Some Results Concerning the Partial Differential Equations Describing the Flow of Holes and Electrons in Semiconductors

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The subject equations are investigated with the aim of establishing some general properties of the flow fields which they describe, including the existence or non-existence of classes of exact solutions having certain formal properties. The results include a number of geometric characteristics of the vector fields involved, a suggestive reformulation of the partial differential equations restricting carrier concentration and electrostatic potential, and several classes of exact solutions involving arbitrary constants and/or functions. Of particular interest is a family of solutions in closed form for the steady-state, no-recombination case involving an arbitrary harmonic function in three dimensions.

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A. Introduction

THIS paper is concerned with the system of relations describing the flow of holes and electrons in the interior of a homogeneous semiconductor subject to the assumption of constant temperature, electrical neutrality, and constant difference in concentrations of ionized donor and acceptor centers. These relations are:

$$\text{div } \overset{\circ}{\|}_p = -e\left[\mathcal{R} + \frac{\partial p}{\partial t}\right] \tag{1}$$

$$\text{div } \overset{\circ}{\|}_n = e\left[\mathcal{R} + \frac{\partial n}{\partial t}\right] \tag{2}$$

$$\mathring{\mathfrak{J}}_p = -\mu_p e\left[p \operatorname{grad} \mathcal{V} + \frac{kT}{e} \operatorname{grad} p\right] \tag{3}$$

$$\mathring{\mathfrak{J}}_n = -\mu_n e\left[n \operatorname{grad} \mathcal{V} - \frac{kT}{e} \operatorname{grad} n\right] \tag{4}$$

$$n - p = n_0 - p_0 \equiv N \text{ (a constant)} \tag{5}$$

$$n,\ p \geq 0 \tag{6}$$

$$\mathring{\mathfrak{J}} = \mathring{\mathfrak{J}}_p + \mathring{\mathfrak{J}}_n \tag{7}$$

wherein

n: concentration of negative carriers (electrons)
p: concentration of positive carriers (holes)
n_0: thermal equilibrium value of n
p_0: thermal equilibrium value of p
$\mathring{\mathfrak{J}}_p$: hole current density vector
$\mathring{\mathfrak{J}}_n$: electron current density vector
$\mathring{\mathfrak{J}}$: total current density vector
t: time variable
e: magnitude of electronic charge
k: Boltzmann's constant
μ_p: hole mobility constant
μ_n: electron mobility constant
T: absolute temperature (assumed constant with time and uniform)
$\mathcal{V}$: potential of electrical intensity field
$\mathcal{R}$: electron-hole recombination rate function (will usually be regarded as depending on $p - p_0$ and $n - n_0$ or equivalent variables).

These relations have fundamental application to transistor electronics, photoelectric effects, and related phenomena. Detailed discussions of their physical bases will be found in References 1 and 3. In brief, (1) and (2) are conservation conditions for the positive and negative carriers; (3) and (4) express the dependence of the local current densities on the electrostatic potential gradient and on the carrier concentration gradients (i.e., on conduction and diffusion); (5) expresses the condition of electrical neutrality under the assumption of a constant difference in concentrations of ionized donor and acceptor centers; and (6) and (7) are self evident.

The present study is directed toward the discovery of (1) general properties of the flow fields inside semiconductors and (2) families of exact solutions to the flow equations. The approach to the latter objective is through the "inverse method" which has proved very useful in the study of various non-linear partial differential equation systems in mechanics. In the inverse method, one proceeds by formal devices suggested by the equations under study to try to find families of solutions to the equations which involve arbitrary constants or, preferably, arbitrary functions. This is done without reference to any preconceived boundary value problems. After a pool of such families of solutions is available, it can be examined from the point of view of finding boundary value problems of interest consistent with any of the solutions in hand. The likelihood of finding solutions of interest in this way is of course greatly enhanced when the solutions involve arbitrary functions. Aside from providing solutions of some useful boundary value problems, the solutions found by the inverse method constitute a reference bank of non-trivial exact solutions against which to check numerical methods and approximation schemes (based, for example, on the assumption that a particular term can be neglected) for solving problems of more immediate practical interest.

J. Bardeen has demonstrated (in Reference 2) how the steady-state behavior of contact-semiconductor combinations can be explained on the basis of the characteristics of (1) the flow field inside the semiconductor and (2) those of the barrier layer at the contact. The present study is concerned in this connection only with the first of these influences. It provides, for example, a complete solution for the spherically symmetric flow field without recombination for arbitrary currents—a generalization of the zero-total current solution given by Bardeen. In the absence of surface recombination this spherically symmetric solution provides the hemispherically symmetric flow field in the neighborhood of a point contact on a plane surface and remote from other electrodes or surfaces. This spherically symmetric solution is contained as a particular case in a family of solutions involving an arbitrary harmonic function in three dimensions. Other choices of the harmonic function can be made to yield flow fields associated with numerous electrode configurations of immediate practical interest, for example that of the type-A transistor.

The objective of the present paper is to find (or establish the non-existence of) broad classes of solutions, and not to undertake detailed studies of any particular solutions. Such detailed studies of particular cases from the family of solutions mentioned above (and from other families found in this study) will form the subject matter of papers dealing with specific flow field configurations. However, in order to illustrate the interpretation of mathematical arbitrary constants in terms of basic physical parameters, the analysis of the spherically symmetric solution mentioned above is carried up to the point of actual substitution of numerical values in the formulae.

Note: In the following, functions and constants described as "arbitrary" are to be considered as being subject nevertheless to the restrictions implied by (6). In any particular case it is an elementary matter to determine these restrictions and we shall not usually carry out this detail. Also, "arbitrary" functions are subject to appropriate differentiability conditions readily evident in any particular case.

B. Some Properties of the Current Density Vector Fields

Several interesting properties of the current density vector fields $\mathring{\mathfrak{J}}_p$, $\mathring{\mathfrak{J}}_n$, and $\mathring{\mathfrak{J}}$ are easily found from (3)–(5).

It is evident that (3) and (4) can be rewritten as

$$\mathring{\mathfrak{J}}_p = -e\mu_p\, p \operatorname{grad}\left(\mathcal{V} + \frac{kT}{e}\ln p\right) \tag{8}$$

and

$$\mathring{\mathfrak{J}}_n = -e\mu_n\, n \operatorname{grad}\left(\mathcal{V} - \frac{kT}{e}\ln n\right). \tag{9}$$

From (3), (4), and (7) we have

$$\mathring{\mathfrak{J}} = -e(\mu_n n + \mu_p p) \operatorname{grad}\mathcal{V} + kT \operatorname{grad}(\mu_n n - \mu_p p) \tag{10}$$

which because of (5) can be rewritten as

$$\mathring{\mathfrak{J}} = -e\,(\mu_n\, n + \mu_p\, p) \operatorname{grad}\left[\mathcal{V} - \frac{kT}{e}\frac{\mu_\mu - \mu_p}{\mu_n + \mu_p}\ln(\mu_n\, n + \mu_p\, p)\right]. \tag{11}$$

Now (8), (9) and (11) are all of the form

$$\mathbf{u} = \phi \operatorname{grad}\psi$$

and hence obviously satisfy the condition

$$\mathbf{u}\cdot\operatorname{curl}\mathbf{u} = 0.$$

Therefore we have

Theorem 1: $\mathring{\mathfrak{J}}_p$, $\mathring{\mathfrak{J}}_n$, and $\mathring{\mathfrak{J}}$ are surface-normal vector fields.

From (8)–(10) we find, using (5)

$$\operatorname{curl}\mathring{\mathfrak{J}}_p = -e\mu_p \operatorname{grad} p \times \operatorname{grad}\mathcal{V}, \tag{12}$$

$$\text{curl } \mathring{\|}_n = -e\mu_n \text{ grad } p \times \text{grad } \mathcal{V}, \tag{13}$$

and

$$\text{curl } \mathring{\|} = -e(\mu_n + \mu_p) \text{ grad } p \times \text{grad } \mathcal{V}, \tag{14}$$

whence

Theorem 2:

$$\frac{\text{curl } \mathring{\|}_p}{\mu_p} = \frac{\text{curl } \mathring{\|}_n}{\mu_n} = \frac{\text{curl } \mathring{\|}}{\mu_n + \mu_p}.$$

That is, curl $\mathring{\|}_p$, curl $\mathring{\|}_n$, and curl $\mathring{\|}$ are constant multiples of one another.

and

Theorem 3: $\mathring{\|}_p$, $\mathring{\|}_n$, and $\mathring{\|}$ are irrotational if and only if

$$\text{grad } p = 0 \qquad (p = p(t))$$

or

$$\text{grad } \mathcal{V} = 0 \qquad (\mathcal{V} = \mathcal{V}(t))$$

or

$$\mathcal{V} = \mathcal{V}(p, t).$$

The following interesting relations can be obtained from (8) and (9) (they are really consequences of Theorem 1):

$$\text{curl } \mathring{\|}_p = \text{grad } \ln p \times \mathring{\|}_p \tag{15}$$

and

$$\text{curl } \mathring{\|}_n = \text{grad } \ln n \times \mathring{\|}_n. \tag{16}$$

Now from (3) — (5) we find

$$\mathring{\|}_p \times \mathring{\|}_n = e\mu_n\mu_p kT(n + p) \text{ grad } p \times \text{grad } \mathcal{V} \tag{17a}$$

$$= \tfrac{1}{2}\mu_n\mu_p kT(n + p) \text{ grad } (n + p) \times \text{grad } \mathcal{V} \tag{17b}$$

$$= \tfrac{1}{4}e\mu_n\mu_p kT \text{ grad } (n + p)^2 \times \text{grad } \mathcal{V} \tag{17c}$$

$$= \tfrac{1}{4}e\mu_n\mu_p kT \text{ curl } [(n + p)^2 \text{ grad } \mathcal{V}] \tag{17d}$$

and

$$\frac{\mathring{\|}_p}{\mu_p e} - \frac{\mathring{\|}_n}{\mu_n e} = \text{grad}\left[N\mathcal{V} - \frac{kT}{e}(n + p)\right] \tag{18}$$

and

$$\frac{\mathring{\|}_p}{\mu_p e} + \frac{\mathring{\|}_n}{\mu_n e} = -(n + p) \text{ grad } \mathcal{V}. \tag{19}$$

[*Note:* As is suggested by (18) and (19), the total carrier concentration

$$\mathcal{P} \equiv n + p = N + 2p = 2n - N \; (\mathcal{P} \geq |N|)$$

will frequently appear as the "natural" concentration variable in the relations with which we shall be working. Hence, expressions involving p, or p and n will often be replaced in the sequel by their equivalents in terms of the variable $\mathcal{P}$. It will be noted that

$$\text{grad } \mathcal{P} = 2 \text{ grad } p = 2 \text{ grad } n.]$$

Equations (17) and (19) yield at once the following theorems:

Theorem 4: The vector field

$$\overset{\circ}{\|}_p \times \overset{\circ}{\|}_n = \overset{\circ}{\|} \times \overset{\circ}{\|}_n = \overset{\circ}{\|}_p \times \overset{\circ}{\|}$$

is solenoidal.

Theorem 5: The vector field

$\left(\dfrac{\overset{\circ}{\|}_p}{\mu_p} - \dfrac{\overset{\circ}{\|}_n}{\mu_n}\right)$ is irrotational with a potential $(-eN\mathcal{V} + kT\mathcal{P})$.

Theorem 6: The vector field

$\left(\dfrac{\overset{\circ}{\|}_p}{\mu_p} + \dfrac{\overset{\circ}{\|}_n}{\mu_n}\right)$ is surface-normal (to the surfaces of constant $\mathcal{V}$).

Theorem 7: $\overset{\circ}{\|}_p$, $\overset{\circ}{\|}_n$, $\overset{\circ}{\|}$, grad $\mathcal{V}$, and grad p are coplanar vectors.

Theorem 8: The flow lines of any two of the fields $\overset{\circ}{\|}_p$, $\overset{\circ}{\|}_n$, and $\overset{\circ}{\|}$ coincide if and only if

$$\text{grad } p = 0 \qquad (p = p(t))$$

or

$$\text{grad } \mathcal{V} = 0 \qquad (\mathcal{V} = \mathcal{V}(t))$$

or

$$\mathcal{V} = \mathcal{V}(p, t).$$

Also, from (17) and (19) we obtain the curious relations:

$$\frac{\overset{\circ}{\|}_p}{\mu_p} \times \frac{\overset{\circ}{\|}_n}{\mu_n} = -\frac{kT}{2} \text{ grad } \mathcal{P} \times \left(\frac{\overset{\circ}{\|}_p}{\mu_p} + \frac{\overset{\circ}{\|}_n}{\mu_n}\right) \tag{20a}$$

$$= -\frac{kT}{2} \mathcal{P} \text{ curl} \left(\frac{\overset{\circ}{\|}_p}{\mu_p} + \frac{\overset{\circ}{\|}_n}{\mu_n}\right) \tag{20b}$$

$$= -\frac{kT}{2} \text{ curl} \left[\mathcal{P}\left(\frac{\overset{\circ}{\|}_p}{\mu_p} + \frac{\overset{\circ}{\|}_n}{\mu_n}\right)\right]. \tag{20c}$$

Finally, by taking the divergence of (7) and making use first of (1) and (2) and then of (5), we obtain:

[*Theorem 9:* The vector field $\overset{\circ}{\|}$ is solenoidal.

C. Formulation of Partial Differential Equation System Restricting $\mathcal{P}$ and $\mathcal{V}$

A very convenient formulation of the partial differential equations restricting $\mathcal{P}$ and $\mathcal{V}$ is suggested by (18) and (19). Taking the divergence of these equations and substituting (1) and (2) into the results we obtain:

$$\text{div grad}\left(N\mathcal{V} - \frac{kT}{e}\mathcal{P}\right) = -\alpha\left(\mathcal{R} + \frac{1}{2}\frac{\partial\mathcal{P}}{\partial t}\right) \tag{21}$$

and

$$\text{div}\,(\mathcal{P}\text{ grad }\mathcal{V}) = \beta\left(\mathcal{R} + \frac{1}{2}\frac{\partial\mathcal{P}}{\partial t}\right) \tag{22}$$

wherein for brevity we have set

$$\alpha \equiv \frac{1}{\mu_p} + \frac{1}{\mu_n}$$

and

$$\beta \equiv \frac{1}{\mu_p} - \frac{1}{\mu_n}$$

and shall henceforth assume $\beta \neq 0$, i.e., $\mu_p \neq \mu_n$. Equations (21) and (22) yield immediately a derived equation not containing explicitly the terms introduced by recombination and time variations:

$$\text{div grad}\left(N\mathcal{V} - \frac{kT}{e}\mathcal{P}\right) = -\frac{\alpha}{\beta}\,\text{div}\,(\mathcal{P}\text{ grad }\mathcal{V}) \tag{23a}$$

or

$$\text{div}\left[\left(N + \frac{\alpha}{\beta}\mathcal{P}\right)\text{grad }\mathcal{V} - \frac{kT}{e}\text{grad }\mathcal{P}\right] = 0 \tag{23b}$$

or

$$\text{div}\left(\left(\mathcal{P} + \frac{\beta N}{\alpha}\right)\text{grad}\left[\mathcal{V} - \frac{\beta kT}{\alpha e}\ln\left(\mathcal{P} + \frac{\beta N}{\alpha}\right)\right]\right) = 0. \tag{23c}$$

Either the set (21) and (22) or one of the forms of (23) together with either (21) or (22) constitutes a basic set of two partial differential equations determining $\mathcal{P}$ and $\mathcal{V}$. We are here considering $\mathcal{R}$ as $\mathcal{R}(\mathcal{P})$.

It will be observed that (23) is equivalent to the condition

$$\operatorname{div} \mathring{I} = 0 \tag{24}$$

established as Theorem 9.

(In terms of $\mathcal{P}$, (10) becomes

$$\mathring{I} = -\frac{e(\mu_n - \mu_p)}{2}\left[\left(\frac{\alpha}{\beta}\mathcal{P} + N\right)\operatorname{grad}\mathcal{V} - \frac{kT}{e}\operatorname{grad}\mathcal{P}\right)\right]. \tag{25}$$

In most of the following sections we shall find it expedient to consider separately the cases $N \neq 0$ and $N = 0$ (associated respectively with semiconductors of the extrinsic and intrinsic conductivity types). *For the case $N \neq 0$*, use will be made frequently of new dependent variables $\mathcal{U}$ and $\mathcal{H}$ defined by:

$$\mathcal{U} \equiv \frac{kT}{eN}\mathcal{P} \tag{26}$$

$$\mathcal{H} \equiv \mathcal{V} - \frac{kT}{eN}\mathcal{P} = \mathcal{V} - \mathcal{U}. \tag{27}$$

That is,

$$\mathcal{P} \equiv \frac{eN}{kT}\mathcal{U} \tag{28}$$

$$\mathcal{V} \equiv \mathcal{U} + \mathcal{H} \tag{29}$$

will be substituted into relations involving $\mathcal{P}$ and $\mathcal{V}$ to obtain the corresponding relations in terms of $\mathcal{U}$ and $\mathcal{H}$. Incidentally, it will be noted that $\mathcal{U}$ and $\mathcal{H}$ have the dimensions of voltage.

In terms of $\mathcal{U}$ and $\mathcal{H}$ the basic equations (21)–(23) can be written:

$$\operatorname{div}\operatorname{grad}\mathcal{H} = -\frac{\alpha}{N}\left[\mathcal{R} + \frac{eN}{2kT}\frac{\partial\mathcal{U}}{\partial t}\right] \tag{30}$$

$$\operatorname{div}\left[\mathcal{U}\operatorname{grad}(\mathcal{U} + \mathcal{H})\right] = \frac{\beta kT}{eN}\left[\mathcal{R} + \frac{eN}{2kT}\frac{\partial\mathcal{U}}{\partial t}\right] \tag{31}$$

$$\operatorname{div}\left[\operatorname{grad}\mathcal{H} + \frac{\alpha e}{\beta kT}\mathcal{U}\operatorname{grad}(\mathcal{U} + \mathcal{H})\right] = 0 \tag{32}$$

wherein $\mathcal{R}$ will be considered as $\mathcal{R}(\mathcal{U})$.

It will be observed that, in the absence of recombination and time variation, (30)–(32) reduce to

$$\operatorname{div}\operatorname{grad}\mathcal{H} = 0 \tag{33}$$

and $[N \neq 0]$

$$\operatorname{div}\left[\mathcal{U}\operatorname{grad}(\mathcal{U} + \mathcal{H})\right] = 0 \tag{34}$$

The elegant form of this set of equations furnished the original motivation for the introduction of the variables $\mathcal{U}$ and $\mathcal{H}$. The comparable equations for $N = 0$ are

$$\text{div grad } \mathcal{P} = 0 \tag{35}$$

$$[N = 0]$$

$$\text{div } [\mathcal{P} \text{ grad } \mathcal{V}] = 0. \tag{36}$$

D. The Recombination Rate Function $\mathcal{R}$

In order to avoid undue confusion in the sequel we shall at this point make some clarifying remarks concerning the function $\mathcal{R}$. As was stated in the Introduction, we basically regard $\mathcal{R}$ as a function of $p - p_0$ and $n - n_0$. However, because of (5), any expression in $p - p_0$ and $n - n_0$ can be replaced by one in which (say) p is the only field variable quantity. It is then convenient to regard $\mathcal{R}$ as a function of p and write it $\mathcal{R}(p)$. When dealing with expressions in terms of $\mathcal{P}$ and of $\mathcal{U}$, it is convenient to regard $\mathcal{R}$ as a function of one of these variables and to indicate this fact by writing $\mathcal{R}(\mathcal{P})$ or $\mathcal{R}(\mathcal{U})$. When we do this we do not mean that $\mathcal{R}(\mathcal{P})$ (say) is the same algebraic function of $\mathcal{P}$ as $\mathcal{R}(p)$ is of p, but rather that $\mathcal{R}(p)$ is the function of p obtained when one substitutes $\mathcal{P} = N + 2p$ into $\mathcal{R}(\mathcal{P})$.

For example, *for constant mean lifetime recombination*

$$\mathcal{R}(p) \equiv \frac{1}{\tau_0}(p - p_0) \tag{37a}$$

$$\mathcal{R}(\mathcal{P}) \equiv \frac{1}{2\tau_0}(\mathcal{P} - \mathcal{P}_0) \tag{37b}$$

$$\mathcal{R}(\mathcal{U}) \equiv \frac{eN}{2\tau_0 kT}(\mathcal{U} - \mathcal{U}_0) \tag{37c}$$

with τ_0 constant;
and *for mass-action recombination*

$$\mathcal{R}(p) \equiv \frac{1}{n_0\tau_0}[p(p + N) - p_0 n_0] \tag{38a}$$

$$\mathcal{R}(\mathcal{P}) \equiv \frac{1}{2\tau_0(\mathcal{P}_0 + N)}(\mathcal{P}^2 - \mathcal{P}_0^2) \tag{38b}$$

$$\mathcal{R}(\mathcal{U}) \equiv \frac{e^2N^2}{2k^2T^2\tau_0(\mathcal{P}_0 + N)}(\mathcal{U}^2 - \mathcal{U}_0^2). \tag{38c}$$

E. Addition of Arbitrary Time Functions to $\mathcal{V}$ and $\mathcal{H}$

Since only the gradient of $\mathcal{V}$ appears in the basic equations (21) and (22), it is evident that if

$$\mathcal{V} = \mathcal{V}(x, y, z, t)$$

and

$$\mathcal{P} = \mathcal{P}(x, y, z, t)$$

are a pair of functions satisfying (21) and (22), then so also are

$$\tilde{\mathcal{V}} = \mathcal{V}(x, y, z, t) + \tilde{m}(t)$$

and

$$\tilde{\mathcal{P}} = \mathcal{P}(x, y, z, t)$$

where $\tilde{m}(t)$ is an arbitrary time function.
And since $\mathcal{V} = \mathcal{U} + \mathcal{H}$, if

$$\mathcal{H} = \mathcal{H}(x, y, z, t)$$

and

$$\mathcal{U} = \mathcal{U}(x, y, z, t)$$

are a pair of functions satisfying (30)–(32), so also are

$$\tilde{\mathcal{H}} = \mathcal{H}(x, y, z, t) + \tilde{m}(t)$$

and

$$\tilde{\mathcal{U}} = \mathcal{U}(x, y, z, t).$$

These arbitrary additive functions with zero gradients are physically trivial in that they merely reflect the arbitrariness of the reference voltage level. They will, however, be retained for the sake of formal completeness whenever they appear in the subsequent analyses.

F. Summary of Solutions for No Recombination or Time Variation

The next ten sections of this paper (Sections G–Q) contain a sequence of detailed analyses in which is determined the existence or non-existence of solution fields having certain prescribed formal properties. In most of these studies time variability and recombination are admitted and the analysis includes the establishment of the class of recombination rate functions $\mathcal{R}$ consistent with the property under consideration. In those cases where solutions are found to exist, they are expressed in the simplest convenient terms: in closed form, or as solutions of an ordinary differential equation, or as solutions of a single partial differential equation. The solutions found usually involve arbitrary constants and/or arbitrary functions of various kinds.

The present section is intended to provide a skimpy but compact sampling of the results obtained in the next ten sections. It will be confined to a simple listing of solutions found and furthermore will contain only the forms to which these solutions reduce when recombination and time variation are excluded. (Some solutions are lost under this reduction.) A heading will indicate the section(s) from which the solution comes as well as the formal property associated with each solution.

For the sake of conciseness and simplicity the symbols denoting arbitrary constants and functions in this section are independent of those employed in the later sections. They are to be interpreted as follows:

A, B: arbitrary constants
$h(x, y, z)$: any harmonic function
(or with subscript)

$(\tilde{\mathcal{V}}, \tilde{\mathcal{P}})$: any given solution field

[G. grad $\mathcal{V} = 0$]

$$\begin{bmatrix} \mathcal{V} = A \\ \mathcal{P} = h(x, y, z) \end{bmatrix}$$

[H, I. grad $\mathcal{P} = 0$]

$$\begin{bmatrix} \mathcal{V} = h(x, y, z) \\ \mathcal{P} = A \end{bmatrix}$$

[J. grad $\mathcal{H} = 0$, $N \neq 0$]

$$\begin{bmatrix} \mathcal{V} = A + \sqrt{h(x, y, z)} \\ \mathcal{P} = \dfrac{Ne}{kT} \sqrt{h(x, y, z)} \end{bmatrix}$$

[K, L. $\mathcal{V} = \mathcal{V}(\mathcal{P})$, $N \neq 0$]

$$(A \neq 0) \begin{bmatrix} \mathcal{V} = h(x, y, z) + A\Lambda\left[\dfrac{B - h(x, y, z)}{A}\right] \\ \mathcal{P} = \dfrac{Ne}{kT} A\Lambda\left[\dfrac{B - h(x, y, z)}{A}\right] \end{bmatrix}$$

(For definition of function Λ see Equation (87) and Figs. 1 and 2.)

[K, M. $\mathcal{V} = \mathcal{V}(\mathcal{P}),\ N = 0$]

$$\begin{cases} \mathcal{V} = A \ln h(x, y, z) + B \\ \mathcal{P} = h(x, y, z) \end{cases}$$

[N, O. $\operatorname{grad} \mathcal{P} \cdot \operatorname{grad} \mathcal{V} = 0$]

$$\begin{cases} \mathcal{V} = h_1(x, y, z) \\ \mathcal{P} = h_2(x, y, z) \\ \quad \text{provided} \\ \operatorname{grad} h_1(x, y, z) \cdot \operatorname{grad} h_2(x, y, z) = 0 \end{cases}$$

[N. $\operatorname{grad} \mathcal{U} \cdot \operatorname{grad} \mathcal{H} = 0,\ N \neq 0$]

$$\begin{cases} \mathcal{V} = \sqrt{h_1(x, y, z)} + h_2(x, y, z) \\ \mathcal{P} = \dfrac{Ne}{kT} \sqrt{h_1(x, y, z)} \\ \quad \text{provided} \\ \operatorname{grad} h_1(x, y, z) \cdot \operatorname{grad} h_2(x, y, z) = 0 \end{cases}$$

[P. $\operatorname{grad} \mathcal{P} \cdot \operatorname{grad} h = 0$]

$$\begin{cases} \mathcal{V} = \tilde{\mathcal{V}} + h(x, y, z) \\ \mathcal{P} = \tilde{\mathcal{P}} \\ \quad \text{provided} \\ \operatorname{grad} \tilde{\mathcal{P}} \cdot \operatorname{grad} h(x, y, z) = 0. \end{cases}$$

G. Solutions with $\mathcal{V} = \mathcal{V}(t)$

Our point of view in general is that $\mathcal{P}$ and $\mathcal{V}$ (or $\mathcal{U}$ and $\mathcal{H}$) are functions of three space coordinates and time, so that $\mathcal{V} = \mathcal{V}(t)$ implies for example that $\dfrac{\partial \mathcal{V}}{\partial x} = \dfrac{\partial \mathcal{V}}{\partial y} = \dfrac{\partial \mathcal{V}}{\partial z} = 0$. That is to say, we now seek solutions for which everywhere

$$\operatorname{grad} \mathcal{V} = 0. \tag{39}$$

From (21) and (22) this condition gives us the following restrictions on

$\mathcal{P}$ (and none on $\mathcal{V}(t)$):

$$\text{div grad } \mathcal{P} = 0 \tag{40}$$

and

$$\mathcal{R}(\mathcal{P}) + \frac{1}{2}\frac{\partial \mathcal{P}}{\partial t} = 0. \tag{41}$$

By operating with div grad on (41) we obtain

$$\mathcal{R}''(\mathcal{P}) = 0$$

(we consistently use primes to denote differentiation with respect to the argument of a function of a single variable—e.g.,

$$\left.\mathcal{R}''(\mathcal{P}) \equiv \frac{d^2\mathcal{R}(\mathcal{P})}{d\mathcal{P}^2}\right)$$

whence,

$$2\mathcal{R}(\mathcal{P}) = A\mathcal{P} + B \tag{42}$$

(A, B: arbitrary constants). Substituting (42) into (41) we obtain

$$\frac{\partial \mathcal{P}}{\partial t} + A\mathcal{P} = -B$$

whence

$$\mathcal{P} = c(x, y, z)\epsilon^{-At} - B/A \qquad (A \neq 0) \tag{43a}$$

or

$$\mathcal{P} = c(x, y, z) - Bt \qquad (A = 0). \tag{43b}$$

From (36) it follows that

$$\text{div grad } c(x, y, z) = 0, \tag{44}$$

that is, c must be harmonic.

In brief, if $\mathcal{R}(\mathcal{P})$ is of the form given in (42), any $\mathcal{V}(t)$ and (43) constitute solutions to the flow equations for any harmonic $c(x, y, z)$. Other forms of $\mathcal{R}(\mathcal{P})$ admit no solutions with $\mathcal{V} = \mathcal{V}(t)$.

It is evident that when recombination is absent time variation is also absent, and vice versa. The solutions reduce in this case to:

$$\mathcal{V} = C \qquad (C\text{: arbitrary constant}) \tag{45}$$

$$\mathcal{P} = c(x, y, z). \tag{46}$$

H. Solutions with $\mathcal{P} = \mathcal{P}(t)$, $N \neq 0$

The condition

$$\text{grad}\ \mathcal{P} = 0 \tag{47}$$

yields from (21) and (22)

$$\left(N + \frac{\alpha}{\beta}\mathcal{P}\right) \text{div grad}\ \mathcal{V} = 0 \tag{48}$$

and

$$\mathcal{P}\ \text{div grad}\ \mathcal{V} = \beta\left[\mathcal{R}(\mathcal{P}) + \frac{1}{2}\frac{d\mathcal{P}}{dt}\right]. \tag{49}$$

Two cases arise for $\mathcal{R} \not\equiv 0$:
Case 1:

$$\mathcal{P} = -\frac{\beta N}{\alpha} \tag{50}$$

and

$$\text{div grad}\ \mathcal{V} = -\frac{\alpha}{N}\mathcal{R}(\mathcal{P})\Big|_{\mathcal{P} = -\frac{\beta N}{\alpha}} \tag{51}$$

Case 2:

$$\mathcal{R}(\mathcal{P}) + \frac{1}{2}\frac{d\mathcal{P}}{dt} = 0$$

or

$$D - t = \int \frac{d\mathcal{P}}{2\mathcal{R}(\mathcal{P})} \qquad (D\text{: arbitrary constant}) \tag{52}$$

and

$$\text{div grad}\ \mathcal{V} = 0. \tag{53}$$

When recombination is absent, these cases reduce to:

$$\mathcal{P} = E \qquad (E\text{: arbitrary constant}) \tag{54}$$

and (53).
When time variation is absent, Case 2 again yields (53) and (54).

It should be recalled that $\mathcal{V}$ can depend on t as well as x, y, z; so that arbitrary functions of t play the role of arbitrary constants in (51) and (53), whenever time variation is allowed.

I. Solutions with $P = P(t)$, $N = 0$

For $N = 0$, only Case 2 of the previous section occurs, because the condition $\mathcal{P} = 0$ (implying no carriers!) is of no interest.

J. Solutions with $\mathcal{H} = \mathcal{H}(t)$, $N \neq 0$

For grad $\mathcal{H} = 0$, (30) and (32) yield:

$$\mathcal{R}(\mathcal{U}) + \frac{eN}{2kT}\frac{\partial \mathcal{U}}{\partial t} = 0 \tag{55}$$

and

$$\text{div grad } \mathcal{U}^2 = 2 \text{ div } \mathcal{U} \text{ grad } \mathcal{U} = 0. \tag{56}$$

Taking the div grad of (55) multiplied by $\mathcal{U}$ we obtain

$$\text{div grad } \mathcal{U}\mathcal{R}(\mathcal{U}) = 0$$

whence, because of (56)

$$\frac{4kT}{eN} \mathcal{U}\mathcal{R}(\mathcal{U}) = F\mathcal{U}^2 + G \qquad (F, G\text{: arbitrary constants})$$

or

$$\frac{4kT}{eN} \mathcal{R}(\mathcal{U}) = F\mathcal{U} + G\mathcal{U}^{-1}. \tag{57}$$

Substituting this permitted form for the recombination rate function into (55) we obtain

$$\frac{\partial \mathcal{U}^2}{\partial t} + F\mathcal{U}^2 = -G \tag{58}$$

whence

$$\mathcal{U} = \sqrt{f(x, y, z)\epsilon^{-Ft} - G/F} \qquad (F \neq 0) \tag{59a}$$

or

$$\mathcal{U} = \sqrt{f(x, y, z) - Gt} \qquad (F = 0). \tag{59b}$$

From (56), $f(x, y, z)$ is subject to

$$\text{div grad } f(x, y, z) = 0. \tag{60}$$

In summary, if and only if $\mathcal{R}(\mathcal{U})$ has the form (57), there are solutions for which $\mathcal{H} = \mathcal{H}$ (t) (arbitrary). The $\mathcal{U}$ is given by (59) in which f is an arbitrary harmonic function of x, y, z.

In terms of $\mathcal{P}$ and $\mathcal{V}$ these solutions are given by:

$$\mathcal{R}(\mathcal{P}) = \frac{F}{4}\mathcal{P} + \left(\frac{eN}{kT}\right)^2 \frac{G}{4}\mathcal{P}^{-1}, \tag{61}$$

$$\mathcal{P} = \frac{eN}{kT}\sqrt{f(x, y, z)\epsilon^{-Ft} - G/F} \qquad (F \neq 0) \tag{62a}$$

or

$$\mathcal{P} = \frac{eN}{kT}\sqrt{f(x, y, z) - Gt} \qquad (F = 0), \tag{62b}$$

and

$$\mathcal{V} = \mathcal{H}(t) + \sqrt{f(x, y, z)\epsilon^{-Ft} - G/F} \quad (F \neq 0) \tag{63a}$$

or

$$\mathcal{V} = \mathcal{H}(t) + \sqrt{f(x, y, z) - Gt} \qquad (F = 0). \tag{63b}$$

For no recombination ($\mathcal{R} \equiv 0$), these results specialize to (59b), (60), (62b), and (63b) with G set equal to zero. It should be noted (see (55)) that absence of time variation implies absence also of recombination.

K. Solutions with $\mathcal{V} = \mathcal{V}(\mathcal{P}, t)$, grad $\mathcal{P} \neq 0$

In Theorems 3 and 8 of Section B we have shown that some very interesting properties are implied by the condition

$$\text{grad}\,\mathcal{V} \times \text{grad}\,\mathcal{P} = 0. \tag{64}$$

In sections G–I we have treated the cases grad $\mathcal{V} = 0$ and grad $\mathcal{P} = 0$. We now turn to the remaining possibility leading to (64):

$$\mathcal{V} = \mathcal{V}(\mathcal{P}, t) \text{ with grad } \mathcal{P} \neq 0. \tag{65}$$

Substitution of (65) into (23b) leads to

$$\left[\left(N + \frac{\alpha}{\beta}\mathcal{P}\right)\frac{\partial \mathcal{V}}{\partial \mathcal{P}} - \frac{kT}{e}\right] \text{div grad}\,\mathcal{P} + \frac{\partial}{\partial \mathcal{P}}\left[\left(N + \frac{\alpha}{\beta}\mathcal{P}\right)\frac{\partial \mathcal{V}}{\partial \mathcal{P}} - \frac{kT}{e}\right](\text{grad}\,\mathcal{P})^2 = 0. \tag{66}$$

Two cases arise.

Case 1:

$$\left(N + \frac{\alpha}{\beta}\mathcal{P}\right)\frac{\partial \mathcal{V}}{\partial \mathcal{P}} - \frac{kT}{e} = 0.$$

This condition clearly satisfies (66) and leads to

$$\mathcal{V}(\mathcal{P}, t) = g(t) + \frac{\beta kT}{\alpha e} \ln \left| \mathcal{P} + \frac{\beta N}{\alpha} \right| \qquad (g(t)\text{: arbitrary function}). \qquad (67)$$

The restriction on $\mathcal{P}$ is then provided by the result of substituting (67) into (21):

$$\text{div grad} \left[\mathcal{P} - \frac{\beta N}{\alpha} \ln \left| \mathcal{P} + \frac{\beta N}{\alpha} \right| \right] = \alpha \left[\mathcal{R}(\mathcal{P}) + \frac{1}{2} \frac{\partial \mathcal{P}}{\partial t} \right]. \qquad (68)$$

Any $\mathcal{P}$ satisfying (68) constitutes with (67) a solution having the property desired.

If (65) is substituted into (25) it will be found that the condition

$$\left(N + \frac{\alpha}{\beta} \mathcal{P} \right) \frac{\partial \mathcal{V}}{\partial \mathcal{P}} - \frac{kT}{e} = 0$$

is equivalent to $\overset{\circ}{\|} = 0$, so that Case 1 is characterized by zero total current.

Case 2:

$$\left(N + \frac{\alpha}{\beta} \mathcal{P} \right) \frac{\partial \mathcal{V}}{\partial \mathcal{P}} - \frac{kT}{e} \neq 0.$$

In this case (66) can be written in the form

$$\frac{\text{div grad } \mathcal{P}}{(\text{grad } \mathcal{P})^2} = - \frac{\partial}{\partial \mathcal{P}} \ln \left[\left(N + \frac{\alpha}{\beta} \mathcal{P} \right) \frac{\partial \mathcal{V}}{\partial \mathcal{P}} - \frac{kT}{e} \right] = \phi(\mathcal{P}, t). \qquad (69)$$

From (69) it follows that $\mathcal{P}$ must be of the form $\mathcal{P}\ (h, t)$ with

$$\text{div grad } h(x, y, z, t) = 0. \qquad (70)$$

In summary we have

Theorem 10: If $\mathcal{V} = \mathcal{V}(\mathcal{P}, t)$ with grad $\mathcal{P} \neq 0$, then either $\overset{\circ}{\|} = 0$ or $\mathcal{V} = \mathcal{V}(h, t)$ and $\mathcal{P} = \mathcal{P}(h, t)$ with div grad $h(x, y, z, t) = 0$.

We shall investigate the restrictions on the functions $h(x, y, z, t)$, $\mathcal{V}(h, t)$, and $\mathcal{P}(h, t)$ in the next two sections.

Theorem 10 remains unchanged if recombination is absent. If time variation is absent, it simply drops t as a variable in the functions mentioned in the theorem. If both recombination and time variation are absent, the theorem can be strengthened to:

Theorem 11: If both recombination and time variation are absent and $\mathcal{V} = \mathcal{V}(\mathcal{P})$, then $\mathcal{V} = \mathcal{V}(h)$ and $\mathcal{P} = \mathcal{P}(h)$ with div grad $h(x, y, z) = 0$.

L. Solutions with $\mathcal{V} = \mathcal{V}(h, t)$, $\mathcal{P} = \mathcal{P}(h, t)$, grad $\mathcal{P} \neq 0$, div grad $h = 0$, $N \neq 0$

For formal reasons we shall work, not with the conditions $\mathcal{P} = \mathcal{P}(h, t)$ and $\mathcal{V} = \mathcal{V}(h, t)$, but with the equivalent conditions

$$\mathcal{U} = \mathcal{U}(h, t) \text{ and } \mathcal{H} = \mathcal{H}(h, t). \tag{71}$$

The condition grad $\mathcal{P} \neq 0$ now implies $\dfrac{\partial \mathcal{U}}{\partial h} \neq 0$.

Substitution of (79) into (30) and (32) yields—after use of (70):

$$\frac{\partial^2 \mathcal{H}}{\partial h^2} (\text{grad } h)^2 = -\frac{\alpha}{N}\left[\mathcal{R}(\mathcal{U}) + \frac{eN}{2kT}\frac{\partial \mathcal{U}}{\partial t} + \frac{eN}{2kT}\frac{\partial \mathcal{U}}{\partial h}\frac{\partial h}{\partial t}\right] \tag{72}$$

and

$$\frac{\partial}{\partial h}\left(\left[\frac{\beta kT}{\alpha e} + \mathcal{U}\right]\frac{\partial \mathcal{H}}{\partial h} + \mathcal{U}\frac{\partial \mathcal{U}}{\partial h}\right) = 0. \tag{73}$$

From (73) we get

$$\frac{\partial \mathcal{H}}{\partial h} = \frac{j(t) - \mathcal{U}\dfrac{\partial \mathcal{U}}{\partial h}}{\dfrac{\beta kT}{\alpha e} + \mathcal{U}} \tag{74}$$

($j(t)$: arbitrary function)

which yields upon substitution into (72):

$$\frac{\partial}{\partial h}\left[\frac{j(t) - \mathcal{U}\dfrac{\partial \mathcal{U}}{\partial h}}{\dfrac{\beta kT}{\alpha e} + \mathcal{U}}\right](\text{grad } h)^2 = -\frac{\alpha}{N}\left[\mathcal{R}(\mathcal{U}) + \frac{eN}{2kT}\left(\frac{\partial \mathcal{U}}{\partial h}\frac{\partial h}{\partial t} + \frac{\partial \mathcal{U}}{\partial t}\right)\right] \tag{75}$$

in which $\mathcal{U}$, $\dfrac{\partial \mathcal{U}}{\partial h}$, and $\dfrac{\partial^2 \mathcal{U}}{\partial h^2}$ are, of course, functions of h and of t.

In determining the combined implications of (75) and (70) three cases arise according to whether or not $\dfrac{\partial^2 \mathcal{H}}{\partial h^2} = 0$ or grad (grad $h)^2 = 0$.

Case 1:

$$\frac{\partial^2 \mathcal{H}}{\partial h^2} \neq 0, \qquad \text{grad (grad } h)^2 \neq 0.$$

In this case no satisfactory interpretation has been found when time variability is present.

When time variation is absent, we work with the conditions

$$\mathcal{U} = \mathcal{U}(h); \qquad \mathcal{H} = \mathcal{H}(h)$$

with

$$\text{div grad } h(x, y, z) = 0 \tag{76}$$

and arrive at counterparts of (74) and (75):

$$\mathcal{H}' = \frac{H - \mathcal{U}\mathcal{U}'}{\gamma + \mathcal{U}} \qquad \left(\gamma \equiv \frac{\beta kT}{\alpha e}\right) \qquad (H\text{: arbitrary constant}) \tag{77}$$

and

$$\mathcal{H}'' \, (\text{grad } h)^2 = \left(\frac{H - \mathcal{U}\mathcal{U}'}{\gamma + \mathcal{U}}\right)' (\text{grad } h)^2 = -\frac{\alpha}{N}\, \mathcal{R}(\mathcal{U}). \tag{78}$$

From (78) it is evident that $\mathcal{R} \neq 0$ implies $\mathcal{H}'' \neq 0$ and grad $h \neq 0$. So we have

$$(\text{grad } h)^2 = \frac{-\dfrac{\alpha}{N}\, \mathcal{R}(\mathcal{U})}{\left(\dfrac{S - \mathcal{U}\mathcal{U}'}{\gamma + \mathcal{U}}\right)'}, \tag{79}$$

which is of the form

$$[\text{grad } h(x, y, z)]^2 = \phi(h). \tag{79a}$$

Now from (79a) follows

$$\text{grad } h \times \text{grad } (\text{grad } h)^2 = 0 \tag{80}$$

which implies that the vector lines of the field grad h are all straight. Since h is harmonic, this restricts the choice of h to the potential fields associated with a uniform parallel flow, a straight line source, or a point source. Hence, for suitably chosen rectangular coordinates (x, y, z), circular cylindrical coordinates (ρ, θ, z) or spherical polar coordinates (r, θ, ϕ), the only possibilities, are respectively

$$h = x \rightarrow (\text{grad } h)^2 = 1 \tag{81a}$$

or

$$h = \ln \frac{1}{\rho} \rightarrow (\text{grad } h)^2 = \frac{1}{\rho^2} = \epsilon^{2h} \tag{81b}$$

or

$$h = \frac{1}{r} \rightarrow (\text{grad } h)^2 = \frac{1}{r^4} = h^4. \tag{81c}$$

The possibility $h = x$ violates one defining condition for the present case (i.e., grad (grad $h)^2 \not\equiv 0$) and hence will be left for consideration in Case 3. The remaining two possibilities lead respectively to the following forms of ordinary differential equation for the determination of $\mathfrak{U}(h)$:

$$\left(\frac{S - \mathfrak{U}\mathfrak{U}'}{\gamma + \mathfrak{U}}\right)' + \frac{\alpha}{N} \epsilon^{-2h} \mathfrak{R}(\mathfrak{U}) = 0 \tag{82b}$$

or

$$\left(\frac{S - \mathfrak{U}\mathfrak{U}'}{\gamma + \mathfrak{U}}\right)' + \frac{\alpha}{N} \frac{1}{h^4} \mathfrak{R}(\mathfrak{U}) = 0. \tag{82c}$$

Given any $\mathfrak{U}(h)$ satisfying one of these equations, the associated $\mathfrak{H}(h)$ is obtained by integration from (77):

$$\mathfrak{H}(h) = \int \left(\frac{H - \mathfrak{U}\mathfrak{U}'}{\gamma + \mathfrak{U}}\right) dh + J \qquad (J\text{: arbitrary constant}). \tag{83}$$

It is evident from (72) that Case 1 does not exist if both recombination and time variation are absent.

Case 2:

$$\frac{\partial^2 \mathfrak{H}}{\partial h^2} = 0$$

In considering this case we shall exclude the condition $\dfrac{\partial \mathfrak{H}}{\partial h} = 0$ because it has been included in Section J.

From the condition $\dfrac{\partial^2 \mathfrak{H}}{\partial h^2} = 0$ we have

$$\mathfrak{H} = k(t)h + \ell(t) \qquad (k(t), \ell(t)\text{: arbitrary functions}) \tag{84}$$

with $k \not\equiv 0$. This shows that $\mathfrak{H}$ itself is a harmonic function and we can without loss of generality use it in place of h.

Equations (74) and (75) now yield the two conditions on $\mathfrak{U}(\mathfrak{H}, t)$, $\mathfrak{R}(\mathfrak{U})$, and $\mathfrak{H}(x, y, z, t)$:

$$\frac{j(t) - \mathfrak{U} \dfrac{\partial \mathfrak{U}}{\partial \mathfrak{H}}}{\mathfrak{U} + \gamma} = 1 \tag{85}$$

and

$$\mathcal{R}(\mathcal{U}) + \frac{eN}{2kT}\left(\frac{\partial \mathcal{U}}{\partial \mathcal{H}}\,\frac{\partial \mathcal{H}}{\partial t} + \frac{\partial \mathcal{U}}{\partial t}\right) = 0. \tag{86}$$

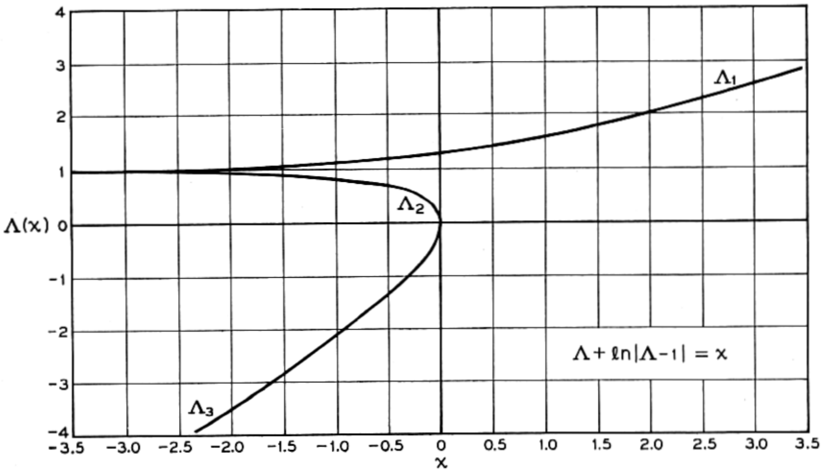

Fig. 1—The transcendental function Λ (x).

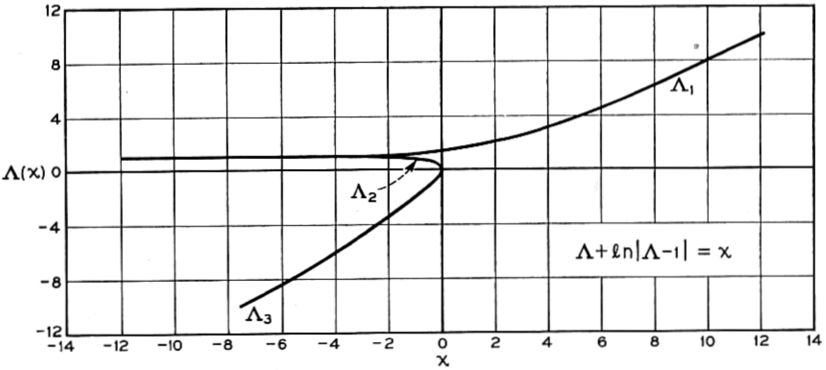

Fig. 2—The transcendental function Λ (x).

For the integration of (85) we need the transcendental algebraic function of a single real variable defined by

$$\Lambda(x) + \ln \mid \Lambda(x) - 1 \mid = x. \tag{87}$$

This function is plotted in Figs. 1 and 2. It will be observed that x is always a single-valued function of Λ; while Λ is a single-valued function of x for $x > 0$, a double-valued function for $x = 0$, and a triple-valued function for

$x < 0$. The single-valued monotone functions Λ_1, Λ_2, and Λ_3 are defined respectively by the restrictions $\Lambda > 1$, $1 > \Lambda \geq 0$, and $\Lambda \leq 0$. When Λ is used without subscript it is implied that either Λ_1, Λ_2, or Λ_3 can be used. It will be useful to remember that

$$\Lambda'(x) = \frac{\Lambda(x) - 1}{\Lambda(x)}. \tag{88}$$

In terms of the function Λ, (85) integrates to

$$\mathcal{U}(\mathcal{H}, t) = [j(t) - \gamma]\Lambda\left[\frac{m(t) - \mathcal{H}}{j(t) - \gamma}\right] \qquad (j \neq \gamma) \tag{89a}$$

(m(t): arbitrary function)

or

$$\mathcal{U}(\mathcal{H}, t) = m(t) - \mathcal{H} \qquad (j = \gamma). \tag{89b}$$

The latter case ($j = \gamma$) corresponds to $\mathcal{V} = \mathcal{V}(t)$ and hence was included in Section G. Therefore in the following we shall consider only $j \neq \gamma$.

Now by making use of (89a) and (88), (86) can be rewritten in the form:

$$\frac{2kT}{eN}\frac{\mathcal{U}\mathcal{R}(\mathcal{U})}{\mathcal{U} - j + \gamma} + \frac{j'\mathcal{U}^2}{(j - \gamma)(\mathcal{U} - j + \gamma)} = \frac{\partial\mathcal{H}}{\partial t} + \frac{j'(m - \mathcal{H})}{j - \gamma} - m' \tag{90}$$

$$\left(\text{primes denoting here } \frac{d}{dt}\right).$$

We now observe that the right side of (90) is harmonic, while the left side is a function only of $\mathcal{H}$ and t. From this it follows that the right side can be written in the form:

$$\frac{\partial\mathcal{H}}{\partial t} + \frac{j'(m - \mathcal{H})}{j - \gamma} - m' = q(t)\left[\frac{m - \mathcal{H}}{j - \gamma}\right] + r(t). \tag{91}$$

From (90), (91) and (89a) follows

$$\frac{2kT}{eN}\mathcal{R}(\mathcal{U}) = -\frac{j'}{j - \gamma}\mathcal{U} + \frac{\mathcal{U} - j + \gamma}{\mathcal{U}}\left(r + \frac{q}{j - \gamma}\mathcal{U} + q\ln\left|\frac{\mathcal{U}}{j - \gamma} - 1\right|\right). \tag{92}$$

Since (92) is of the form

$$\mathcal{R}(\mathcal{U}) = \phi(\mathcal{U}, t),$$

The result of taking $\left(\frac{\partial}{\partial t}\right)_{\mathcal{U} = \text{const}}$ of the right side must be zero identically in $\mathcal{U}$. Making use of the algebraic lemma that

$$A + Bx + \frac{C}{x} + \left(D + \frac{E}{x}\right) \ln \left| \frac{x}{F} - 1 \right| \equiv 0$$

implies $A = B = C = D = E = 0$, we arrive at two possibilities:

Possibility 1:

$$(q(t) = r(t) = 0 \quad \text{and} \quad j - \gamma = K\epsilon^{-Lt})$$

(K, L: arbitrary constants)

This yields from (92), (89a) and (91):

$$\frac{2kT}{eN} \mathcal{R}(\mathcal{U}) = L\mathcal{U} \tag{93}$$

$$\mathcal{U}(\mathcal{H}, t) = K\epsilon^{-Lt}\Lambda\left[\frac{1}{K}\epsilon^{Lt}(m(t) - \mathcal{H})\right] \tag{94}$$

and

$$\mathcal{H}(x, y, z, t) = \epsilon^{-Lt} s(x, y, z) + \epsilon^{-Lt} \int \epsilon^{Lt} \left[Lm(t) + m'(t)\right] dt \tag{95}$$

where $s(x, y, z)$ is any harmonic function.

Possibility, 2:

$$(j(t) = M, q(t) = Q, r(t) = R)$$

(M, Q, R: arbitrary constants)

This yields from (92), (89a) and (91):

$$\frac{2kT}{eN} \mathcal{R}(\mathcal{U}) = \frac{\mathcal{U} - M + \gamma}{\mathcal{U}} \cdot \left(R + \frac{Q}{M - \gamma}\mathcal{U} + Q \ln \left| \frac{\mathcal{U}}{M - \gamma} - 1 \right| \right) \tag{96}$$

$$\mathcal{U}(\mathcal{H}, t) = (M - \gamma)\Lambda\left[\frac{m(t) - \mathcal{H}}{M - \gamma}\right] \tag{97}$$

and

$$\mathcal{H}(x, y, z, t) = \epsilon^{Qt/(M-\gamma)} u(x, y, z) + \epsilon^{Qt/(M-\gamma)} \int \epsilon^{-(Qt/(M-\gamma))} \left[R + m'(t) + \frac{Q}{M - \gamma} m(t)\right] dt \tag{98}$$

where $u(x, y, z)$ is any harmonic function.

In the absence of recombination, Possibilities 1 and 2 lead to the same result: Equation (97) and

$$\mathcal{H}(x, y, z, t) = u(x, y, z) + m(t). \tag{99}$$

In the absence of time variation, (86) shows that recombination is necessarily absent, too, so the results reduce to

$$\mathcal{U}(\mathcal{H}) = \tilde{A}\Lambda\left(\frac{\tilde{B} - \mathcal{H}}{\tilde{A}}\right) \tag{100}$$

with $\mathcal{H}(x, y, z)$ any harmonic function and $\tilde{A}$ and $\tilde{B}$ arbitrary constants. This solution for the case grad $\mathcal{H} \neq 0$, together with that given by (59b) and (60) (with $G = 0$) for the case grad $\mathcal{H} = 0$, constitute a veritable gold mine of useful solutions because of the arbitrary harmonic function involved. An example involving a particular choice of $\mathcal{H}$ will be examined in Section R.

Case 3:

$$\frac{\partial^2\mathcal{H}}{\partial h^2} \neq 0, \qquad \text{grad (grad h)}^2 = 0.$$

In this case (grad $h)^2$ is a function of t so that (75) can be written in the form

$$\frac{\partial h}{\partial t} = \phi(h, t)$$

From this it follows (because div grad $h = 0$) that

$$h(x, y, z, t) = \bar{a}(t)\bar{b}(x, y, z) + \bar{c}(t) \tag{101}$$

with

$$\text{div grad } \bar{b}(x, y, z) = 0. \tag{102}$$

The condition grad (grad $h)^2 = 0$ now requires further that

$$\text{grad (grad } \bar{b})^2 = 0. \tag{103}$$

But any $\bar{b}(x, y, z)$ satisfying both (100) and (101) can, by suitable choice of axes, be written

$$\bar{b} = Sx \qquad (S\text{: constant}).$$

This leaves us with exactly the same totality of solutions as we could have obtained by setting $\mathcal{U} = \mathcal{U}(x, t)$, $\mathcal{H} = \mathcal{H}(x, t)$ in the first place. So we replace h by x in (74) and (75) and obtain:

$$\frac{\partial\mathcal{H}}{\partial x} = \frac{j(t) - \mathcal{U}\dfrac{\partial\mathcal{U}}{\partial x}}{\gamma + \mathcal{U}} \tag{104}$$

and

$$\frac{\partial}{\partial x}\left[\frac{j(t) - \mathfrak{U}\,\dfrac{\partial \mathfrak{U}}{\partial x}}{\gamma + \mathfrak{U}}\right] + \frac{\alpha}{N}\left[\mathcal{R}(\mathfrak{U}) + \frac{eN}{2kT}\frac{\partial \mathfrak{U}}{\partial t}\right] = 0. \tag{105}$$

Any $\mathfrak{U}(x, t)$ satisfying (105) can be substituted into (104) to obtain $\mathcal{H}(x, t)$ from

$$\mathcal{H}(x, t) = \bar{f}(t) + \int^{(x)} \frac{j(t) - \mathfrak{U}\,\dfrac{\partial \mathfrak{U}}{\partial x}}{\gamma + \mathfrak{U}}\,dx \tag{106}$$

($\bar{f}(t)$: arbitrary function).

If recombination is absent, $\mathcal{R}(\mathfrak{U})$ disappears from (105). If time variation is absent, $\dfrac{\partial \mathfrak{U}}{\partial t}$ disappears from (103) and $j(t)$ and $\bar{f}(t)$ are replaced by arbitrary constants. In the latter case, the standard change of variables

$$\mathcal{W}(\mathfrak{U}) \quad \text{for} \quad \frac{d\mathfrak{U}}{dx} \qquad \mathcal{W}(\mathfrak{U})\frac{d}{d\mathfrak{U}} \quad \text{for} \quad \frac{d}{dx} \tag{107}$$

reduces the solution of the second order equation (105) to the solution of a first-order equation followed by a quadrature. If both recombination and time variation are absent, the substitution (107) reduces the solution of (105) to two quadratures.

A set of equations equivalent to the steady-state $\left(\dfrac{\partial}{\partial t} \equiv 0\right)$ forms of (104) and (105) has been the subject of an extensive numerical investigation by W. van Roosbroeck (Reference 1) for the recombination rate functions given in (37) and (38).

M. Solutions with $\mathcal{V} = \mathcal{V}(h, t)$, $\mathcal{P} = \mathcal{P}(h, t)$, Grad $\mathcal{P} \neq 0$, Div Grad $h = 0$, $N = 0$

For these conditions (21) and (23b) yield

$$\frac{\partial^2 \mathcal{P}}{\partial h^2}(\text{grad } h)^2 = \frac{\alpha e}{kT}\left[\mathcal{R}(\mathcal{P}) + \frac{1}{2}\left(\frac{\partial \mathcal{P}}{\partial h}\frac{\partial h}{\partial t} + \frac{\partial \mathcal{P}}{\partial t}\right)\right] \tag{107}$$

and

$$\frac{\partial}{\partial h}\left[\frac{\partial \mathcal{P}}{\partial h} - \frac{\alpha e}{\beta kT}\mathcal{P}\frac{\partial \mathcal{V}}{\partial h}\right](\text{grad } h)^2 = 0. \tag{108}$$

Since we do not here allow grad $h = 0$, (108) implies

$$\frac{\partial \mathcal{V}}{\partial h} = \frac{\gamma \left[\frac{\partial \mathcal{P}}{\partial h} - \bar{g}(t)\right]}{\mathcal{P}} \qquad \gamma \equiv \frac{\beta kT}{\alpha e} \qquad (\bar{g}(t)\text{: arbitrary function}) \tag{109}$$

Case 1:

$$\frac{\partial^2 \mathcal{P}}{\partial h^2} \neq 0, \qquad \text{grad } (\text{grad } h)^2 \neq 0.$$

In this case, as in the associated case in Section L, the implications of (107) together with

$$\text{div grad } h(x, y, z, t) = 0$$

are not known when time variation is present.

When time variation is absent, we work with the conditions

$$\mathcal{P} = \mathcal{P}(h) \quad \text{and} \quad \mathcal{V} = \mathcal{V}(h)$$

with

$$\text{div grad } h(x, y, z) = 0$$

and arrive at counterparts of (107) and (108):

$$\mathcal{P}'' \cdot (\text{grad } h)^2 = \frac{\alpha e}{kT} \mathcal{R}(\mathcal{P}) \tag{110}$$

and

$$\left(\mathcal{P}' - \frac{1}{\gamma} \mathcal{P}\mathcal{V}'\right)' = 0. \tag{111}$$

Proceeding as in the analysis of Case 1 of Section L, we infer that h must be of the kind given by (81b) or (81c). The associated second-order differential equations restricting $\mathcal{P}(h)$ are then, respectively:

$$\mathcal{P}'' - \frac{\alpha e}{kT} \epsilon^{-2h} \mathcal{R}(\mathcal{P}) = 0 \tag{112a}$$

and

$$\mathcal{P}'' - \frac{\alpha e}{kT} \frac{1}{h^4} \mathcal{R}(\mathcal{P}) = 0. \tag{112b}$$

The $\mathcal{V}(h)$ associated with any solution of (112) can be obtained by integration from

$$\mathcal{V}(h) = \tilde{C} + \int \frac{\gamma \mathcal{P}' - \tilde{D}}{\mathcal{P}} \, dh \qquad (\tilde{C}, \tilde{D}\text{: arbitrary constants}). \tag{113}$$

It will be noted from (110) that simultaneous absence of recombination and time variation is inconsistent with the defining conditions of this case.

Case 2:

$$\frac{\partial^2 \mathcal{P}}{\partial h^2} = 0.$$

We shall exclude the possibility of $\frac{\partial \mathcal{P}}{\partial h} = 0$ because it is included in Section I. Then, proceeding as in Case 2 of Section L, we conclude that $\mathcal{P}$ itself is a harmonic function and can be used in place of h. (107) and (109) then become·

$$\mathcal{R}(\mathcal{P}) + \frac{1}{2}\frac{\partial \mathcal{P}}{\partial t} = 0 \tag{114}$$

and

$$\frac{\partial \mathcal{V}}{\partial \mathcal{P}} = \frac{\gamma[1 - \tilde{g}(t)]}{\mathcal{P}} \tag{115}$$

or

$$\mathcal{V}(\mathcal{P}, t) = \gamma[1 - \tilde{g}(t)] \ln \mathcal{P} + \tilde{\jmath}(t) \qquad (\tilde{\jmath}(t)\text{: arbitrary function}). \tag{116}$$

Because $\frac{\partial \mathcal{P}}{\partial t}$ is harmonic and a function of $\mathcal{P}$, we have

$$2\mathcal{R}(\mathcal{P}) = -\frac{\partial \mathcal{P}}{\partial t} = E\mathcal{P} - \tilde{F}$$

$$(\tilde{E}, \tilde{F}\text{: arbitrary constants})$$

whence

$$2\mathcal{R}(\mathcal{P}) = \tilde{E}\mathcal{P} - \tilde{F} \tag{117}$$

and

$$\mathcal{P}(x, y, z, t) = \epsilon^{-\tilde{E}t}\tilde{m}(x, y, z) - \frac{\tilde{F}}{\tilde{E}} \qquad (E \neq 0) \tag{118a}$$

or

$$\mathcal{P}(x, y, z, t) = \tilde{m}(x, y, z) + \tilde{F}t \qquad (\tilde{E} = 0) \tag{118b}$$

where $\tilde{m}(x, y, z)$ is an arbitrary *harmonic* function.

If recombination is absent, these results specialize to (116) and (118b) with $\bar{F} = 0$. If time variation is absent it follows from (114) that recombination is absent, too, and the results specialize to

$$\mathcal{V}(\mathcal{P}) = \tilde{G} + \tilde{\tilde{H}} \ln \mathcal{P} \tag{119}$$

with $\mathcal{P}(x, y, z)$ any harmonic function and $\tilde{G}$ and $\tilde{\tilde{H}}$ arbitrary constants. These solutions play the same role for the intrinsic semiconductor ($N = 0$) that (100) does for the extrinsic ($N \neq 0$).

Case 3:

$$\frac{\partial^2 \mathcal{P}}{\partial h^2} \neq 0, \qquad \text{grad (grad } h)^2 = 0.$$

In this case it can be shown, just as in Case 3 of Section L, that no generality is lost by considering $\mathcal{P} = \mathcal{P}(x, t)$ and $\mathcal{V} = \mathcal{V}(x, t)$ in place of $\mathcal{P}(h, t)$ and $\mathcal{V}(h, t)$. Equations (107) and (109) then become

$$\frac{\partial^2 \mathcal{P}}{\partial x^2} = \frac{\alpha e}{kT}\left[\mathcal{R}(\mathcal{P}) + \frac{1}{2}\frac{\partial \mathcal{P}}{\partial t}\right] \tag{120}$$

and

$$\frac{\partial \mathcal{V}}{\partial x} = \frac{\gamma\left[\dfrac{\partial \mathcal{P}}{\partial x} - \bar{g}(t)\right]}{\mathcal{P}}. \tag{121}$$

Any solution of (120) when substituted into (121) gives an associated $\mathcal{V}$ from

$$\mathcal{V}(x, t) = \bar{q}(t) + \gamma \int^{(x)} \frac{\dfrac{\partial \mathcal{P}}{\partial x} - \bar{g}(t)}{\mathcal{P}}\, dx. \tag{122}$$

If recombination is absent, $\mathcal{R}(\mathcal{P})$ merely vanishes from (120). If time variation is absent, the functions $\bar{g}(t)$ and $\bar{q}(t)$ are replaced by arbitrary constants and the standard change of variables

$$\begin{aligned} \mathbf{u}(\mathcal{P}) \quad &\text{for} \quad \frac{d\mathcal{P}}{dx} \\ \mathbf{u}(\mathcal{P})\frac{d}{d\mathcal{P}} \quad &\text{for} \quad \frac{d}{dx} \end{aligned} \tag{123}$$

leads to a solution of (120) in two quadratures. An equivalent solution is given by W. van Roosbroeck in Reference 1. From (120) it follows that recombination and time variation cannot simultaneously be absent for Case 3.

N. Construction of Solutions from Orthogonal Harmonic Fields, $N \neq 0$

There are many known examples of pairs of harmonic functions $h_1(x, y, z)$ and $h_2(x, y, z)$ that have orthogonal vector fields—that is, for which

$$\text{grad } h_1 \cdot \text{grad } h_2 = 0 \tag{124}$$

with grad $h_1 \neq 0$ and grad $h_2 \neq 0$. [E.g., the real and imaginary parts of any analytic function of a complex variable.] From any such pair of functions we can construct the following solutions of (33) and (34):

$$\mathcal{U} = h_1 ; \qquad \mathcal{H} = h_2 - h_1 \tag{125}$$

and

$$\mathcal{U} = \sqrt{h_1} ; \qquad \mathcal{H} = h_2 . \tag{126}$$

In terms of $\mathcal{P}$ and $\mathcal{V}$ these solutions are

$$\mathcal{P} = \frac{Ne}{kT} h_1 ; \qquad \mathcal{V} = h_2 \tag{127}$$

and

$$\mathcal{P} = \frac{Ne}{kT} \sqrt{h_1} ; \qquad \mathcal{V} = \sqrt{h_1} + h_2 . \tag{128}$$

The validity of the solution (125) is seen from (33) and this expanded form of (34):

$$\mathcal{U} \text{ div grad } \mathcal{U} + \text{grad } \mathcal{U} \cdot \text{grad } (\mathcal{U} + \mathcal{H}) = 0. \tag{129}$$

Similarly, the validity of (126) follows from (33) together with a different expansion of (34):

$$\text{div grad } \mathcal{U}^2 + 2 \text{ grad } \mathcal{U} \cdot \text{grad } \mathcal{H} = 0. \tag{130}$$

It is evident that a given h_1 and h_2 can be interchanged in the above solutions to yield different solutions, and also that any given h_1 or h_2 can be replaced by an arbitrary constant multiple of itself plus a second arbitrary constant.

O. Construction of Solutions from Orthogonal Harmonic Fields, $N = 0$

We can write the differential equation system for the intrinsic semiconductor [(35) and (36)] in the form:

$$\text{div grad } \mathcal{P} = 0 \tag{131}$$

$$\mathcal{P} \text{ div grad } \mathcal{V} + \text{grad } \mathcal{P} \cdot \text{grad } \mathcal{V} = 0. \tag{132}$$

From these we verify the solution:

$$\mathcal{P} = h_1 ; \qquad \mathcal{V} = h_2 \tag{133}$$

for any harmonic h_1 and h_2 satisfying (124).

The solutions given by (127) and (133) have the property

$$\text{grad } \mathcal{P} \cdot \text{grad } \mathcal{V} = 0$$

and so may be considered, in a sense, complementary to the solutions in Sections L and M for which

$$\text{grad } \mathcal{P} \times \text{grad } \mathcal{V} = 0.$$

P. Superposition of a Harmonic $\mathcal{H}$ Field, N. $\neq 0$

Inspection of the equation system [(33), (130)] reveals the following superposition theorem for obtaining new solutions from some known solutions *for the case of no recombination or time variation:*

[*Theorem 12:* If $[\tilde{\mathcal{U}}, \tilde{\mathcal{H}}]$ is a known solution and if h is any harmonic function such that grad $\tilde{\mathcal{U}} \cdot$ grad $h = 0$, then $[\mathcal{U}, \tilde{\mathcal{H}} + h]$ is also a solution.]

Or, in terms of $\mathcal{P}$ and $\mathcal{V}$:

[*Theorem 12':* If $[\tilde{\mathcal{P}}, \tilde{\mathcal{V}}]$ is a known solution and if h is any harmonic function such that grad $\tilde{\mathcal{P}} \cdot$ grad $h = 0$, then $[\tilde{\mathcal{P}}, \tilde{\mathcal{V}} + h]$ is also a solution.]

In the latter form it is evident from Section O that the theorem holds also for $N = 0$, but does not extend the results of Section O.

Q. A Partial Differential Equation in Terms of $\mathcal{H}$ Alone, $N \neq 0$

For $N = 0$, (21) provides a differential equation involving only one dependent variable—$\mathcal{P}$. We shall now derive an analogous—but vastly more complicated—differential equation *for the case* $N \neq 0$, $\dfrac{\partial \mathcal{U}}{\partial t} = 0$.

For this case (30) and (32) become

$$\text{div grad } \mathcal{H} = -\frac{\alpha}{N} \mathcal{R}(\mathcal{U})$$

and

$$\text{div}\left[\text{grad } \mathcal{H} + \frac{1}{\gamma} \mathcal{U} \text{ grad } (\mathcal{U} + \mathcal{H})\right] = 0,$$

or in terms of a familiar vector symbolism

$$\nabla^2 \mathcal{H} = -\frac{\alpha}{N} \mathcal{R}(\mathcal{U}) \tag{134}$$

and

$$\nabla \cdot \left[\nabla \mathcal{H} + \frac{1}{\gamma} \mathcal{U} \nabla (\mathcal{U} + \mathcal{H}) \right] = 0. \tag{135}$$

Now let $\mathcal{S}(\mathcal{U})$ be the inverse function to $\mathcal{R}(\mathcal{U})$, i.e. the function such that

$$\mathcal{S}(\mathcal{R}(\mathcal{U})) \equiv \mathcal{U}.$$

Then from (134) we have

$$\mathcal{U} = \mathcal{S}\left(-\frac{N}{\alpha} \nabla^2 \mathcal{H} \right). \tag{136}$$

Substitution of (136) into (135) yields after some computation

$$\begin{aligned} \mathcal{S}\mathcal{S}' \nabla^2 (\nabla^2 \mathcal{H}) - \frac{N}{\alpha} (\mathcal{S}\mathcal{S}'' + \mathcal{S}'^2)(\nabla \nabla^2 \mathcal{H})^2 \\ + \mathcal{S}' \nabla \mathcal{H} \cdot \nabla \nabla^2 \mathcal{H} - \frac{\alpha}{N} (\mathcal{S} + \gamma) \nabla^2 \mathcal{H} = 0 \end{aligned} \tag{137}$$

where $\mathcal{S}'(\psi) \equiv \frac{d}{d\psi} \mathcal{S}(\psi)$, etc.

$\mathcal{S}$, $\mathcal{S}'$, $\mathcal{S}''$ are considered as given functions of $\left(-\frac{N}{\alpha} \nabla^2 \mathcal{H} \right)$.

The simplest meaningful choice of $\mathcal{S}$ is

$$\mathcal{S}\left(-\frac{N}{\alpha} \nabla^2 \mathcal{H} \right) = \frac{\alpha}{N} \tilde{J} \cdot \left(-\frac{N}{\alpha} \nabla^2 \mathcal{H} \right) + \tilde{K} \tag{138}$$

(J, $\tilde{K}$: prescribed constants)

corresponding to constant mean lifetime recombination. For this $\mathcal{S}$, (137) specializes to

$$\begin{aligned} \tilde{J}(\tilde{K} - J\nabla^2 \mathcal{H}) \nabla^2 (\nabla^2 \mathcal{H}) - \tilde{J}^2 (\nabla \nabla^2 \mathcal{H})^2 \\ + J \nabla \mathcal{H} \cdot \nabla \nabla^2 \mathcal{H} - (\gamma + \tilde{K} - J \nabla^2 \mathcal{H}) \nabla^2 \mathcal{H} = 0. \end{aligned} \tag{139}$$

If any $\mathcal{H}$ can be found satisfying (139), the associated $\mathcal{U}$ is given (from (136)) by

$$\mathcal{U} = \tilde{J} \nabla^2 \mathcal{H} + \tilde{K}.$$

R. Sample Application of the Results of Section L: Spherical Symmetry, $N \neq 0$

As an example of the solutions included in the results of Section L we consider the case of a spherically symmetric field about a point (or spherical) source of current.

We take, as the most general harmonic function having spherical symmetry,

$$\mathcal{H} = \tilde{L}\frac{1}{r} + \tilde{M} \tag{140}$$

($\tilde{L}$, $\tilde{M}$: arbitrary constants).

For the time being we shall assume $\tilde{L} \neq 0$ and $\tilde{M} \neq 0$. Then from (100) and (28) and (29) we have

$$\mathcal{V} = \tilde{A}\Lambda\left(\frac{\tilde{B} - \tilde{M} - \tilde{L}/r}{\tilde{A}}\right) + M + \frac{L}{r} \tag{141}$$

and

$$\mathcal{P} = \frac{Ne}{kT}\tilde{A}\Lambda\left(\frac{\tilde{B} - M - \tilde{L}/r}{\tilde{A}}\right). \tag{142}$$

In terms of $\mathcal{V}$ and $\mathcal{P}$, (3) and (4) can be written

$$\mathring{\|}_p = \frac{-\mu_p e}{2}\left[(\mathcal{P} - N)\operatorname{grad}\mathcal{V} + \frac{kT}{e}\operatorname{grad}\mathcal{P}\right] \tag{143}$$

and

$$\mathring{\|}_n = \frac{-\mu_n e}{2}\left[(\mathcal{P} + N)\operatorname{grad}\mathcal{V} - \frac{kT}{e}\operatorname{grad}\mathcal{P}\right] \tag{144}$$

which yield upon substitution of (141) and (142):

$$\mathring{\|}_p = \frac{1}{2}\mu_p eN\tilde{L}\left(\frac{e}{kT}\tilde{A} - 1\right)\frac{1}{r^2}\mathbf{r}_1 \tag{145}$$

and

$$\mathring{\|}_n = \frac{1}{2}\mu_n eN\tilde{L}\left(\frac{e}{kT}\tilde{A} + 1\right)\frac{1}{r^2}\mathbf{r}_1 \tag{146}$$

where $\mathbf{r}_1$ is the unit radial vector. The total current density is obtained by adding (151) and (152):

$$\mathring{\|} = \frac{1}{2}eN\tilde{L}\left[(\mu_n + \mu_p)\frac{e}{kT}\tilde{A} + (\mu_n - \mu_p)\right]\frac{1}{r^2}\mathbf{r}_1. \tag{147}$$

The currents flowing are obtained from the current densities from the relation

$$I = \Omega r^2 \, \mathring{||} \cdot \mathbf{r}_1$$

where Ω is the solid angle (with respect to the origin) within which the flow field lies. (If the current source is surrounded by the homogeneous semiconductor, $\Omega = 4\pi$; if it lies on a flat surface of a large slab, $\Omega = 2\pi$, etc.) So we have

$$I_p = \frac{1}{2} \Omega \mu_p e N \tilde{L} \left(\frac{e}{kT} \tilde{A} - 1 \right) \tag{148}$$

$$I_n = \frac{1}{2} \Omega \mu_n e N \tilde{L} \left(\frac{e}{kT} \tilde{A} + 1 \right) \tag{149}$$

$$I = \frac{1}{2} \Omega e N \tilde{L} \left[(\mu_n + \mu_p) \frac{e}{kT} \tilde{A} + (\mu_n - \mu_p) \right]. \tag{150}$$

We shall now obtain expressions for the mathematical parameters $\tilde{B}$, $\tilde{A}$, $\tilde{L}$, and $\tilde{M}$ in terms of meaningful physical quantities: I_p, I_n, $\mathcal{V}_\infty$ and $\mathcal{P}_\infty$. (Subscript ∞ refers to values of variables as r becomes very large.) We shall take our reference voltage as the voltage "at infinity" so that $\mathcal{V}_\infty = 0$. Setting $1/r = 0$ in (141) and (142) we obtain

$$0 = \tilde{A} \Lambda \left(\frac{\tilde{B} - \tilde{M}}{\tilde{A}} \right) + \tilde{M}$$

and

$$\mathcal{P}_\infty = \frac{Ne}{kT} \tilde{A} \Lambda \left(\frac{\tilde{B} - \tilde{M}}{\tilde{A}} \right)$$

from which follows (for $\tilde{A} \neq 0$)

$$\tilde{B} = \tilde{A} \ln \left| \frac{\tilde{M}}{\tilde{A}} + 1 \right| \tag{151}$$

and

$$\tilde{M} = -\frac{kT}{eN} \mathcal{P}_\infty . \tag{152}$$

From (148) and (149) we readily find (for $\tilde{L} \neq 0$):

$$\tilde{A} = \frac{kT}{e} \frac{\mu_p I_n + \mu_n I_p}{\mu_p I_n - \mu_n I_p} \tag{153}$$

and

$$\tilde{L} = \frac{\mu_p I_n - \mu_n I_p}{\Omega \mu_n \mu_p eN}. \tag{154}$$

Finally we substitute (152) and (153) into (154) to get

$$\tilde{B} = \frac{kT}{e} \frac{\mu_p I_n + \mu_n I_p}{\mu_p I_n - \mu_n I_p} \ln \left| \frac{(N - \mathcal{P}_\infty)\mu_p I_n + (N + \mathcal{P}_\infty)\mu_n I_p}{N(\mu_p I_n - \mu_n I_p)} \right| \tag{155}$$

Equations (152)–(155) give the desired expressions for $\tilde{M}$, $\tilde{A}$, $\tilde{L}$ and $\tilde{B}$ in terms of I_p, I_n, and $\mathcal{P}_\infty$ for $\mathcal{V}_\infty = 0$ if $\tilde{A} \neq 0$ and $\tilde{L} \neq 0$.

For $\tilde{A} = 0$ we can repeat the above steps using

$$\mathcal{V} = \tilde{B} \tag{156}$$

and

$$\mathcal{P} = \frac{Ne}{kT}(\tilde{B} - \tilde{M} - \tilde{L}/r) \tag{157}$$

in place of (141) and (142). The result for $\tilde{L} \neq 0$ and $\mathcal{V}_\infty = 0$ is

$$I_p = -\tfrac{1}{2}\Omega\mu_p eN\tilde{L} \tag{158}$$

$$I_n = \tfrac{1}{2}\Omega\mu_n eN\tilde{L} \tag{159}$$

$$I = \tfrac{1}{2}\Omega eN(\mu_n - \mu_p)\tilde{L} \tag{160}$$

with

$$\tilde{B} = 0 \tag{161}$$

$$\tilde{M} = -\frac{kT}{eN}\mathcal{P}_\infty \tag{162}$$

and

$$\tilde{L} = \frac{-2I_p}{\Omega\mu_p eN} = \frac{2I_n}{\Omega\mu_n eN} = \frac{\mu_p I_n - \mu_n I_p}{\Omega\mu_p\mu_n eN}. \tag{163}$$

It is evident that $\tilde{A} = 0$ implies $\mathcal{V}$ = constant and $\mu_p I_n + \mu_n I_p = 0$.

The condition $L = 0$ makes $\mathcal{H}$ = constant, so we use (62b) and (63b) and set

$$\mathcal{V} = \tilde{Q} + \sqrt{\tilde{R} + \tilde{S}/r} \tag{164}$$

and

$$\mathcal{P} = \frac{Ne}{kT}\sqrt{\tilde{R} + \tilde{S}/r} \qquad (\tilde{Q}, \tilde{R}, \tilde{S}\text{: arbitrary constants}). \tag{165}$$

From (143) and (144) we obtain

$$I_p = -\frac{\Omega N e^2 \mu_p}{4kT} \tilde{S} \tag{166}$$

$$I_n = -\frac{\Omega N e^2 \mu_n}{4kT} \tilde{S} \tag{167}$$

and

$$I = -\frac{\Omega N e^2}{4kT} (\mu_n + \mu_p) \tilde{S}. \tag{168}$$

From (164) and (165) we readily obtain for $\mathcal{V}_\infty = 0$:

$$R = \left(\frac{kT}{Ne} \mathcal{P}_\infty\right)^2 \tag{169}$$

and

$$\tilde{Q} = -\frac{kT}{Ne} \mathcal{P}_\infty . \tag{170}$$

It is evident that $\tilde{L} = 0$ corresponds to the case $\mathcal{H}$ = constant and implies $\mu_p I_n - \mu_n I_p = 0$.

The foregoing now provides a formal solution with $\mathcal{V}_\infty = 0$ for every assignment of values to $\mathcal{P}_\infty$, I_p, and I_n. There remains the question of the requirements imposed by the condition

$$n, p \geq 0$$

which is equivalent to

$$\mathcal{P} \geq | N | . \tag{171}$$

This implies first of all that $\mathcal{P}_\infty$ must be chosen $\geq | N |$.

It is instructive to look first at the case $\tilde{L} = 0$. Equation (165) shows immediately that (171) requires the choice of the positive sign for the radical for $N > 0$ and the negative for $N < 0$ to avoid $\mathcal{P}_\infty < | N |$. We further find by substitution of (166) and (169) into (165) that (171) requires

$$r \geq \left[\frac{4}{\Omega k T \mu_p (\mathcal{P}_\infty^2 - | N |^2)}\right] N I_p . \tag{172}$$

The bracketed factor is positive. Since we are interested only in non-negative values of r, (172) imposes no restriction if I_p is zero or not of the same sign as N. However, for N and I_p of the same sign, (172) establishes an inner radius inside which the solution does not satisfy (171). This may be regarded as establishing the minimum radius for an inner spherical electrode for prescribed I_p and $\mathcal{P}_\infty$, or alternatively as limiting the possible choices of I_p and $\mathcal{P}_\infty$ for prescribed inner electrode radius. Had we chosen the constants $\tilde{Q}$, $\tilde{R}$ and $\tilde{S}$ so as to obtain prescribed values of $\mathcal{P}$ and $\mathcal{V}$ at a preselected electrode radius r_0, restrictions analogous to (172) on the *maximum* radius would appear.

For the case $\tilde{A} = 0$ the restriction analogous to (172) is

$$r \geq -\left[\frac{2}{\Omega\mu_p kT(\mathcal{P}_\infty - |N|)}\right] I_p. \tag{173}$$

Since the bracketed factor is positive, (173) provides no restriction for $I_p \geq 0$, but for $I_p < 0$ establishes a minimum radius of the kind just discussed.

For $\tilde{L}, \tilde{A} \neq 0$, the analog of (172) and (173) is

$$r \geq \frac{\tilde{L}/\tilde{A}}{\Lambda^{-1}\left(\frac{kT\mathcal{P}_\infty}{Ne\tilde{A}}\right) - \Lambda^{-1}\left(\frac{kT\,|N|}{Ne\tilde{A}}\right)} \tag{174}$$

where $\tilde{A}$ and $\tilde{L}$ are given by (153) and (154) and Λ^{-1} denotes the inverse function of Λ—i.e.,

$$\Lambda^{-1}[\Lambda(x)] \equiv x$$

or

$$\Lambda^{-1}(\Lambda) = \Lambda + \ln|\Lambda - 1|.$$

Equation (174) is a minimum radius restriction of the same kind as those obtaining for $\tilde{A} = 0$, and $\tilde{L} = 0$, but the relationship between the minimum radius r_0 and $\mathcal{P}_\infty$, I_p and I_n is considerably more complicated than in the more degenerate cases.

It will be noted that the relation

$$\frac{kT}{eN}\frac{\mathcal{P}_\infty}{\tilde{A}} = \Lambda\left(\frac{\tilde{B} - \tilde{M}}{\tilde{A}}\right)$$

(with $\tilde{A}$, $\tilde{B}$, $\tilde{M}$ given in terms of $\mathcal{P}_\infty$, I_p, I_n by (152), (153), and (154)) determines which function (Λ_1, Λ_2, or Λ_3) is to be used for Λ in any given case, because any assigned value ($\neq 0$) is taken on by one and only one of (Λ_1, Λ_2, Λ_3).

If surface recombination is negligible as well as interior recombination, this spherically symmetric solution is of use in the study of "point" contacts on a plane surface of a semiconductor. [Fig. 3 and Ref. 2.]

The results of this section can easily be duplicated for any other choice of the harmonic function $\mathcal{H}$ to obtain a great variety of specimen solutions.

Solutions based on $\mathcal{H}$'s having a single source singularity (such as the example above) will contain four mathematical parameters, and hence will permit arbitrary selection (subject to (6)) of the physical parameters, I_p, I_n, $\mathcal{P}_\infty$, and $\mathcal{V}_\infty$. However, solutions based on $\mathcal{H}$'s having more than one source singularity will provide only a subset of the possible assignments of the physical parameters. For example, the harmonic function associated with the electrostatic field produced by two separate point charges each equidistant from two parallel infinite plane conductors provides solutions of

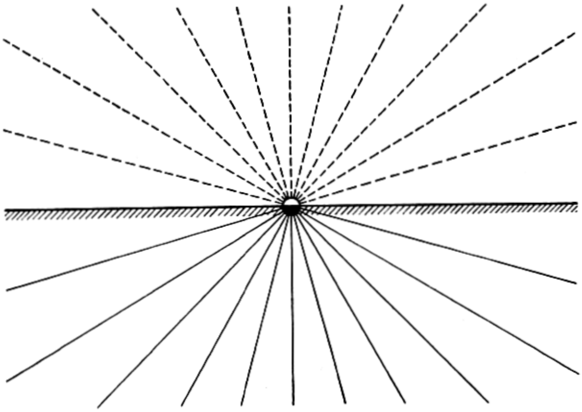

Fig. 3—Point source flow field, useful in connection with point contact theory

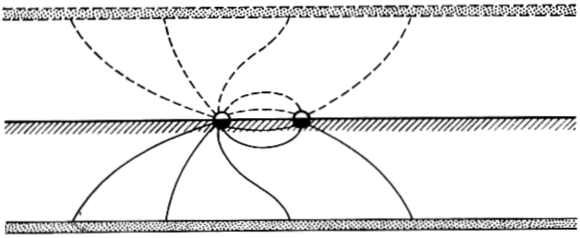

Fig. 4—Two-source flow field between conducting planes, useful in connection with Type A transistor theory.

interest in connection with the type A transistor configuration (Fig. 4). However, the family of solutions obtained contains only a five-parameter subset of the six-parameter family obtainable by arbitrary assignment of I_{p1}, I_{p2}, I_{n1}, I_{n2}, $\mathcal{P}_\infty$, and $\mathcal{V}_\infty$.

S. Sample Application of the Results of Section M: Spherical Symmetry, $N = 0$

We now round out the considerations of Section R by exhibiting the related solutions for $N = 0$ (i.e., the intrinsic semiconductor).

In accordance with the results of Section M, we choose for $\mathcal{P}$ the most general harmonic function with spherical symmetry:

$$\mathcal{P} = \hat{A}\frac{1}{r} + \hat{B} \qquad (\hat{A}, \hat{B}\text{: arbitrary constants}). \quad (175)$$

From (119) then, <u>*for* $\hat{A} \neq 0$</u>

$$\mathcal{V} = \tilde{H} \ln\left(\hat{A}\frac{1}{r} + \hat{B}\right) + \tilde{G} \quad (176)$$

and from (175), (176), (143), and (144)

$$I_p = \frac{1}{2}\Omega\mu_p e\hat{A}\left(\tilde{H} + \frac{kT}{e}\right) \quad (177)$$

$$I_n = \frac{1}{2}\Omega\mu_n e\hat{A}\left(\tilde{H} - \frac{kT}{e}\right) \quad (178)$$

$$I = \frac{1}{2}\Omega e\hat{A}\left[(\mu_n + \mu_p)\tilde{H} - (\mu_n - \mu_p)\frac{kT}{e}\right]. \quad (179)$$

From (177) and (178) we obtain

$$\hat{A} = \frac{\mu_n I_p - \mu_p I_n}{\Omega\mu_p\mu_n kT} \quad (180)$$

and

$$\tilde{H} = \frac{kT}{e}\frac{\mu_n I_p + \mu_p I_n}{\mu_n I_p - \mu_p I_n}, \quad (181)$$

and from (175) and (176) for $\mathcal{V}_\infty = 0$:

$$\hat{B} = \mathcal{P}_\infty \quad (182)$$

and

$$\tilde{G} = -\tilde{H}\ln\hat{B} = -\frac{kT}{e}\frac{\mu_n I_p + \mu_p I_n}{\mu_n I_p - \mu_p I_n}\ln\mathcal{P}_\infty. \quad (183)$$

The condition $\mathcal{P}_\infty \geq |N| = 0$ introduces the restriction (for $\hat{A} \neq 0$):

$$r \geq \left[\frac{1}{\Omega\mu_p\mu_n kT\mathcal{P}_\infty}\right](\mu_n I_p - \mu_p I_n). \quad (184)$$

Evidently this implies no real restriction for $\mu_n I_p - \mu_p I_n < 0$ (i.e., $\hat{A} < 0$), but introduces a minimum radius—of the same kind we have already discussed—when $\mu_n I_p - \mu_p I_n > 0$ (i.e., $\hat{A} > 0$).

For $\hat{A} = 0$, $\mathcal{P}$ is constant and, by Section I, $\mathcal{V}$ is harmonic. So we set

$$\mathcal{P} = \mathcal{P}_\infty > 0 \tag{185}$$

and

$$\mathcal{V} = \hat{C}\frac{1}{r} + \hat{D} \tag{186}$$

and obtain from (143) and (144)

$$I_p = \tfrac{1}{2}\Omega\mu_p e\hat{C}\mathcal{P}_\infty \tag{187}$$

and

$$I_n = \tfrac{1}{2}\Omega\mu_n e\hat{C}\mathcal{P}_\infty . \tag{188}$$

From (187) and (188):

$$\hat{C} = \frac{2I_p}{\Omega\mu_p e\mathcal{P}_\infty} = \frac{2I_n}{\Omega\mu_n e\mathcal{P}_\infty} = \frac{\mu_n I_p + \mu_p I_n}{\Omega\mu_n\mu_p e\mathcal{P}_\infty} \tag{189}$$

and from (186) for $\mathcal{V}_\infty = 0$,

$$\hat{D} = 0. \tag{190}$$

Evidently $\hat{A} = 0$ is associated with the condition

$$\mu_n I_p - \mu_p I_n = 0.$$

T. Summary List of Symbols

Coordinate Systems:

(x, y, z): ordinary rectangular cartesian coordinates.
(ρ, θ, z): ordinary circular cylindrical coordinates.
(r, θ, ϕ): ordinary spherical polar coordinates.
$\mathfrak{r}_1$: unit radial vector in (r, θ, ϕ).
t : time variable.

Physical Variables:

n : concentration of negative carriers (electrons).
p : concentration of positive carriers (holes).
$\mathcal{P}$: total carrier concentration $\equiv n + p$.
$\mathcal{U}: \equiv \frac{kT}{eN}\mathcal{P} \qquad (N \neq 0)$.
$\mathcal{R}$: recombination rate function.
$\mathcal{V}$: electrostatic potential.

$\mathcal{H}: \equiv \mathcal{V} - \mathcal{U} = \mathcal{V} - \frac{kT}{eN}\mathcal{P} \qquad (N \neq 0).$

$\overset{\circ}{||}$: total current density vector.

$\overset{\circ}{||}_n$: electron current density vector.

$\overset{\circ}{||}_p$: hole current density vector.

subscript "0": designates thermal equilibrium values.

subscript "∞": designates values "at infinity".

Physical Constants:

T: absolute temperature.

e : magnitude of electronic charge.

k : Boltzmann's constant.

μ_n: electron mobility constant.

μ_p: hole mobility constant.

$\alpha : \equiv 1/\mu_p + 1/\mu_n$.

$\beta : \equiv 1/\mu_p - 1/\mu_n$. (Assumed $\neq 0$)

$\gamma : \equiv \frac{\beta k T}{\alpha e}$

$N: \equiv n_0 - p_0$.

Other Constants and Functions:

$A, B, \cdots, Z$ ((except I, N, and T)),

$\tilde{A}, \tilde{B}, \cdots, \tilde{Z}$,

$\hat{A}, \hat{B}, \cdots, \hat{Z}$: arbitrary constants

$a, b, \cdots, z$ ((except $e, h, k, n, p, r, t, x, y, z$)),

$\tilde{a}, \tilde{b}, \cdots, \tilde{z}$: arbitrary functions of variables designated (e.g., $j(t)$).

h, h_1, h_2 : harmonic functions of variables designated at place of usage.

Λ: $\Lambda(x)$ is defined by the relation $\Lambda(x) + \ln|\Lambda(x) - 1| \equiv x$.

(See Figs. 1 and 2.)

$\mathcal{S}$: $\mathcal{S}(\mathcal{U})$ is defined by $\mathcal{S}[\mathcal{R}(\mathcal{U})] \equiv \mathcal{U}$.

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