The Reproduction of Magnetically Recorded Signals

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For certain speech studies at the Bell Telephone Laboratories, it has been necessary to design some rather specialized magnetic recording equipment. In connection with this work, it has been found experimentally and theoretically that introducing a spacing of d inches between the reproducing head and the recording medium decreases the reproduced voltage by 54.6 (d/λ) decibels when the recorded wavelength is λ inches. For short wavelengths this loss is many decibels even when the effective spacing is only a few ten-thousandths of an inch. On this basis it is argued that imperfect magnetic contact between reproducing head and recording medium may account for much of the high-frequency loss which is experimentally observed.

Introduction

Within the last few years there has been increasing use of magnetic recording in various telephone research applications (examples are various versions of the sound spectrograph used in studies of speech and noise). Some of these uses[1] have required a reproducing head spaced slightly out of contact with the recording medium. Experimental studies were made to determine the effect of such spacing and the results were found to be expressible in an unexpectedly simple form. The general equation derived is believed to be fundamental to the recording problem and to account for much of the high-frequency loss that is found in both in- and out-of-contact systems.

This paper discusses results of the experimental study and presents for comparison some theoretical calculations based on an idealized model.

Measurements of Spacing Loss

In order to measure the effect of spacing between the reproducing head and the medium, an experiment was set up as indicated in Fig. 1. The recording medium used was a 0.0003 inch plating of cobalt-nickel alloy[2] on the flat surface of a brass disc approximately 13 inches in diameter by $\frac{1}{4}$ inch thick.

This disc was made with considerable care to insure that the recording surface was as nearly plane and smooth as possible and that it would turn reasonably true in its bearings. Speeds of 25 and 78 rpm were provided.

[1] R. C. Mathes, A. C. Norwine, and K. H. Davis, "Cathode-Ray Sound Spectroscope," *Jl. Acous. Soc. Am.*, *21*, 527 (1949).
[2] Plating was done by the Brush Development Company.

The ring-type record-reproduce head shown in Fig. 1 was lapped slightly to obtain a reasonably good fit with the surface of the disc.

A single-frequency recording was made with the head in contact with the disc using a-c. bias in the usual way. Then the open circuit reproduced signal level was measured, first with the head in contact, and then after introducing paper shims of various thickness between the reproducing head and the medium. Thus the effect of spacing was measured at a particular frequency and recording speed. The signal was then erased and the process was repeated for other recorded frequencies and for several record-reproduce speeds. Measurements were also made for cases in which the recording and reproducing speeds were different. Considerable care was required to keep the disc and head sufficiently clean so that reproducible results could be obtained.

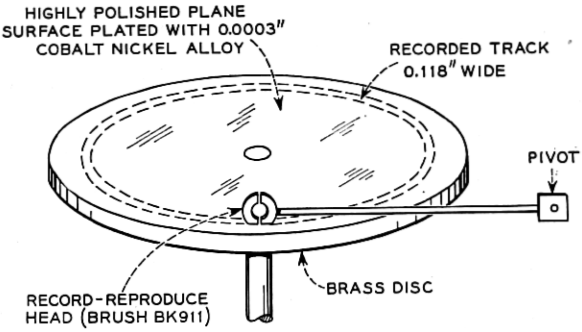

Fig. 1—Mechanical arrangement of recording set up. The one head served for recording, playback, and erase.

Figure 2 shows typical response curves measured at 21 in./sec. with the reproducing head in contact and with 0.004 inch spacing. The difference between these two curves will be called the spacing loss corresponding to this spacing and speed. From these data and more of the same sort it is found that, within experimental error, spacing loss can be very simply related to spacing and the recorded wavelength, λ, by the empirical equation,

$$\text{Spacing loss} = 55(d/\lambda) \text{ decibels} \qquad (1)$$

where spacing loss is the number of decibels by which the reproduced level is decreased when a spacing of d inches is introduced between the reproducing head and a magnetic medium on which a signal of wavelength λ inches has been recorded.

The fact that this expression fits the experimental data reasonably well is indicated in Fig. 3 where spacing loss data measured at a number of different speeds, frequencies, and spacings are plotted against d/λ.

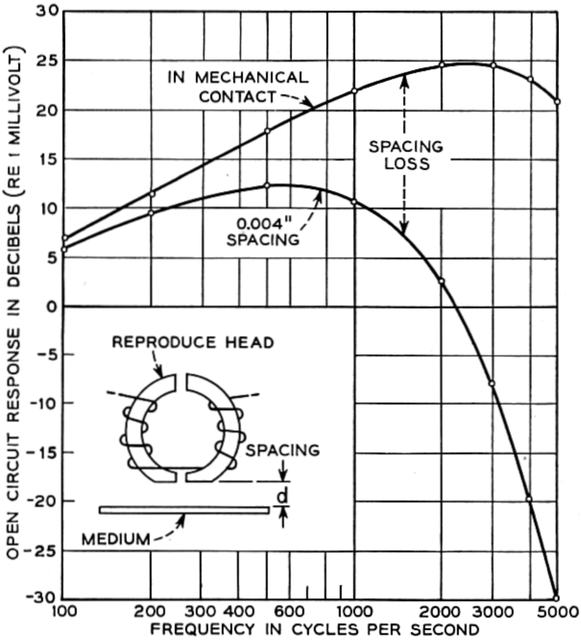

Fig. 2—Response curves taken at 21 in./sec. Recordings were made with head in contact and were played back first with head in contact and then with a spacing of 4 mils between head and disc.

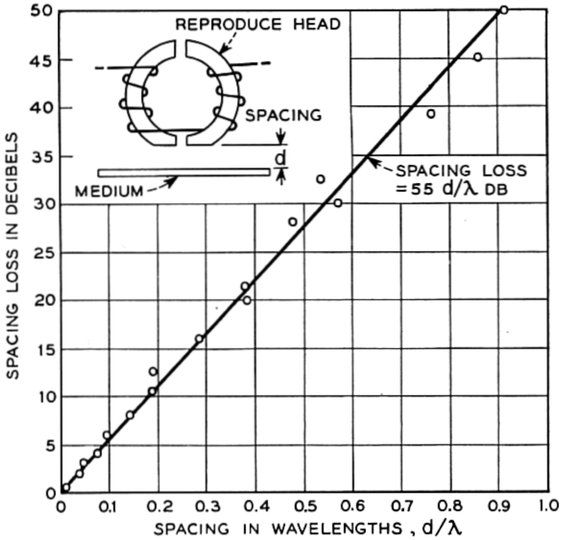

Fig. 3—Data obtained as in Fig. 2 show spacing loss approximately equal to $55(d/\lambda)$ decibels.

Implications of the Experiment

In this section it will be assumed that equation (1) holds true in all cases where the spacing d is sufficiently small and the recorded track is sufficiently wide so that end effects are negligible. If this is true, as it seems experimentally to be, then it is indeed surprising how great can be the effect of even a very small spacing when the recorded wavelength is small. For example, take the case of a 7500 cps signal recorded at 7.5 in./sec. in which case the wavelength is 0.001 inch. A particle of dust which separated the tape from the reproducing head by one-thousandth of an inch would decrease the reproduced level by 55 db. A spacing of 0.0001 inch would produce a quite noticeable 5.5 db effect and even at 0.00001 inch spacing the 0.55 db loss would be measurable in a carefully controlled experiment.

In view of the magnitudes involved, it seems probable that this spacing loss may play a significant role even in cases where the reproducing head is supposed to be in contact with the medium. For example, it has been known for some time that chattering of the tape on the reproducing head or changes in the degree of contact due to imperfect smoothness of the tape can result in amplitude modulation of the reproduced signal and thereby give rise to "modulation noise" or "noise behind the signal."

With the aid of equation (1) it is possible to estimate the magnitude of the noise provided some assumption is made about the wave form of the modulation. To take a simple case, suppose that the roughness of the tape were such as to sinusoidally modulate the spacing by a very small amount and at a low frequency. The reproduced signal would then be modulated and would contain a sideband on each side of the center frequency. The energy in these two sidebands constitutes the modulation noise in this case. If it is required that this noise be 40 db down on the signal, then one can calculate the maximum permissible excursion of the tape away from the reproducing head. This turns out to be $1.1(10)^{-5}$ cm. or about one-sixth of the wavelength of the red cadmium line! Of course, the one mil wavelength assumed in this example is about as short as is often used and the effect becomes less severe as the wavelength is increased. This is one of the reasons that speeds greater than 7.5 in./sec. are used for highest quality reproduction.

One can also make some rough qualitative inferences about the effect of the thickness of the recording medium on the shape of the response curve. As can be seen from equation (1) or from Fig. 2, low frequencies can be reproduced with very little loss in amplitude in spite of considerable spacing between the reproducing head and the medium while high frequencies (i.e. short wavelengths) may be appreciably attenuated by even 0.0001 inch spacing between the head and the medium. With this in mind it is easy to see that at high frequencies only a thin layer of the medium nearest the reproducing head will contribute to the reproduced signal. In this case (short λ) increasing the thickness of the medium beyond a certain amount can have no effect on the reproduced level simply because the added part of the tape is too far from the head to make its effect felt. Consider the effect of increasing the thickness of the medium from 0.3 mil to 0.6 mil when the wavelength is one mil. Since the spacing loss for 0.3 mil spacing at $\lambda = 1$ mil is 16.5 db, the signal contributed by the lower half of a medium 0.6 mils thick cannot be less than 16.5 db lower than that contributed by the upper half and hence the increase in thickness can do no more than to raise the reproduced level by 1.2 db.

At a lower frequency for which $\lambda = 100$ mil, however, the corresponding spacing loss is only 0.165 db and in this case the two halves of the tape can contribute almost equally with the result that doubling the thickness of the medium can almost double the reproduced signal voltage.

Qualitatively, then, one might expect that increasing the thickness of the recording medium, other things being equal, would increase the response to low frequencies and leave the high frequency response relatively unaltered. This is in agreement with data published by Kornei.[3]

The estimates of magnitudes just given rest on assumptions which cannot be proved except by further experiments. It has been implicitly assumed, for example, that the medium is uniformly magnetized throughout its thickness and this may not be the case. It does seem perfectly safe, however, to conclude that at a wavelength of one mil that part of the medium which lies deeper than about 0.3 mil from the surface cannot contribute appreciably to the reproduced signal. Furthermore, as the wavelength is decreased beyond this point the thickness of the effective part of the tape decreases in inverse proportion to λ with the result that the available flux also decreases. For this reason the "ideal" response curve cannot continue indefinitely to rise at 6 db per octave as it does at low frequencies. In fact, when the effective part of the tape becomes thin enough, the available flux will decrease at 6 db per octave and just cancel the usual 6 db per octave rise, giving an "ideal" response curve which rises 6 db per octave at low frequencies but which eventually becomes flat, neither rising nor falling with further increase in frequency.

Spacing loss may contribute in still another way to the frequency response characteristic of a magnetic recording system in which the reproducing head makes contact with the medium. It is well known to those who work

[3] Otto Kornei, "Frequency Response of Magnetic Recording," *Electronics*, p. 124, August, 1947.

with magnetic structures such as are used in transformers and the like that intimate mechanical contact between two parts of a magnetic circuit does not imply intimate magnetic contact. In fact, even when great care is taken in fitting such parts together, measurements invariably show an effective air gap between them and the effective width of this gap usually amounts to appreciably more than one mil. One reason for this is that the permeability of soft materials such as are used in the cores of transformers and reproducing heads is very sensitive to strain. Even the light cold working which a surface receives in being ground flat is sufficient to impair very seriously the permeability of a thin surface layer.

In view of this it is to be expected that the magnetic contact between reproducing head and medium is less than perfect. If cold working during the fabrication of the head or due to abrasion by the recording medium should result in an effective air space between head and medium amounting to as much as one mil, the effect on frequency response would be pronounced indeed. At a recording speed of 7.5 in./sec. this amount of spacing would cause a loss of 7.3 db at 1000 cps, 14.6 db at 2000 cps, 21.9 db at 3000 cps, 29.2 db at 4000 cps, etc.

It seems certain that in a practical recording system some loss of this sort must occur. The problem of determining the magnitude of the loss or in other words the amount of the effective spacing in a practical case is, however, a difficult one. So far, no direct experimental method for its determination has been found.

Theoretical Calculations for an Idealized Case

In the preceding section an experimentally determined spacing loss function has been discussed. It was shown that as the reproducing head is moved away from the recording medium the reproduced signal level decreases. This means that the magnetic flux through the head decreases. If the distribution of magnetization in the recording medium were known, it should be possible to compute the flux through the head and thereby to derive the spacing loss function on a theoretical basis. Unfortunately it seems almost impossible to do this calculation in an exact way because very little is known about the magnetization pattern in the medium and because the geometry of the usual ring type head makes the boundary value problem an exceedingly difficult one to solve.

It is possible, however, to obtain a solution for an idealized case which bears at least some resemblance to the practical situation and this solution will be presented. The results must, of course, be viewed with due skepticism until they can be proved experimentally or else recalculated on the basis of better initial assumptions. It is hoped, however, that in some measure they may serve as a guide to a better understanding of the magnetic reproducing process.

The Idealized Recording Medium

The problem will be reduced to two dimensions by assuming an infinitely wide and infinitely long tape of finite thickness δ. A rectangular coordinate system will be chosen in such a way that the central plane midway between the upper and lower surfaces of the recording medium lies in the x–y plane. It will be assumed that the medium is sinusoidally magnetized in such a way that in the medium the intensity of magnetization is given by

$$I_x = I_m \sin (2\pi x/\lambda)$$

$$I_y = I_z = 0. \tag{2}$$

Equations (2) say that the recording is purely longitudinal. In a practical case, of course, the recorded signal is neither purely longitudinal nor purely perpendicular but rather contains components of both sorts. In Appendix I it is shown that the frequency response does not depend on the relative amounts of these two components and hence that the computed results are equally valid whether the recorded signal is purely longitudinal, purely perpendicular, or a mixture of the two.

Appendix II contains calculations for the case of a round wire sinusoidally magnetized along its axis, and for a plated wire. These results, though much different in mathematical form, are shown to be very similar to the results for a flat medium.

The Idealized Reproducing Head

Figure 4 shows a semi-practical version of the sort of idealized reproducing head which will be treated.

It consists of a bar of core material with a single turn of exceedingly fine wire around it. This head is imagined to be spaced d inches above the surface of the recording medium. If the dimensions of the bar are made large enough, the amount of flux through it will obviously be as great as could be made to pass through any sort of head which makes contact with only one side of the tape and so the open circuit reproduced voltage per turn is as high as can be obtained with any practical head.

Suppose a very narrow gap is introduced in this head where the single turn coil was and that the magnetic circuit is completed by a ring of core material as shown in Fig. 5.

If the permeability of the head is very high and the gap very small then the flux which passed through the single turn coil of Fig. 4 will now pass through the ring of Fig. 5 and can be made to thread through a coil of many turns wound on the ring. In so far as this is true, calculations based on this bar type head are applicable to ring type heads.

If the bar of Fig. 4 is now allowed to become infinite in length, width, and thickness, the flux density in it can be computed and the flux per unit width can be evaluated. This calculation is outlined in Appendix I. If the tape moves past the head with a velocity v in the x direction, the repro-

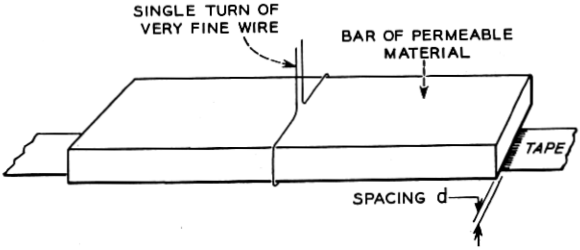

Fig. 4—Idealized bar-type reproducing head.

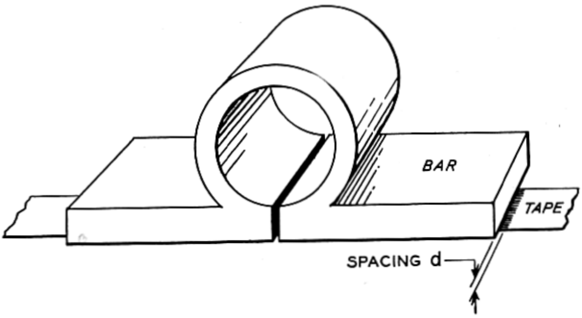

Fig. 5—Idealized ring-type reproducing head.

duced voltage should be proportional to the rate of change of flux. In the appendix this is shown to be

$$\frac{d\phi_x}{dt} = -\frac{\mu}{\mu + 1} 4\pi W v I_m (1 - e^{-2\pi\delta/\lambda}) e^{-2\pi d/\lambda} \cos(\omega t) \tag{3}$$

where $\frac{d\phi_x}{dt}$ is the rate of change of flux in W cm. width of the reproducing head measured in Maxwells per sec.,

μ is the permeability of the reproducing head,

W is the width in cm. of the reproducing head (and of the recorded track in a practical case),

v is the velocity in cm./sec. with which the recording medium passes the *reproducing* head.
I_m is the peak value of the sinusoidal intensity of magnetization in the recording medium measured in gauss,
δ is the thickness of the recording medium measured in the same units as λ,
λ is the recorded wavelength measured in any convenient units,
d is the effective spacing between the reproducing head and the surface of the recording medium measured in the same units as λ, and
ω is 2π times the reproduced frequency in cycles per sec.

Note that equation (3) applies to a ring type head with no back gap. If the head has a back gap then not all the available flux will thread through the ring. Some of it will return to the medium through the scanning gap and hence will not contribute to the reproduced voltage. This does not affect the shape of the frequency response curve but does contribute a constant multiplying factor (less than unity) to the right hand side of equation (3). The value of this factor depends on the reluctances of the gaps and of the magnetic parts of the reproducing head. If the reluctance of the magnetic parts is negligible and the reluctance of the back gap is equal to the reluctance of the front gap then the available flux will divide equally in the two gaps and the factor will be one-half. This factor will not be considered further in this paper because it does not contribute to the shape of the response curve but only to the absolute magnitude of the reproduced voltage. It could be interpreted as reducing the effective number of turns on the reproducing head to a value somewhat lower than the actual number of turns.

Spacing Loss

The term $e^{-2\pi d/\lambda}$ tells how the reproduced voltage depends on spacing. In order to compare this computed effect with the experimentally observed one it is necessary to put it in decibel form by computing twenty times the $\log_{10}$ of $e^{-2\pi d/\lambda}$. This gives

$$\text{Spacing Loss} = 54.6\,(d/\lambda)\text{ decibels}.$$

This agrees very well indeed with the experimentally determined equation (1) in which the constant is 55 instead of the computed 54.6. The computed spacing loss function is plotted in Fig. 6.

Thickness Loss

The effect of the thickness of the recording medium shows up in the term $(1 - e^{-2\pi\delta/\lambda})$. At low frequencies for which the wavelength is much greater than the thickness of the medium this reduces to $2\pi\delta/\lambda$. In this case the reproduced voltage is proportional to the thickness of the medium and to frequency. This is the familiar six db per octave characteristic.

At high frequencies, however, when $\lambda \ll \delta$ the term reduces to unity

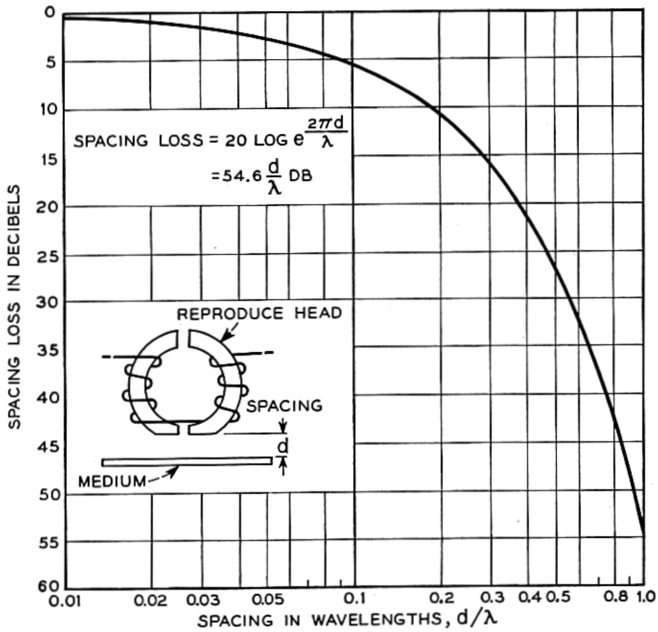

Fig. 6—Computed spacing loss as a function of d/λ.

and the computed "ideal" response is flat with frequency and independent of the thickness of the medium.

If the term $(1 - e^{-2\pi\delta/\lambda})$ is rewritten as

$$(2\pi\delta/\lambda)\left[\frac{1 - e^{-2\pi\delta/\lambda}}{2\pi\delta/\lambda}\right]$$

then the part in parenthesis accounts for a 6 db per octave characteristic and the part in brackets accounts for a loss *with respect to this 6 db per octave characteristic*. This loss, which will be called Thickness Loss[4], is given by

[4] It seems somewhat awkward to speak of "Thickness Loss" when nothing is actually lost by making the medium thick. The only excuse for this way of splitting the terms is that it makes for ease in comparing measured and computed curves.

$$\text{Thickness Loss} = 20 \log_{10} \frac{2\pi\delta/\lambda}{1 - e^{-2\pi\delta/\lambda}} \text{ db} \tag{5}$$

where λ is the recorded wavelength and δ is the thickness of the recording medium. This function is plotted in Fig. 7.

Comparison with Experiment

The most elementary consideration of the magnetic recording process indicates that when the recording signal current is held constant the open circuit reproduced voltage should be a function of frequency, increasing by 6 db for each octave increase in frequency. Experimental response curves tend to show this 6 db per octave characteristic when the recorded wave-

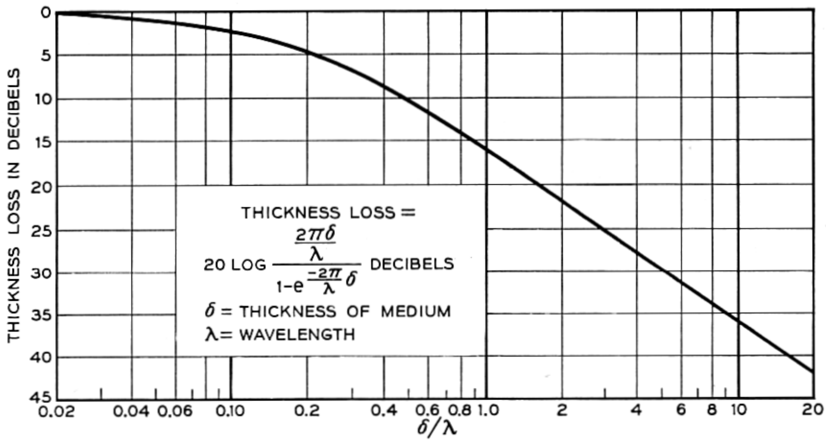

Fig. 7—Computed thickness loss as a function of δ/λ.

length is moderately long and the frequency moderately low. This makes it possible to draw a 6 db per octave line on the measured response characteristic in such a way as to coincide with the low-frequency part of the measured response characteristic. As the frequency is increased the measured curve tends to fall more and more below the 6 db per octave line. This is because several kinds of loss come into play as the wavelength decreases or as the frequency increases. Among these losses are:

1. Self demagnetization,
2. Eddy current and other losses in the recording and reproducing heads, and
3. Gap loss due to the finite scanning slit in the reproducing head.

The work presented in the first sections of this paper indicates that the following two kinds of loss should be added to this list:

4. Spacing loss due to imperfect magnetic contact between the reproducing head and the recording medium, and
5. Thickness loss.

Of these five losses three can be evaluated quantitatively either by direct measurement or by calculation from theory. The remaining two are self-demagnetization and spacing loss.

In this section the known losses will be evaluated for a particular recording system. This leads to a response curve which can be compared with the measured curve. The difference between the two curves should be due to self-demagnetization and to spacing loss provided the above list of losses is complete.

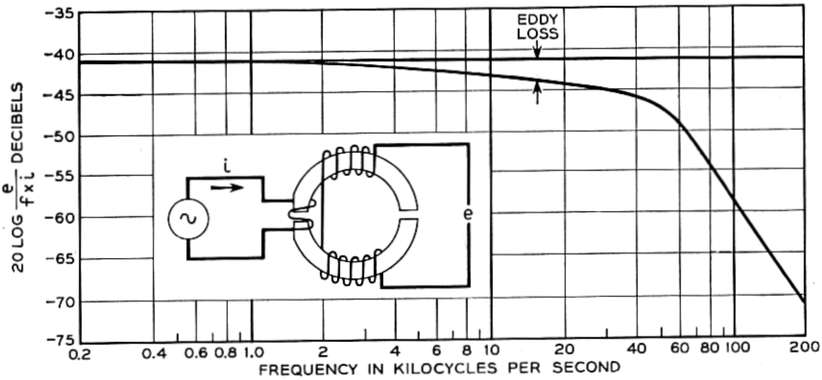

Fig. 8—Measured eddy current loss as a function of frequency.

The recording system used is the one shown in Fig. 1 with the speed set at 15.5 in./sec. for both recording and reproducing. A constant signal current of 0.1 ma was used for recording with the 55 kc bias adjusted to give maximum open circuit reproduced voltage.

Eddy current losses were measured as indicated in Fig. 8 by sending a measured constant current i through a small auxiliary winding around the pole tip and measuring the open circuit voltage developed across the normal winding of the head. Any departure of this measured voltage from a 6 db per octave increase with increasing frequency is due to losses in the head which will be loosely called eddy current losses. Other kinds of loss may enter into this measurement (as, for example, loss due to the self-capacitance of the winding) but in the frequency range of interest, eddy losses predominate.

By a completely different sort of measurement,[5] J. R. Anderson has arrived at a similar value for eddy current loss in this type of head and has shown that approximately the same loss occurs in both the recording and the reproducing process. For this reason it seems proper to assume that eddy currents account for just twice the loss measured by the method of Fig. 8.

The loss due to the finite gap in the reproducing head is computed from the well known relation.[6]

$$\text{Gap loss} = 20 \log_{10} \frac{\pi g/\lambda}{\sin (\pi g/\lambda)}$$

where g is the effective gap width in inches and λ is the recorded wavelength in inches.

Thickness loss is computed from equation (5). It must be remembered that this loss was derived on the assumption of uniform magnetization throughout the thickness of the recording medium. This may be a fairly good approximation to the actual state of affairs for a thin medium such as the one being considered, but obviously if the thickness of the medium is large compared with the width of the recording gap then the recording field will not penetrate uniformly through the medium and the derived thickness loss function will not apply.

The derived equation (3) indicates that at low frequencies the reproduced voltage should be proportional to the thickness of the medium. If the thickness of the medium is increased beyond the limit to which the recording field can penetrate, this will no longer be the case and further increase in thickness will have no effect on the response.

Data presented by Kornei[3] on the cobalt-nickel plating being considered here shows that the low-frequency response is approximately proportional to the thickness of the medium for values of thickness between 0.075 mil and 0.5 mil. This may be taken as an indication of approximately uniform penetration through these thicknesses and hence tends to indicate that the derived thickness loss function should be applicable in the case of the 0.3 mil plating being considered here.

The effects of these losses are shown in Fig. 9 along with measured frequency response data. Consider first the experimentally measured response

[5] In unpublished work, J. R. Anderson of the Bell Telephone Laboratories has made use of the fact that eddy losses depend on frequency while all other magnetic recording losses depend on wavelength. By recording a single frequency and playing back at various speeds he determined the loss on playback. By recording various frequencies with recording speed adjusted to give constant recorded wavelength and using a single playback speed he evaluates the eddy loss in the recording process.

[6] S. J. Begun, "Magnetic Recording," p. 84, Murray Hill Books, Inc., New York.

data shown as circles falling near the lowest curve. Some of the measured points have been omitted to avoid crowding but enough remain to show the trend. At low frequencies these points fall along a line of approximately 6 db per octave.

A straight 6 db per octave line labeled 1 has been drawn through these points and extended as shown in the figure. This line is the base from which the various losses must be subtracted. Curve 2 shows the effect of subtracting the computed thickness loss. When eddy losses and gap loss are

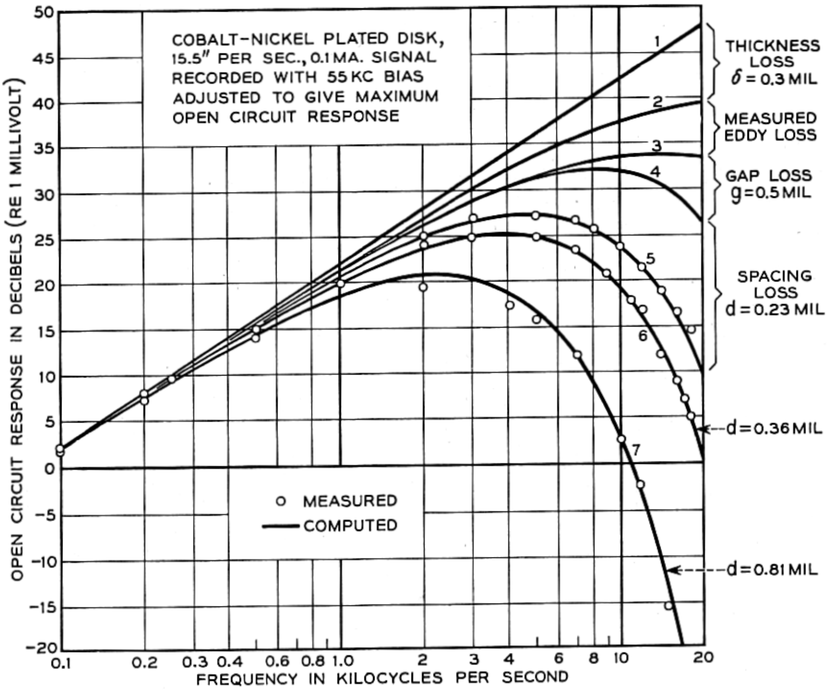

Fig. 9—Computed response curves and measured response points.

also taken into account, curve 4 is obtained. The difference between this curve and the lowest measured response points is presumably due either to self-demagnetization, to spacing loss, or perhaps to both.

There is one clue which may be of help in deciding how much of this loss should be attributed to self-demagnetization and how much to spacing loss. This clue comes from the fact that the form of the spacing loss function is known. Any part of the loss which is due to spacing must follow the equation

$$\text{Spacing Loss} = 54.6\ (d/\lambda)\ \text{db}$$

whereas there is reason to believe that the effects of self-demagnetization cannot possibly account for more than something like ten or fifteen db loss and hence could not follow the equation given above.

In view of this it seems reasonable to try as a first guess the assumption that all the unexplained loss is due to spacing.

If this assumption properly accounts for the shape of the measured response curve there will be at least some reason to suppose it may be correct; particularly so if the required amount of effective spacing seems reasonable.

The lowest solid curve, No. 7 of Fig. 9, has been computed on this basis. That is, a spacing loss corresponding to 0.81 mil effective spacing has been subtracted from curve 4. It is seen that this computed response curve fits reasonably well with the measured points. Furthermore, 0.81 mil effective spacing corresponds to quite reasonably good magnetic contact.

If this interpretation of the measured data is correct then it is obvious that the high-frequency response could be improved a great deal if more intimate magnetic contact between the reproducing head and the recording medium could be achieved. To this end an attempt was made to lap the surface of the head in such a way as to remove material very gently and slowly. After lapping, the response was appreciably improved as indicated by the set of measured points around curve 6. This curve was computed assuming an effective spacing of 0.36 mil. Note that the computed curve now fits the measured points very well indeed.

After still more lapping,[7] the measured response points around curve 5 were obtained. In this case it is necessary to assume only 0.23 mil effective spacing in order to account for the measured curve. Further lapping failed to give further improvement in response but a defect in the head which may account for this has since been found and it is believed that with great care one might actually measure something very close to curve 4.

To summarize, this is what seems to have been found. It is possible to compute a response curve taking into account gap loss, eddy current losses, and thickness loss. If this curve is compared with the final measured response curve it is found that the measured curve gives less high-frequency response than was computed. The difference between the two curves is just the right sort of function of frequency and of just the right magnitude to be accounted for by an effective spacing of 0.00023 inch between the reproducing head and the recording medium. It seems probable that the effective spacing could not have been much smaller than this value and therefore it may be correct to assume that practically all the unexplained

[7] After each lapping it was found that smaller values of bias current sufficed to give maximum reproduced voltage. This is presumably because the improved magnetic contact made the bias current more effective.

In the present case this leads to

$$
\begin{aligned}
dH_x &= -(4\pi I_m/\lambda)\,\frac{(x_0 - x)}{(x_0 - x)^2 + (z_0 - z)^2}\cos(2\pi x/\lambda)\,dx\,dz \\
dH_z &= -(4\pi I_m/\lambda)\,\frac{(z_0 - z)}{(x_0 - x)^2 + (z_0 - z)^2}\cos(2\pi x/\lambda)\,dx\,dz
\end{aligned}
\tag{8}
$$

The total field at (x_0, z_0) is obtained by integrating with respect to x over the range $-\infty$ to $+\infty$ and with respect to z over the range $-\delta/2$ to $+\delta/2$. In carrying out the integration over x it is convenient to make the substitution

$$
\begin{aligned}
(x_0 - x)/(z_0 - z) &= p \\
dx &= -(z_0 - z)\,dp
\end{aligned}
\tag{9}
$$

Neglecting terms which obviously integrate to zero, this gives

$$
\begin{aligned}
H_x &= (4\pi I_m/\lambda)\sin(2\pi x_0/\lambda)\int_{-\delta/2}^{\delta/2}\left[\int_{\infty}^{-\infty}\frac{p\sin[2\pi(z_0 - z)p/\lambda]}{1 + p^2}\,dp\right]dz \\
H_z &= (4\pi I_m/\lambda)\cos(2\pi x_0/\lambda)\int_{-\delta/2}^{\delta/2}\left[\int_{\infty}^{-\infty}\frac{\cos[2\pi(z_0 - z)p/\lambda]}{1 + p^2}\,dp\right]dz \\
& z_0 \geq z
\end{aligned}
\tag{10}
$$

The integrals in brackets can be found in tables.[10] Carrying out the integration gives

$$
\begin{aligned}
H_x &= -(4\pi^2 I_m/\lambda)\sin(2\pi x_0/\lambda)\int_{-\delta/2}^{\delta/2} e^{-2\pi(z_0 - z)/\lambda}\,dz \\
H_z &= -(4\pi^2 I_m/\lambda)\cos(2\pi x_0/\lambda)\int_{-\delta/2}^{\delta/2} e^{-2\pi(z_0 - z)/\lambda}\,dz \\
& z_0 \geq z
\end{aligned}
\tag{11}
$$

which integrate to

$$
\begin{aligned}
H_x &= -2\pi I_m \sin(2\pi x_0/\lambda)e^{-2\pi z_0/\lambda}[e^{\pi\delta/\lambda} - e^{-\pi\delta/\lambda}] \\
H_z &= -2\pi I_m \cos(2\pi x_0/\lambda)e^{-2\pi z_0/\lambda}[e^{\pi\delta/\lambda} - e^{-\pi\delta/\lambda}] \\
& z_0 \geq \delta/2
\end{aligned}
\tag{12}
$$

[10] D. Bierens de Haan, "Nouvelles Tables D'Integrales Définies," p. 223, Leide, Engels, 1867.

Below the recording medium, that is for $z_0 \leq -\delta/2$,

$$\begin{aligned} H_x &= -2\pi I_m \sin(2\pi x_0/\lambda) e^{+2\pi z_0/\lambda} [e^{\pi\delta/\lambda} - e^{-\pi\delta/\lambda}] \\ H_z &= 2\pi I_m \cos(2\pi x_0/\lambda) e^{+2\pi z_0/\lambda} [e^{\pi\delta/\lambda} - e^{-\pi\delta/\lambda}] \\ & z_0 \leq -\delta/2 \end{aligned} \tag{13}$$

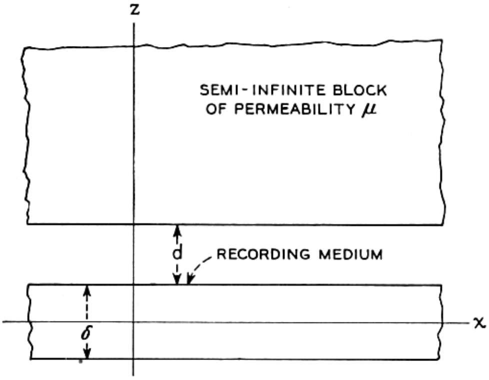

Fig. 11—Flat tape under idealized reproducing head.

Inside the recording medium,

$$\begin{aligned} H_x &= -(4\pi^2 I_m/\lambda) \sin(2\pi x_0/\lambda) \\ &\quad \cdot \left[\int_{-\delta/2}^{z_0} e^{-2\pi(z_0 - z)/\lambda} \, dz + \int_{z_0}^{\delta/2} e^{+2\pi(z_0 - z)/\lambda} \, dz \right] \\ H_z &= -(4\pi^2 I_m/\lambda) \cos(2\pi x_0/\lambda) \\ &\quad \cdot \left[\int_{-\delta/2}^{z_0} e^{-2\pi(z_0 - z)/\lambda} \, dz - \int_{z_0}^{\delta/2} e^{+2\pi(z_0 - z)/\lambda} \, dz \right] \end{aligned} \tag{14}$$

which integrate to

$$\begin{aligned} H_x &= -2\pi I_m \sin(2\pi x_0/\lambda)[2 - e^{-\pi\delta/\lambda}(e^{-2\pi z_0/\lambda} + e^{2\pi z_0/\lambda})] \\ H_z &= 2\pi I_m \cos(2\pi x_0/\lambda) e^{-\pi\delta/\lambda}(e^{-2\pi z_0/\lambda} - e^{2\pi z_0/\lambda}) \\ & \delta/2 \geq z_0 \geq -\delta/2 \end{aligned} \tag{15}$$

The Fields in and Under the Reproducing Head

The idealized reproducing head amounts simply to a semi-infinite block of high permeability material with a flat face spaced a distance d above the surface of the recording medium as shown in Fig. 11.

The problem of most interest is that of finding the x component of magnetic induction, B_x, at any point (x_0, z_0) in the idealized head and integrating this with respect to z_0 to determine the total flux passing through unit width (in the y direction) of a plane $x = x_0$. This plane will then be allowed to move with a velocity v by putting $x_0 = vt$ and the time rate of change of flux will be computed. Except for the effects of eddy currents, self demagnetization, gap loss, etc. (which are treated separately) this rate of change of flux should be proportional to the open circuit reproduced voltage. This is the only result of which direct use will be made but for the sake of completeness all the field components will be evaluated not only in the idealized head but also at all other points.

This problem is completely analogous to the problem of a point charge in front of a semi-infinite dielectric treated by Abraham and Becker[11] and can be solved by use of the method of images.

The Field Inside the High Permeability Head

By analogy with the treatment of Abraham and Becker, the value of B in the high permeability head is computed as though this head filled all space and as though the recording medium were polarized to a value $2\mu/(\mu + 1)$ times the actual value of polarization present. This gives directly from equations (12),

$$\begin{aligned} B_x &= -[2\mu/(\mu + 1)]2\pi I_m \sin (2\pi x_0/\lambda)e^{-2\pi z_0/\lambda}(e^{\pi\delta/\lambda} - e^{-\pi\delta/\lambda}) \\ B_z &= -[2\mu/(\mu + 1)]2\pi I_m \cos (2\pi x_0/\lambda)e^{-2\pi z_0/\lambda}(e^{\pi\delta/\lambda} - e^{-\pi\delta/\lambda}) \\ &\qquad z_0 \geq d + \delta/2 \end{aligned} \tag{16}$$

The Field Below the Reproducing Head

Again by analogy with the treatment of Abraham and Becker, the field outside the idealized head is computed as though no head were present. The field is that due to the actual magnetized medium plus the field due to an image of the medium (centered about $z = 2d + \delta$). The intensity of magnetization of the image medium is $-(\mu - 1)/(\mu + 1)$ times the intensity of magnetization of the actual medium.

The field due to the image medium is computed from equations (13) after suitable modification. The required modifications are:

1. Multiply the right hand sides by $-(\mu - 1)/(\mu + 1)$ to take account of the magnitude and sign of the image magnetization as just discussed, and
2. Replace z_0 by $z_0 - (2d + \delta)$ to take account of the position of the image.

[11] M. Abraham and R. Becker, *The Classical Theory of Electricity and Magnetism.* p. 77, Blackie and Son Limited, London, 1937.

This gives the field due to the image plane as

$$H_{xi} = 2\pi I_m \frac{\mu - 1}{\mu + 1} \sin (2\pi x_0/\lambda) e^{2\pi(z_0 - 2d - \delta)/\lambda}(e^{\pi\delta/\lambda} - e^{-\pi\delta/\lambda})$$

$$H_{zi} = -2\pi I_m \frac{\mu - 1}{\mu + 1} \cos (2\pi x_0/\lambda) e^{2\pi(z_0 - 2d - \delta)/\lambda}(e^{\pi\delta/\lambda} - e^{-\pi\delta/\lambda}) \tag{17}$$

$$z_0 \leq d + \delta/2$$

To this must be added the field due to the real medium which is given by equations (12) when $\delta/2 \leq z_0 \leq d + \delta/2$, by equations (15) when $-\delta/2 \leq z_0 \leq \delta/2$, and by equations (13) when $z_0 \leq -\delta/2$.

Performing this addition gives the following results:

Between the head and the recording medium,

$$H_x = -2\pi I_m \sin (2\pi x_0/\lambda) e^{-2\pi z_0/\lambda}(e^{\pi\delta/\lambda} - e^{-\pi\delta/\lambda}) \cdot \left[1 - \frac{\mu - 1}{\mu + 1} e^{-2\pi(2d + \delta - 2z_0)/\lambda}\right]$$

$$H_z = -2\pi I_m \cos (2\pi x_0/\lambda) e^{-2\pi z_0/\lambda}(e^{\pi\delta/\lambda} - e^{-\pi\delta/\lambda}) \cdot \left[1 + \frac{\mu - 1}{\mu + 1} e^{-2\pi(2d + \delta - 2z_0)/\lambda}\right] \tag{18}$$

$$d + \delta/2 \geq z_0 \geq \delta/2$$

Inside the recording medium,

$$H_x = -2\pi I_m \sin (2\pi x_0/\lambda) \cdot \left[2 - e^{-\pi\delta/\lambda}(e^{2\pi z_0/\lambda} + e^{-2\pi z_0/\lambda}) - \frac{\mu - 1}{\mu + 1} e^{-2\pi(2d + \delta - z_0)/\lambda}(e^{\pi\delta/\lambda} - e^{-\pi\delta/\lambda})\right]$$

$$H_z = -2\pi I_m \cos (2\pi x_0/\lambda) \cdot \left[e^{-\pi\delta/\lambda}(e^{2\pi z_0/\lambda} - e^{-2\pi z_0/\lambda}) + \frac{\mu - 1}{\mu + 1} e^{-2\pi(2d + \delta - z_0)/\lambda}(e^{\pi\delta/\lambda} - e^{-\pi\delta/\lambda})\right] \tag{19}$$

$$\delta/2 \geq z_0 \geq \delta/2$$

Below the recording medium,

$$H_x = -2\pi I_m \sin (2\pi x_0/\lambda) e^{2\pi z_0/\lambda}(e^{\pi\delta/\lambda} - e^{-\pi\delta/\lambda}) \cdot \left[1 - \frac{\mu - 1}{\mu + 1} e^{-2\pi(2d + \delta)/\lambda}\right]$$

$$H_z = 2\pi I_m \cos (2\pi x_0/\lambda) e^{2\pi z_0/\lambda}(e^{\pi\delta/\lambda} - e^{-\pi\delta/\lambda}) \cdot \left[1 - \frac{\mu - 1}{\mu + 1} e^{-2\pi(2d + \delta)/\lambda}\right] \tag{20}$$

$$z_0 \leq -\delta/2$$

THE FLUX PER UNIT WIDTH IN THE IDEALIZED REPRODUCING HEAD

The desired flux per unit width is computed from

$$\phi_x = \int_{d+\delta/2}^{\infty} B_x \, dz \tag{21}$$

where B_x is given by equation (16). Performing the indicated integration gives

$$\phi_x = -\frac{2\mu}{\mu + 1} 2\pi\delta I_m \sin(2\pi x_0/\lambda) \left[\frac{1 - e^{-2\pi\delta/\lambda}}{2\pi\delta/\lambda}\right] e^{-2\pi d/\lambda} \tag{22}$$

If the reproducing head moves past the recording medium with a velocity v so that $x_0 = vt$,

$$\frac{d\phi_x}{dt} = -\frac{\mu}{\mu + 1} 4\pi v I_m (1 - e^{-2\pi\delta/\lambda}) e^{-2\pi d/\lambda} \cos(\omega t) \tag{23}$$

where ω is 2π times the reproduced frequency. This is the result for unit width of the reproducing head. For a width of W cm.,

$$\frac{d\phi_x}{dt} = -\frac{\mu}{\mu + 1} 4\pi W v I_m (1 - e^{-2\pi\delta/\lambda}) e^{-2\pi d/\lambda} \cos(\omega t) \tag{24}$$

THE CASE OF PERPENDICULAR MAGNETIZATION

Equation (23) was derived for the case of pure longitudinal magnetization as defined by equations (6). It will now be shown that this same result is obtained for $d\phi_x/dt$ if the magnetization is purely perpendicular, that is if

$$\begin{aligned} I_z &= -I_m \cos(2\pi x/\lambda) \\ I_x &= I_y = 0 \end{aligned} \tag{25}$$

In this case the divergence of I is zero except at the surface of the tape and this magnetization is equivalent to a surface distribution of magnetic charge on the top and bottom surfaces of the tape. The magnitude of this charge density is just equal to I_z so that on the top surface of the tape there is a surface density of charge given by

$$\sigma = -I_m \cos(2\pi x/\lambda) \qquad \text{at } z = \delta/2 \tag{26}$$

and on the bottom surface of the tape there is a surface density of charge given by

$$\sigma = I_m \cos(2\pi x/\lambda) \qquad \text{at } z = -\delta/2 \tag{27}$$

Since the permeability of the recording medium is assumed to be unity, this problem reduces to that of finding $d\phi_x/dt$ due to two infinitely thin tapes of the sort to which equation (23) applies. One of these tapes is at $z = \delta/2$ and the other at $z = -\delta/2$.

The problem then is to rewrite equation (23) for a very thin tape and in terms of surface density of charge. As δ approaches zero, equation (23) reduces to

$$\frac{d\phi_x}{dt} = -\frac{\mu}{\mu + 1} 4\pi v I_m (2\pi\delta/\lambda) e^{-2\pi d/\lambda} \cos(\omega t) \tag{28}$$

From equation (7), the volume density of charge in this tape is

$$\rho = -(2\pi I_m/\lambda) \cos(2\pi x/\lambda)$$

But as δ approaches zero, the longitudinally magnetized tape to which equation (28) applies becomes equivalent to a surface distribution of magnetic charge of surface density equal to $\delta\rho$. This amounts, for the thin longitudinally magnetized tape, to a surface charge density of

$$\sigma_1 = -(2\pi\delta/\lambda) \cos(2\pi x/\lambda) \tag{29}$$

But the charge density on the top side of the perpendicularly magnetized tape is given by equation (26). Comparing these two values shows that the surface charge density in the thin longitudinally magnetized tape is just $2\pi\delta/\lambda$ times as great as the surface charge density on top of the perpendicularly magnetized medium. This means that $d\phi_x/dt$ due to the top side of the perpendicularly magnetized tape can be obtained by dividing the right hand side of equation (28) by $2\pi\delta/\lambda$. This gives

$$\frac{d\phi_x}{dt} = -\frac{\mu}{\mu + 1} 4\pi v I_m e^{-2\pi d/\lambda} \cos(\omega t) \tag{30}$$

due to the top side of the tape.

The contribution from the bottom side is obtained from equation (30) by replacing d by $d + \delta$ (since the bottom side is spaced $d + \delta$ from the reproducing head) and changing the sign. Adding these two contributions gives for the total

$$\frac{d\phi_x}{dt} = -\frac{\mu}{\mu + 1} 4\pi v I_m (1 - e^{-2\pi\delta/\lambda}) e^{-2\pi d/\lambda} \cos(\omega t) \tag{31}$$

This is the same as equation (23) and so the desired result has been established.

Note from equations (6) and (24) that in order to get the same result for the perpendicular and longitudinal cases it was necessary to assume a 90-degree phase difference between I_x and I_z. The usual type of recording head lays down a pattern of magnetization which is neither purely perpendicular nor purely longitudinal but the two components are always in phase. This means that the two contributions to $d\phi_x/dt$ add as vectors at 90 degrees. If the intensity of magnetization in the recording medium is held constant while the relative values of perpendicular and longitudinal components are changed, the only effect on the reproduced signal is a change of phase.

APPENDIX II

THE FIELD DUE TO A ROUND WIRE

In Appendix I the field due to a sinusoidally magnetized flat medium such as a tape has been calculated and the rate of change of flux in an

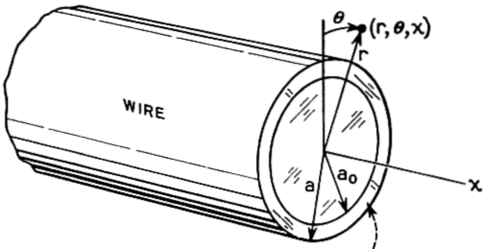

Fig. 12—Coordinate system for round wire calculations.

idealized reproducing head has been evaluated. The analogous calculations for a round wire have also been carried through and it is the purpose of this section to present some of the results. The derivation of these results seems too tedious and long to be presented here.

The Recording Medium

Let the recording medium be a wire, the axis of which lies along the x axis as shown in Fig. 12. Let the radius of the wire be a. To take account of plated wires as well as solid magnetic ones, let the wire have a nonmagnetic core of radius a_0. Let the cylindrical shell between a_0 and a be magnetized sinusoidally in the x direction so that

$$I_x = I_m \text{ Sin } (2\pi x/\lambda)$$
$$I_r = I_\theta = 0 \tag{32}$$

By putting $a_0 = 0$ in the expressions which follow it will be possible to obtain the result for a solid magnetic wire.

The Field in Free Space

If no reproducing head is present to disturb the field distribution, the computed field components at a point (x_0, r_0) are

$$\begin{aligned} H_x &= -4\pi I_m \text{ Sin } (2\pi x_0/\lambda)K_0(2\pi r_0/\lambda)[(2\pi a/\lambda)I_1(2\pi a/\lambda) \\ &\qquad - (2\pi a_0/\lambda)I_1(2\pi a_0/\lambda)] \\ H_r &= -4\pi I_m \text{ Cos } (2\pi x_0/\lambda)K_1(2\pi r_0/\lambda)[(2\pi a/\lambda)I_1(2\pi a/\lambda) \\ &\qquad - (2\pi a_0/\lambda)I_1(2\pi a_0/\lambda)] \\ &\qquad r_0 \geq a \end{aligned} \tag{33}$$

A discussion and tabulation of the I and K functions can be found in Watson's "Theory of Bessel Functions."[12]

The field due to a solid magnetic wire is obtained by setting $a_0 = 0$ in equations (33). This gives

$$\begin{aligned} H_x &= -4\pi I_m \text{ Sin } (2\pi x_0/\lambda)(2\pi a/\lambda)K_0(2\pi r_0/\lambda)I_1(2\pi a/\lambda) \\ H_r &= -4\pi I_m \text{ Cos } (2\pi x_0/\lambda)(2\pi a/\lambda)K_1(2\pi r_0/\lambda)I_1(2\pi a/\lambda) \\ &\qquad r_0 \geq a \end{aligned} \tag{34}$$

The Rate of Change of Flux in an Idealized Head

It has not been possible to carry out the calculations for an idealized head which is a satisfactory approximation to the grooved ring-type head often used in wire recording. The results presented below will apply only to reproducing heads which completely surround the wire. In this case the idealized head is an infinitely large block of core material of permeability μ pierced by a cylindrical hole of radius R in which the wire is centered as shown in Fig. 13. At any point (x_0, r_0) in the permeable medium the components of flux density can be shown to be

$$\begin{aligned} B_x &= \alpha H_x \\ B_r &= \alpha H_r \\ r_0 &\geq R \end{aligned} \tag{35}$$

where

$$\alpha = \frac{\mu}{(\mu - 1)(2\pi R/\lambda)I_0(2\pi R/\lambda)K_1(2\pi R/\lambda) + 1} \tag{36}$$

and H_x and H_r are given by equation (33).

[12] G. N. Watson, "A Treatise on the Theory of Bessel Functions," p. 79, 361, 698, Cambridge Univ. Press, 1922.

The total flux through a plane $x = x_0$ in the permeable medium is obtained by integrating

$$\phi_x = \int_R^\infty B_x(2\pi r)\, dr \tag{37}$$

This gives

$$\phi_x = -2\lambda^2 \alpha I_m \sin(2\pi x_0/\lambda)(2\pi R/\lambda)K_1(2\pi R/\lambda)[(2\pi a/\lambda)I_1(2\pi a/\lambda) - (2\pi a_0/\lambda)I_1(2\pi a_0/\lambda)] \tag{38}$$

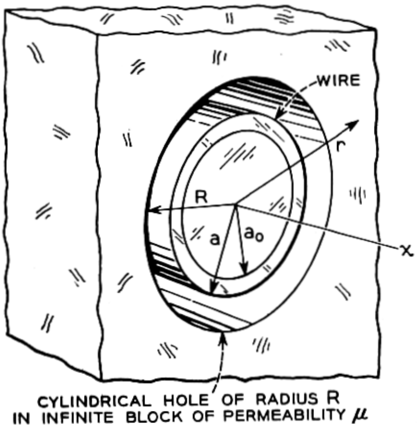

Fig. 13—Round wire surrounded by idealized reproducing head consisting of an infinite block of core material of permeability μ.

If the plane $x = x_0$ moves with a velocity v with respect to the wire so that $x_0 = vt$, then

$$\frac{d\phi_x}{dt} = -4\pi\lambda\alpha v I_m \cos(\omega t)(2\pi R/\lambda)K_1(2\pi R/\lambda)[(2\pi a/\lambda)I_1(2\pi a/\lambda) - (2\pi a_0/\lambda)I_1(2\pi a_0/\lambda)] \tag{39}$$

where $\omega = 2\pi f$ and f is the reproduced frequency.

SPECIAL CASES

Equation (39) can be used to compute the response of a simple reproducing head consisting of a single turn of very fine[13] wire as shown in Fig. 14.

In this case $\mu = 1$ and equation (36) shows that $\alpha = 1$. Furthermore if the wire is solid so that $a_0 = 0$, equation (39) reduces to

[13] Unless the diameter of the wire is small compared to the recorded wavelength there will be additional loss not accounted for by 39.

$$\frac{d\phi_x}{dt} = -4\pi\lambda v I_m \cos(\omega t)(2\pi R/\lambda)(2\pi a/\lambda)K_1(2\pi R/\lambda)I_1(2\pi a/\lambda) \quad (40)$$

As λ approaches infinity, $K_1(2\pi R/\lambda)$ approaches $\lambda/2\pi R$ and $I_1(2\pi a/\lambda)$ approaches $\pi a/\lambda$ so that, for very long wavelengths, equation (40) reduces to

$$\frac{d\phi_x}{dt} = -4\pi I_m v(2\pi/\lambda)(\pi a^2)\cos(\omega t) \quad (41)$$

This relation (which could have been derived in a much simpler manner) should be useful for the experimental determination of the intensity of magnetization, I_m.

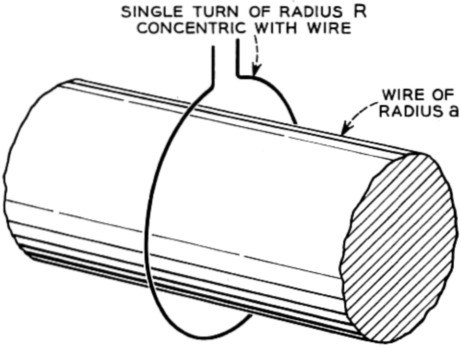

Fig. 14—Elementary reproducing head consisting of a single turn of wire.

Another case of some interest corresponds to a high permeability reproducing head which surrounds the wire. In this case μ is great enough so that equation (36) reduces to

$$\alpha = \frac{1}{(2\pi R/\lambda)I_0(2\pi R/\lambda)K_1(2\pi R/\lambda)} \quad (42)$$

If it is assumed, in addition, that the wire is solid so that $a_0 = 0$, then equations (42) and (39) give

$$\frac{d\phi_x}{dt} = -4\pi\lambda v I_m \cos(\omega t)(2\pi a/\lambda)I_1(2\pi a/\lambda)/I_0(2\pi R/\lambda) \quad (43)$$

Comparison between Round Wire and Flat Medium Response

It is interesting to compare equation (43) with equation (24) to see how the response characteristic of a round wire compares with that of a tape.

Assuming $\mu \gg 1$, the appropriate equation for the flat medium is

$$\frac{d\phi_x}{dt} = -4\pi W v I_m \cos(\omega t)(1 - e^{-(2\pi\delta/\lambda)})e^{-2\pi d/\lambda} \quad (44)$$

To compare equations (43) and (44), consider first the limiting cases of very long and very short wavelength. As λ approaches infinity they reduce to

$$\frac{d\phi_x}{dt} = -\pi a^2(8\pi^2 v/\lambda)I_m \cos(\omega t) \tag{45}$$

for the wire and

$$\frac{d\phi_x}{dt} = -\delta W(8\pi^2 v/\lambda)I_m \cos(\omega t) \tag{46}$$

for the tape.

These two expressions are identical provided the cross section area of the wire, (πa^2), is the same as that of the recorded track on the tape, (δW).

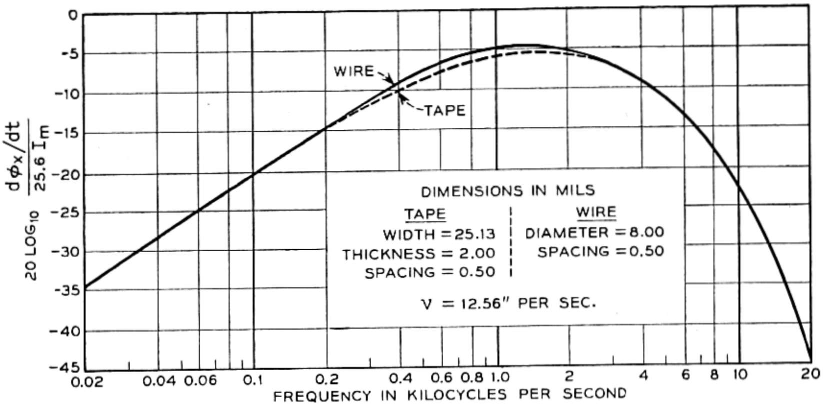

Fig. 15—Computed responses for wire and tape showing that the responses are very similar provided the dimensions of the wire and tape are suitably related.

As λ approaches zero, the two expressions reduce to

$$\frac{d\phi_x}{dt} = -4\pi v(2\pi a)\sqrt{R/a}\,e^{-2\pi(R-a)/\lambda}I_m \cos(\omega t) \tag{47}$$

for the wire, and

$$\frac{d\phi_x}{dt} = -4\pi v(W)e^{-2\pi d/\lambda}I_m \cos(\omega t) \tag{48}$$

for the tape.

Suppose that the reproducing head makes reasonably good contact with the wire so that $\sqrt{R/a} \doteq 1$. In this case equations (47) and (48) are identical provided the circumference of the wire, ($2\pi a$), is the same as the width of the recorded track on the tape and provided also the effective spacing between reproducing head and medium is the same in the two cases, ($d = R - a$). In both cases only a thin surface layer of the recording medium is effective in producing high frequency response. For this reason the high-frequency response is independent of the "thickness" of the medium and is directly proportional to the "width" of the track provided $2\pi a$ is interpreted as the width of track on a wire.

The comparisons which have just been made indicate that if the dimensions of a wire and of a tape are suitably related, the two media should give identical response at very high and very low frequencies provided they are equally magnetized. The dimensional requirements are

$$\begin{aligned} \pi a^2 &= \delta W, \\ 2\pi a &= W, \text{ and} \\ R - a &= d \end{aligned} \tag{49}$$

In order to show how the computed responses compare at intermediate frequencies, numerical calculations have been made for a special case in which equations (49) are satisfied. The case chosen is that of a wire 8 mils in diameter moving at a velocity of 12.56 in./sec. past a reproducing head which is effectively one half mil out of contact with the wire ($R - a = 0.5(10)^{-3}$ in.). By equations (49) the corresponding flat medium is a tape which is 2 mils thick and 25.13 mils wide. The tape is assumed to be moving with a velocity of 12.56 in./sec. past a reproducing head which is also effectively one half mil out of contact ($d = 0.5(10)^{-3}$ in.). In this case the numerical constants in equations (43) and (44) are equal. That is,

$$8\pi^2 av = 4\pi Wv = 25.6 \text{ cm.}^2/\text{sec.}$$

and the quantity to be computed and compared for the two cases is

$$20 \log_{10} \frac{d\phi_x/dt}{25.6 I_m}$$

The computed curves are shown in Fig. 15 from which it can be seen that they coincide at low and high frequencies as planned and that furthermore they differ by no more than 1.5 db in the middle range of frequencies.

As has been pointed out, equation (43) applies only to the unusual case in which the head completely surrounds the wire. The similarity of the two curves of Fig. 15, however, suggests a way of computing approximately the response to be expected when the wire head makes contact with only a part of the circumference of the wire. It suggests that the computation be carried out as though the wire were a flat medium of suitably chosen dimensions. In order to make the high frequency end come out right one would expect that W in equation (44) should be given a value equal to the length of the arc of contact between the wire and the head. To make the low frequency end come out right, δ must be given a value which makes the cross section area of the tape equal to that of the wire, i.e. such that $\delta W = \pi a^2$.