Telephone Traffic Time Averages

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This paper describes the determination of the first four semi-invariants of the distribution of the average, over an arbitrary time interval, of traffic carried by a telephone system with an infinite number of trunks, during a period of statistical equilibrium. Both finite and infinite numbers of independent call sources are considered, and the distribution function of call holding times is left general.

1. Introduction

FOR mathematical studies of telephone traffic, like those of call loss or delay which are used in trunking engineering, the traffic is considered as a flow of probability in time. In the period of most importance, the busy hour, this flow is usually regarded as stationary; that is to say, the probability of a given number of busy trunks, or the probability of delay of an incoming call (or any other probability of the system which comes in question) is taken as independent of the particular moment in the busy hour at which the system is examined. The system is said to be in statistical equilibrium.

For such theoretical studies, the statistical quantities which determine these probabilities, like the rate at which calls appear, are of course taken as given, but in the application they must be determined by observations, such as those being taken in the current extensive program of traffic measurements. Here a difficulty appears. To abridge the extensive amount of observational material, either measurements are made of traffic averages over periods small compared to the busy hour (but not small enough to be neglected) or the measurements of continuous recorders are averaged by hand. It may be noticed here that for application of the results given below the traffic averages obtained by measurements must be those of a continuous device which records all traffic changes and not, as in some measuring devices, those obtained from a number of "looks" at points within the averaging interval. But to use these measurements in determining the traffic parameters by standard sampling theory, a corresponding theoretical study of the averages is necessary.

Such a study, within limits to be described presently, is given here. No attempt is made to describe the sampling studies possible from the results reached. These seem to be of many kinds, not necessary to describe, but for concreteness it may be mentioned that the most important, at the moment, seems to be that of setting confidence limits for the average traffic.

The most important of the limits to this study are those implied by the assumptions of statistical equilibrium with fixed average, and an infinite number of trunks. The former limits application to periods in which, roughly speaking, average traffic is neither rising nor falling; the latter is justified only by the extreme mathematical difficulties produced by assuming otherwise. The traffic variable is the number of busy trunks in a period of statistical equilibrium. For pure chance call input, the call holding time characteristic is left arbitrary throughout the development, but main interest lies in the two extreme cases of constant holding time and exponential holding time, which are examined in detail.* For calls from a limited number of sources, results are obtained only for exponential holding time.

More precisely, if $N(t)$ is the random variable for the number of busy trunks at time t, the variable studied, the average number of calls in an interval of length T, is

$$M(T) = \frac{1}{T} \int_0^T N(t)\, dt \tag{1}$$

The question is: What are the statistical properties of $M(T)$?

The results given are the first four cumulants (semi-invariants) of $M(T)$, which seem to have the simplest expressions. For the convenience of the reader it may be noticed that the first cumulant is the mean, the second the second moment about the mean which is the variance, the third the third moment about the mean, and the fourth the fourth moment about the mean less three times the square of the variance.

In all cases the mean of $M(T)$ is the mean of $N(t)$ and for pure chance call input is called b, the average number of calls in unit average holding time, h.

The other cumulants for pure chance call input, k_n, have the general expression

$$k_n = b\,\frac{n(n-1)}{T^n} \int_0^T dx\, g(x)(T-x)x^{n-2}; \qquad n = 2, 3, 4$$

with

$$g(x) = \frac{1}{h} \int_x^\infty f(t)\, dt$$

* F. W. Rabe [6] has reported results for these two cases for relatively long averaging intervals, which are verified below. I owe my interest in this problem to a report on Rabe's work made by Messrs. Gibson, Hayward and Seckler in a probability colloquium at Bell Telephone Laboratories initiated and directed by Roger Wilkinson.

and $f(t)$ the probability that a call lasts at least t, that is, the distribution function of holding times. The specializations of this, for constant holding time and exponential holding time, appear in section 4. The results for finite source input have a similar character.

The procedure in obtaining these is as follows. The cumulants are determined from the ordinary moments (about the origin) and the latter are determined by the integration of expectations. Thus the first moment, the mean is determined from

$$E[M(T)] = \frac{1}{T} \int_0^T E[N(t)]\, dt = E[N(t)] \tag{2}$$

where $E(x)$ is written for the expectation or mean of x.

Similarly the second moment is given by

$$E[M^2(T)] = \frac{1}{T^2} \int_0^T \int_0^T E[N(t)N(u)]\, dt\, du \tag{3}$$

and so on for higher moments. Correlation effects appear in (3) in $E[N(t)N(u)]$ and are included in the development by formulation of transition probabilities, that is, those probabilities determining the traffic flow in time. The transition probability $P_{jk}(t)$ is defined as the probability of transition in t from j calls in progress (busy trunks) to k calls in progress, and fixes the inter-relatedness of call probabilities at different time epochs. Only for large values of t are these probabilities independent.

Hence, the first task is to determine these simple transition probabilities, then those of double and triple transitions, then the expected values of pairs, triples and quadruples of numbers of busy trunks, and finally the moments.

2. Transition Probabilities

For exponential holding time, and infinite sources, infinite trunks, these probabilities have already been determined by Conny Palm [5]. Palm's work has been summarized both by Feller [1] and by Jensen [3], and describes the whole process, not merely the equilibrium condition. For the equilibrium condition, a different procedure,* similar to that used by Newland [4] for another purpose, allows the assumption of a more general holding time characteristic.

* Thanks are due S. O. Rice for suggesting this, as well as for many corrections and improvements. I also have had the advantage of a careful reading of the mss. by E. L. Kaplan.

For infinite sources, and calls arriving individually and collectively at random with average density a, the well-known formula for the probability that exactly k calls arrive in time interval t is the Poisson

$$\pi_k(t) = e^{-at}(at)^k/k! \tag{4}$$

Then, if $P_{ij}(t;\, k)$ is the conditional probability of transition from i to j when k calls arrive in time t,

$$P_{ij}(t) = \sum_{k=0}^{\infty} P_{ij}(t;\, k)\pi_k(t) \tag{5}$$

Consider $P_{ij}(t;\, 0)$, that is the (conditional) transition probabilities when no calls arrive. Let the probability that a call lasts at least t be $f(t)$, so that the average holding time h is given by

$$h = \int_0^{\infty} u[-f'(u)]\, du = \int_0^{\infty} f(u)\, du \tag{6}$$

The i calls initially in process are independent of each other. Select one of them and suppose the time from its arrival (its age) is t_1. Then the probability that it will also exist t units later is the conditional probability $f(t + t_1)/f(t_1)$. Since in equilibrium conditions all moments of arrival have equal probability, the corresponding probability for an arbitrary call is

$$g(t) = \int_0^{\infty} f(t + t_1)\, dt_1 \div \int_0^{\infty} f(t_1)\, dt = \frac{1}{h}\int_t^{\infty} f(u)\, du \tag{7}$$

Hence the transitional probability $P_{ij}(t;\, 0)$ is the binomial expression

$$P_{ij}(t;\, 0) = \binom{i}{j} g^j(1 - g)^{i-j} \tag{8}$$

and its generating function is

$$P_i(t, x;\, 0) = \sum P_{ij}(t;\, 0)x^j = [1 + (x - 1)g]^i \tag{9}$$

In (8) and (9), for brevity, the argument of g is omitted.

Now, suppose one call arrives in interval t. The moment of arrival is uniformly distributed in t; that is, if u_1 is the moment of arrival,

$$Pr(u < u_1 < u + du) = du/t$$

and the probability that a call arriving at an arbitrary moment will be in existence at time t is, say,

$$Q(t) = \int_0^{t} f(t - u)\, \frac{du}{t} = \frac{1}{t}\int_0^{t} f(u)\, du = \frac{h}{t}\,(1 - g(t)) \tag{10}$$

The corresponding generating function is

$$1 - Q(t) + xQ(t) = 1 + (x - 1)Q(t)$$

and, since calls arriving are independent, the generating function for k calls arriving is

$$[1 + (x - 1)Q]^k$$

and

$$P_i(t, x; k) = [1 + (x - 1)g]^i[1 + (x - 1)Q]^k \tag{11}$$

Hence, finally by (5),

$$\begin{aligned} P_i(t; x) &= \sum P_{ij}(t)x^j \\ &= [1 + (x - 1)g]^i \sum_{k=0}^{\infty} [1 + (x - 1)Q]^k \frac{e^{-at}(at)^k}{k!} \\ &= [1 + (x - 1)g]^i \exp (x - 1) \text{ at } Q \\ &= [1 + (x - 1)g]^i \exp (x - 1)\, ah\, (1 - g) \end{aligned} \tag{12}$$

The last step uses (10).

This is the generating function for the simplest transition probabilities, and is quite like Palm's result; indeed, for exponential holding time $g = f = e^{-t/h}$. The probabilities themselves are obtained by expansion of the generating function in powers of x, or by substituting g for $e^{-t/h}$ in Palm's result. But they are not needed here; the generating function is most apt for determining the averages of interest, as will appear.

Before going on to the other transition probabilities, it is interesting to notice certain checks of equation (12). In statistical equilibrium the traffic has Poisson density (Palm l.c.) that is, in the present notation

$$Pr(N(t) = k) = e^{-b}b^k/k!$$

where $b = ah$. This of course is independent of time. Then, if $N(0)$ has this density, so should $N(t)$ as determined from $N(0)$ and the transition probabilities implicit in (12). This is verified by

$$\begin{aligned} \sum P_i(t, x)e^{-b}b^i/i! &= \exp (x - 1)b(1 - g) \sum [1 + (x - 1)g]^i \frac{e^{-b}b^i}{i!} \\ &= \exp [(x - 1)b(1 - g) - b + b + (x - 1)bg] \\ &= \exp (x - 1)b. \end{aligned} \tag{13}$$

Also, $g(0) = 1$ and $g(\infty) = 0$ so that

$$P_i(0, x) = [1 + (x - 1)]^i = x^i \tag{14}$$

$$P_i(\infty, x) = \exp(x - 1)b \tag{15}$$

showing that in zero time no transit to another state is possible, and in infinite time the equilibrium probabilities are reached no matter what the initial state has been.

Finally, in a Markov process (cf. Feller [2], Chap. 15) the simple transition probabilities alone are needed since

$$P_{ijk}(t, u) = P_{ij}(t)P_{jk}(u)$$

A test for this is the Chapman-Kolomogorov equation, namely

$$P_{ik}(t + u) = \sum_j P_{ij}(t)P_{jk}(u)$$

Using (12), the corresponding relation of generating functions is

$$[1 + (x - 1)g(t + u)]^i \exp b(x - 1)[1 - g(t + u)]$$
$$= [1 + (x - 1)g(t)g(u)]^i \exp b(x - 1)[1 - g(t)g(u)];$$

so the process is Markovian only if

$$g(t + u) = g(t)g(u)$$

which is true only for exponential holding time.

For the next transition probability $P_{ijk}(t, u)$, consider first the condition in which no call arrives in the whole interval $t + u$. As before

$$P_{ij}(t) = \binom{i}{j} g_t^j (1 - g_t)^{i-j}$$

where for convenience g_t is written for $g(t)$. For the next transit, however, there is a difference, namely

$$P_{jk}(u) = \binom{j}{k} \left(\frac{g_{t+u}}{g_t}\right)^k \left(1 - \frac{g_{t+u}}{g_t}\right)^{j-k}$$

since g_{t+u}/g_t is the conditional probability that a call which has lasted t will last u more; $P_{jk}(u)$ is the conditional probability of a transit from j to k in u, given the transit i to j in t.

The generating function for the double transition probabilities in this case is, then,

$$\sum_j \sum_k P_{ijk}(t, u; 0)x^j y^k = [1 + (x - 1)g_t + x(y - 1)g_{t+u}]^i \tag{16}$$

Now suppose a single call arrives at random in interval t. As before, the probability that it will occupy a trunk at time t is $Q(t) = ht^{-1}(1 - g(t))$

and the conditional probability that it will also occupy a trunk at time $t + u$ is

$$\frac{1}{t}\int_0^t f(t + u - x)\, dx \div Q(t)$$

or

$$\frac{g(u) - g(t + u)}{1 - g(t)} = R(t, u), \quad \text{say.}$$

The corresponding generating function, with x and y the indicators of calls at t and $t + u$, resp. is

$$1 - Q(t) + Q(t)[1 - R(t, u)]x + Q(t)R(t, u)xy$$

or

$$1 + (x - 1)(1 - g(t))h/t + x(y - 1)[g(u) - g(t + u)]h/t$$

The generating function for c calls in this interval is this expression raised to the c'th power, since calls arrive independently; and since c calls arrive with probability $e^{-at}(at)^c/c!$, the generating function for calls arriving in this interval is

$$\begin{aligned}\sum [1 + (x - 1)Q + x(y - 1)QR]^c e^{-at}(at)^c/c! \\ = \exp b[(x - 1)(1 - g(t)) + x(y - 1)(g(u) - g(t + u))]\end{aligned} \tag{17}$$

For brevity Q and R have been written for $Q(t)$ and $R(t, u)$.

Finally the generating function for calls arriving in t, $t + u$, is

$$\exp b(y - 1)(1 - g(u)) \tag{18}$$

Hence

$$\begin{aligned}\sum_j \sum_k P_{ijk}(t, u)x^j y^k &= [1 + (x - 1)g(t) + x(y - 1)g(t + u)]^i \\ &\cdot \exp b[(x - 1)(1 - g(t)) + (y - 1)(1 - g(u)) \\ &\qquad + x(y - 1)(g(u) - g(t + u))]\end{aligned} \tag{19}$$

By similar argument, the generating function for triple transition probabilities is

$$\begin{aligned}&\sum_j \sum_k \sum_\ell P_{ijk\ell}(t, u, v)x^i y^k z^\ell \\ &= [1 + (x - 1)g_t + x(y - 1)g_{t+u} + xy(z - 1)g_{t+u+v}]^i \\ &\quad \cdot \exp b((x - 1)(1 - g_t) + (y - 1)(1 - g_u) + \\ &\quad (z - 1)(1 - g_v) + x(y - 1)(g_u - g_{t+u}) + \\ &\quad y(z - 1)(g_v - g_{u+v}) + xy(z - 1)(g_{u+v} - g_{t+u+v}))\end{aligned} \tag{20}$$

3. Expected Correlations

Correlation expectations, like $E[N(t)N(u)]$ in equation (3), are needed for evaluation of the moments of $M(T)$. They may be determined from the transition probability generating functions, if it is agreed, as a matter only of convenience, that the time epochs t, u, v, etc. are in that order ($t \leq u \leq v \leq \cdots$). Since, on the assumption of statistical equilibrium, the call probabilities at the first epoch t, are independent of its value, as already noticed, this value may be taken as zero without loss of generality.

Thus for the second moment it is sufficient to determine

$$\varphi(u) = E[N(0)N(u)] = \sum ip_i \sum jP_{ij}(u) \tag{21}$$

with $p_i = Pr[N(0) = i] = e^{-b}b^i/i!$

Write

$$G_u(x, y) = \sum p_i x^i \sum P_{ij}(u) y^j$$

By (12), this is the same as

$$G_u(x, y) = \exp b[x - 1 + y - 1 + (x - 1)(y - 1)g(u)]$$

or

$$H_u(x, y) = G_u(x + 1, y + 1) \equiv \exp b(x + y + xyg(u))$$

and

$$\begin{aligned} \varphi(u) &= \left.\frac{\partial^2 H}{\partial x \partial y}\right|_{x,y=0} \\ &= b^2 + bg(u) \end{aligned} \tag{22}$$

In the same way the second order correlation expectation, that is

$$\varphi(u, v) = E[N(0)N(u)N(u + v)],$$

is obtained from

$$G_{u,v}(x, y, z) = \sum p_i x^i \sum \sum P_{ijk}(u, v) y^j z^k$$

and

$$\begin{aligned} H_{u,v}(x, y, z) &= G_{u,v}(x + 1, y + 1, z + 1) \\ &= \exp b(x + y + z + xyg(u) + yzg(v) + x(y + 1)zg(u + v)) \end{aligned}$$

Hence

$$\varphi(u, v) = b^3 + b^2[g(u) + g(v) + g(u + v)] + bg(u + v) \tag{23}$$

Finally, the third order correlation turns out to be

$$\begin{aligned}\varphi(u, v, w) &= E[N(0)N(u)N(u+v)N(u+v+w)] \\ &= b^4 + b^3[g(u) + g(v) + g(w) \\ &\quad + g(u+v) + g(v+w) + g(u+v+w)] \\ &\quad + b^2[g(u+v) + g(v+w) + 2g(u+v+w)] \\ &\quad + b^2[g(u)g(w) + g(u+v)g(v+w) \\ &\quad + g(v)g(u+v+w)] + bg(u+v+w)\end{aligned} \tag{24}$$

As will appear, the arrangement of terms in (22), (23) and (24) corresponds to the expansion of ordinary moments in terms of cumulants (semi-invariants); e.g. (24) corresponds to $m_4 = b^4 + 6b^2k_2 + 4bk_3 + 3k_2^2 + k_4$ with k_i the i'th cumulant (for the Poisson of mean b, $k_i = b$).

4. Moments

Moments are obtained from these results by integrations. As already noted, equation (2), the first moment is b for any holding time distribution.

Since there are two ways of ordering the epochs t, u, the second moment is

$$\begin{aligned}E[M^2(T)] &= \frac{2}{T^2}\int_0^T dt \int_0^t du\, \varphi(t-u) \\ &= b^2 + \frac{2b}{T^2}\int_0^T dt \int_0^t du\, g(t-u) \\ &= b^2 + \frac{2b}{T^2}\int_0^T dx\, g(x)(T-x)\end{aligned} \tag{25}$$

The last step is by the formula for reversing the order of integration indicated by

$$\int_0^T dt \int_0^t du = \int_0^T du \int_u^T dt$$

The variance or second central moment, which is also the second cumulant k_2, is then

$$\begin{aligned}\mathrm{Var}\,[M(T)] &= E[(M(T) - b)^2] \\ &= E[M^2(T)] - b^2 \\ &= \frac{2b}{T^2}\int_0^T dx\, g(x)(T-x)\end{aligned} \tag{26}$$

Since there are $3! = 6$ ways of ordering 3 epochs, the third moment may be written

$$
\begin{aligned}
E[M^3(T)] &= \frac{6}{T^3}\int_0^T dt \int_0^t du \int_0^u dv\, \varphi(t-u, u-v) \\
&= b^3 + \frac{6b^2}{T^3}\int_0^T dt \int_0^t du \int_0^u dv[g(t-w) + g(u-v) + g(t-v)] \\
&\quad + \frac{6b}{T^3}\int_0^T dt \int_0^t du \int_0^u dv\, g(t-v)
\end{aligned}
$$

Here the first triple integral is immediately evaluated by use of the identity

$$
\begin{aligned}
2\int_0^T dt \int_0^t du \int_0^u dv[g(t-u) + g(u-v) + g(t-v)] \\
&= \int_0^T \int_0^T \int_0^T dt\, du\, dv\, g(|\, t - v\,|) \\
&= 2T \int_0^T dx\, g(x)(T-x) \\
&= T^3\, k_2/b
\end{aligned}
$$

The last triple integral, by successive inversions of integration order, turns out to be

$$
\frac{6b}{T^3}\int_0^T dx\, g(x)(T-x)x
$$

Hence finally

$$
E[M^3(T)] = b^3 + 3bk_2 + \frac{6b}{T^3}\int_0^T dx\, g(x)(T-x)x \tag{27}
$$

and

$$
\begin{aligned}
k_3 &= E[(M(T) - b)^3] \\
&= E[M^3(T)] - 3b\, E[M^2(T)] + 2b^3 \\
&= E[M^3(T)] - 3b\, k_2 - b^3 \\
&= \frac{6b}{T^3}\int_0^T dx\, g(x)(T-x)x
\end{aligned} \tag{28}
$$

The fourth moment is given by

$$E[M^4(T)] = \frac{24}{T^4}\int_0^T dt \int_0^t du \int_0^u dv \int_0^v dw\, \varphi(t-u, u-v, v-w)$$

$$= b^4$$

$$+ \quad \frac{24}{T^4}\Big\{b^3 \int_0^T \int_0^t \int_0^u \int_0^v dt\, du\, dv\, dw[g(t-u) + g(t-v)$$

$$+ g(t-w) + g(u-v) + g(u-w) + g(v-w)]$$

$$+ b^2 \int_0^T \int_0^t \int_0^u \int_0^v dt\, du\, dv\, dw\, [g(t-v) + g(u-w) + 2g(t-w)]$$

$$+ b^2 \int_0^T \int_0^t \int_0^u \int_0^v dt\, du\, dv\, dw[g(t-u)g(v-w) +$$

$$g(t-v)g(u-w) + g(t-w)g(u-v)]$$

$$+ b \int_0^T \int_0^t \int_0^u \int_0^v dt\, du\, dv\, dw[g(t-w)]\Big\}$$

Employing the identities

$$4 \int_0^T \int_0^T \int_0^T \int_0^T dt\, du\, dv\, dw\, [g(t-u) + g(t-v) + g(t-w)$$

$$+ g(u-v) + g(u-w) + g(v-w)]$$

$$= \int_0^T \int_0^t \int_0^u \int_0^v dt\, du\, dv\, dw\ g(|\, t-u\,|)$$

$$= 2T^2 \int_0^T dx\ g(x)(T-x) = T^4 k_2/b,$$

$$8 \int_0^T \int_0^t \int_0^u \int_0^v dt\, du\, dv\, dw\, [g(t-u)\, g(v-w) + g(t-v)\, g(u-w)$$

$$+ g(t-w)\, g(u-v)]$$

$$= \int_0^T \int_0^T \int_0^T \int_0^T dt\, dw\, dv\, dw\ g(|\, t-u\,|)g(|\, v-w\,|)$$

$$= 4\left[\int_0^T dx\ g(x)(T-x)\right]^2 = T^4 k_2^2/b^2,$$

and successive inversion of order of integration, the final result turns out to be

$$E[M^4(T)] = b^4 + 6b^2 k_2 + 4bk_3 + 3k_2^2 + \frac{12b}{T^4} \int_0^T dx\, g(x)(T - x)x^2 \tag{29}$$

and

$$\begin{aligned} k_4 &= E[(M(T) - b)^4] - 3E[(M(T) - b)^2] \\ &= \frac{12b}{T^4} \int_0^T dx\, g(x)(T - x)x^2 \end{aligned} \tag{30}$$

It is a tempting surmise that

$$k_n = b\, \frac{n(n-1)}{T^n} \int_0^T dx\, g(x)(T - x)\, x^{n-2}$$

but this has not been proved. Note that for $g(x) = 1$, $k_n = b$, the cumulant of the Poisson, as it should.

For the two cases of chief interest, constant and exponential holding times, the function $g(x)$, *in average holding time units* (that is, $x = t/h$) is given by

c.h.t.	$g(x) = 1 - x$	$x < 1$
	$= 0$	$x > 1$
e.h.t.	$g(x) = e^{-x}$	

and the results are as follows:

Cumulant	Constant Holding Time T < 1	T > 1
k_2	$b(1 - T/3)$	$bT^{-1}(1 - 1/3T)$
k_3	$b(1 - T/2)$	$bT^{-2}(1 - 1/2T)$
k_4	$b(1 - 3T/5)$	$bT^{-3}(1 - 3/5T)$

	Exponential Holding Time
k_2	$2bT^{-2}[T - 1 + e^{-T}]$
k_3	$6bT^{-3}[T - 2 + (T + 2)e^{-T}]$
k_4	$12bT^{-4}[2T - 6 + (T^2 + 4T + 6)e^{-T}]$

It may be worth noting that, if the surmise is correct, for constant holding time

$$\begin{aligned} k_n &= b\left[1 - \frac{n-1}{n+1} T\right] & T < 1 \\ &= \frac{b}{T^{n-1}}\left[1 - \frac{n-1}{n+1}\frac{1}{T}\right] & T > 1 \end{aligned}$$

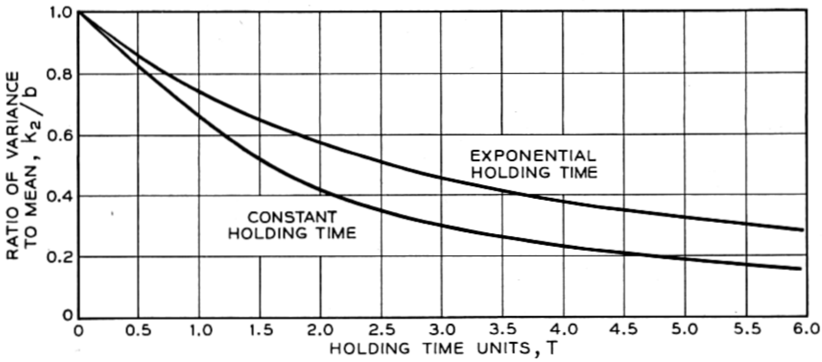

FIG. 1.—Comparison of variances of average traffic for constant and exponential holding times.

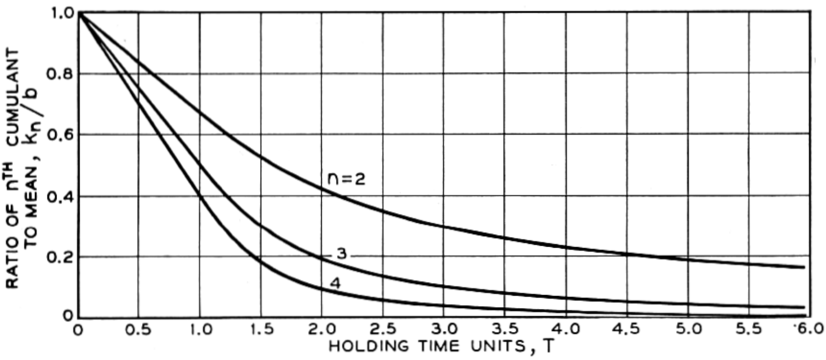

FIG. 2.—Cumulants k_2, k_3, and k_4 for constant holding time.

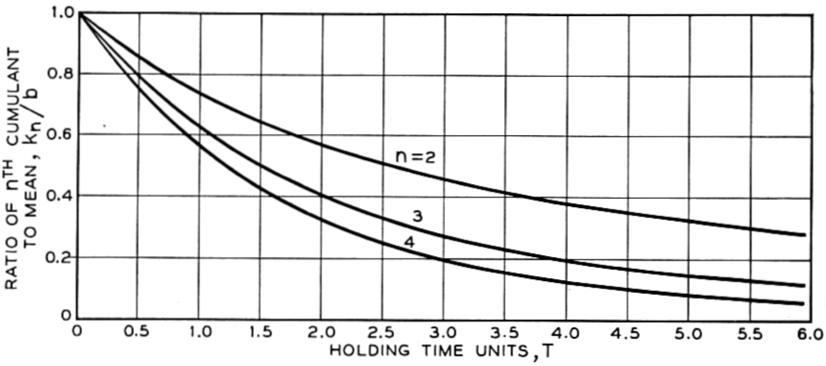

FIG. 3.—Cumulants k_2, k_3 and k_4 for exponential holding time.

and for exponential holding time

$$k_n = b\,\frac{n(n-1)}{T^n}\,[(n-2)!\,T - (n-1)! + e^{-T}\,(T+\alpha)^{n-2}]$$

where in the last term $(T + \alpha)^{n-2}$ is a symbolic expression or shorthand for

$$(T+\alpha)^{n-2} = \sum_0^{n-2} \binom{n-2}{m} T^{n-2-m}\,\alpha_m$$

and $\alpha_m = (m + 1)!$; e.g.

$$(T+\alpha)^3 = T^3 + 6T^2 + 18T + 24$$

For small values of T, the two cases coalesce ($e^{-x} \approx 1 - x$) and at $T = 0$ approach b as they should. For large values of T, and constant holding time,

$$k_n \sim b/T^{n-1}, \qquad (n = 2, 3, 4);$$

for exponential holding time

$$k_n \sim n!b/T^{n-1}, \qquad (n = 2, 3, 4).$$

For $n = 2$, these results agree with Rabe [6].

As T increases, for either holding time, the cumulants are progressively smaller, and the approximation of the distribution of $M(T)$ by a normal curve (which has all cumulants, except the first and second, zero) improves. This is what follows from the central limit theorem if the subdivision of T into a large number of intervals results in mutually independent random variables (cf. Rice [7] 3.9).

Figure 1 shows a comparison of the variances (k_2) for the two holding time cases. Figure 2 shows a comparison of the cumulants k_2, k_3 and k_4 for constant holding time, and Fig. 3 shows the same thing for exponential holding time.

5. Finite Sources—Exponential Holding Time

The generating function for transitional probabilities for N subscribers, each originating calls independently with probability λ, and for exponential holding time, as given by Jensen (l.c.) is as follows:

$$P_i(t, x) = [1 + q_1(x-1)]^i[1 + q_0(x-1)]^{N-i} \qquad (31)$$

with

$$q_0 = p - pe^{-(\lambda+\gamma)t}$$

$$q_1 = p + q \quad \text{"}$$

$$p = 1 - q = \lambda/(\lambda+\gamma)$$

$$\gamma = 1/h$$

It should be noticed that for $t = \infty$, $q_0 = q_1 = p$ and

$$P_i(\infty, x) = [1 + p(x - 1)]^N \tag{32}$$

The right hand side is the binomial generating function and, as independent of i, is the generating function for the statistical equilibrium probabilities; that is

$$Pr\ [N(t) = k] = \binom{N}{k} p^k\, q^{N-k}$$

Also the process is Markovian since

$$\sum_k x^k \sum_j P_{ij}(t)P_{jk}(u) = \sum_j P_{ij}(t)[1 + q_{1u}(x - 1)]^j\, [1 + q_{0u}(x - 1)]^{N-j}$$

$$= [1 + (q_{0u} + q_{1t}\,q_{1u} - q_{1t}\,q_{0u})(x - 1)]^i$$

$$[1 + (q_{0u} + q_{0t}\,q_{1u} - q_{0t}\,q_{0u})(x - 1)]^{N-i}$$

and

$$q_{0u} + q_{1t}\,q_{1u} - q_{1t}\,q_{0u} = q_{1,\,t+u}$$

$$q_{0u} + q_{0t}q_{1u} - q_{0t}\,q_{0u} = q_{0,\,t+u}$$

Here it has been convenient to indicate by the double subscript the dependence of q_0 and q_1 on a time variable.

Moments are obtained by the process given in detail for the infinite source case. For brevity it is convenient to use the binomial cumulants which are as follows

$$\kappa_2 = Npq$$

$$\kappa_3 = Npq(q - p)$$

$$\kappa_4 = Npq(1 - 6pq)$$

and the modified time variable $T_1 = (\lambda + \gamma)T$. Then the results are

$$k_2 = 2T_1^{-2}\kappa_2[T_1 - 1 + e^{-T_1}]$$

$$k_3 = 6T_1^{-3}\kappa_3[T_1 - 2 + (T_1 + 2)e^{-T_1}]$$

$$k_4 = 12T_1^{-4}((\kappa_4 + \kappa_2^2 N^{-1})[2T_1 - 6 + (T_1^2 + 4T_1 + 6)e^{-T_1}]$$

$$- \kappa_2^2 N^{-1}[1 - (T_1^2 + 2)e^{-T_1} + e^{-2T_1}])$$

These of course bear a strong resemblance to the infinite source case (exponential holding time), to which they converge.

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