Electron Streams in a Diode*

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A general solution of the electron stream equations is developed for a parallel plane diode, under the assumption that the electron velocity is single valued. This solution contains all particular solutions. It serves to unify the wave theory and the particle theory of electron flow, and it is an approximation for multi-velocity streams over a wide range of conditions.

Introduction

THE theory of an electron stream flowing in a diode has received much attention;[1–13] because the tetrodes, pentodes and other modern tubes are cascade arrangements of individual diodes. The theory of the diode is the foundation for considering the circuit characteristics and the noise characteristics of these tubes. In earlier days when communication channels were confined to relatively low frequencies, an electron could traverse a diode in a short period of time compared to an oscillation of any electrical signal, and the theory could be developed rather simply from the known d-c equations. But in these days of high and ultra-high frequencies, the situation is quite different. A signal voltage may oscillate several times while an electron is traversing a diode, and the electron stream flows in bunches or waves. The present article is primarily concerned with this more complicated type of flow. It is confined to the case of parallel plane electrodes, and developed under the usual assumption that the electron velocity is a single valued function of space and time. It is shown to be an approximate solution for physical electron streams over a wide range of conditions.

Particular solutions for an electron stream under small signal conditions are given in various published articles. These theories approach the subject in two different manners. In one approach attention is confined to the motions of electrons as individual particles,[1–3] and the other approach may best be described as a wave theory of electron streams. But the two lines of approach have not hitherto given identical results, and the disagreement can probably be attributed to neglected factors in the wave theory.

The present article considers electron streams without regard to any other than a mathematical approach to the subject. The differential equations are linear in the derivatives, and they should therefore have a general solution that contains all particular solutions. The theory seeks and obtains

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this general solution.† It involves a wave equation, and the results are in exact agreement with the small-signal calculations for the motions of electrons as individual particles. It is therefore believed that the general solution reconciles the two lines of approach to the theory of electron streams.

With this solution available, the situation is comparable to that encountered in two-dimensional potential theory; assignment of definite functions to two arbitrary functions gives a solution for a particular problem in electron streams, but it is then difficult to determine just what problem has been solved. In the case of small signals the general solution does not greatly shorten the calculations, and it probably should not be regarded as a labor saving tool in comparison to any particular solution when the latter is already known. It is more probable that the broader solution will serve as a guide for general reasonings about electron streams, and as a guide to approximations that can be used in particular problems.

1. The Parallel Plane Diode

The diode of this article is shown in Fig. 1. It is two parallel planes indicated as (a) and (b), and separated by a distance l. The first plane (a) may be a thermionic cathode that emits electrons, or it may be a grid through which a stream of electrons is injected into the diode. The second plane (b) may be a metallic plate that receives the electrons after they have traversed the diode space, or it may be a grid that permits the electron stream to pass out of the diode. The dimensions of the diode are assumed small compared to the electromagnetic wave-length at any frequency involved, that is, small compared to the velocity of light divided by the frequency; and the separation of the planes is assumed small compared to their lateral dimensions. Under these conditions the electric intensity is parallel to the x-axis, and the electrons move in that direction only.

The electron stream injected through the first plane may vary with time, both in charge density and electron velocity; and the voltages at the two planes may also vary with time. The total current flowing in the diode space is then the sum of two components: a conduction current resulting from the motion of electrons, and a displacement current arising from the time rate of change of electric intensity. The displacement current can flow even when there are no electrons in the diode space; it is then the familiar a-c. current, flowing between two plates of a condenser. But, when electrons are present, the two currents interact with each other and they both flow in a complicated manner.

† H. W. König also demonstrated the existence of a general solution; and he developed the solution for the particular case of a sinusoidal current.[5]

The determining conditions that can be measured in any physical circuit associated with the diode are: the total current, the conduction current at the first plane, and the electron velocity at that plane. Then, for conveniently considering the diode as a circuit element, it has been shown by others[8] that we should be able to calculate the conduction current at the second plane, the electron velocity at that plane, and the resultant voltage across the diode. From the viewpoint of circuit theory, these last three quantities may be considered as dependent variables whose solutions should be sought in terms of the initial conditions. But an electron stream flows according to its own nature, with little regard for circuit theory, its fundamental equa-

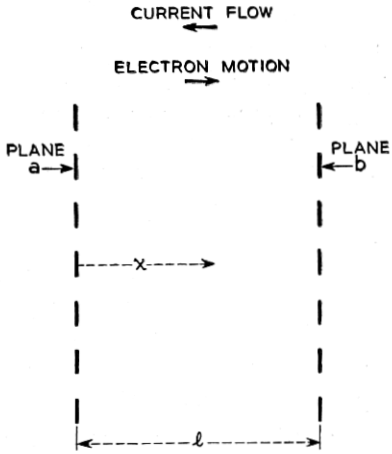

Fig. 1—Parallel plane diode with a first plane (a) and a second plane (b).

tions involve electric intensity and electron velocity as the dependent variables, and the general theory must therefore be developed in terms of these naturally occurring quantities. But it should be noted that the desired circuit relations can always be calculated from these fundamental variables.

1.1 Units and Symbols

The equations are written in practical electrical units, centimeters, grams and seconds. In this system of units, the permittivity ϵ of a vacuum is $\frac{10^{-11}}{36\pi}$, and the acceleration constant η of an electron is approximately $1.77 \cdot 10^{15}$. To conform with circuit convention, the total current and the conduction current are measured in the negative x-direction, that is, opposite to the motion of the electrons; all other directed quantities are measured in the positive x-direction.

The symbols are introduced and defined as needed. The following partial list is included to give the reader a general idea of the symbolism:

General Symbols

a,b	Subscripts referring to the diode planes
x	Distance from the first plane
l	Length of diode space
t	Time
τ	D-c. transit time to x
T	D-c. transit-time across the diode
ω	2π Frequency
j	$\sqrt{-1}$
β	$j\omega T$
ρ	Space-charge density
ζ	D-c. space-charge factor
ϵ	Permittivity of vacuum, $\frac{10^{-11}}{36\pi}$
η	Acceleration constant of an electron $1.77 \cdot 10^{15}$.

Symbols in Section 2—The General Solution

V	Voltage
E	Electric intensity
U	Idealized electron velocity
Q	Conduction current density
I	Total current density
I_D	D-c. component of I
I_A	Oscillating component of I
S	$\epsilon E + \int I\,dt$
S	$\epsilon E + \int I_A dt$
$F_1(S), F_2(S)$	Arbitrary functions of S
$A(S), B(S)$	Finite arbitrary functions of S.

Symbols in Section 3—Small Signal Theory

In this section the capital letters V, E, U, Q and I change their meaning and indicate only the d-c. components of their quantities; and the small letters v, e, u, q and i then indicate the amplitudes of the corresponding a-c. components. This section also uses the following special symbols:

A, B	Arbitrary constants
$A^* B^* \ldots I^*$	Coefficients for the circuit theory of a diode.

Symbols in Section 4—Physical Electron Streams

This section returns to the symbolism of the general theory; and the capital letters E, U, Q and I indicate total quantities. It also uses the following special symbols:

ν	Actual electron velocity
$\overline{U}$	Average of ν
N	Mass density of electrons
n	Partial density in a range $d\nu$
P	Momentum density of electrons
K	Kinetic energy density of electrons.

2. The General Solution for an Electron Stream

The present theory of electron streams is a solution of two partial differential equations, in which electron velocity U and electric intensity E appear as the dependent variables.

The equation for electron velocity U is based on an idealism that is commonly used in vacuum tube theory. It assumes that the electron velocity is a single-valued function of space and time or, stated in other words, it assumes that all electrons in any plane normal to the x-axis have the same velocity. The variable U may then be regarded as a continuous function of x and t, which is everywhere equal to the velocities of the individual electrons. The differential equation for U follows at once from the fundamental mechanics of electron motion, which states that for any individual electron

$$\frac{dU}{dt} = -\eta E \tag{1}$$

where η is the acceleration constant of an electron, and the small relativity terms are neglected. Then, since U is regarded as a continuous function of x, its total derivatives may be written in terms of partial derivatives, and

$$U \frac{\partial U}{\partial x} + \frac{\partial U}{\partial t} = -\eta E \tag{2}$$

which is here regarded as the fundamental equation for electron velocity. It is of course based on an idealizing assumption that imposes limitations on the general theory, and these limitations are discussed in a concluding section of the article, where it is shown that the idealized velocity is an approximation for the average velocity in physical electron streams.

The differential equation for the electric intensity E is given by the theory of electromagnetism. It follows from this fundamental theory that the total current density I flowing in the diode is a function of time alone; it has the same value at all planes along the x-axis, and is given by

$$I = -\rho U - \epsilon \frac{\partial E}{\partial t} \tag{3}$$

The first term is the conduction current density, the second term is the displacement current density, and I is measured according to circuit conventions in the direction opposite to the motion of the electrons. The charge density ρ is

$$\rho = \epsilon \frac{\partial E}{\partial x} \tag{4}$$

and its substitution in (3) gives

$$U \frac{\partial E}{\partial x} + \frac{\partial E}{\partial t} = -\frac{I}{\epsilon} \tag{5}$$

which is the differential equation for electric intensity.

Before passing, it should be noted that the conduction current density Q, measured according to circuit conventions, is

$$Q = -\rho U = -\epsilon U \frac{\partial E}{\partial x} \tag{6}$$

The two differential equations for U and E are now repeated as a group

$$U \frac{\partial U}{\partial x} + \frac{\partial U}{\partial t} = -\eta E \tag{2}$$

$$U \frac{\partial E}{\partial x} + \frac{\partial E}{\partial t} = -\frac{I}{\epsilon} \tag{5}$$

and in this group the total current density I may be regarded as any known or arbitrarily assigned function of time. These are the basic equations whose solution is sought in the present theory of electron streams. They are a description of the whole diode space, and they tell how U and E occur and vary with time throughout that whole space. They are first order equations, linear in their derivatives, and it is known from the theory of differential equations that their general solution is the complete solution, and that it will contain two arbitrary functions. So if we find a solution containing two arbitrary functions, we may be quite sure that it is the complete solution. The equations can be solved by the Lagrange method, as outlined in Appendix I. But that is a rather abstract operation, and the solution is here obtained by another method that has more physical meaning and is really equivalent to the Lagrange method.

For any individual electron, (2) and (5) may be written in the form of total differential equations (7) and (8)

$$\frac{dU}{dt} = -\eta E \tag{7}$$

$$\epsilon \frac{dE}{dt} = -I \tag{8}$$

where for that individual electron

$$U = \frac{dx}{dt} \tag{9}$$

and x is the coordinate of the electron. This group of total differential equations describes U and E only in the immediate vicinity of the one moving electron, and it is therefore a restricted picture in comparison to the one given by the original partial differential equations. It should, however, be clearly understood that we are not seeking the solution of this group of total differential equations; we are merely using them as aids for solving the original equations.

Equation (8) may be written in the form

$$\frac{d}{dt}\left[\epsilon E + \int I\, dt\right] = 0 \tag{10}$$

the bracketed term is regarded as a new variable S, that is,

$$S = \epsilon E + \int I\, dt \tag{11}$$

and (10) says that S is an invariant for any individual electron, it remains constant as the electron moves along. The solution of (10) is

$$S = C_1 \tag{12}$$

where C_1 is any arbitrary constant.

Turning now to (7) it may be written in the form

$$\frac{dU}{dt} = -\frac{\eta}{\epsilon}\left[S - \int I\, dt\right] \tag{13}$$

and its solution for any particular electron—remembering that S is an invariant for that electron—is

$$U = C_2 - \frac{\eta}{\epsilon}\left[St - \iint I\, dt\right] \tag{14}$$

where C_2 is an arbitrary constant. [In repeated integrations with respect to time, the increment dt is written only once, it being understood that dt is repeated in each integration.] Now any arbitrary function F_1 of S is a constant for the particular electron under consideration, so we may replace C_2 by $F_1(S)$ and write

$$U = F_1(S) - \frac{\eta}{\epsilon}\left[St - \iint I\,dt\right] \tag{15}$$

For the same electron, (9) may now be written in the form

$$\frac{dx}{dt} = F_1(S) - \frac{\eta}{\epsilon}\left[St - \iint I\,dt\right] \tag{16}$$

and its solution is

$$x = C_3 + F_1(S)t - \frac{\eta}{\epsilon}\left[\frac{St^2}{2} - \iiint I\,dt\right] \tag{17}$$

where C_3 is an arbitrary constant that may again be replaced by an arbitrary function F_2 of S, and

$$x = F_2(S) + F_1(S)t - \frac{\eta}{\epsilon}\left[\frac{St^2}{2} - \iiint I\,dt\right] \tag{18}$$

By considering one individual electron we have thus arrived at two general relations (15) and (18) which, taken together, describe U and E as functions of x and t. Now the reader will probably be much surprised, as was the writer, to learn that these two equations when standing alone are not solutions of the group of total differential equations (7), (8) and (9). The solution of that group is (12), (15) and (18). In other words, the two general relations are solutions of the total differential equations only in the very special case of S equal to a constant. But this constant may have any value, and the general relations therefore apply to all electrons in the diode space.

We are therefore practically forced to the conclusion that (15) and (18) are the solution of the broader group of partial differential equations (2) and (5), and this turns out to be true. This solution, which is here rewritten,

$$U = F_1(S) - \frac{\eta}{\epsilon}\left[St - \iint I\,dt\right] \tag{15}$$

$$\left.\begin{aligned} x &= F_2(S) + F_1(S)t - \frac{\eta}{\epsilon}\left[\frac{St^2}{2} - \iiint I\,dt\right] \\ S &= \epsilon E + \int I\,dt \end{aligned}\right\} \tag{18}$$

moreover contains two arbitrary functions, and it is therefore the general and complete solution for an idealized stream flowing in a diode.

As the solution stands, I is an arbitrary function of time, and $F_1(S)$ and $F_2(S)$ are perfectly arbitrary. They correspond to all possible determining conditions: to all the d-c., a-c. and transient conditions that are possible in the idealized diode, and to all the purely mathematical conditions that cannot be realized in any physical sense.

With this complete solution available, the situation is analogous in many respects to that encountered in the solution of potential problems in two-dimensional space. We can find a particular solution by merely assigning definite functions to the three arbitrary functions $I(t)$, $F_1(S)$ and $F_2(S)$; but we then encounter the difficult task of finding out just what problem has been solved.

As a simple example of the general method, the reader may be interested in arbitrarily setting I, $F_1(S)$ and $F_2(S)$ equal to zero. He will then find that the resulting expressions, (15) and (18), are actual solutions of the partial differential equations, and that they represent a transient electron stream that can flow for a short period of time in a diode space.

2.1 The General Solution in the Presence of a Direct Current

In the majority of circuits that are of practical interest, there is a continuous direct current flowing in a diode, and the arbitrary functions then assume a more restricted form. In such cases the total current density I may be considered as the sum of a d-c. component, which for the time being is indicated as I_D, and a transient or alternating component I_A.

Then we have the condition that

$$I_D > 0 \tag{19}$$

and also the condition that U and x must be finite in any physical tube. Now consider (15) for U and note that

$$\begin{aligned} \int I\, dt &= I_D t + \int I_A\, dt \\ \iint I\, dt &= \frac{I_D t^2}{2} + \iint I_A\, dt \end{aligned} \tag{20}$$

The bracketed factor in (15) thus contains power terms in t, which becomes infinite as t approaches infinity. The function $F_1(S)$ must therefore be of such form that it cancels these terms and causes U to remain finite. Inspection shows that $F_1(S)$ must consequently be of the form

$$F_1(S) = A(S) + g + g_1 S + g_2 S^2 \tag{21}$$

where $A(S)$ is an arbitrary finite function of S, g is an arbitrary constant, and the coefficients g_1 and g_2 have such values that the power terms in t cancel out in (15). The finite function may, for example, be a sinusoidal function of S, or a series of such sinusoidal terms. The values of the coefficients are easily calculated, and when the resultant expression for $F_1(S)$ is substituted in (15), it may be written:

$$U = g + A(S) + \frac{\eta}{\epsilon}\left[\frac{\bar{S}^2}{2I_D} + \iint I_A\, dt\right] \tag{22}$$

where $\bar{S}$ is S with the I_D term omitted, that is,

$$\bar{S} = \epsilon E + \int I_A\, dt \tag{23}$$

In a similar manner it may be shown that, for x to be finite, (18) assumes the form

$$x = k + B(S) - \frac{g + A(S)}{I_D}\bar{S} - \frac{\eta}{\epsilon}\left[\frac{\bar{S}^3}{6I_D^2} - \iiint I_A\, dt\right] \tag{24}$$

where k is an arbitrary constant, and $B(S)$ is any arbitrary finite function of S. Then (22) and (24) constitute the general solution when a continuous direct current is flowing in the diode. They are mathematical means for shortening the calculations in the presence of the direct current.

It is believed that the general solution presented in this section will serve as a guide for reasoning about electron streams, and as a guide that can be used in particular problems. It should also be an aid for considering the large signal theory of electron streams. But it is here advisable to confine attention to a less ambitious program, and apply the method to the case of small signals. The results will not be entirely new, but they will bring out certain important features of the general solution.

3. The Small Signal Theory of Electron Streams

The small signal theory is developed as follows: The value of each dependent variable, in any plane normal to the x-axis, is regarded as the sum of two components: a value that does not vary with time and is therefore called the d-c. component, and a value that does vary with time and is called the a-c. component. All of these components may vary with x, that is, with the exception of the total current density which is a function of time alone. Corresponding to small signal circuit theory, it is also assumed that the a-c. quantities are small compared to the d-c. quantities, and that the squares and products of the a-c. quantities are negligible in comparison to their first order values. For such signals the circuit equivalent of a diode is completely determined by its performance at single frequencies, and this permits the solution to be developed in terms of simple sinusoidal functions of time.

New symbols are needed for the small signal theory, and to avoid an undue number of subscripts they are introduced in the following manner: In the preceding general equations the dependent variables were indicated by capital letters; in the following small signal theory, the same capital letters are used to indicate the d-c. components, and the corresponding small letters then indicate the complex amplitudes of the a-c. components. This gives the following list of symbols:

	DC Component	Amplitude of AC Component
Total current density	I	i
Conduction current density	Q	q
Voltage	V	v
Electric intensity	E	e
Electron velocity	U	u

This symbolism has the disadvantage of using e to indicate both electric intensity and the base of the natural logarithms, but the duplication causes no serious confusion, for the meaning of the symbol is always evident from the text. As examples of the new nomenclature, the conduction current density is now $Q + qe^{j\omega t}$, and the electron velocity is $U + ue^{j\omega t}$. It should be noted that the a-c. amplitude in each of these expressions is a complex space-varying amplitude, which is sometimes called the space part of the a-c. component.

Before passing it is well to write the following useful relations, which follow immediately from the fundamental equations (5) and (6):

$$
\begin{aligned}
q &= i + j\omega\epsilon e \\
e &= \frac{j(i - q)}{\epsilon\omega}
\end{aligned}
\tag{25}
$$

their substitution in (11) and (23) give

$$
\begin{aligned}
S &= \epsilon E + It - \frac{jq}{\omega} e^{j\omega t} \\
\bar{S} &= \epsilon E - \frac{jq}{\omega} e^{j\omega t}
\end{aligned}
\tag{26}
$$

With this introduction to the change in symbolism, we now express the general solution (22) and (24) in terms of the new symbols, and neglect all second order terms in the oscillating components. Each part of the general solution then separates into two equations, one for the d-c. quantities, and another for the a-c. quantities. The resulting equations for the d-c. components are

$$U = g + \frac{\eta \epsilon E^2}{2I} \tag{27}$$

$$x = k - \frac{g\epsilon E}{I} - \frac{\eta \epsilon^2 E^3}{6I^2} \tag{28}$$

and the equations for the a-c. components are

$$ue^{j\omega t} = A(S) - \frac{\eta}{\epsilon}\left[\frac{j\epsilon E}{\omega I} q + \frac{i}{\omega^2}\right] e^{j\omega t} \tag{29}$$

$$0 = B(S) - \frac{\epsilon E}{I} A(S) + j\left[\frac{U}{\omega I} q + \frac{\eta}{\epsilon \omega^3} i\right] e^{j\omega t} \tag{30}$$

where in the last equation g has been replaced by its value from (27).

3.1 The D-c. Components of the Electron Stream

We first consider the d-c. components in (27) and (28). It is easily shown that they obey the primitive differential equations

$$\begin{aligned} U \frac{\partial U}{\partial x} &= -\eta E \\ U \frac{\partial E}{\partial x} &= -I/\epsilon \end{aligned} \tag{31}$$

which are the static equations for a diode, when it is idling in the absence of an a-c. signal. Their solutions are given in various published articles, and they are available without further calculations.[7, 8] These d-c. components are involved in the subsequent development of the a-c. theory, and the latter requires certain d-c. relations. These relations are briefly presented without derivations as follows:

The current density I and the d-c. voltages at the two diode planes are assumed to be known quantities. Then the d-c. velocities at those planes are also known quantities, because their values are given by the simple relations

$$U_a = \sqrt{2\eta V_a}, \qquad U_b = \sqrt{2\eta V_b} \tag{32}$$

where it is assumed that the original source of electrons is at zero voltage.

The d-c. transit time plays an important role in the small signal theory.

The transit time τ from the first plane to any coordinate x is

$$\tau = \int_0^x \frac{dx}{U} \tag{33}$$

and

$$\frac{\partial \tau}{\partial x} = \frac{1}{U} \tag{34}$$

The total transit time T across the diode space also plays an important role; it is usually expressed in terms of a so-called space charge factor ζ, whose value is given by[8]

$$\zeta \left(1 - \frac{\zeta}{3}\right)^2 = \frac{4}{9} \frac{I}{I_m} \tag{35}$$

Here I is the actual d-c. current; and I_m is the maximum current that could be projected across the diode when its planes are at the voltages V_a and V_b, that is, I_m is the space charge limited current

$$I_m = \frac{2\epsilon}{9} \sqrt{\frac{2}{\eta}} \frac{(\sqrt{V_a} + \sqrt{V_b})^3}{l^2} \tag{36}$$

Then the total transit time T is given by

$$T = \frac{2l}{\left(1 - \frac{\zeta}{3}\right)\left(U_a + U_b\right)} \tag{37}$$

It also follows that I can be expressed in the form

$$I = \frac{2\epsilon\zeta}{\eta T^2} (U_a + U_b) \tag{38}$$

Certain equations for the d-c. electric intensity are also required. They are

$$\begin{aligned} E &= E_a - \frac{I\tau}{\epsilon} \\ E_a &= \frac{1}{\eta T} (U_a - U_b) + \frac{IT}{2\epsilon} \\ E_b &= \frac{1}{\eta T} (U_a - U_b) - \frac{IT}{2\epsilon} \end{aligned} \tag{39}$$

They complete the list of d-c. relations required in the following small signal theory.

3.2 The A-c. Components of the Electron Stream

We now return to the a-c. equations for the electron stream, (29) and (30). In (29), the arbitrary function $A(S)$ must obviously involve an exponential function of $j\omega t$, and it must therefore be of the form

$$A(S) = Ae^{\frac{j\omega S}{I}} \tag{40}$$

where A is an arbitrary constant. Then the substitution of (26) gives

$$A(S) = A \text{ exp.} \left[j\omega\left(t + \frac{\epsilon E}{I}\right) + \frac{q}{I} e^{j\omega t} \right] \tag{41}$$

The term in q/I is a second-order term that may be neglected, and $\frac{\epsilon E}{I}$ can be replaced by its value from (39) that is,

$$\frac{\epsilon E}{I} = \frac{\epsilon Ea}{I} - \tau \tag{42}$$

where τ is the d-c. transit time to any coordinate x. The resultant exponential factor in $\frac{\epsilon Ea}{I}$ can then be included in the arbitrary constant A, and this gives

$$A(S) = Ae^{j\omega(t-\tau)} \tag{43}$$

The substitution of this function in (29) now gives the following relation for the amplitudes of the a-c. components

$$u = Ae^{-j\omega\tau} - \frac{j\eta E}{\omega I} q - \frac{\eta}{\epsilon\omega^2} i \tag{44}$$

The arbitrary function $B(S)$ may be treated in a similar manner, and (30) then gives the complex amplitude of the conduction current density

$$q = \frac{j\omega I}{U}\left(B - A\frac{\epsilon E}{I}\right) e^{-j\omega\tau} - \frac{\eta I}{\epsilon\omega^2 U} i \tag{45}$$

where B is another arbitrary constant.

The substitution of this value of q in (25) and (44) also gives the amplitudes of electron velocity and electric intensity.

$$u = \left[B\frac{\eta E}{U} + A\left(1 - \frac{\eta\epsilon E^2}{IU}\right)\right] e^{-j\omega\tau} - \frac{\eta}{\epsilon\omega^2}\left[1 - \frac{j\eta E}{\omega U}\right] i \tag{46}$$

$$e = \frac{I}{\epsilon U}\left[B - A\frac{\epsilon E}{I}\right] e^{-j\omega\tau} + \frac{j}{\epsilon\omega}\left[1 + \frac{\eta I}{\epsilon\omega^2 U}\right] i \tag{47}$$

The amplitude of the a-c. voltage in the diode space is also required, and it is derived from its expression

$$v = v_a - \int_0^x e\,dx \tag{48}$$

where e has its value (47), and the integration can be performed by remembering that $\frac{\partial \tau}{\partial x}$ is $1/U$. This gives

$$\begin{aligned} v = v_a + \frac{1}{\epsilon\omega}\Bigg[\left(j\epsilon AE - A\frac{I}{\omega} - jBI\right)(e^{-j\omega\tau} - 1) \\ - jAI\tau - j\left(x + \frac{\eta I\tau}{\epsilon\omega^2}\right)i\Bigg] \end{aligned} \tag{49}$$

We are now in a position to examine the character of the electron stream, and for this purpose we write the conduction current density in its complete form $qe^{j\omega t}$, that is,

$$qe^{j\omega t} = \frac{j\omega I}{U}\left(B - A\frac{\epsilon E}{I}\right)e^{j\omega(t-\tau)} - \frac{\eta I}{\epsilon\omega^2 U}\,ie^{j\omega t} \tag{50}$$

The phase angle of the first term involves the d-c.transit time τ, which is a function of x, so this term is a wave traveling in the x-direction. Its amplitude involves the d-c. quantities U and E, and its amplitude varies with x. The velocity of the wave is given by

$$\text{Wave velocity} = \left(\frac{\partial \tau}{\partial x}\right)^{-1} \tag{51}$$

and from (34)

$$\text{Wave Velocity} = U \tag{52}$$

That is the velocity of the conduction current wave is equal to the d-c. component of electron velocity.

The second term in (50) is an oscillation that has the same phase throughout the diode space, and its amplitude also varies with x. The a-c. conduction current is thus a wave of electric charge traveling at a finite velocity plus an oscillating charge that is in phase over the entire diode space.

An inspection of equations (46), (47) and (49) shows that the other a-c. components are of the same general form; each of them is a wave traveling in the x-direction plus an oscillation that is in phase over the entire diode. This clear-cut disclosure of the dual nature of an electron stream is an important contribution of the general theory.

The formal solution for small signals is really completed with the derivation of the preceding general equations for the a-c. amplitudes. But there still remains the rather tedious process of deriving the relations for circuit calculations as outlined in the following section, and they give a direct comparison with previous theories of electron streams.

3.3 Small Signal Equations for Circuit Calculations

Llewellyn[3] has shown that the treatment of a diode as a circuit element requires certain variables at the second plane to be expressed in terms of their values at the first plane; that is, the circuit theory requires three equations of the form

$$\begin{aligned} v_b - v_a &= A^*i + B^*q_a + C^*u_a \\ q_b &= D^*i + E^*q_a + F^*u_a \\ u_b &= G^*i + H^*q_a + I^*u_a \end{aligned} \tag{53}$$

where the starred coefficients are known functions of the d-c. components.

The derivation of these relations from the preceding general equations is outlined as follows: the first step is the evaluation of the arbitrary constants A and B. This is done by substituting the values at the first plane in (44) and (45), and then solving for A and B, which gives

$$A = u_a + \frac{j\eta E_a}{\omega I} q_a + \frac{\eta}{\epsilon\omega^2} i \tag{54}$$

$$B = \frac{\epsilon E_a}{I} A - \frac{jU_a}{\omega I} q_a - \frac{j\eta}{\epsilon\omega^3} i \tag{55}$$

These expressions, and the values at the second plane, are then substituted in the equations for the a-c. amplitudes (45), (46) and (49); and they immediately give the desired relations. They do, however, involve the inconvenient electric intensities E_a and E_b, and these quantities are replaced by their values from (39).

These simple but rather tedious substitutions are illustrated by the following derivation of q_b—, which is brief enough to be included for that purpose. The first step is the substitution of the values at the second plane in (45); this gives

$$q_b = \frac{j\omega I}{U_b}\left(B - \frac{\epsilon E_b}{I} A\right) e^{-\beta} - \frac{\eta I}{\epsilon\omega^2 U_b} i \tag{56}$$

where β is $j\omega T$. It is now advantageous to replace B by its value from (55), and

$$q_b = j\omega\epsilon\left(\frac{E_a - E_b}{U_b}\right) Ae^{-\beta} + \frac{U_a}{U_b} e^{-\beta} q_a + \frac{\eta I}{\epsilon\omega^2 U_b}(e^{-\beta} - 1)i \tag{57}$$

The inconvenient electric intensities are now easily eliminated by substitutions from the d-c. equations (39), which give

$$q_b = \frac{I\beta e^{-\beta}}{U_b} A + \frac{U_a}{U_b} e^{-\beta} q_a + \frac{\eta I}{\epsilon \omega^2 U_b} (e^{-\beta} - 1)i \tag{58}$$

The value of A is now introduced from (54), with E_a again eliminated by (39), and a grouping of terms then gives the final equation

$$q_b = \frac{\eta I}{\epsilon \omega^2 U_b} (\beta e^{-\beta} + e^{-\beta} - 1)i + \left(1 - \frac{\eta I T^2}{2\epsilon U_b}\right) e^{-\beta} q_a + \frac{I\beta e^{-\beta}}{U_b} u_a \tag{59}$$

This equation gives the following values of starred coefficients:

$$D^* = \frac{\eta I}{\epsilon \omega^2 U_b} (\beta e^{-\beta} + e^{-\beta} - 1) \tag{60}$$

$$E^* = \left(1 - \frac{\eta I T^2}{2\epsilon U_b}\right) e^{-\beta}$$

$$F^* = \frac{I\beta}{U_b} e^{-\beta}$$

These coefficients may now be rewritten in any desired form; and, to conform with previous articles on electron streams, we replace ω by its equivalent expression $\frac{-j\beta}{T}$; and we also express I in terms of the space charge factor ζ from (38), that is,

$$I = \frac{2\epsilon\zeta}{\eta T^2} (U_a + U_b) \tag{61}$$

These substitutions then give the coefficients in the form

$$D^* = 2\zeta \left(\frac{U_a + U_b}{U_b}\right) \frac{1 - e^{-\beta} - \beta e^{-\beta}}{\beta^2}$$

$$E^* = \frac{1}{U_b} [U_b - \zeta(U_a + U_b)] e^{-\beta} \tag{62}$$

$$F^* = \frac{2\epsilon\zeta}{\eta T^2} \left(\frac{U_a + U_b}{U_b}\right) \beta e^{-\beta}$$

This is obviously a longer mode of expression, but it has two advantages: it is convenient for circuit calculations, and it permits a direct comparison with previous articles on electron streams.

The equations for u_b and $(v_b - v_a)$ are obtained by similar substitutions in (46) and (49); and the three equations are then written in the symbolic form (53), with the values of the starred coefficients abbreviated and assembled as follows:

$$\begin{aligned} \alpha_1 &= 1 - e^{-\beta} - \beta e^{-\beta} \\ \alpha_2 &= 1 - e^{-\beta} \\ \alpha_3 &= 2 - 2e^{-\beta} - \beta - \beta e^{-\beta} \end{aligned} \tag{63}$$

$$\begin{aligned} A^* &= \frac{T^2}{2\epsilon\beta}(U_a + U_b)\left[1 - \frac{\zeta}{3}\left(1 - \frac{12\alpha_3}{\beta^3}\right)\right] \\ B^* &= \frac{T^2}{\epsilon\beta^3}[(\alpha_1 - \beta\alpha_2)U_a - \alpha_1 U_b + \alpha_1\zeta(U_a + U_b)] \\ C^* &= -\frac{2\alpha_1\zeta}{\eta\beta^2}(U_a + U_b) \\ D^* &= \frac{2\alpha_1\zeta}{\beta^2}\left(\frac{U_a + U_b}{U_b}\right) \\ E^* &= \frac{1}{U_b}[U_b - \zeta(U_a + U_b)]e^{-\beta} \\ F^* &= \frac{2\epsilon\zeta}{\eta T^2}\left(\frac{U_a + U_b}{U_b}\right)\beta e^{-\beta} \\ G^* &= -\frac{\eta T^2}{\epsilon\beta^3 U_b}[(\alpha_1 - \alpha_2\beta)U_b - \alpha_1 U_a + \alpha_1\zeta(U_a + U_b)] \\ H^* &= -\frac{\eta T^2}{2\epsilon}\left(\frac{U_a + U_b}{U_b}\right)(1 - \zeta)\frac{e^{-\beta}}{\beta} \\ I^* &= \frac{1}{U_b}[U_a - \zeta(U_a + U_b)]e^{-\beta} \end{aligned} \tag{64}$$

With the exception of a difference in symbols, these coefficients are identically the same as those obtained by Llewellyn and Peterson[3, 8] from calculations on the motions of electrons as individual particles, and this correspondence apparently reconciles the wave theory and the particle theory of electron streams. The correspondence is largely the result of a new feature in the wave theory, that is, the expression of the electron stream as the sum of two components, a wave travelling with a finite velocity and an oscillation that is in phase over the entire diode space.

Llewellyn and Peterson have derived the circuit equivalents of electronic tubes from the values of the starred coefficients, and these equivalents are well known in the electronic art.[8] The present section confirms these relations, as derived for an idealized electron stream. The validity of this idealization is considered in the following section.

4. Physical Electron Streams

[This section returns to the symbolism of the general theory; and the capital letters, V, E, U, Q and I now indicate total values.]

The preceding general solution for an electron stream is based on idealism, namely, the assumption that the electron velocity is a single-valued function of space and time. The stream then obeys the differential equations (2) and (5), and the theory is a general solution of these fundamental equations. But it is well known that the velocity in a physical electron stream is not single valued:[10–13] Electrons are emitted from their original source with slightly different velocities; and some electrons acquire energy from the high-frequency electric field and overtake their slower neighbors. These factors cause the velocity to be a multi-valued function of space, and the electrons have various velocities in any plane normal to the axis of the diode. The present section derives the differential equations for a multi-velocity stream, and compares them with the idealized equations (2) and (5).

For this purpose, the actual velocity of an electron is indicated as v. It is also convenient to develop the equations in terms of mass, so we let N equal the mass density of electrons at any coordinate x. The fractional mass density of electrons with velocities lying in any range from v to $v + dv$ may likewise be expressed as ndv, where n is a function of v, x and t; and it follows that

$$N = \int_{-\infty}^{+\infty} ndv. \tag{65}$$

The momentum density P of the electrons is then given by

$$P = \int_{-\infty}^{+\infty} nvdv \tag{66}$$

and their kinetic energy density K is

$$K = \int_{-\infty}^{+\infty} \frac{nv^2}{2} dv \tag{67}$$

It also follows that the average electron velocity $\bar{U}$ is given by

$$\bar{U} = \frac{P}{N} \tag{68}$$

This is the new mechanical variable in the theory of physical electron streams.

The differential equation for the electric intensity is now easily derived from the fundamental electromagnetic equation

$$\epsilon \frac{\delta E}{\delta t} - Q = -I \tag{69}$$

The conduction current density Q is

$$Q = -\frac{eP}{m} = -\bar{U}\frac{eN}{m} = -\bar{U}\epsilon\frac{\delta E}{\delta x} \tag{70}$$

and its substitution in (69) gives

$$\bar{U}\frac{\delta E}{\delta x} + \frac{\delta E}{\delta t} = -\frac{I}{\epsilon} \tag{71}$$

which is the analogue of (5).

The mechanical equation for the physical stream is obtained from the Liouville theorem. In the diode regions with which we shall be concerned, the individual electrons are so far apart that their microscopic forces are negligible, the electrons flow freely under the action of the macroscopic forces, and they therefore obey the Liouville theorem for particle motion. This theorem states that

$$\frac{dn}{dt} = 0 \tag{72}$$

that is, n remains constant as we travel along with any particular electron. This equation may also be written in terms of partial derivatives of n

$$\frac{\delta n}{\delta x}\frac{dx}{dt} + \frac{\delta n}{\delta v}\frac{dv}{dt} + \frac{\delta n}{\delta t} = 0 \tag{73}$$

and the substitution of the values of the total derivatives then gives

$$v\frac{\delta n}{\delta x} - \eta E\frac{\delta n}{\delta v} + \frac{\delta n}{\delta t} = 0 \tag{74}$$

The mechanical relations are obtained by integrating this equation with respect to v. It is first multiplied by dv and then integrated as follows:

$$\int_{-\infty}^{+\infty}\frac{\delta n}{\delta x}\,v\,dv - \eta E\int_{-\infty}^{+\infty}\frac{\delta n}{\delta v}\,\delta v + \int_{-\infty}^{+\infty}\frac{\delta n}{\delta t}\,dv = 0 \tag{75}$$

The second integral reduces to the difference in the values of n at $v = +\infty$, and $v = -\infty$. It vanishes because there are no electrons with infinite velocities. The differential operators may also be moved outside the other integrals, to give

$$\frac{\delta}{\delta x}\int_{-\infty}^{+\infty} nv\,dv + \frac{\delta}{\delta t}\int_{-\infty}^{+\infty} n\,dv = 0 \tag{76}$$

then from (65) and (66)

$$\frac{\delta N}{\delta t} = -\frac{\delta P}{\delta x} \tag{77}$$

Equation (74) is next multiplied by vdv, and a similar integration gives

$$\frac{\delta P}{\delta t} = -\eta NE - 2\frac{\delta K}{\delta x} \tag{78}$$

With these relations we are now in a position to derive the differential equation for $\bar{U}$.

This mechanical equation is obtained by first writing the obvious equality

$$\bar{U}\frac{\delta \bar{U}}{\delta x} + \frac{\delta \bar{U}}{\delta t} = \bar{U}\frac{\delta \bar{U}}{\delta x} + \frac{\delta}{\delta t}\left(\frac{P}{N}\right) \tag{79}$$

Then partial differentiation of the last term gives

$$\bar{U}\frac{\delta \bar{U}}{\delta x} + \frac{\delta \bar{U}}{\delta t} = \bar{U}\frac{\delta \bar{U}}{\delta x} + \frac{1}{N}\frac{\delta P}{\delta t} - \frac{P}{N^2}\frac{\delta N}{\delta t} \tag{80}$$

and, when the resultant time derivatives are replaced by (77) and (78)

$$\bar{U}\frac{\delta \bar{U}}{\delta x} + \frac{\delta \bar{U}}{\delta t} = \bar{U}\frac{\delta \bar{U}}{\delta x} - \eta E - \frac{2}{N}\frac{\delta K}{\delta x} + \frac{P}{N^2}\frac{\delta P}{\delta x} \tag{81}$$

the substitution of $N\bar{U}$ for P then gives the final differential equation for $\bar{U}$, which may be written in the form

$$\bar{U}\frac{\delta \bar{U}}{\delta x} + \frac{\delta \bar{U}}{\delta t} = -\eta E - \frac{2}{N}\frac{\delta}{\delta x}\left[K - \frac{N\bar{U}^2}{2}\right] \tag{82}$$

It is the analogue of equation (2).

The two sets of equations are now assembled and written in a form suitable for comparison. The equations for the idealized stream are

$$U\frac{\delta E}{\delta x} + \frac{\delta E}{\delta t} = -\frac{I}{\epsilon} \tag{5}$$

$$\frac{1}{2}\frac{\delta U^2}{\delta x} + \frac{\delta U}{\delta t} = -\eta E \tag{2}$$

and the analogous equations for the physical stream are

$$\bar{U}\frac{\delta E}{\delta t} + \frac{\delta E}{\delta t} = -\frac{I}{\epsilon} \tag{71}$$

$$\frac{1}{2}\frac{\delta \bar{U}^2}{\delta x} + \frac{2}{N}\frac{\delta}{\delta x}\left[K - \frac{N\bar{U}^2}{2}\right] + \frac{\delta \bar{U}}{\delta t} = -\eta E \tag{82}$$

When U is set equal to $\overline{U}$, the first equations in the two sets are identical; and in this respect the theory of the idealized stream corresponds to that of the physical stream. But the second equation for the physical stream then differs from its analogue by the inclusion of an additional term

$$\frac{2}{N}\frac{\delta}{\delta x}\left[K - \frac{N\overline{U}^2}{2}\right] \tag{83}$$

The bracketed quantity in this term is the difference between the actual kinetic energy density and the kinetic energy density calculated as if the electrons were all moving with their average velocity $\overline{U}$. It is often a small term that can be neglected, and the physical stream is then approximately described by the idealized equations (2) and (5).

It is, however, rather obvious that there are cases in which this approximation cannot be made. It is invalid in the region between a thermionic cathode and its voltage minimum, where the electrons are traveling in both directions along the x-axis, and cause K and $\frac{N\overline{U}^2}{2}$ to have appreciably different values. So, when the first plane of the diode is a space-change-limited cathode, the idealized theory can apply only in the region beyond the voltage minimum. This difficulty is usually overcome by considering the virtual cathode as the first plane of the diode. In all other regions the electrons are normally traveling in one direction only, and the idealized equations are then an approximation for the physical stream over a wide range of conditions.

The nature of this approximation is seen more clearly by considering the electrons to be uniformly distributed over a velocity range s, where s is a function of x and t. Then the mechanical equation (82) is

$$\frac{1}{2}\frac{\delta\overline{U}^2}{\delta x} + \frac{1}{8}\frac{\delta s^2}{\delta x} + \frac{\delta\overline{U}}{\delta t} = -\eta E \tag{84}$$

Under the usual conditions encountered in electronic tubes, $\frac{s^2}{8}$ is small compared to $\frac{U^2}{2}$, and its gradient may be neglected in comparison to that of $\frac{\overline{U}^2}{2}$.

The approximation can also be considered in a more rigorous manner as follows: The velocity spread s may be expressed in electron volts by the relation

$$s^2 \doteq \frac{\eta\phi^2}{2V} \tag{85}$$

where ϕ is the spread measured in electron volts, and V is the voltage in the stream. Then (84) may be written in the form

$$U \frac{\delta U}{\delta x} + \frac{\delta U}{\delta t} = \eta \frac{\delta V}{\delta x} - \frac{\eta}{8} \frac{\phi}{V} \frac{\delta \phi}{\delta x} + \frac{\eta}{16} \left(\frac{\phi}{V}\right)^2 \frac{\delta V}{\delta x} \quad (86)$$

The last term is small compared to $\eta \frac{\delta V}{\delta x}$ and may be neglected, and it follows that the idealized theory is an approximation for physical electron streams when

$$\frac{1}{8} \frac{\phi}{V} \frac{\delta \phi}{\delta x} \ll \frac{\delta V}{\delta x} \quad (87)$$

it being understood that the inequality holds for the gradients of the d-c. components, and also for the gradients of the a-c. components of ϕ and V. This requirement is satisfied over a wide range of conditions, and the idealized equations are applicable in a corresponding manner.*

It is thought that these considerations explain why the single-velocity theory of electron streams is so successful in explaining the characteristics of electronic tubes.[8, 9]

Conclusion

It is believed that the preceding pages serve to unify our theories of electron streams in some such manner as follows:

(1) They develop the general solution for a single velocity stream, and this solution contains all particular solutions.

(2) The small signal theory is considered in detail as a special case of the general solution, and the a-c. stream is shown to be the sum of two components: a wave traveling with a finite velocity plus an oscillation that is in phase over the entire diode space.

(3) The wave expression gives identically the same results as previous calculations based on the motions of electrons as individual particles.

(4) The idealized stream is shown to be an approximation for multi-velocity streams over a wide range of conditions, and this correspondence explains why the single velocity theory is so successful in describing the characteristics of electronic tubes.

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* It should be noted that this requirement is not satisfied by a velocity-modulated stream of small current density in a long, field-free drift space.

APPENDIX I—THE LAGRANGE SOLUTION

Lagrange has shown that any partial differential equation of the first order, linear in its derivatives, is equivalent to a group of total differential equations. The Lagrange equations corresponding to (2) and (5) are

$$\frac{dx}{U} = \frac{dt}{1} = \frac{dU}{-\eta E} \tag{88}$$

$$\frac{dx}{U} = \frac{dt}{1} = \frac{\epsilon dE}{-I} \tag{89}$$

or, taken together,

$$\frac{dx}{U} = \frac{dt}{1} = \frac{dU}{-\eta E} = \frac{\epsilon dE}{-I} \tag{90}$$

Now we can find three independent solutions of this group. One solution is

$$\epsilon E + \int I dt = c_1 \tag{91}$$

The first member of this solution is indicated as S; then the other solutions are

$$U + \frac{\eta S t}{\epsilon} - \frac{\eta}{\epsilon} \iint I dt = c_2 \tag{92}$$

$$x - Ut - \frac{\eta S t^2}{2\epsilon} + \frac{\eta}{\epsilon}\left[t \iint I dt - \iiint I dt\right] = c_3 \tag{93}$$

Since each of these quantities is a constant, we may set any one of them equal to an arbitrary function of another, and the resulting equation is also a solution of (90). We can, however, obtain only two independent solutions in this manner, and we naturally choose the two simplest combinations, that is,

$$U + \frac{\eta S}{\epsilon} t - \frac{\eta}{\epsilon} \iint I dt = F_1(S) \tag{94}$$

$$x - Ut - \frac{\eta S}{2\epsilon} t^2 + \frac{\eta}{\epsilon}\left[t \iint I dt - \iiint I dt\right] = F_2(S) \tag{95}$$

where $F_1(S)$ and $F_2(S)$ are arbitrary functions of S. These equations contain two arbitrary functions; they are solutions of the Lagrange equations (88) and (89), and they therefore constitute the general and complete solution of the partial differential equations (2) and (5). With the exception of a slight difference in form, this solution is identically the same as the one given in Section 2.

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