

$$f(p) = pr + f_1(p),$$

where $r > 0$, and $f_1(p)$ is PR.

15.17 We shall use all of these in the next section, save 15.13. Our aim is to prove properties analogous to 15.11, $\cdots$, 15.16 for PR matrices and operators.

The reader familiar with the Brune process[2] for realization of a 2-pole will remember the importance of the properties 15.11, $\cdots$, 15.16 for the success of that process. Correspondingly, we must establish the analogs of these properties to implement the general Brune process for $2n$-poles.

XVI. POSITIVE REAL OPERATORS

16.0 An operator $Z(p)$ from $\mathbf{K}$ to $\mathbf{V}$ will be called positive real (PR) if in some real coordinate frame the matrix $[Z(p)]$ is a PR matrix in the sense of 1.1—that is

(i) $[Z(p)]$ has rational elements $Z_{rs}(p)$

(ii) $\overline{Z_{rs}(p)} = Z_{rs}(\bar{p})$

(iii) $Z_{rs}(p) = Z_{sr}(p)$

(iv) For any real $k \epsilon \mathbf{K}$ and any $p \epsilon \Gamma_+$

$$\mathrm{Re}(Z(p)k, k) \geq 0.$$

We intend in this section to establish for PR operators the properties listed below. By subtracting 0.9 from the designation of each property one obtains the designation of the analogous property of a PR scalar function, stated earlier.

16.01 $Z(p)$ has no poles in Γ_+.

16.02 If $\mathrm{Re}(Z(p)k, k) = 0$ for some $p \epsilon \Gamma_+$, then $Z(p)k \equiv 0$ for all p.

16.03 If it exists, $Z^{-1}(p) = Y(p)$ is PR.

16.04 If $Z(p)$ has a pole at $p = p_0$, it has one at $p = \bar{p}_0$.

16.05 If $Z(p)$ has a pole at $p = i\omega_0$, that pole is simple* and

$$Z(p) = \frac{2p}{p^2 + \omega_0^2} R + Z_1(p),$$

where R is real, symmetric, and semi-definite, not zero, and $Z_1(p)$ is PR.

* i.e., of order one.

16.06 If $Z(p)$ has a pole at $p = \infty$, that pole is simple and

$$Z(p) = pR + Z_1(p)$$

where $R = R' = \bar{R}$, $R \geq 0$ and $Z_1(p)$ is PR.

16.07 There is property of rational scalar functions $f(p)$, whether PR or not, that is essential in the Brune theory: the existence of a finite integer, the degree of f. Each step in the Brune reduction of $f(p)$ leaves an unreduced portion which is of lower degree than the function upon which the step was performed. The finiteness of the original degree of f then guarantees the termination of the process in finitely many steps.

There exists also for rational matrices (and operators) a concept of degree. This degree plays the same role in the general Brune process for $2n$-poles as the degree of a scalar function does in the process for 2-poles. To define this degree and develop its properties requires an excursion into classical algebra. Since we shall not need these ideas until Part II we defer further discussion of them to that part.

16.1 If $Z(p)$ is PR it follows at once that the matrix $[Z(p)]$ is PR in any real frame.

Proof: Two such matrices are related by

$$[Z(p)]_1 = [U][Z(p)][U]'$$

where U is real, by 11.22 and the argument in 8.6. The PR properties of $[Z(p)]$ are obviously preserved by this operation.

16.11 If $Z(p)$ is PR, then

$$Z(p) = Z'(p) = \overline{Z^*(p)} = \overline{Z(\bar{p})}.$$

Proof: Use 16.0 and 14.03 in a real frame.

16.12 If $Z(p)$ is PR, then for any given $k \epsilon \mathbf{K}$ the function

$$J_k(p) = (Z(p)k, k)$$

is a PR scalar function. It follows that the limitation in (iv) of 16.0 to real k is a simplification, not a restriction.

Proof: $J_k(p)$ is independent of coordinate representation. By use of a real frame, (i) of 16.0 implies (i) of 15.0.

By 14.01 and 16.11

$$\overline{J_k(p)} = (Z^*(p)k, k) = (Z(\bar{p})k, k) = J_k(\bar{p}).$$

This is (ii) of 15.0. For any k, 14.12 and (iv) of 16.0 imply (iii) of 15.0.

16.13 Conversely to 16.12, if $Z(p)$ is symmetric and $J_k(p)$ is PR for every real k, then $Z(p)$ is PR, and $J_k(p)$ is PR for all k.

Proof: $J_k(p)$ is rational so (i) of 16.0 holds in any frame by 14.13. Clearly (iv) of 16.0 holds.

Now for real k, by (10) of 10.33 and 14.02

$$J_k(\bar{p}) = \overline{J_k(p)} = \overline{(Z(p)k, k)}.$$

Hence $Z(\bar{p}) = \overline{Z(p)}$ by 14.13. This is (ii) of 16.0, and (iii) there holds by hypothesis.

16.2 *Proof of* 16.01: By 15.11 and 16.12, $J_k(p)$ has no poles in Γ_+. This is 16.01 by the definition 14.3 of pole.

16.21 *Corollary:* Any PR $Z(p)$ can be considered as defined throughout Γ_+: for any k, $J_k(p)$ is defined throughout Γ_+ by 16.2. For each p, as a function of k, $J_k(p)$ defines $Z(p)$ (14.13).

16.3 *Proof of* 16.03: In any frame $[Z^{-1}(p)] = [Z(p)]^{-1} = [Y(p)]$ consists of rational elements, by direct calculation of the inverse matrix. In a real frame $[Y(p)] = [Z^{-1}(p)]$ is symmetric and real for real p by the same argument (both facts are also deducible geometrically). Hence we have the duals of (i), (ii) and (iii) of 16.0 for $Y(p)$. Clearly $Y(p)$ is defined throughout Γ_+.

Now suppose that for some $v \epsilon \mathbf{V}$ and some $p_0 \epsilon \Gamma_+$ we have

$$\mathrm{Re}\overline{(v, Y(p_0)v)} < 0.$$

Then there is a $k \epsilon \mathbf{K}$ such that $v = Z(p_0)k$. Therefore

$$\mathrm{Re}\overline{(Z(p_0)k, k)} = \mathrm{Re}(Z(p_0)k, k) < 0.$$

Since this is impossible, we have the dual of (iv) of 16.0 for $Y(p)$ and $Y(p)$ is PR.

16.4 *Proof of* 16.04: This is immediate from 15.14, 14.3, and 16.12.

16.5 *Proofs of* 16.05 *and* 16.06: Suppose $Z(p)$ has a pole at $p = i\omega_0$. Then $(Z(p)k, k)$ does and that pole is simple by 15.15 and 16.12. Then by 14.3 we can write

$$Z(p) = \frac{1}{p - i\omega_0} R_0 + Z_0(p)$$

where $Z_0(p)$ is regular at $p = i\omega_0$. Now $Z_0(p)$ has a pole at $p = -i\omega_0$ by 16.5, so a similar argument gives

$$Z(p) = \frac{1}{p - i\omega_0} R_0 + \frac{1}{p + i\omega_0} R_1 + Z_1(p), \tag{1}$$

where $Z_1(p)$ has no pole at $i\omega_0$ or $-i\omega_0$. The symmetry of Z and linear independence of the terms above then imply the symmetry of R_0, R_1 and $Z_1(p)$.

For any $k \epsilon \mathbf{K}$, now,

$$(Z(p)k, k) = \frac{1}{p - i\omega_0}(R_0 k, k) + \frac{1}{p + i\omega_0}(R_1 k, k) + (Z_1(p)k, k).$$

Applying 16.12 and 15.15,

$$(R_0 k, k) = (R_1 k, k) \geq 0$$

for all k. Hence $R_0 = R_1 = R$ (say) and R is semi-definite. Also, $(Z_1(p)k, k)$ appears as the residue $f_1(p)$ in 15.15 and is therefore PR. Then $Z_1(p)$ is PR by 16.13. With R_0 and R_1 identified, (1) above is the expansion given in 16.05. We have now proved all of 16.05 save the reality of R. But

$$\frac{2p}{p^2 + \omega_0^2} R$$

is PR, by 16.13, hence is real for real p. Therefore R is real.

The proof of 16.06 is similar.

16.6 To prove 16.02 we appear to digress somewhat, by first completing the proof of the fundamental lemma of 12.0. It was established in Section 13 that the matrix $[Z_1(p)]$ describing $M(p)$ in the chosen basis is PR. The case in which it is nonsingular (i.e., $m \neq 0$, cf. 12.56, 12.57) remains to be examined.

16.61 If $[Z_1(p)]$ is nonsingular then its inverse is PR (16.3). Then for any $v \epsilon \mathbf{V}_2$,

$$\overline{(v, k)} = \overline{(v, Y(p)v)} \tag{2}$$

is PR (16.12 dual). By 12.61, for any $u \epsilon \mathbf{V}_L$, the values of the function $F_u(p)$ are the values of (2) for some $v \epsilon \mathbf{V}_2$. Hence $F_u(p)$ is PR. This is P6(A) and P7(A) for L.

16.62 To settle P5 for L in 12.0, consider $p \epsilon \Gamma_L'$ and

$$[v, k] \epsilon L(p), \qquad [u, j] \epsilon L(p),$$

where u and v are real. Then, say,

$$v = v_0 + v_1,$$

where $v_0 \epsilon V_{L0}$, $v_1 \epsilon V_2$. But then

$$v = \bar{v} = \bar{v}_0 + \bar{v}_1$$

and, because $\mathbf{V}_{L0}$ and $\mathbf{V}_2$ are real, $\bar{v}_0 = v_0$, $\bar{v}_1 = v_1$, and these vectors are real. Using similar reasoning for u,

$$(v, j) = (v_1, Y(p)u_1), \qquad (u, k) = (u_1, Y(p)v_1), \tag{3}$$

by 12.61. The equality $(u, k) = (v, j)$ now follows from (3) and the duals of 16.11, 14.11. Hence we have P5(A) for L and 12.0 is proved.

16.7 We now prove an important

Lemma: Let $Z(p)$ be a PR operator from $\mathbf{K}$ to $\mathbf{V}$. Let Γ_L be the set of p where $Z(p)$ is defined and has a rank equal to its nominal rank. Let L be the correspondence with domain Γ_L and pairs

$$[Z(p)k, k], \qquad k \epsilon \mathbf{K}_L .$$

Then L satisfies P1, $\cdots$, P7.

Proof: L satisfies P1 and P2 (6.3). Γ_L satisfies P4 by the argument of 12.52. Then L satisfies P4, for by 16.11

$$\overline{Z(p)k} = Z(\bar{p})\bar{k}.$$

L satisfies P5(I) by 14.11 and 16.11. Γ_L satisfies P6 by 12.52. Then L satisfies P6(I) and P7(I) by 16.12. The fundamental lemma, 12.0, now proves that L satisfies all the postulates.

16.71 We call a correspondence satisfying all the postulates PR.

16.72 *Proof of* 16.02: Suppose $\text{Re}(Z(p_0)k, k) = 0$ for some $p_0 \epsilon \Gamma_+$. Because this function of p is PR (16.12) we have

$$J_k(p) = (Z(p)k, k) \equiv 0.$$

Hence $k \epsilon \mathbf{K}_1 = \mathbf{K}_{L0}$ (12.12, 12.55). Hence $[0, k] \epsilon L(p)$ for every $p \epsilon \Gamma_L$. That is

$$Z(p)k = 0 \quad \text{for} \quad p \epsilon \Gamma_L .$$

16.73 *Corollary:* If $Z(p_0)k = 0$ for some $p_0 \epsilon \Gamma_+$, then $Z(p)k \equiv 0$. For the hypothesis here implies that of 16.72. This is the analog of 15.12; the result of 16.02 is stronger.

16.8 An important consequence of 16.7 is the

*Lemma**: If $Z(p)$ is PR and of rank m, then there exists a real coordinate frame in which the matrix $[Z(p)]$ is an $m \times m$ nonsingular PR matrix $[Z_1(p)]$ bordered by zeros.

---

* Proved by Cauer[5].

Proof: Consider the PR correspondence L defined by $Z(p)$. Then $\mathbf{V}_{L0} = 0$, because $Z(p)0 = 0$ for every $p \epsilon \Gamma_L$. Consider the real frame of 13.02. $[Z(p)]$ in this frame takes any of $k_{r+m+1}, \cdots, k_n$ into 0 because these span $\mathbf{K}_{L0}$. Within $\mathbf{K}_2$, $[Z(p)]$ must describe the same operation as the $[Z_1(p)]$ of 12.54. Because $[Z(p)]$ is symmetric the lemma follows.

XVII. THE JUXTAPOSITION OF CORRESPONDENCES

17.0 This section and the next will consider ways of constructing new correspondences from old. This will provide the basis of the necessity proof of Section 19.

17.01 It is obvious that if two physical networks are set side by side and their accessible terminals regarded as the terminals of a single larger network, that enlarged network is again a physical network. This is the gist of the present section.

17.1 Suppose that

$$\mathbf{V} = \mathbf{V}_1 \oplus \mathbf{V}_2, \qquad \mathbf{K} = \mathbf{K}_1 \oplus \mathbf{K}_2,$$

where $\mathbf{K}_i = \mathbf{V}_i^*$ and all spaces are real (10.6). Let E_1 project on $\mathbf{V}$ along $\mathbf{V}_2$ (Halmos[9], par. 33) and $E_2 = 1 - E_1$ project on $\mathbf{V}_2$ along $\mathbf{V}_1$. Then E_i^* projects on $\mathbf{K}_i$ along $\mathbf{K}_j$, $j \neq i$ (Halmos[9], loc. cit.). It is easily verified that $E_i = \overline{E}_i$, $E_i^* = \overline{E}_i^*$, from the analog of 14.02 for dimensionless operators.

Considering $\mathbf{V}_i$ and $\mathbf{K}_i$ as separate spaces, let L_i be a geometrical linear correspondence between them with frequency domain Γ_i, $i = 1, 2$.

Consider the correspondence L between $\mathbf{V}$ and $\mathbf{K}$ defined by

(i) The frequency domain $\Gamma_L = \Gamma_1 \cap \Gamma_2$

(ii) $[v, k] \epsilon L(p)$ if and only if $[E_i v, E_i^* k] \epsilon L_i(p)$, $\qquad i = 1, 2$.

In (ii), of course, we regard $E_i v$ and $E_i^* k$ as elements of $\mathbf{V}_i$, $\mathbf{K}_i$.

17.11 L so defined is called the juxtaposition of L_1 and L_2.

17.2 *Lemma:* L is PR if and only if each of L_1 and L_2 is PR.

17.21 *Proof of* "if": It is clear that L satisfies P1 and P2. Further notation is now simplified if we put $L_1 = M$, $L_2 = N$. Consider the manifolds

$$\mathbf{V}_M \oplus \mathbf{V}_N, \qquad \mathbf{V}_{M0} \oplus \mathbf{V}_{N0}, \qquad \mathbf{K}_M \oplus \mathbf{K}_N, \qquad \mathbf{K}_{M0} \oplus \mathbf{K}_{N0},$$

where $\mathbf{V}_M \subseteq \mathbf{V}_1$ is the manifold of voltages admitted by $L_1 = M$ considered as a correspondence between $\mathbf{V}_1$ and $\mathbf{K}_1$, and $\mathbf{V}_{M0}$ the manifold of voltages $v \epsilon \mathbf{V}_1$ such that $[v, 0] \epsilon L_1(p)$ for all $p \epsilon \Gamma_1$. Dual definitions for $\mathbf{K}_M$, $\mathbf{K}_{M0}$, and symmetrical ones for $\mathbf{V}_N$, $\cdots$, $\mathbf{K}_{N0}$ need not be repeated.

It is clear from these definitions that the four manifolds above are, in the order listed, the manifolds

$$\mathbf{V}_L, \mathbf{V}_{L0}, \mathbf{K}_L, \mathbf{K}_{L0}$$

for L. Now, for example,

$$(\mathbf{K}_{L0})^0 = (\mathbf{K}_{M0} \oplus \mathbf{K}_{N0})^0 = (\mathbf{K}_{M0})^0 \cap (\mathbf{K}_{N0})^0$$

by 10.6. This last manifold, *in* $\mathbf{V}$, is $(\mathbf{V}_M \oplus \mathbf{V}_2) \cap (\mathbf{V}_N \oplus \mathbf{V}_1)$, byP3 for M and N, and by 10.6. But by direct calculation

$$(\mathbf{V}_M \oplus \mathbf{V}_2) \cap (\mathbf{V}_N \oplus \mathbf{V}_1) = \mathbf{V}_M \oplus \mathbf{V}_N = \mathbf{V}_L .$$

The dual of this result then completes P3 for L.

P4 for L is immediate because the E_i and E_i^* are real.

The duality of the decompositions of $\mathbf{V}$ and $\mathbf{K}$ implies the identity

$$(v, k) = (E_1 v, E_1^* k) + (E_2 v, E_2^* k)$$

(that is $E_1 E_2 = E_2 E_1 = 0$, and dually. This is Halmos[9], par. 33). All of P5, P6, and P7 for L follow at once from this identity.

17.22 The "only if" of 17.2 is a special case of the result of Section 18. Its proof will be deferred to 18.4.

17.23 It is obvious that the notion of juxtaposition and the lemma of 17.2 extend to juxtapositions of more than two correspondences.

17.3 Even without the "only if" part of 17.2, we have enough for the following characterization of PR correspondences:

Theorem: A correspondence L is PR if and only if it is the juxtaposition of

(i) a correspondence defined by a nonsingular PR matrix between a $\mathbf{V}_1$ and a $\mathbf{K}_1 = \mathbf{V}_1^*$,

(ii) a correspondence consisting of short circuits: that is of pairs $[0, k]$ for all $k \epsilon \mathbf{K}_2$ and all p,

(iii) a correspondence consisting of open circuits: that is, of pairs $[v, 0]$ for all $v \epsilon \mathbf{V}_3$ and all p.

Proof: If L is PR, the decomposition indicated is that of 13.1, 13.11, 13.12. If L is the juxtaposition indicated, then it is PR by 16.6 and the "if" in 17.1, provided the short and open circuits are PR correspondences. The verification of the postulates for these latter is easy and will be omitted.

17.31 The labor of considering PR correspondences instead of matrices has yielded the disappointingly simple result of 17.3. We have already been warned of this, however, by our knowledge of the properties of physical networks (2.9).

XVIII. THE OPERATION OF RESTRICTION

18.0 In addition to juxtaposition, which is an operation on correspondences clearly motivated by physical considerations, there is an operation, here called restriction, which has important use in the next section. There the physical meaning of the operation will become clear.

18.1 Let **V** and **K** = **V*** be a pair of dual spaces. Let **U** and **J** = **U*** be another pair. Suppose that C is a given fixed linear operation from **J** to **K**: given any $j \epsilon \mathbf{J}$, there is a unique $k(j) \epsilon \mathbf{K}$, written

$$k(j) = Cj,$$

such that if $k_r = Cj_r$, $r = 1, 2$, then

$$a_1k_1 + a_2k_2 = C(a_1j_1 + a_2j_2)$$

for any complex scalars a_1 , a_2 .

18.11 Let $(v, k)_1$ denote the scalar product between **V** and **K**, and $(u, j)_2$ that between **U** and **J**. Given C, and any $v \epsilon \mathbf{V}$, let us find that unique vector $u(v) \epsilon \mathbf{U}$ for which

$$(u(v), j)_2 = (v, Cj)_1 \tag{1}$$

for every $j \epsilon \mathbf{J}$. That such a vector $u(v)$ exists and is unique follows from 10.13 when we notice that the right-hand side of (1) defines a function conjugate linear in j. Now for fixed j, the right-hand side of (1) is linear in v, hence so also is the left side. That is, there is a linear operation C^* from **V** to **U** such that

$$u(v) = C^*v.$$

The following chart illustrates the situation:

$$\begin{array}{ccc} \mathbf{V} & & \mathbf{K} \\ C^* \downarrow & & \uparrow C \\ \mathbf{U} & & \mathbf{J} \end{array}$$

18.12 We suppose now that C takes real j into real k, i.e., that C is real. Then by (1)

$$(\overline{C^*v}, j)_2 = \overline{(C^*v, j)_2} = \overline{(v, C\bar{j})_1} = (\bar{v}, Cj)_1 .$$

By comparison with (1), we have

$$\overline{C^* v} = C^* \bar{v}.$$

Hence C^* also takes real vectors into real vectors and is real.

18.2 Now let L be a PR correspondence between $\mathbf{V}$ and $\mathbf{K}$. We define one, say M, between $\mathbf{U}$ and $\mathbf{J}$, as follows: For each $p \epsilon \Gamma_L$, let $M(p)$ consist of all pairs

$$[u, j]$$

such that $u = C^* v$ and

$$[v, Cj] \epsilon L(p).$$

This definition can be illustrated by enlarging the chart of 18.11:

$$\begin{array}{ccc} \mathbf{V} & \xleftarrow{L} & \mathbf{K} \\ C^* \downarrow & & \uparrow C \\ \mathbf{U} & \xleftarrow[M]{} & \mathbf{J} \end{array}$$

The u's corresponding to $j \epsilon \mathbf{J}$ can be constructed by going around through C, L and C^*. This then defines a direct mapping from $\mathbf{J}$ to $\mathbf{U}$.

18.21 We call the M defined by 18.2 a restriction of L, since its pairs are images under C^* and C^{-1} (which is not defined over all of $\mathbf{K}$) of a restricted set of pairs drawn from L.

18.22 Clearly there is a dual operation defined by an operator D from $\mathbf{U}$ to $\mathbf{V}$. We might distinguish the operation of 18.2 by calling it a current restriction, its dual by calling it a voltage restriction.

18.23 The restriction M of L is defined by lists $M(p)$ which exist for any $p \epsilon \Gamma_L$. The frequency domain of M has not yet been specified, however.

18.3 *Theorem:* If L is PR, then there is a frequency domain Γ_M for M such that M is PR.

Proof: P1 and P2 for M are evident at once, for any $p \epsilon \Gamma_L$. The remainder of the proof is divided among 18.31, $\cdots$, 18.37 below.

18.31 For P3, let $\mathbf{J}_M$ be all $j \epsilon \mathbf{J}$ such that $Cj \epsilon \mathbf{K}_L$. Then, given $j \epsilon \mathbf{J}_M$, for each $p \epsilon \Gamma_L$ there is a v such that

$$[v, Cj] \epsilon L(p),$$

whence

$$[C^* v, j] \epsilon M(p).$$

Therefore $\mathbf{J}_M(p)$, the space of currents admitted by M at frequency p, coincides with the fixed $\mathbf{J}_M$ at each $p \epsilon \Gamma_L$.

Clearly $\mathbf{J}_M$ is a real linear manifold.

18.32 Consider now $\mathbf{U}_{M0}(p)$: if $[u, 0] \epsilon M(p)$, then there is a v such that $u = C^*v$ and

$$[v, C0] = [v, 0] \epsilon L(p).$$

Hence $v \epsilon \mathbf{V}_{L0}(p) = \mathbf{V}_{L0}$ for each $p \epsilon \Gamma_L$. Therefore, for each $p \epsilon \Gamma_L$,

$$\mathbf{U}_{M0}(p) \subseteq C^* \mathbf{V}_{L0}. \tag{2}$$

Now suppose, conversely, that $p \epsilon \Gamma_L$ and $v \epsilon \mathbf{V}_{L0} = \mathbf{V}_{L0}(p)$. Then $[v, 0] \epsilon L(p)$. Now $0 = C0$, so $[v, C0] \epsilon L(p)$. Hence $[C^*v, 0] \epsilon M(p)$, so $C^*v \epsilon \mathbf{U}_{M0}(p)$. This proves the inequality opposite to that of (2), so for $p \epsilon \Gamma_L$

$$\mathbf{U}_{M0}(p) = C^* \mathbf{V}_{L0} = \mathbf{U}_{M0}, \tag{3}$$

a fixed space.

18.33 Now consider $(\mathbf{U}_{M0})^0$. If $j \epsilon (\mathbf{U}_{M0})^0$, then

$$(u, j)_2 = 0$$

for every $u \epsilon \mathbf{U}_{M0}$. That is, by (3),

$$(C^*v, j)_2 = (v, Cj)_1 = 0$$

for every $v \epsilon \mathbf{V}_{L0}$. Therefore $Cj \epsilon (\mathbf{V}_{L0})^0 = \mathbf{K}_L$, and $j \epsilon \mathbf{J}_M$ by 18.31. That is, we have proved

$$\mathbf{J}_M \supseteq (\mathbf{U}_{M0})^0,$$

and, combining 18.31 with this and (3),

$$\mathbf{J}_M(p) = \mathbf{J}_M \supseteq (\mathbf{U}_{M0}(p))^0 = (\mathbf{U}_{M0})^0. \tag{4}$$

This is the weak form P3′(I) of 12.0 for M. It is as far as we can go with P3 at the moment.

18.34 Consider P4. If for $p \epsilon \Gamma_L$ we have

$$[u, j] \epsilon M(p)$$

then $[v, Cj] \epsilon L(p)$ and $u = C^*v$. But then $[\bar{v}, C\bar{j}] \epsilon L(\bar{p})$ and $\bar{u} = C^*\bar{v}$, by 18.12. Then however

$$[\bar{u}, \bar{j}] \epsilon M(\bar{p})$$

by definition of M. This is P4.

18.35 Consider P5(I): if

$$[u_r, j_r] \epsilon M(p),$$

where j_r is real, $r = 1, 2$, then

$$(u_r, j_s)_1 = (C^* v_r, j_s)_1 = (v_r, Cj_s)_1, \tag{5}$$

where $[v_r, Cj_r] \epsilon L(p)$. Since Cj_r is real

$$(v_1, Cj_2)_1 = (v_2, Cj_1)_1$$

by P5(I) for L. This with (5) for $r \neq s$ proves P5(I) for M.

18.36 Fix a $j \epsilon \mathbf{J}_M$ and for each $p \epsilon \Gamma_L$ a $u(p)$ such that

$$[u(p), j] \epsilon M(p).$$

Then $u(p) = C^* v(p)$ and

$$[v(p), Cj] \epsilon L(p),$$

for some $v(p)$. Then as in (5) above

$$(u(p), j)_2 = (v(p), Cj)_1 .$$

P6(I) and P7(I) for L then imply that P6(I) and P7(I) hold for M, using Γ_L for Γ_M in P6.

18.37 We now have M satisfying the hypotheses of 12.0. Therefore there is a Γ_M such that M satisfies all the postulates. This is 18.3.

18.4 *Proof of* "only if" *in* 17.2: Suppose that L between $\mathbf{V}$ and $\mathbf{K}$ is the juxtaposition of L_1 between $\mathbf{V}_1$ and $\mathbf{K}_1$, L_2 between $\mathbf{V}_2$ and $\mathbf{K}_2$. Let, say, $\mathbf{U} = \mathbf{V}_1$ and $\mathbf{J} = \mathbf{K}_1$. Let C be the identity map from $\mathbf{K}_1$ to $\mathbf{K}$: if $j \epsilon \mathbf{J} = \mathbf{K}_1$, then Cj is just j considered as a vector in $\mathbf{K}$. Then C is real. It is easily computed that C^* is E_1.

Consider the restriction M of L based on this C. Its pairs for $p \epsilon \Gamma_M \subseteq \Gamma_L$ are all the pairs $[u, j]$ such that $j = E^* j \epsilon \mathbf{K}_L$ and $u = Ev$, where

$$[v, j] \epsilon L(p). \tag{6}$$

But then

$$[u, j] = [Ev, E^* j]$$

and this is in $L_1(p)$ by (6) and the definition of juxtaposition. Therefore the list $M(p)$ is contained in $L_1(p)$.

Suppose that $[u, j] \epsilon L_1(p)$. We have $[0, 0] \epsilon L_2(p)$ so by P2 and the definition of juxtaposition

$$[u, j] \epsilon L(p).$$

But then $j = E^*j$, $u = Eu$, and by definition of M

$$[u, j] \epsilon M(p).$$

Therefore for every $p \epsilon \Gamma_M$, $M(p) = L_1(p)$. Therefore there is a frequency domain (Γ_M) for L_1 such that L_1 is PR.

XIX THE NECESSITY PROOF

19.0 Fortunately for this section, those parts of network theory which we require have recently been very succinctly stated by J. L. Synge[12]. We shall paraphrase them here, referring the reader to the source[12] for details of definition.

19.01 First, we observe that in Cauer's definition[5], which we shall repeat in detail below, an ideal transformer with m windings is a $2m$-pole whose terminal pairs are the termini of the respective windings.

A system of m coupled coils is a $2m$-pole with similarly defined terminal pairs.

19.02 Given a $2n$-pole **N** which is a finite passive network, let us adjoin ideal transformers as in Figure 1(b). We then draw the ideal graph of this network. Adjoin to the graph ideal *generator branches* $\gamma_1, \cdots, \gamma_n$, γ_r between T_r and T'_r, $1 \leq r \leq n$. Let β_r be the ideal branch representing the transformer winding between T_r and T'_r, $1 \leq r \leq n$. Enumerate the remaining branches of the graph $\beta_{n+1}, \cdots, \beta_b$.

19.03 The branch γ_r is in a mesh with β_r and no other branches. Let us call this the r-th external mesh. Any basic set of meshes must include each of these.

19.04 Let $\ell_1, \cdots, \ell_n$ be the currents in the generator branches, $k_1, \cdots, k_b$ the currents in the branches $\beta_1, \cdots, \beta_b$ and

$$[\ell] = [\ell_1, \cdots, \ell_n, k_1, \cdots, k_b], \qquad [k] = [k_1, \cdots, k_b].$$

Let $w_1, \cdots, w_n$ be the voltages across the generator branches, $v_1, \cdots, v_b$ the currents in the $\beta_1, \cdots, \beta_b$ and

$$[w] = [w_1, \cdots, w_n, v_1, \cdots, v_b], \qquad [v] = [v_1, \cdots, v_b].$$

19.05 Let us choose a basic set of meshes, let $j_1, \cdots, j_s$ be the respective mesh currents, and

$$[j] = [j_1, \cdots, j_s].$$

Let

$$[u] = [u_1, \cdots, u_s]$$

be the s-tuple of mesh voltages. We suppose that $j_1, \cdots, j_n, u_1, \cdots, u_n$ refer respectively to the n external meshes. (Cf. 19.03.)

19.06 The results of Synge[12] can now be stated as follows:

There exists a real constant matrix $[C_1]$ of s columns and $b + n$ rows (having, in fact, elements which are $+1$, -1, or 0) such that for any $[j]$

$$[\ell] = [C_1][j] \tag{1}$$

is a set of branch currents satisfying Kirchoff's node law, and for any $[w]$

$$[u] = [C_1]'[w] \tag{2}$$

is a set of mesh voltages satisfying Kirchoff's mesh law. Furthermore, given any $[\ell]$ which satisfies the node law, there is a $[j]$ such that (1) holds.

19.07 If we interpret the $[\ell]$, $[j]$, etc., as representations in real bases then $[C_1]$ is real and $[C_1]' = [C_1]^*$.

19.08 The matrix $[C_1]$ has the form

$$[C_1] = \begin{array}{|c|c|} \hline C_2 & 0 \\ \hline 0 & C \\ \hline \end{array}$$

where $[C_2]$ is an $n \times n$ diagonal matrix (having diagonal elements ± 1, in fact).

Proof: By construction, $j_1, \cdots, j_n$ are mesh currents in the external meshes. These are then equal, save for sign, to the currents $\ell_1, \cdots, \ell_n$ in the generator branches.

19.09 By 19.08, (1), and the definitions in 19.04,

$$[k] = [C][j], \qquad [u] = [C]'[v],$$

and by 19.07, $[C]' = [C]^*$.

19.1 Let us suppose that we have enumerated the branches $\beta_{n+1}, \cdots, \beta_b$ in 19.02 in such a way that $\beta_{n+1}, \cdots, \beta_c$ are all the two poles in the graph, $\beta_{c+1}, \cdots, \beta_d$ are all the branches containing coils which are magnetically coupled, and $\beta_{d+1}, \cdots, \beta_b$ the remaining ideal branches of ideal transformers.

Let $[Z_d(p)]$ be the $(d - n) \times (d - n)$ impedance matrix relating the voltages across the branches $\beta_{n+1}, \cdots, \beta_d$ to the currents in them when we consider the individual two-poles and the system of coupled coils as separate unconnected networks. Then $[Z_d(p)]$ is composed of a $(c - n) \times (c - n)$ diagonal matrix in the upper left field and a $(d - c) \times (d - c)$ matrix in the lower right, with zeros elsewhere.

19.11 The diagonal part of $[Z_d(p)]$ has elements drawn from the following list:

(i) $f(p) = \rho$

(ii) $f(p) = \delta p$

(iii) $f(p) = \lambda p$

where ρ, δ, λ are non-negative constants, possibly different for each branch.

19.12 It is shown in texts on electromagnetic theory that the matrix representing a system of coupled coils is of the form

$$p[G],$$

where $[G]$ is a real, constant, symmetric, and semi-definite matrix. The lower right field of $[Z_d(p)]$ is then such a matrix.

19.13 It is obvious from this description that $[Z_d(p)]$ is PR. It therefore describes a PR correspondence between $(d - n)$-tuples of current and voltage.

19.2 We must at last consider ideal transformers in detail. Let $\mathbf{V}_1$ and $\mathbf{K}_1$ be m-dimensional spaces represented as aggregates of m-tuples.

Let $\rho_1, \rho_2, \cdots, \rho_m$ be m real numbers. Let $\mathbf{V}_T$ consist of all m-tuples $[a] = [a_1, \cdots, a_m] \epsilon \mathbf{V}_1$ such that

$$\frac{a_1}{\rho_1} = \frac{a_2}{\rho_2} = \cdots = \frac{a_m}{\rho_m}.$$

We interpret these relations as follows:

(a) If any $\rho_r = 0$, then $a_r = 0$

(b) If any two ρ_r, ρ_s are not zero, then

$$\frac{a_r}{\rho_r} = \frac{a_s}{\rho_s}$$

(c) If only one $\rho_r \neq 0$, then a_r is arbitrary.

Let $\mathbf{K}_T$ consist of all m-tuples $[b] = [b_1, \cdots, b_m] \epsilon \mathbf{K}_1$ such that

$$\rho_1 b_1 + \rho_2 b_2 + \cdots + \rho_m b_m = 0.$$

$\mathbf{V}_T$ and $\mathbf{K}_T$ are linear manifolds.

Let $[L_T]$ be the concrete linear correspondence defined by the list $[L_T](p)$ which consists for each complex p of all pairs $[[a], [b]]$ where $[a]\epsilon\mathbf{V}_T$, $[b]\epsilon\mathbf{K}_T$.

The correspondence described by $[L_T]$ is what Cauer[5] defines as an ideal transformer. He shows, loc. cit., how it can be defined as the limiting case of a physical transformer.

There is also a dual kind of device, described by a correspondence admitting all $[b]\epsilon\mathbf{K}_1$ for which

$$\frac{b_1}{\lambda_1} = \frac{b_2}{\lambda_2} = \cdots = \frac{b_m}{\lambda_m}$$

and all $[a]\epsilon\mathbf{V}_1$ for which

$$\lambda_1 a_1 + \cdots + \lambda_m a_m = 0.$$

This also is an ideal transformer obtainable as a limiting case of a physical one.

19.21 The correspondence L_T is PR.

Proof: We observe that $\mathbf{V}_T = (\mathbf{K}_T)^0$, for let $[a]\epsilon\mathbf{V}_T$, $[b]\epsilon\mathbf{K}_T$, and let t be the common value of the a_r/ρ_r. Then

$$(a, b) = \Sigma a_r\bar{b}_r = t\Sigma\rho_r\bar{b}_r = t\overline{(\Sigma\rho_r b_r)} = 0.$$

The postulates are now all easily proved. We omit the details.

19.3 Let $\mathbf{V}$ and $\mathbf{K}$ be b-dimensional spaces. We interpret the b-tuples $[v]$ and $[k]$ of 19.04 as representing vectors $v\epsilon\mathbf{V}$, $k\epsilon\mathbf{K}$ in a real frame.

Let L be the correspondence between $\mathbf{V}$ and $\mathbf{K}$ formed by juxtaposing

(i) the correspondence described by $[Z_d(p)]$ relating components with indices in the range $n + 1$ to d,

(ii) the several correspondences described by ideal transformers, relating components with indices in the ranges 1 to n and $d + 1$ to b.

L is PR because it is the juxtaposition of PR correspondences.

19.31 Let $\mathbf{U}$ and $\mathbf{J}$ be $s - n$-dimensional spaces. We interpret the $[u]$ and $[j]$ of 19.04 as representing $u\epsilon\mathbf{U}$, $j\epsilon\mathbf{J}$ in a real frame.

19.32 Let C be the operation from $\mathbf{J}$ to $\mathbf{K}$ whose matrix in our chosen frames is $[C]$. Then C^* operates from $\mathbf{V}$ to $\mathbf{U}$ with the matrix $[C]^* = [C]'$. By these definitions, C is real. Let M be the correspondence between $\mathbf{U}$ and $\mathbf{J}$ obtained by restricting L with C. Then there is a frequency domain Γ_M such that M is PR (18.3).

19.4 By 19.09, $[M]$ in our chosen frame is the correspondence established between mesh currents and mesh voltages by the network of the

$2n$-pole **N**. When this network operates as a $2n$-pole, the only mesh voltages which are not zero are those relating to the external meshes, since there are no internal sources of voltage. We must now account for this.

19.41 Let $\mathbf{V}_2$, $\mathbf{K}_2$ be n-dimensional spaces. Choose a real frame and let D be the operation which takes

$$[a_1, \cdots, a_n] \epsilon \mathbf{V}_2 \tag{3}$$

into

$$[a_1, \cdots, a_n, 0, \cdots, 0] \epsilon \mathbf{U} \tag{4}$$

in the frame of 19.31. Then D is real and D^* in the chosen frames takes

$$[b_1, \cdots, b_s] \epsilon \mathbf{J} \tag{5}$$

into

$$[b_1, \cdots, b_n] \epsilon \mathbf{K}_2 . \tag{6}$$

19.42 We interpret the n-tuples (3) and (6) as voltages and currents in the external meshes of **N**. Their relations to (4) and (5) are consistent with this interpretation.

Let us restrict M by D, to get a correspondence M_1 between $\mathbf{V}_2$ and $\mathbf{K}_2$. In our chosen frame, the passage to $[M_1]$ corresponds, by (3) and (4) of 19.41, to considering mesh voltages in **N** which vanish for every internal mesh, and, correspondingly letting the mesh currents adjust themselves to this situation. We of course observe only the external mesh currents (6).

19.43 M was PR. So, therefore is M_1 (18.3 dual). Since $[M_1]$ is the correspondence established by the physically realizable $2n$-pole **N**, the necessity of P1, $\cdots$, P7 for formal realizability is established.

XX. APPENDIX TO PART I

20.0 We must prove 7.22 and those assertions of 10.6 which are not covered in Halmos[9]. These concern reality.

20.1 Let $\mathbf{V}_1$ be a real manifold and

$$\mathbf{V} = \mathbf{V}_1 \oplus \mathbf{V}_2, \qquad \mathbf{K} = \mathbf{K}_1 \oplus \mathbf{K}_2$$

where $\mathbf{K}_1 = (\mathbf{V}_2)^0$, etc. The basis (14) of 10.6 exists by Halmos[9], par. 19. We show that it can be chosen to be real. We have linearly independent vectors

$$v_1, \cdots, v_r, v_{r+1}, \cdots, v_n,$$

where the first r span $\mathbf{V}_1$, the last $n - r$, $\mathbf{V}_2$. Let

$$v_s = u_s + iw_s, \qquad 1 \leq s \leq n,$$

where u_s, w_s are real (10.42). Since $\mathbf{V}_1$ is real and a linear manifold,

$$u_s = \tfrac{1}{2}(v_s + \bar{v}_s) \epsilon \mathbf{V}_1, \qquad 1 \leq s \leq r,$$

and, similarly, $w_s \epsilon \mathbf{V}_1$, $1 \leq s \leq r$. Among the $2n$ real vectors

$$u_1, u_2, \cdots, u_r, w_1, \cdots, w_r, u_{r+1}, \cdots, u_n, w_{r+1}, \cdots, w_n, \qquad (1)$$

the first $2r$ are in $\mathbf{V}_1$, and they span $\mathbf{V}_1$ because the v_s, $1 \leq s \leq r$, can be constructed from them. The whole list (1) spans $\mathbf{V}$, because from it all the v_s, $1 \leq s \leq n$, can be constructed. Since the $v_s \epsilon \mathbf{V}_2$ do not use in their construction any of the first $2r$ vectors (1), it follows that the last $2(n - r)$ vectors in that list must contain a set spanning $\mathbf{V}_2$. The reality of the vectors (1) then establishes the existence of a real basis, say,

$$v_1', \cdots, v_r', v_{r+1}', \cdots, v_n' \qquad (2)$$

which provides a basis in $\mathbf{V}_1$ and $\mathbf{V}_2$.

20.11 We now have 7.22. The unique dual basis

$$k_1', \cdots, k_n'$$

to (2) is real by 10.41. Hence all of $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{K}_1$, $\mathbf{K}_2$ are real. The proof of 10.6 is then complete.

20.2 If in a real basis (2) (dropping primes)

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n,$$

that is, if

$$[v] = [a_1, \cdots, a_n],$$

then by (5) of 10.3

$$\bar{v} = \bar{a}_1 v_1 + \cdots + \bar{a}_n v_n,$$

hence

$$[\bar{v}] = [\bar{a}_1, \cdots, \bar{a}_n].$$

The geometrical conjugation of 10.3 is therefore simply the concrete one of 7.2 in any real basis. This proves the remark of 10.35.

BIBLIOGRAPHY

1. M. Bayard, "Synthèse des Réseaux Passifs a un Nombre Quelconque de Paires de Bornes Connaissant Leurs Matrices d'Impedance ou d'Admittance," *Bulletin, Société Francaise des Electriciens*, **9**, 6 series, Sept. 1949.
2. O. Brune, *Jour. Math. and Phys., M.I.T.*, **10**, Oct. 1931, pp. 191–235.
3. W. Cauer, *Ein Reaktanztheorem, Sitzungsberichte Preuss. Akad. Wissenschaft*, Heft 30/32, 1931.
4. W. Cauer, "Die Verwirklichung von Wechselstromswiderständen vorgeschriebener Frequenzabhängigkeit," *Archiv für Elektrotechnik*, **17**, 1926.
5. W. Cauer, "Ideale Transformatoren und Lineare Transformationen," *Elektrische Nachrichten-Technik*, **9**, May, 1932.
6. S. Darlington, *Journal of Mathematics and Physics, M.I.T.* **18**, No. 4, Sept. pp. 257–353.
7. R. M. Foster, *Bell System Tech. J.*, April, 1924, pp. 259–267.
8. C. M. Gewertz, *Network Synthesis*, Baltimore, 1933.
9. P. R. Halmos, *Finite Dimensional Vector Spaces*, Princeton, 1942.
10. Y. Oono, "Synthesis of a Finite 2n-Terminal Network by a Group of Networks Each of Which Contains Only One Ohmic Resistance," *Jour. Inst. Elec. Comm. Eng. of Japan*, March, 1946. Reprinted in English in the *Jour. Math. and Phys., M.I.T.*, **29**, Apr., 1950.
11. Y. Oono, "Synthesis of a Finite 2n-Terminal Network as the Extension of Brune's Theory of Two-Terminal Network Synthesis," *Jour. Inst. Elec. Comm. Eng. of Japan*, Aug., 1948.
12. J. L. Synge, "The Fundamental Theorem of Electrical Networks," *Quarterly of Applied Mathematics*, **9**, No. 2, July, 1951.
13. R. Bott, and R. J. Duffin, "Impedance Synthesis Without the Use of Transformers," *Jour. Appl. Phys.*, **20**, Aug., 1949, p. 816.
14. H. W. Bode, *Network Analysis and Feedback Amplifier Design*, New York, 1945.