Theory of the Flow of Electrons and Holes in Germanium and Other Semiconductors

By W. VAN ROOSBROECK

(*Manuscript Received Mar. 30, 1950*)

A theoretical analysis of the flow of added current carriers in homogeneous semiconductors is given. The simplifying assumption is made at the outset that trapping effects may be neglected, and the subsequent treatment is intended particularly for application to germanium. In a general formulation, differential equations and boundary-condition relationships in suitable reduced variables and parameters are derived from fundamental equations which take into account the phenomena of drift, diffusion, and recombination. This formulation is specialized so as to apply to the steady state of constant total current in a single cartesian distance coordinate, and properties of solutions which give the electrostatic field and the concentrations and flow densities of the added carriers are discussed. The ratio of hole to electron concentration at thermal equilibrium occurs as parameter. General solutions are given analytically in closed form for the intrinsic semiconductor, for which the ratio is unity, and for some limiting cases as well. Families of numerically obtained solutions dependent on a parameter proportional to total current are given for n-type germanium for the ratio equal to zero. The solutions are utilized in a consideration of simple boundary-value problems concerning a single plane source in an infinite filament.

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1. Introduction

In a semiconductor there are current carriers of two types: electrons in the conduction band, and positive holes in the filled valence band; and the increase of their concentrations in the volume of the semicon-

ductor over the concentrations which obtain at thermal equilibrium is fundamental to a number of related phenomena, of which transistor action is a familiar instance. In an *n*-type semiconductor, for example, in which the carriers are predominantly electrons, the carrier concentrations are increased by the introduction of holes which, through a process of space-charge neutralization, produce additional electrons in the same numbers and concentrations. The bulk conductivity of the semiconductor is thereby so increased that power gain is obtainable.[1] Holes can be introduced by the local application of heat, or by irradiation with light, X-rays, or high-velocity electrons—in fact, by any agency which transfers electrons from the highest filled band to the conduction band. They can be introduced also through an emitter, which may be a positively biased point contact or a positively biased $p - n$ junction[3], as exemplified in the transistor. In this case the emitter introduces holes, which flow into the volume of the semiconductor[4], by the removal of electrons from the filled band.[2, 5] Entirely analogous considerations apply to the introduction of electrons into a *p*-type semiconductor.[6]

In their flow in a semiconductor, added electrons and holes are subject to drift under electrostatic fields and to diffusion in the presence of concentration gradients as a consequence of their random thermal motions. They are subject also to recombination, which results in concentration gradients in source-free regions even for the steady state in one dimension, or which augments those which may otherwise be associated with the time-dependence of the flow, or with its geometry in the steady state. From fundamental equations which take into account these phenomena of drift, diffusion, and recombination, for the existence of each of which there is experimental evidence[1], general differential equations and boundary-condition relationships in suitable reduced or dimensionless variables and parameters may be derived, and solutions which give the concentrations and flow densities of added carriers obtained for various cases of physical interest.

This paper presents results of a theoretical analysis, along these lines, of the flow of electrons and holes in semi-conductors. The treatment is intended particularly for application to germanium. An initial formulation,

[1] W. Shockley, G. L. Pearson and J. R. Haynes, *B. S. T. J.* 28, (3), 344–366 (1949).
[2] J. Bardeen and W. H. Brattain, *Phys. Rev.* 74 (2), 230–231 (1948); W. H. Brattain and J. Bardeen, *Phys. Rev.* 74 (2) 231–232 (1948).
[3] W. Shockley, G. L. Pearson and M. Sparks, *Phys. Rev.* 76 (1), 180 (1949); W. Shockley, *B. S. T. J.* 28 (3), 435–489 (1949).
[4] E. J. Ryder and W. Shockley, *Phys. Rev.* 75 (2), 310 (1949); J. N. Shive, *Phys. Rev.* 75 (4), 689–690 (1949); J. R. Haynes and W. Shockley, *Phys. Rev.* 75 (4), 691 (1949).
[5] J. Bardeen and W. H. Brattain, *Phys. Rev.* 75 (8), 1208–1225 (1949); *B. S. T. J.* 28 (2), 239–277 (1949).
[6] W. G. Pfann and J. H. Scaff, *Phys. Rev.* 76 (3), 459 (1949); R. Bray, *Phys. Rev.* 76 (3), 458 (1949).

which retains, wherever convenient, such generality as is instructive per se or of manifest utility, is specialized so as to apply to the steady state of constant current in a single cartesian distance coordinate. For the intrinsic semiconductor, general analytical solutions are obtainable in closed form, and such solutions are given, as well as general solutions obtained numerically for n-type germanium in which the hole concentration at thermal equilibrium may be neglected compared to the electron concentration. Solutions for these cases are given explicitly for each of two recombination laws: recombination according to a mass-action law, and recombination such that the mean lifetime of the added carriers is constant. Methods are described for the fitting of boundary conditions, and the following relatively simple boundary-value problems are considered: a source at the end of a semi-infinite semi-conductor filament; and a single source in a doubly-infinite filament.

To indicate the presumed scope and application of the results obtained, it may suffice to outline briefly the principal assumptions on which they are based and the approximations employed: The assumption is made at the outset that trapping effects may be neglected, which provides the important simplification that the recombination rates of holes and electrons are equal at all times. One justification for this is the circumstance that the fairly high hole mobilities found by G. L. Pearson from Hall-effect and conductivity measurements[7] are no larger than those found by J. R. Haynes from transit times under pulse conditions[1]. With hole trapping, holes injected in a pulse would initially fill traps; and if there were subsequent relatively slow release of the holes from the traps, an apparent reduction of mobility would be manifest. It is further assumed that substantially all donor and acceptor impurities are ionized. With the assumption that the semi-conductor is homogeneous in its bulk, and free from grain boundaries[8] or rectifying barriers, the assumption of the electrical neutrality of the semiconductor, or of the neglect of space charge, is in general an excellent approximation: Small departures from electrical neutrality in the volume would vanish rapidly, with time constant equal to that for the dielectric relaxation of charge, which for germanium equals $1.5 \cdot 10^{-12}$ sec per ohm cm of resistivity[9] and is in general small compared with the mean lifetime of added carriers. A uniform local departure from electrical neutrality in germanium of only one per cent in relative concentration would produce appreciable changes in field in a

[7] G. L. Pearson, *Phys. Rev.* 76 (1), 179–180 (1949).

[8] G. L. Pearson, *Phys. Rev.* 76 (3), 459 (1949); W. E. Taylor and H. Y. Fan, paper OA5, and N. H. Odell and H. Y. Fan, paper OA2 of the 1950 Annual Meeting of the American Physical Society, February 3, 1950.

[9] A value of 16.6 for the dielectric constant of germanium is obtained from optical data of H. B. Briggs: *Phys. Rev.* 77 (2), 287 (1950).

mean free path for the carriers, equal to $1.1 \cdot 10^{-5}$ cm at room temperature, which would even preclude the applicability of the fundamental equations employed. In qualitative terms, the conductivity of the semiconductor is sufficiently large that the currents which commonly occur are produced by moderate fields whose maximum gradients are relatively small. Space charge may persist in the steady state, but then only in surface regions whose thickness[10] in germanium is generally less than about 10^{-4} cm and whose effects may be dealt with through suitable boundary conditions.

The steady-state solutions, in their qualitative aspects, are illustrative of the phenomena taken into consideration. In an extrinsic semiconductor, if the concentrations of added carriers are not too large, the solutions for moderate and large fields are in general approximately ohmic in their local behavior. The effect of diffusion is then comparatively small, and the added carriers largely drift under a field which varies with distance through the increased conductivity which these recombining carriers themselves produce. Diffusion effects are incident in addition to this behavior, and become pronounced for large concentrations or small applied fields. For example, solutions which specify the concentrations of added holes as functions of distance, for different total currents or applied fields in a source-free region, all approach a common solution for large hole concentrations, regardless of applied field; those for the hole current and the electrostatic field behave similarly. This behavior results from diffusion in conjunction with the increase in conductivity. Another example is that of the solutions for zero total current: As the result of diffusion in conjunction with recombination, a flow of added holes can occur along a semi-conductor filament with no flow of current. It is, of course, accompanied by an equal electron flow, so that the hole and electron currents cancel, and occurs in any open-circuited semi-conductor filament which adjoins a region in which added holes flow. It can also be realized by suitable irradiation of an end of a filament, with no applied field. A closely related effect is illustrated in the flow of holes injected through a point-contact emitter into a semi-conductor filament along which a sweeping field is applied: Some of the holes will flow against the field, an appreciable proportion, unless the current in the filament is sufficiently large. As a further example, if the mobilities of holes and electrons were equal, the electrostatic field would be given by Ohm's law as the total current

[10] The (largest) distance over which the increment in electrostatic potential exceeds kT/e may be expressed in units of the length $L_d \equiv (kT\epsilon/8\pi n_i e^2)^{\frac{1}{2}}$, where n_i is the thermal-equilibrium concentration of electrons (or holes) in the intrinsic semiconductor; see the paper of reference 3, also W. Schottky and E. Spenke, *Wiss. Veröff. Siemens-Werken* 18 (3), 1–67 (1939). This distance increases with resistivity, never exceeding the value 1.4 L_d for the intrinsic semiconductor. In high back voltage n-type germanium, it exceeds about 0.5 L_d, and L_d for germanium is about $7.4 \cdot 10^{-5}$ cm at room temperature.

divided by the local increased conductivity. With electrons more mobile than holes, this ohmic field is modified by a contribution which is directed away from a hole source and proportional to the magnitude of the concentration gradient divided by the local conductivity. This contribution gives a non-vanishing electrostatic field for zero total current.

The intrinsic semiconductor has, as the result of a conductivity which is everywhere proportional to the concentration of carriers of either type, the property that the flow in it is as if the added carriers were actuated entirely by diffusion, with only the carriers normally present drifting under a field equal to the unmodulated applied field. The extrinsic semiconductor becomes in effect intrinsic if the concentrations of carriers are sufficiently increased, by whatever means, the ohmic contribution to the current density of either electrons or holes then becoming proportional to the total current density and, in this case, negligible compared with the contribution due to diffusion. It may, for example, be expected that the transport velocity of added carriers in an extrinsic semiconductor can be increased by an increase in the applied field only if the consequent joule heating does not unduly modify the semiconductor in the intrinsic direction.

General solutions for the steady state in one dimension are obtainable analytically in closed form for a number of important special cases. Aside from that for which diffusion is neglected, they include the general cases for no recombination, for the intrinsic semiconductor, and for zero total current, and the limiting cases of small and of large concentrations of added carriers. W. Shockley has made use of small-concentration theory in an analysis of $p - n$ junctions[3]. J. Bardeen and W. H. Brattain have given solutions for the steady-state hole flow in three dimensions, neglecting recombination, in the neighborhood of a point-contact emitter.[5, 11] Transient solutions are obtainable analytically for the intrinsic semiconductor for constant mean lifetime, and for the extrinsic semiconductor if the concentrations of added carriers are sufficiently small that the change in conductivity is negligible. For concentrations unrestricted in magnitude, Conyers Herring has described a general method for graphical or numerical construction of transient solutions in one dimension from a first-order partial differential equation appropriate to the case for which diffusion is neglected in the extrinsic semi-conductor, and has given some solutions so obtained, with estimates of the effect of diffusion. Reference might be made to his paper[12] also for discussion of various physical con-

[3] loc. cit.
[5] loc. cit.
[11] See the paper of J. Bardeen in this issue.
[12] Conyers Herring, *B. S. T. J.* 28 (3), 401–427 (1949).

siderations and of certain interesting transient effects. Steady-state alternating-current theory for relatively small total hole concentrations in the n-type semiconductor has been used to describe the action of the filamentary transistor[13] for which diffusion may in general be neglected.[1]

The steady-state solutions in one dimension apply to single-crystal semiconductor filaments, and for critical comparisons between theory and experiment, the ideal one-dimensional geometry should be simulated as closely as possible. Experimental estimates of hole concentrations and flows are frequently obtained from measurements of potentials and conductances of point contacts along a filament[1]. These estimates require a knowledge of the dependence of the current-voltage characteristics of point contacts on hole concentration. Theory for this dependence has been presented by J. Bardeen[11], and the determination of hole concentrations by means of the solutions here given should provide an essential adjunct to this point contact theory for its comparison with experiment.

2. General formulation

2.1 *Outline*

The formulation of the general problem is initiated by writing the fundamental equations for the time-dependent flow of holes and electrons in a source-free region of a homogeneous semiconductor under the assumption that there is no trapping. Conditions for their validity are discussed. Neglecting changes in the concentrations of ionized donors and acceptors, the fundamental equations are expressed in reduced or dimensionless form by suitable transformations of the dependent and independent variables. They are simplified so that the general problem is formulated by means of second-order partial differential equations in two dependent variables, one for concentration and the other for electrostatic potential; corresponding equations are derived for the intrinsic semiconductor. Various properties of the equations are adduced. For the flow in one dimension, a differential equation in the hole concentration is given for the n-type semiconductor, accompanied by expressions for the electrostatic field and hole flow density, as well as by some boundary-condition relationships involving specification of the latter. The equations for this case are found to depend on three parameters: the ratio of electron to hole mobility; a reduced concentration of holes at thermal equilibrium; and a parameter which fixes the total current density.

The recombination of holes and electrons is specified by means of a

[1] loc. cit.

[11] loc. cit.

[13] W. Shockley, G. L. Pearson, M. Sparks, and W. H. Brattain, *Phys. Rev.* 76 (3), 459 (1949).

suitable function of the concentration of the added carrier, whose form is specified for two recombination laws: recombination according to a mass-action law, and recombination characterized by constant mean lifetime. It is shown that essentially the same reduced equations apply to the case for which recombination is neglected.

Second-order differential equations in the hole concentration for the n-type semiconductor with the thermal-equilibrium value of the hole concentration assumed negligible compared to the electron concentration, and for the intrinsic semiconductor, are then written for the steady state of constant current in one dimension. These are converted into first-order equations which have, as dependent variable a reduced concentration gradient G, and as independent variable a reduced concentration of added holes, ΔP. Boundary conditions are expressed as relationships between these variables. Properties of the general solutions and of the boundary conditions are accordingly examined in the $(\Delta P, G)$-plane. It is found that there are two intersecting solutions through the $(\Delta P, G)$-origin, which is a saddle-point of the differential equation, and that these are the solutions for field directed respectively towards and away from sources in semi-infinite regions which have sources only to one side. They are called field-opposing and field-aiding solutions, and possess two degrees of freedom. Solutions which do not intersect at the origin are asymptotic to these, possess three degrees of freedom, and are called solutions of the composite type. This is the general type, and applies to a finite region in distance at both ends of which boundary conditions are specified. The region may, for example, be one between a source and either another source, a sink, a non-rectifying electrode, or a surface upon which recombination takes place. While the analysis of composite cases is straightforward, the present treatment is confined to the simpler cases of field opposing and field aiding, the latter being the one most generally applicable to experiments in hole injection. Also, where the differential equations involved are linear, solutions for composite cases can be written as linear combinations of field-aiding and field-opposing solutions.

From the properties of the curves in the $(\Delta P, G)$-plane is determined the qualitative behavior of the hole concentration at a hole source at the end of a semi-infinite filament as the total current is indefinitely increased.

2.2 *Fundamental equations for the flow of electrons and holes*

The equations for the flow in three dimensions of electrons and holes in a homogeneous semiconductor contain, as principal dependent variables, the hole and electron concentrations, p and n, the flow densities $\mathbf{J}_p$ and $\mathbf{J}_n$, and the electrostatic field, $\mathbf{E}$, or potential, V. With no trapping,

the equations may be written in a symmetrical form, so that they are applicable to either an n-type, a p-type, or an intrinsic semiconductor, as follows:

$$
(1)\quad
\begin{cases}
\dfrac{\partial p}{\partial t} = -[p/\tau_p - g_0] - \operatorname{div} \mathbf{J}_p \\[2ex]
\dfrac{\partial n}{\partial t} = -[n/\tau_n - g_0] - \operatorname{div} \mathbf{J}_n \\[2ex]
\mathbf{J}_p \equiv \dfrac{1}{e}\mathbf{I}_p = \mu_p\left[p\mathbf{E} - \dfrac{kT}{e}\operatorname{grad} p\right] = -\mu_p p \operatorname{grad}\left[V + \dfrac{kT}{e}\log p\right] \\[2ex]
\mathbf{J}_n \equiv -\dfrac{1}{e}\mathbf{I}_n = \mu_n\left[-n\mathbf{E} - \dfrac{kT}{e}\operatorname{grad} n\right] \\[2ex]
\qquad\qquad = -\mu_n n \operatorname{grad}\left[-V + \dfrac{kT}{e}\log n\right] \\[2ex]
\operatorname{div} \mathbf{E} = \dfrac{4\pi e}{\epsilon}[(p - p_0) - (n - n_0) + (D^+ - D_0^+) - (A^- - A_0^-)] \\[2ex]
\mathbf{E} = -\operatorname{grad} V.
\end{cases}
$$

In the first two equations, which are the continuity equations for holes and electrons written for a region free from external sources, g_0 is a constant which represents the thermal rate of generation of hole-electron pairs per unit volume; for cases in which hole-electron pairs are produced also by penetrating radiation, appropriate source terms in the form of identical functions of the space and time coordinates can be included on the right in the respective equations. The mean lifetimes of holes and electrons, τ_p and τ_n, are in general considered to be concentration-dependent and, since trapping is neglected, the quantities p/τ_p and n/τ_n are equal, being the rate at which holes and electrons recombine. Evaluated for the normal semiconductor, or the semiconductor at thermal equilibrium with no injected carriers, they equal g_0.

The equations for $\mathbf{J}_p$ and $\mathbf{J}_n$, which are vectors whose magnitudes equal, respectively, the numbers of holes and of electrons which traverse unit area in unit time, are diffusion equations of M. von Smoluchowski, written for hole flow and for electron flow[14]. Of the type frequently employed, after C. Wagner, in theories of rectification, each expresses the dependence of the flow density on the electrostatic field and on the concentration gradient, the diffusion constant for holes or electrons having been expressed in terms of the mobility, μ_p or μ_n, in accordance with the

[14] S. Chandrasekhar, *Rev. Mod. Phys.* 15, 1–89 (1943).

well-known relationship of A. Einstein[15]. In them, e denotes the magnitude of the electronic charge; T is temperature in degrees absolute; and k is Boltzmann's constant. With transport velocity defined as flow density divided by concentration, the product of the mobility and the quantity in square brackets in the expression for $\mathbf{J}_p$ or $\mathbf{J}_n$ on the extreme right gives the corresponding velocity potential, which is thus proportional to the sum of an electrostatic potential and a diffusion potential.

The next to last equation is Poisson's equation, which relates the divergence of the field to the net electrostatic charge. Here ϵ is the dielectric constant; p_0 and n_0 are the concentrations of holes and electrons at thermal equilibrium, in the normal semiconductor. The concentrations of ionized donor and acceptor impurities at thermal equilibrium are represented by D_0^+ and A_0^-, while D^+ and A^- are dependent variables which denote the respective concentrations in general of ionized donors and acceptors in the semiconductor with added carriers. As shown in the Appendix, variations in D^+ and A^- may be neglected if the impurity centers are substantially all ionized in the normal semiconductor, despite the effect large concentrations of added carriers may have on the equilibria[16].

The expression of the electrostatic field as the gradient of a potential according to the last equation is consistent with the circumstance that the effects of magnetic fields, with none applied, are in general quite negligible.

Subtracting the first continuity equation from the second, it is found that

$$\text{div}\,(\mathbf{J}_p - \mathbf{J}_n) = -\frac{\partial}{\partial t}(p - n), \tag{2}$$

since, with no trapping, p/τ_p equals n/τ_n. Neglecting changes in the concentrations of ionized donors and acceptors, this equation and Poisson's equation give

$$\mathbf{J}_p - \mathbf{J}_n = \mathbf{J} - \frac{\epsilon}{4\pi e}\frac{\partial \mathbf{E}}{\partial t}; \qquad \mathbf{I}_p + \mathbf{I}_n = \mathbf{I} - \frac{\epsilon}{4\pi}\frac{\partial \mathbf{E}}{\partial t}, \tag{3}$$

where $\mathbf{J}$ and $\mathbf{I}$ are solenoidal vector point functions, in general time-dependent. The latter is the total current density, and the term which follows it in (3) gives the displacement current density.

[15] A. Einstein, *Annalen der Physik* 17, 549–560 (1905); Müller-Pouillet, *Lehrbuch der Physik*, Braunschweig, 1933, IV (3), 316–319.

[16] It has been found from measurements of the temperature dependence of the conductivity and Hall coefficient that the energy of thermal ionization of the donors in n-type germanium of relatively high purity is only about $10^{-2}eV$, whence most of the donors are ionized at room temperature: G. L. Pearson and W. Shockley, *Phys. Rev.* 71 (2), 142 (1947).

It may be well to point out that the validity of the diffusion equations depends on two assumptions, which, while hardly restrictive in general for homogeneous semiconductors, indicate the nature of the generalizations which might otherwise be necessary[14]. The first assumption is that there are no appreciable time changes in the dependent variables in the relaxation time for the conductivity, or the time of the elementary fluctuations. This is tantamount to the requirement that the carriers undergo many collisions in the time intervals of interest. The second assumption is that the changes in the carriers' electrostatic potential energy over distances equal to the mean free path are small compared with the average thermal energy. In accordance with this assumption, very large fields in the electrically neutral semiconductor for which the carriers are not substantially in thermal equilibrium with the lattice are ruled out. The neglect of space charge then in general validates the two assumptions, if the resistivity is not too small, since the neglect of changes in the dependent variables which occur in the dielectric relaxation time obviates their change in the relaxation time for conductivity; and the neglect in the steady state of appreciable variations in electrostatic potential, and thus in the other dependent variables, in the distance[10] L_d , obviates their variation in a mean free path. The dielectric relaxation time for germanium, $1.5 \cdot 10^{-12}$ sec per ohm cm of resistivity, in high back voltage material exceeds the relaxation time for conductivity, which is about $1.0 \cdot 10^{-12}$ sec; and in semi-conductors in which the mobilities and the conductivity are smaller than the comparatively large values for germanium, the dielectric relaxation time may be appreciably larger than the relaxation time for conductivity. Similarly, L_d for germanium is about 7 times the mean free path, and this ratio, which is essentially inversely proportional to the square root of the product of mobility and intrinsic conductivity, may be appreciably larger for other semiconductors.

If, on the other hand, it should be desired to consider space-charge effects in germanium, the diffusion equations may be of rather marginal applicability, and the use of their appropriate generalization indicated, since with L_d equal to 7 mean free paths, appreciable space-charge variation of potential, corresponding to a field which is not small compared with the free-path thermal-energy equivalent of about 3500 volt cm^{-1}, may occur in at least one of the free paths. For example, diode theory, rather than diffusion theory, provides the better approximation for the characteristics of germanium point-contact rectifiers, and is particularly applicable to those from low resistivity material for which the potential variation is largely confined to one mean free path or less[17].

[10] loc. cit.
[14] loc. cit.
[17] H. C. Torrey and C. A. Whitmer, "Crystal Rectifiers," New York, 1948, Sec. 4.3.

Neglecting space charge, Poisson's equation becomes simply the condition of electrical neutrality:

$$(p - p_0) - (n - n_0) = 0, \tag{4}$$

assuming substantially complete ionization of donors and acceptors. Similarly, equations (3) become

$$\mathbf{J}_p - \mathbf{J}_n = \mathbf{J}; \qquad \mathbf{I}_p + \mathbf{I}_n = \mathbf{I}. \tag{5}$$

With electrical neutrality, the two continuity equations merge into one: Since derivatives of p equal the corresponding ones of n,

$$\begin{aligned} \operatorname{div} \mathbf{J}_p &= -[p/\tau_p - g_0] - \frac{\partial p}{\partial t} \\ = \operatorname{div} \mathbf{J}_n &= -[n/\tau_n - g_0] - \frac{\partial n}{\partial t}. \end{aligned} \tag{6}$$

The neutrality condition in conjunction with the two equations obtained by substituting for $\mathbf{J}_p$ and $\mathbf{J}_n$ from the diffusion equations in (6) thus provide three equations for the determination of p, n, and $\mathbf{E}$ or V.

It is instructive to rewrite equations (6) in accordance with

$$\begin{cases} \operatorname{div} \mathbf{J}_p = \mathbf{s} \cdot \operatorname{grad} p \\ = \operatorname{div} \mathbf{J}_n = \mathbf{s} \cdot \operatorname{grad} n, \\ \mathbf{s} \equiv \left[\dfrac{\partial \mathbf{J}_p \cdot \mathbf{i}}{\partial x} \Big/ \dfrac{\partial p}{\partial x}\right]\mathbf{i} + \left[\dfrac{\partial \mathbf{J}_p \cdot \mathbf{j}}{\partial y} \Big/ \dfrac{\partial p}{\partial y}\right]\mathbf{j} + \left[\dfrac{\partial \mathbf{J}_p \cdot \mathbf{k}}{\partial z} \Big/ \dfrac{\partial p}{\partial z}\right]\mathbf{k}, \end{cases} \tag{7}$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors in the directions of the respective axes. The velocity $\mathbf{s}$, which is given as well by the expression for electrons analogous to that written for holes, may be defined alternatively as follows: Suppose, for definiteness, that the second-order system of equations (4) and (6) have been solved, so that the concentrations and flow densities are known in terms of the cartesian coordinates x, y, and z, and the time t. The x-component of $\mathbf{s}$ is then the partial derivative with respect to p of the x-component of $\mathbf{J}_p$ in which x has been replaced by the proper function of p, y, z, and t, and similarly for the other components. Thus, with $\mathbf{s}$ a known function, p or n may be considered to satisfy the first-order partial differential equation obtained by substituting from (7) in (6), from which it is evident that $\mathbf{s}$ is the velocity with which concentration transients are propagated[18]. This velocity, which is here called the differential

[18] The identification of $\mathbf{s}$ as this propagation velocity follows the example of C. Herring, in whose method for solving the transient constant-current problem in one dimension the velocity depends in a known manner on concentration only, through the neglect of diffusion, so that the general solution of the differential equation in which thus neither independent variable x nor t occurs explicitly may be obtained; cf. reference 12, pp. 412 ff.

transport velocity and loosely referred to as the transport velocity of added carriers, of course differs in general from the transport velocity proper, defined as the ratio of flow density to concentration; its general definition, which is applicable to the steady state, has been introduced to facilitate later interpretations.

2.3 *Reduction of the fundamental equations to dimensionless form*

2.31 *The general case*

In order to obtain solutions in forms which exhibit such generality as they may possess, the fundamental equations are to advantage written in terms of dimensionless dependent and independent variables which are the original variables measured in suitable units. Through formal consideration of the equations (1), in conjunction with (3) or with (4) and (6), these units can be so chosen that the system of reduced equations will exhibit independent parameters on which it may be considered to depend. The best choice of suitable units is by no means unique; those choices which have been made are natural ones, in that they have been found to result in greater formal simplicity and ease of interpretation in the theory than others which may be equally valid in principle.

The choice for an n-type semiconductor consists in definitions of dimensionless variables and parameters as follows:

$$
(8)\quad \left[\begin{array}{l}
X \equiv x/L_p,\ Y \equiv y/L_p,\ Z \equiv z/L_p;\ L_p \equiv \left[\dfrac{kT\mu\tau}{e}\right]^{\frac{1}{2}} = [D_p\tau]^{\frac{1}{2}} \\
U \equiv t/\tau \\
P \equiv p/(n_0 - p_0);\ P_0 \equiv p_0/(n_0 - p_0) = g_0\tau/(n_0 - p_0) \\
N \equiv n/(n_0 - p_0);\ N_0 \equiv n_0/(n_0 - p_0) \\
\mathbf{C} \equiv \mathbf{I}/I_0 = \mathbf{E}_a/E_0 = \mu\mathbf{E}_a/[D_p/\tau]^{\frac{1}{2}};\ I_0 \equiv \sigma_0 E_0;\ E_0 \equiv kT/eL_p \\
\mathbf{C}_p \equiv \mathbf{I}_p/I_0 \\
\mathbf{C}_n \equiv -\mathbf{I}_n/I_0 \\
\mathbf{F} \equiv \mathbf{E}/E_0 \\
W \equiv V/E_0L_p = eV/kT \\
Q \equiv \tau/\tau_p\,.
\end{array}\right.
$$

The rectangular cartesian space coordinates are x, y, and z. The quantity τ is the mean lifetime of holes for concentrations of added holes small compared with the thermal-equilibrium electron concentration, n_0; and σ_0 is the conductivity of the normal semiconductor. The hole mobility,

originally μ_p, is denoted by μ for simplicity. If b is the ratio of electron to hole mobility, σ_0 is given in general by

$$(9) \quad \sigma_0 = \mu e(bn_0 + p_0) = M_0 \, b\mu e(n_0 - p_0), \qquad M_0 \equiv 1 + \frac{b+1}{b} P_0,$$

the symbol M_0 being introduced for brevity. If $p_0 << n_0$, M_0 is unity and σ_0 equals $b\mu e n_0$.

The independent dimensionless distance variables are X, Y and Z, where the distance unit, L_p, is a diffusion length for a hole for the mean lifetime, τ, the diffusion constant for holes being D_p. This mean lifetime is the unit for the independent dimensionless time variable, U. The hole and electron concentrations are measured in units of the excess in concentration of electrons over holes[19], $n_0 - p_0$, the reduced variables being P and N, respectively. The reduced total current $\mathbf{C}$ is total current density measured in units of the current density I_0 which flows in the semiconductor with no added carriers under the characteristic field E_0, which is a field such that a carrier would expend the energy kT in drifting with it through the distance L_p. A more illuminating alternative description is that $\mathbf{C}$ is the ratio of the average drift velocity of holes under the applied or asymptotic field, $\mathbf{E}_a$, to the hole diffusion velocity $(D_p/\tau)^{\frac{1}{2}}$. The field $\mathbf{E}_a$ is that which produces the current density $\mathbf{I}$ in the semiconductor with no added carriers. The corresponding reduced hole and electron flow densities are $\mathbf{C}_p$ and $\mathbf{C}_n$. The electrostatic field measured in units of E_0 is denoted by $\mathbf{F}$, and W is the corresponding reduced electrostatic potential. The lifetime ratio Q is a function of P which characterizes the recombination process. While it appears from experiment that the recombination rate for holes depends on both physical and chemical properties of the semiconductor, in a particular semiconductor at given temperature it may be considered to depend on hole concentration alone.

Representative values for germanium of units in terms of which the dimensionless quantities are defined are as follows: The mean lifetime τ may be of the order of 10^{-5} sec. With a mobility for holes[7] of 1700 cm^2 volt^{-1} sec^{-1} in germanium single crystals at 300 deg abs, the length L_p is then about $2 \cdot 10^{-2}$ cm; the characteristic field, E_0, 1.2 volt cm^{-1}; and the current density I_0, 0.12 ampere cm^{-2} for a resistivity of 10 ohm cm.

With these definitions[20], the fundamental equations for a region free from external sources, neglecting changes in the concentrations of ionized

[7] loc. cit.

[19] The excess in concentration of electrons over holes is of course equal to that of ionized donors over ionized acceptors.

[20] The definitions given appear best if there is a region in which $P - P_0$ is small, with $P_0 \neq 0$. Modified definitions of the reduced flow densities, in which the conductivity σ_0 is replaced by the conductivity $b\mu e(n_0 - p_0)$ due to the excess electrons alone, result in equations obtainable formally by setting M_0 equal to unity.

donors and acceptors and neglecting space charge[21], are given in reduced form as follows:

$$(10)\quad \begin{cases} \dfrac{\partial P}{\partial U} = -[bM_0 \operatorname{div} \mathbf{C}_p + PQ - P_0] \\[2ex] \dfrac{\partial N}{\partial U} = -[bM_0 \operatorname{div} \mathbf{C}_n + PQ - P_0] \\[2ex] \mathbf{C}_p = \dfrac{1}{bM_0}[\mathbf{F}P - \operatorname{grad} P] = -\dfrac{1}{bM_0} P \operatorname{grad}[W + \log P] \\[2ex] \mathbf{C}_n = \dfrac{1}{M_0}[-\mathbf{F}N - \operatorname{grad} N] = -\dfrac{1}{M_0} N \operatorname{grad}[-W + \log N] \\[2ex] (P - P_0) - (N - N_0) = P - N + 1 = 0 \\[1ex] \mathbf{F} = -\operatorname{grad} W, \end{cases}$$

and the reduced form of equations (5) is

$$(11)\qquad \mathbf{C}_p - \mathbf{C}_n = \mathbf{C}.$$

These reduced equations may be simplified and two differential equations in the dependent variables P and W written as follows:

$$(12)\quad \begin{cases} -b\, M_0 \operatorname{div} \mathbf{C}_p = \operatorname{div} P \operatorname{grad}[W + \log P] = [PQ - P_0] + \dfrac{\partial P}{\partial U} \\[2ex] \operatorname{div} \mathbf{C} = 0, \qquad \mathbf{C} = -\Sigma \operatorname{grad}\left[W - \dfrac{b-1}{b+1}\log \Sigma\right], \end{cases}$$

where Σ is the conductivity σ in reduced form:

$$(13)\qquad \Sigma \equiv \frac{\sigma}{\sigma_0} = \frac{bN + P}{bN_0 + P_0} = \frac{1}{M_0}\left[1 + \frac{b+1}{b} P\right].$$

An alternative formulation, due to R. C. Prim, which is obtained by evaluating div $[\mathbf{C}_n \pm b\, \mathbf{C}_p]$, consists of the two equations,

$$(14)\qquad \begin{aligned} \frac{b}{b-1} \operatorname{div}(1 + 2P) \operatorname{grad} W &= -\frac{b}{b+1} \operatorname{div} \operatorname{grad}[W - (1 + 2P)] \\ &= [PQ - P_0] + \frac{\partial P}{\partial U}, \end{aligned}$$

[21] It may be desirable to take space charge into account in cases involving high frequencies or high resistivities. Poisson's equation and equations (3) are in reduced form,

$$P - N + 1 = bM_0\Gamma \operatorname{div} \mathbf{F} \text{ and } \mathbf{C}_p - \mathbf{C}_n = \mathbf{C} - \Gamma \frac{\partial \mathbf{F}}{\partial U}, \text{ where } \Gamma \equiv \epsilon/4\pi\sigma_0\tau.$$

The term containing Γ may often be omitted from one of these equations, depending on the nature of the particular case considered.

in which the use of $(1 + 2P)$ as dependent variable may be desirable. This variable is equal to the concentration of carriers of both kinds divided by the excess of electron concentration over hole concentration, which is a constant.

The expression in the equations which specifies the recombination rate may be written more simply. Since the lifetime ratio Q is unity for $P = P_0$,

$$PQ - P_0 = (P - P_0)R, \tag{15}$$

where R, which will be called the recombination function, depends on P and also equals unity for $P = P_0$. The lifetime ratio and the recombination function which, of course, differ in general, both equal unity for the case of constant mean lifetime. Recombination of holes and electrons at a rate proportional to the product of their concentrations, called mass-action recombination, and recombination characterized by a constant mean lifetime for holes are frequently of interest. For a combination of independent mechanisms of both types, it is easily seen that

$$\begin{cases} Q \equiv \tau/\tau_p = 1 + a\,(p - p_0)/n_0 = 1 + a\,(P - P_0)/(1 + P_0), \\ \qquad\qquad\qquad\qquad\qquad\qquad a \equiv \tau/\tau_v\,, 0 \leq a \leq 1 \\ R = 1 + ap/n_0 = 1 + aP/(1 + P_0), \end{cases} \tag{16}$$

where τ_v is the mean lifetime for small concentrations associated with mass-action recombination alone, so that $a = 0$ for constant mean lifetime, and $a = 1$ for mass-action recombination. If both recombination mechanisms are operative, that of mass-action recombination will, of course, determine the mean lifetime where the concentration of added carriers is sufficiently large.

Recent experiments have shown that the mean lifetime for holes in n-type germanium can be increased materially, to at least 100 microseconds, by minimizing surface recombination through decreases in surface-to-volume ratios.[1] On the other hand, comparatively short mean lifetimes, of the order of one microsecond, occur in p-type germanium produced, for example, from n-type by nucleon bombardment. It should be possible to determine in various cases which recombination law would provide the better approximation by use of the technique of H. Suhl and W. Shockley of hole injection in the presence of a magnetic field[22] or by the photoelectric technique of F. S. Goucher[23].

[1] loc. cit.
[22] H. Suhl and W. Shockley, *Phys. Rev.* 75 (10), 1617–1618; 76 (1), 180 (1949).
[23] F. S. Goucher, paper I 11 of the Oak Ridge Meeting of the American Physical Society, March 18, 1950; *Phys. Rev.* 78 (6), 816 (1950).

It appears that solutions neglecting recombination furnish useful approximations for some applications. If recombination is neglected, by assuming that the mean lifetime is infinite, the definitions (8) of the dimensionless quantities no longer have meaning, but essentially the same differential equations and corresponding boundary-condition equations can still be used. The reduced equations become essentially homogeneous in τ for τ large, and it suffices to suppress the recombination terms, $PQ - P_0$, retaining formally the definitions of the dimensionless quantities in which now τ, and thus L_p and E_0 or I_0 no longer have physical significance. One of these unitary quantities may be chosen arbitrarily. It might be noted that if Poisson's equation is retained the length unit is advantageously chosen as L_d, which gives a dielectric relaxation time for the time unit.[21]

In one cartesian dimension, with total current a function of time only, W may be eliminated by means of the equation for $\mathbf{C}$ in (12) and, upon substituting for it in any of the three remaining equations in (12) and (14), a differential equation for P results which depends on b, P_0, and $\mathbf{C}$ as parameters. Dropping vector notation, this equation is

$$\frac{\partial P}{\partial U} = \frac{\left[(1+2P)\left(1+\frac{b+1}{b}P\right)\right]\frac{\partial^2 P}{\partial X^2} + \frac{b-1}{b}\left[\frac{\partial P}{\partial X}\right]^2 - M_0 C\frac{\partial P}{\partial X}}{\left[1+\frac{b+1}{b}P\right]^2} - (P - P_0)R. \tag{17}$$

Similarly, from (10),

$$\begin{cases} C_p = \dfrac{M_0 CP - (1+2P)\dfrac{\partial P}{\partial X}}{bM_0\left[1+\dfrac{b+1}{b}P\right]} \\[2ex] F = \dfrac{M_0 C - \dfrac{b-1}{b}\dfrac{\partial P}{\partial X}}{1+\dfrac{b+1}{b}P}. \end{cases} \tag{18}$$

The expressions for F and C_p possess some interesting features. That for the reduced field, F, is composed of two terms, the first of which expresses Ohm's law, since C is reduced total current density and the denominator is proportional to the local conductivity. The second term is a contribution which is directed away from a hole source, since b is greater than

unity, or since electrons are more mobile than holes. If b were equal to unity, the field would be independent of the concentration gradient. The second term thus represents a departure from Ohm's law which is due to diffusion and which is associated with the presence of current carriers of differing mobilities. It gives a non-vanishing electrostatic field for the case of zero total current. The two terms in the expression for C_p are likewise ohmic and diffusion terms, but here the diffusion term would be present even if the hole and electron mobilities were equal.

Boundary-condition relationships might be illustrated by some examples for this one-dimensional case. If it be specified that for $U > 0$ a fraction f of the total current to the right of a source at the X-origin, say, be carried by holes, then, from (18),

$$(19)\quad \frac{\partial P}{\partial X} = -M_0 \frac{b + (b+1)P}{1 + 2P}\left[f - \frac{P}{b + (b+1)P}\right]C,$$

$$X = +0, \quad U > 0.$$

The solution in an X-region to the right of the origin may be determined by this condition and an additional one. The simplest is that for the flow in the semi-infinite region, namely $P = P_0$ for $X = \infty$. This relationship holds for some finite X for an idealized non-rectifying electrode there. For the region between the source and a surface at $X = X_a$ on which there is recombination characterized by a hole transport velocity s, which is also the differential transport velocity for s constant, it is clear that $C = 0$, so that, for $X = X_a$,

$$(20)\quad C_p = -\frac{1}{M_0}\frac{(1 + 2P)}{b + (b+1)P}\frac{\partial P}{\partial X} = \frac{1}{bM_0}SP;$$

$$S \equiv s/[D_p/\tau]^{\frac{1}{2}}, \qquad s \equiv J_p/p.$$

Consistently with these examples, boundary conditions may in general be expressed as relationships between P, $\dfrac{\partial P}{\partial X}$, and the parameter C, for given values of X.

A simple transformation of dimensionless quantities serves to extend all of the analytical results which have been given for the n-type semiconductor to the p-type semiconductor: Consider the substitutional transformation which consists in replacing the original dimensional quantities for holes by the corresponding ones for electrons, and vice versa, and in replacing the electrostatic field by its negative. The original set of fundamental equations (1) is invariant under this substitution, which defines an equivalent transformation from the dimensionless quantities of

equations (8) to the desired new set, in which the ratio b of electron to hole mobility is replaced by its reciprocal.

2.32 *The intrinsic semiconductor.*

For the intrinsic semiconductor, in which $p_0 = n_0$, the reduced concentrations given in (8) are inapplicable. As p_0 approaches n_0, these reduced concentrations increase indefinitely, and the equations which those given for the n-type semiconductor approach in the limit are homogeneous in the concentration unit. These limiting equations therefore apply to the intrinsic semiconductor in terms of a concentration unit which may be chosen arbitrarily. The quantity n_0 will be chosen as this unit. Thus, redefining the reduced concentration variables as

$$P \equiv p/n_0, \qquad N \equiv n/n_0; \qquad P = N, \tag{21}$$

from equations (12) and (14) any two of the equations in the dependent variables P and W given by

$$(22) \begin{cases} -(b+1) \operatorname{div} \mathbf{C}_p = \dfrac{2b}{b-1} \operatorname{div} P \operatorname{grad} W \\ \qquad = \dfrac{2b}{b+1} \operatorname{div} \operatorname{grad} P = [PQ - 1] + \dfrac{\partial P}{\partial U}, \\ \mathbf{C}_p = -\dfrac{1}{b+1} P \operatorname{grad} [W + \log P]; \\ \operatorname{div} \mathbf{C} = 0, \qquad \mathbf{C} = -P \operatorname{grad}\left[W - \dfrac{b-1}{b+1} \log P\right], \end{cases}$$

and including the right-hand member which is common at least once, characterize the intrinsic semiconductor[24].

It is noteworthy that one of these equations contains only P as dependent variable, W being absent; and this equation indicates that the spatial distribution of carrier concentration is not subject to drift under the field, but only to a diffusion mechanism with diffusion constant $2D_pD_n/(D_p + D_n)$, where $D_n = bD_p$ is the diffusion constant for electrons.[25] This result is readily accounted for as being due to a conductivity in the intrinsic case which is everywhere proportional to the concentration of carriers of either type, so that $\Sigma = P$. The expression for $\mathbf{C}$

[24] These equations for the intrinsic case were first derived quite unambiguously as those for the special case of the parameter p_o/n_o equal to unity in the general equations written in terms of the concentration unit n_o. This unit is, however, less advantageous than $(n_o - p_o)$ which, in obviating much of the formal dependence on p_o, makes for greater generality.

[25] The equations for the intrinsic case might be written in somewhat simpler form by redefining the length unit in terms of $2D_pD_n/(D_p + D_n)$ as a diffusion constant instead of D_p, but their relationship to those of the general case would then be less evident.

in (22) owes its special form simply to this circumstance, while that for $\mathbf{C}_p$ applies also to the general case, and the differential equation in P is a consequence of the equations in P and W from div $\mathbf{C}$ and div $\mathbf{C}_p$. Or, in more detailed terms, since the ohmic contribution to $\mathbf{C}_p$ must be proportional to $\mathbf{C}$, div $\mathbf{C}_p$ contains only the contribution due to diffusion. This is evident from the relationship obtained from (22),

$$\mathbf{C}_p = \frac{1}{b+1}\left[\mathbf{C} - \frac{2b}{b+1}\,\text{grad}\,P\right], \tag{23}$$

from which it follows also that, despite the dependence of the local field on concentration gradient, the ohmic contribution to the hole flow density is the flow density of holes normally present in the intrinsic semiconductor under the unmodulated applied field.

The equations which have been given for one-dimensional flow in the n-type semiconductor can readily be transformed, in the manner indicated, into the corresponding equations for the intrinsic semiconductor.

2.4 *Differential equations in one dimension for the steady state of constant current and properties of their solutions*

The steady state of constant current in one dimension will be considered explicitly for two limiting cases: the n-type semiconductor with $P_0 = 0$, and the intrinsic semiconductor. These serve to illustrate and delimit the qualitative features of the general case. Furthermore, the case $P_0 = 0$ frequently applies as a good approximation[26], as does the intrinsic case, which is of particular interest not only in itself but also because the extrinsic semiconductor exhibits intrinsic behavior for large concentrations, and because moderate increases in temperature above room temperature, such as joule heating may produce, suffice to bring high back voltage germanium into the intrinsic range of conductivity[27]. The temperature dependence of P_0 and of other reduced quantities is evaluated for germanium in the Appendix.

The ordinary differential equations in the reduced hole concentration, P, for the steady state in one dimension, which result from equations (17) and (22) by equating the time derivatives to zero are as follows:

$$\frac{d^2P}{dX^2} = \frac{C\dfrac{dP}{dX} - \dfrac{b-1}{b}\left[\dfrac{dP}{dX}\right]^2}{[1+2P]\left[1+\dfrac{b+1}{b}P\right]} + \frac{P\left[1+\dfrac{b+1}{b}P\right]}{1+2P}R \tag{24}$$

[26] In n-type germanium of resistivity about 5 ohm cm, for example, the electron concentration exceeds the equilibrium hole concentration by a factor of about 70.

[27] Germanium which is substantially intrinsic at room temperature has been produced: R. N. Hall, paper I5 of the Oak Ridge Meeting of the American Physical Society, March 18, 1950.

for the n-type semiconductor with $P_0 = 0$, and

$$\frac{d^2P}{dX^2} = \frac{b+1}{2b}(P-1)R \tag{25}$$

for the intrinsic semiconductor, with R given as $(1 + aP)$ by (16); P has the same meaning in both equations, the concentration unit being n_0 for each case. With time variations excluded in this way, the parameter C is a constant and the differential equations apply to the steady state of constant current.

Since the equations involve only the single independent variable X which does not appear explicitly, their orders may be reduced by one, in accordance with a well-known transformation, which consists in introducing P as a new independent variable, and

$$G \equiv \frac{dP}{dX} \tag{26}$$

as new dependent variable: Noting that $\frac{d}{dX}$ is equivalent to $G\frac{d}{dP}$, the differential equations become

$$\frac{dG}{dP} = \frac{C - \frac{b-1}{b}G}{[1+2P]\left[1 + \frac{b+1}{b}P\right]} + \frac{P\left[1 + \frac{b+1}{b}P\right]R}{[1+2P]G} \tag{27}$$

for the n-type semiconductor, and

$$\frac{dG}{dP} = \frac{b+1}{2b}\frac{(P-1)R}{G} \tag{28}$$

for the intrinsic semiconductor. These are differential equations of the first order.

The solutions sought in the semi-infinite region, $X > 0$, are those for which $G = 0$ for $\Delta P = 0$, that is, those which pass through the $(\Delta P, G)$ — origin, where ΔP, which denotes $P - P_0$, equals P for the n-type semiconductor and $P-1$ for the intrinsic semiconductor. This condition is that the concentration gradient vanish with the concentration of added holes, as it must for X infinite. It will be shown that the differential equations possess singular points at the $(\Delta P, G)$-origin, and the physical interpretation of the solutions through these singular points will be examined. For this purpose, consider equation (27) for the n-type semiconductor which, in the neighborhood of the origin, assumes the approximate form,

$$\frac{dG}{dP} = \frac{G}{P} = C + \frac{P}{G}, \tag{29}$$

since R is close to unity for P small, whence

$$\frac{dG}{dP} = \frac{G}{P} = \frac{1}{2}\left[C \pm \sqrt{C^2 + 4}\right]. \tag{30}$$

Similarly, for the intrinsic semiconductor, for $P-1$ small,

$$\frac{dG}{d(P-1)} = \frac{G}{P-1} = \pm\sqrt{\frac{b+1}{2b}}. \tag{31}$$

There are thus, in each case, two solutions through the $(\Delta P, G)$-origin, one with a positive derivative and the other with a negative derivative. Consider now the doubly-infinite region with a source at $X = 0$. Then, for $X > 0$, the negative derivatives apply, since the concentration gradient G is negative. Similarly, for $X < 0$, the positive derivatives apply. Now, the value of the current parameter C will be substantially the same in both regions, since it has been assumed that ΔP is small. For C positive, equation (30) for the n-type semiconductor indicates that the magnitude of dG/dP for $X < 0$ exceeds that for $X > 0$, and the situation is reversed if the sign of C is changed. That is, the magnitude of the concentration gradient increases more slowly with concentration for field directed away from a source than for field directed towards a source, which is otherwise plausible. For the intrinsic semiconductor, on the other hand, equation (31) shows that corresponding magnitudes of the concentration gradient are equal and entirely independent of C, a result which the differential equation (28) establishes in general.

It thus appears that a differential equation for the steady state possesses two solutions through the $(\Delta P, G)$-origin, and that one of the solutions corresponds to the case of field directed towards a source, the other to the case of field directed away from a source. Field directed towards a source is called field opposing, while field directed away from a source is called field aiding, the latter being the one commonly dealt with in hole-injection experiments. It should be noted that the cases of field opposing or field aiding can be realized in a given X-region only if it adjoins a semi-infinite region free from sources and sinks. In the region between two sources, neither of these cases applies. L. A. MacColl has shown, through a more detailed consideration of the singularity at the $(\Delta P, G)$-origin, that the two solutions through this point are the only ones through it. The origin is thus a saddle-point of the differential equation, and there exist families of nonintersecting solutions in the $(\Delta P, G)$-plane for which the solutions which intersect at the origin are asymptotes. A solution for an X-region between two sources, for example, is a member of such a family, as is in general any solution determined by boundary conditions at the ends of a finite region in X. Such a solution will be called a solution for a composite case; it approaches asymptotically both a field-opposing and a

field-aiding solution, which is consistent with the qualitative geometry associated with a saddle-point, and with the fact that, in the X-region, a

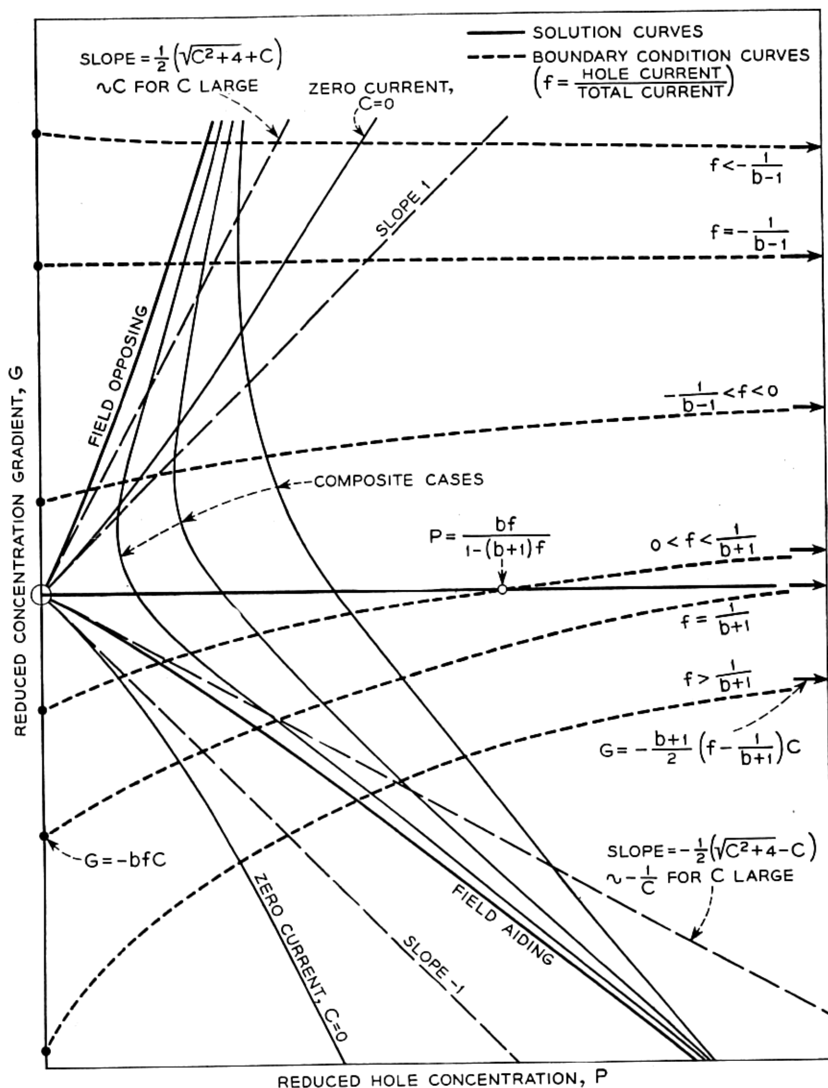

Fig. 1.—Diagrammatic representation in the (P, G)-plane of solutions and boundary conditions for the steady-state one-dimensional flow of holes in an n-type semiconductor.

total current directed away from one source is necessarily directed towards the other. This behavior is illustrated diagramatically for the n-type semiconductor in Fig. 1, which shows, in the (P, G)-plane, solution

curves as well as boundary-condition curves for a source, for a given positive value of C. Those for the intrinsic semiconductor differ only in that the solution curves in the $(P-1, G)$-plane do not depend on C, all being given by the ones for zero total current density, and the corresponding boundary-condition curves are straight lines.

Once a solution, $G(P)$, for field opposing, field aiding, or a composite case, specifying G as a function of P has been obtained, the dependence of P on X is determined by evaluating

$$X = \int_{P^0}^{P} \frac{dP}{G(P)} \tag{32}$$

in accordance with the definition of G, equation (26). For the general composite case, $G(P)$ is that one of the family of solutions for the given C such that the integral between values of P determined by the intersections with the boundary-condition relationships provides the correct interval in X. If P^0 is determined by the condition that for $X = 0$, a fraction f of the total current is carried by holes, then, from (19), P^0 is the point on the solution curve which satisfies either

$$G^0 = -\frac{b + (b+1)P^0}{1 + 2P^0}\left[f - \frac{P^0}{b + (b+1)P^0}\right]C \tag{33}$$

for the n-type semiconductor, or

$$G^0 = -\frac{(b+1)^2}{2b}\left[f - \frac{1}{b+1}\right]C \tag{34}$$

for the intrinsic semiconductor, G^0 being the corresponding value of G.

From the manner of derivation of the boundary conditions (33) and (34), it is evident that they are perfectly general, holding in particular for the cases of field opposing and field aiding, and whatever be the sign of C. The concentration gradient G^0 may be seen to have the correct sign for these cases if it is taken into account that f, defined as C_p/C or I_p/I, may assume any positive or negative value, being positive for field aiding, and negative for field opposing, for which the hole flow is opposite to the applied field. For f negative, the quantities in brackets in equations (33) and (34) are negative. The general principle that the sign of the concentration gradient G is such as to be consistent with the flow of holes from a source requires also that the quantities in brackets be positive for field aiding, or whenever f is positive. For the intrinsic semiconductor, this requires that f for field aiding never be less than $1/(b+1)$. This is clearly a consistent requirement which holds in all generality since, for zero concentration of added holes, or for the normal semiconductor, G^0 vanishes and the ratio of hole current to total current equals $1/(b+1)$.

In the case of the n-type semiconductor, f is not restricted in this way. Consider, for this case, hole injection into the end of a semi-infinite filament, to which the field-aiding solutions apply. As the total current is increased indefinitely, the tangent to the solution in the (P, G)-plane at the origin approaches the P-axis, as does the solution itself, and it is evident from the boundary-condition curves of Fig. 1 that if f is less than $1/(b + 1)$ the hole concentration P^0 at the source approaches as a limit the indicated abscissa of intersection of the appropriate boundary-condition curve with the P-axis, or the value for which the quantity in brackets vanishes. It is similarly evident that P^0 increases indefinitely with total current in either semiconductor if f is greater than or equal to $1/(b + 1)$. This is a result otherwise to be expected from the qualitative consideration that an extrinsic semiconductor becomes increasingly intrinsic in its behavior as the concentration of injected carriers is increased.

Figure 1 serves also to facilitate a count of the number of degrees of freedom which the steady-state solutions possess: Corresponding to values of the concentration and concentration gradient at a point in a semiconductor filament in which added carriers flow, there is a point (P, G) in the half-plane, $P > 0$, of the figure. If the total current density is specified in addition, the value of C and the solution through the point (P, G) are determined. This solution applies in general to a composite case, which therefore possesses three degrees of freedom. That is to say, at a point in a filament, any given magnitudes of both concentration and concentration gradient can be realized for a preassigned total current density by a suitable disposition of sources to the right and left. The cases of field opposing or field aiding, however, possess only two degrees of freedom, since the given concentration and gradient determine the total current density and the solution, which must pass through the origin; and which of the two cases applies depends on whether the point (P, G) lies to the left or to the right of the curves, shown in the figure, for the zero-current solution. Thus, in a filament with a single source of holes, for example, the concentration, concentration gradient, total current density, and any functions of these, such as hole flow density and electrostatic field, are all quantities the specification of any two of which at a point completely determines the solution for a source-free X-region which includes the point.

3. Solutions for the Steady State

For a given value of the current parameter C, solutions for the steady state of constant current in a single cartesian distance coordinate, specifying G in terms of the relative hole concentration P, and P, the reduced hole

flow density C_p, and the reduced electrostatic field F, in terms of reduced distance X are found in general by numerical means, which include numerical integration and the evaluation of appropriate series expansions.

General solutions which have been evaluated numerically for n-type germanium for a number of values of the current parameter are given in the figures. In the limiting cases of P small and P large, analytical approximations for the extrinsic semiconductor are readily obtained, that for P large being derived from an analytical solution for C equal to zero, or zero current. If the steady-state problem for the extrinsic semiconductor is simplified by neglecting either recombination or diffusion, solutions are obtainable which, like the zero-current one, are expressible in closed form.

For the intrinsic semiconductor, the general problem considered in this section is solved quite simply by analytical means. The solution provides, as physical considerations indicate it should, the same analytical approximation for large P as does the zero-current solution for the extrinsic case. It may be well to consider first the intrinsic semiconductor which, aside from the extrinsic semiconductor for the case of zero current, appears to constitute the only analytically solvable steady-state case in one dimension which has physical generality according to the present approach.

3.1 *The intrinsic semiconductor*

Integrating the differential equation (28), it is found that

$$G^2 = \frac{b+1}{b} \int (P - 1) R dP, \tag{35}$$

with R given as $1 + aP$ by (16), for an arbitrary combination of the two recombination mechanisms, assumed independent. Thus

$$G^2 = \frac{b+1}{2b} (P - 1)^2 \left[(1 + a) + \frac{2}{3} a(P - 1) \right] \tag{36}$$

for the cases of field opposing or field aiding, for which $G = 0$ for $P - 1 = 0$; for a composite case, a suitable constant is included on the right-hand side. Excluding composite cases, the root may be taken in (36) and G replaced by its definition, which gives

$$\frac{d(P-1)}{dX} = \pm \left[\frac{b+1}{2b} \right]^{\frac{1}{2}} [P - 1] \left[(1 + a) + \frac{2}{3} a(P - 1) \right]^{\frac{1}{2}} \tag{37}$$

and if the X-origin is selected more or less arbitrarily as the point at which P is infinite, then (37) gives

$$P - 1 = \frac{3(1 + a)}{2a} \operatorname{csch}^2 \left[\frac{(1 + a)(b + 1)}{8b} \right]^{\frac{1}{2}} X \tag{38}$$

provided $a \pm 0$; for mass-action recombination $a = 1$. For $a = 0$ or for constant mean lifetime, (37) gives an exponential dependence of $P - 1$ on X:

$$P - 1 = (P^0 - 1) \exp\left[\pm \left[\frac{b+1}{2b}\right]^{\frac{1}{2}} X\right], \tag{39}$$

where P_0 is the relative hole concentration for $X = 0$. Linear combinations of the two solutions in (39) give solutions for composite cases, since the differential equation from which (39) was derived is linear in P. A similar result does not hold if there is mass-action recombination present, and the more general procedure above referred to must then be followed.

A characteristic feature of these solutions for the intrinsic semiconductor is their independence of the current parameter C, this parameter occurring only through a boundary condition, such as the one given in equation (34) of Section 2.4. They are symmetrical in shape about a source, the dependence of the concentration on the magnitude of the distance from the source being the same for field opposing as for field aiding, which follows quite simply from the symmetrical forms of the solutions, and the condition that the concentration is everywhere continuous.

Equations (22) and (23) of Section 2.32 provide the hole flow density and the electrostatic field for this case. With G given for mass-action recombination or for constant mean lifetime by the appropriate special case of equation (36), and using the positive sign for an X-region to the left of sources and the negative sign for an X-region to the right,

$$\begin{cases} C_p = \dfrac{1}{b+1}\left[C - \dfrac{2b}{b+1} G\right] \\ F = \dfrac{1}{P}\left[C - \dfrac{b-1}{b+1} G\right]. \end{cases} \tag{40}$$

The electrostatic potential, V, is readily expressed in terms of P: From

$$F = -\frac{eL_p}{kT}\frac{dV}{dx} = -\frac{e}{kT}\frac{dV}{dX} = -\frac{e}{kT} G \frac{dV}{dP} \tag{41}$$

and (40), it is found that

$$\frac{e}{kT}\frac{dV}{dP} = \frac{b-1}{b+1}\frac{1}{P} - C\frac{1}{GP}, \tag{42}$$

whence

$$\frac{eV}{kT} = \frac{b-1}{b+1}\log P - C\int \frac{dP}{GP} = \frac{b-1}{b+1}\log P - C\int \frac{dX}{P}, \tag{43}$$

with the integral to be evaluated for the particular case it is desired to consider.

3.2 *The extrinsic semiconductor: n-type germanium*

The evaluation of steady-state solutions for the extrinsic semiconductor involves, as a first step, the determination of G as a function of P from the differential equation (27), which is accomplished by numerical integration and by the use of series expansions. These variables are subsequently found in terms of X in the manner described in Section 2.4. The series expansions, which are Maclaurin's series in P, and series in powers of the current parameter, C, with coefficients functions of P, are given explicitly for the n-type semiconductor in the Appendix; they readily furnish the corresponding series for the p-type semiconductor by means of the transformation discussed at the end of Section 2.31. The Maclaurin's series in P are useful for starting the solutions at the (P, G)-origin. As P increases, these series converge increasingly slowly, and it becomes necessary to extend the solutions by other means. For the larger values of C, however, the numerical integration for the important case of field aiding becomes increasingly difficult, and it is advantageous to use the appropriate series in the current parameter, which converges the more rapidly the larger is C. The first term alone in this series for field aiding gives in closed form the solution for the case in which diffusion is neglected; and the existence of the series itself was, in fact, originally suggested by the form of the solution for this case[28]. Series of this type are given also for field opposing, and it seems probable that such series are obtainable for composite cases as well, though this has not been investigated.

Solutions were evaluated numerically for n-type germanium, by the means described, using the value 1.5 for the mobility ratio[29], b. For the case of mass-action recombination, solutions for values of the current parameter, C, up to 50, specifying $|G|$ in terms of P, are given in Fig. 2, both for field opposing and field aiding. These solutions in the (P, G)-plane are given to permit the fitting of boundary conditions at a hole source, according to a method described in Section 4. Solutions specifying P in terms of X for field aiding are given in Fig. 3, with the X-origin chosen more or less arbitrarily at $P = 100$. The corresponding solutions for the reduced hole flow density, C_p, and the reduced field, F, are given

[28] The solution for this case was communicated by Conyers Herring and is given in his paper of reference 12.

[29] The hole mobility and the value 1.5 for the mobility ratio were determined by G. L. Pearson from the temperature dependence of the conductivity and Hall coefficient in p-type germanium. J. R. Haynes has recently obtained, from drift-velocity measurements, the same hole mobility, but the larger value 2.1 for the ratio of electron mobility in n-type germanium to hole mobility in p-type: Paper L2 of the Chicago Meeting of the American Physical Society, November 26, 1949.

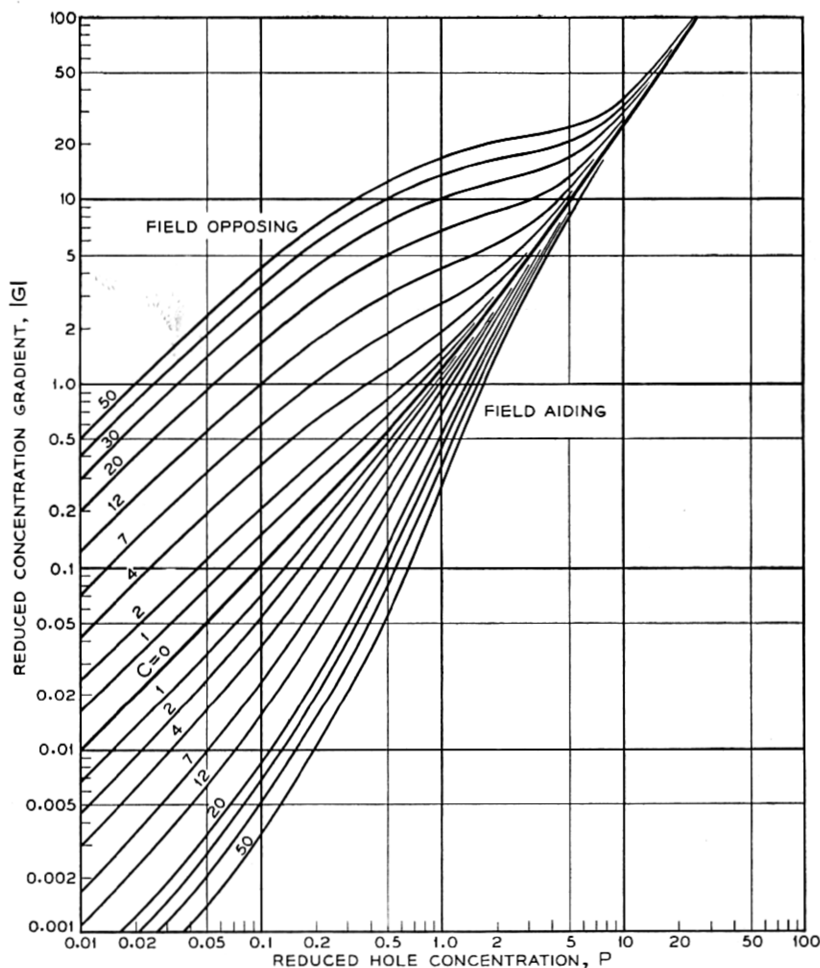

Fig. 2.—The dependence of the reduced concentration gradient on reduced concentration for the steady-state one-dimensional flow of holes with mass-action recombination in n-type germanium.

respectively in Fig. 4 and in Fig. 5. In accordance with equations (10), (18), and (26), the solutions for C_p and F are found from

$$C_p = \frac{1}{b}(FP - G) = \frac{CP - (1 + 2P)G}{b + (b + 1)P}, \tag{44}$$

and

$$F = \frac{C - \dfrac{b - 1}{b} G}{1 + \dfrac{b + 1}{b} P}. \tag{45}$$

The electrostatic potential may be evaluated from F in a manner similar to that followed in the preceding section.

3.3 *Detailed properties of the solutions*

The general solutions given in the figures illustrate certain properties which can be established through the analytical approximations obtainable for small and for large values of the relative concentration of added holes. The principal qualitative properties evident from the figures are:

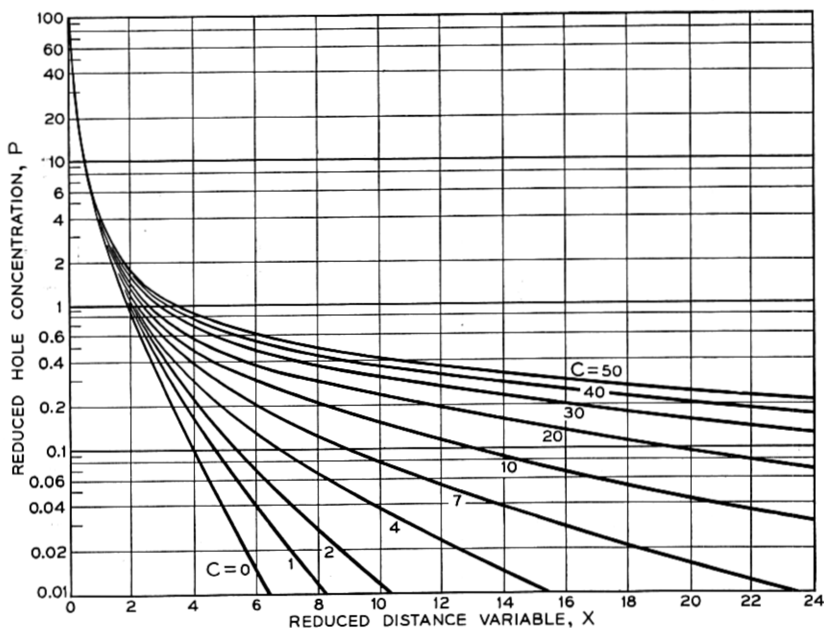

Fig. 3.—The dependence of the reduced concentration on reduced distance for the steady-state one-dimensional flow of holes with mass-action recombination in n-type germanium.

The relative hole concentration, P, and the reduced hole current, C_p, depend exponentially on distance for small concentrations; and for large concentrations all solutions for a given dependent variable run together, independently of the value of the current parameter, and give comparatively rapid variations of hole concentration and current with distance[30]. The property that a common solution independent of total current or

[30] These rapid variations would account for the observation of J. R. Haynes that estimates, for a given emitter current, of hole concentrations or currents in a filament at a point contact removed from the emitter, with no additional applied field, are largely independent of changes in f_e for the emitter.

applied field obtains for large P results from diffusion in conjunction with the increase of conductivity. As may be expected, the solutions for the case of constant mean lifetime also have this property, the recombination law merely affecting the form of the common solution.

In Fig. 6 are shown curves for P, C_p, and F for the case of constant mean lifetime in n-type germanium, evaluated for C equal to 16.3. These curves are intended to illustrate the qualitative differences between the solutions for this case and those for mass-action recombination, which are manifest

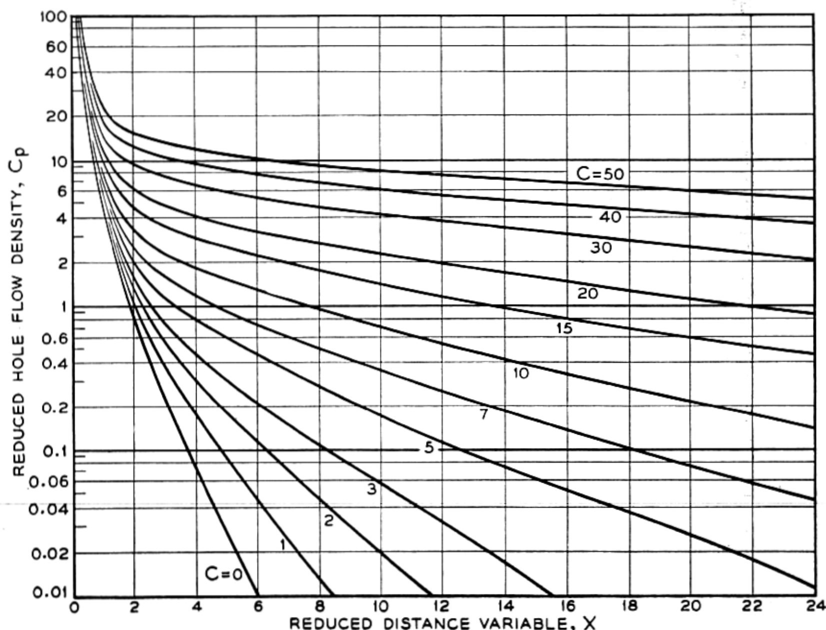

Fig. 4.—The dependence of the reduced hole flow density on reduced distance for the steady-state one-dimensional flow of holes with mass-action recombination in n-type germanium.

primarily at the larger concentrations. The dashed curves in the figure give the corresponding solutions for the case of mass-action recombination; and the X-origins for the two cases have been so chosen that corresponding curves, which exhibit essentially the same dependence on X for small P, coincide in the limit of small P. As the figure shows, constant mean lifetime gives an exponential dependence of P on X for large P, while mass-action recombination gives larger concentration gradients, with an increase of P to indefinitely large values in the neighborhood of a vertical asymptote.

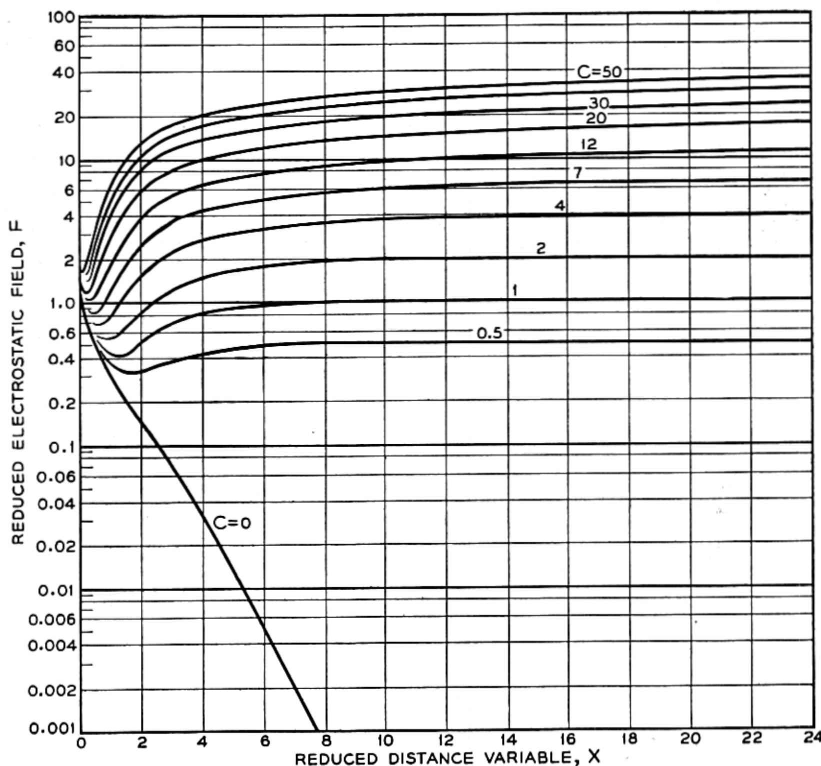

Fig. 5.—The dependence of the reduced electrostatic field on reduced distance for the steady-state one-dimensional flow of holes with mass-action recombination in n-type germanium.

3.31 *The behavior for small concentrations*

The exponential dependence of P and C_p on distance for P small is given for the n-type semiconductor by the analytical approximations,

$$(46)\quad \begin{cases} P = P_s \exp\left[-\tfrac{1}{2}[\pm\sqrt{C^2+4} - C]X\right] \\ C_p = \dfrac{1}{2b}[\pm\sqrt{C^2+4} + C]P, \end{cases}$$

where P_s is a suitable constant. If C is positive, the plus sign holds for field aiding[31] and the minus sign for field opposing. These approximate

[31] It is evident from the curves for C_p in Fig. 4 that the exponential extrapolation back to the emitter location of estimates of hole concentrations or currents at a point contact on a germanium filament lead to values of f_ϵ for the emitter which are too small. Using moderately large injected currents and no additional applied fields, J. R. Haynes once obtained in this way an apparent f_ϵ of about 0.2. From the figure, this is the apparent f_ϵ to be expected for moderate and large values of C for the true f_ϵ equal to unity.

solutions, which hold for any recombination law, are obtained quite simply, by integration, from G in terms of P to the first term of the Maclaurin's expansion, given in the Appendix. It might be noted that for this approximation the electrostatic field is equal to the applied field, so that F equals C.

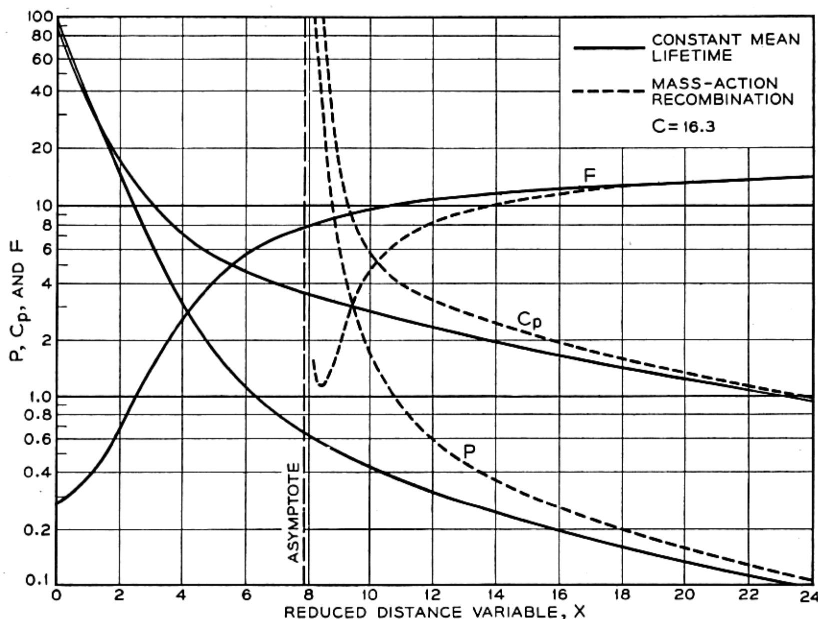

Fig. 6.—The dependence of the reduced hole concentration, hole flow density, and electrostatic field on reduced distance for steady-state one-dimensional hole flow in n-type germanium, for the cases of constant mean lifetime and mass-action recombination.

Since P is small, the transport velocity of holes is equal to their differential transport velocity[32]. Writing the equation for C_p in dimensional form, the transport velocity is found to equal

$$s = \tfrac{1}{2}[\pm\sqrt{(\mu E_a)^2 + 4D_p/\tau} + \mu E_a], \tag{47}$$

with the plus sign for field aiding and the minus sign for field opposing, if the applied field, E_a, is positive. This result is consistent with the

[32] In accordance with equations (7), (8), and (10), the differential transport velocity for the steady state in one dimension may be found from the general formula,

$$bM_0 dC_p/dP = -(P - P_0)R/G.$$

Its equalling the transport velocity proper for P small appears to result from the property of non-composite cases that the dependent variables, for a given C, are all functions of P which do not depend on any quantity determined by the boundary values, a property which composite cases, with their additional degree of freedom, do not possess.

equation for P, which may be written as

$$P = P_s \exp(-x/s\tau). \tag{48}$$

For a large aiding field, s reduces to the velocity of drift under this field while, for a large opposing field, the magnitude of s is approximately $D_p/\mu E_a\tau$. For zero field, s equals the diffusion velocity $(D_p/\tau)^{\frac{1}{2}}$, which is a diffusion distance for a mean lifetime divided by the mean lifetime. This diffusion velocity can be specified in terms of its field equivalent, or the field which gives an equal drift velocity, and for germanium it is found that the equivalent field is about 8 volt cm^{-1} for τ equal to one microsecond and about 2.5 volt cm^{-1} for τ equal to 10 microseconds.

For small concentrations of added holes in the intrinsic semiconductor, or $(P-1) << 1$, equations (38) and (40) give the approximate solutions,

$$\begin{cases} P - 1 = (P^0 - 1) \exp\left[\pm\left[\dfrac{(1+a)(b+1)}{2b}\right]^{\frac{1}{2}} X\right] \\ C_p = \dfrac{1}{b+1}\left[C - \dfrac{2b}{b+1} G\right] \sim \dfrac{1}{b+1} C, \end{cases} \tag{49}$$

the X-origin being selected arbitrarily at the point at which the relative concentration is P^0 according to the approximation. It is evident from the equation for C_p that, for $(P-1)$ small, the transport velocity is the drift velocity under the applied field, which is the velocity of the holes normally present in the semiconductor. The differential transport velocity, obtainable by differentiating the equation for C_p with respect to P and using the differential equation (28), or by writing the exponent in the equation for $(P-1)$ in the form given in (48), is, on the other hand, given by

$$s = \left[\frac{2b}{(1+a)(b+1)}\right]^{\frac{1}{2}} \left[\frac{D_p}{\tau}\right]^{\frac{1}{2}} = \left[\frac{1}{1+a}\frac{2D_pD_n}{D_p+D_n}\Big/\tau\right]^{\frac{1}{2}}, \tag{50}$$

and is a diffusion velocity. This holds for holes added in any concentration if $a = 0$, or for constant mean lifetime, since the first of equations (49) is then the general solution given in (39).

The nature of the flow for small concentrations of added carriers in the general case, which depends on the parameter P_0, is illustrated qualitatively by the n-type and intrinsic cases considered, for which P_0 is respectively zero and infinite. Solutions for the general case are easily evaluated analytically from the linear differential equation which results from (17) if $P-P_0 << \frac{1}{2} + P_0$. It can be shown from the field-aiding steady-state solution that the ratio of the differential transport velocity to the velocity, proportional to C, of drift under the applied field is for $C^2 >> (1 + 2P_0)M_0$ equal to the quantity $1/M_0$. This result is consistent

with those already derived: For large applied aiding fields, the differential transport velocity changes from the drift velocity, for P_0 equal to zero and M_0 unity, to the diffusion velocity given in (50) as P_0 and M_0 increase indefinitely.

3.32 The zero-current solutions and the behavior for large concentrations

The solutions for the intrinsic semiconductor for the current parameter equal to zero are, of course, the same as the general ones given in Section 3.1, since the current parameter does not occur in the differential equation. For the n-type semiconductor, the differential equation (27) becomes an equation of the Bernouilli type for C equal to zero, and may be solved by quadratures. It is then linear in G^2, and gives, for field aiding or field opposing,

$$G^2 = 2\left[\frac{1 + \frac{b+1}{b}P}{1 + 2P}\right]^2 \int_0^P \frac{P(1 + P)(1 + aP)}{1 + \frac{b+1}{b}P}\, dP, \tag{51}$$

expressing the recombination function R according to equation (16) for a combination of the two recombination mechanisms. Writing, for brevity,

$$\begin{cases} \beta \equiv \dfrac{b}{b+1} \\ M \equiv 1 + \dfrac{b+1}{b}P, \end{cases} \tag{52}$$

and evaluating the integral in (51), the following result is obtained:

$$\begin{aligned} G^2 = 2\beta^2\left[\frac{M}{1+2P}\right]^2 \Big[& [1 - a] \\ & \cdot [\beta(M^2 - 1) + (1 - 4\beta)(M - 1) - (1 - 2\beta)\log M] \\ & + a[\tfrac{2}{3}\beta^2(M^3 - 1) + \tfrac{3}{2}(1 - 2\beta)(M^2 - 1) \\ & + (1 - 6\beta + 6\beta^2)(M - 1) - (1 - \beta)(1 - 2\beta)\log M]\Big]. \end{aligned} \tag{53}$$

For P large, this solution gives the approximations,

$$G = \pm\left[\frac{b+1}{2b}\right]^{\frac{1}{2}} P \tag{54}$$

for constant mean lifetime, with $a = 0$, and

$$G = \pm\left[\frac{a(b+1)}{3b}\right]^{\frac{1}{2}} P^{\frac{3}{2}} \tag{55}$$

if there is mass-action recombination present, so that $a \neq 0$. The dependence of P on X for these approximations is readily obtained by integrating the differential equations which result from writing in place of G, its definition, dP/dX; constant mean lifetime gives an exponential dependence. An examination of (54) and (55) in conjunction with the general differential equation (27) shows that, for P large, the dominant term in the differential equation is independent of C. It follows that solutions for all values of C approach a common solution for P large, which is given by (54) or (55). The solutions run together appreciably for P sufficiently large that P and M are substantially proportional, that is, for P large compared with $b/(b + 1)$, which is of order unity. It is to be expected that the approximations (54) and (55) should apply equally well to the intrinsic semiconductor, and this expectation is easily verified by evaluating the integral in equation (35) for the intrinsic semiconductor, for P large, for the two recombination cases here considered.

4. Solutions of Simple Boundary-Value Problems for a Single Source

Among the boundary-value problems whose solutions are useful in the interpretation of data from experiments in hole injection are the following: the semi-infinite filament for field aiding, with holes injected at the end, which constitutes a relatively simple case; and the doubly-infinite filament with a single plane source, with which this section will be primarily concerned.

Consider first the semi-infinite filament, and suppose that it starts at the X-origin and extends over positive X, so that the current parameter is positive for field aiding. If two quantities are specified, namely the current parameter and the fraction f_ϵ of the current carried by holes at the origin or injection point, then the solution of the boundary-value problem is completely determined. It is merely necessary to select the general field-aiding solution for P or C_p in terms of X, for the particular value of the current parameter, and then to determine the X-origin, corresponding to the source, which is simply the X at which the ratio f of C_p to C equals f_ϵ.

Use in the boundary-condition equations (33) and (34) of the approximate expressions given in (54) and (55) for G in terms of P, for large P, permits the complete analytical determination of the dependence of P^0 on total current as this current is indefinitely increased. It was shown in Section 2.4 that, if f_ϵ is less than $1/(b + 1)$ for the n-type semiconductor, P^0 approaches as a limit the value for which G^0 vanishes according to the boundary-condition equation (33); in all other cases for the n-type semiconductor, or if f_ϵ exceeds $1/(b + 1)$ for the intrinsic semiconductor, P^0 increases indefinitely with C. For $f_\epsilon > 1/(b + 1)$, it is readily seen that

P^0 is proportional in the limit to C for constant mean lifetime, and to $C^{\frac{2}{3}}$ for mass-action recombination; and, for $f_\epsilon = 1/(b + 1)$ in the case of the n-type semiconductor, P^0 increases as $C^{\frac{1}{2}}$ for constant mean lifetime, and as $C^{\frac{1}{3}}$ for mass-action recombination.

Consider now the doubly-infinite semiconductor filament with a source at the origin, and suppose that the total injected current at the source is C_ϵ, in reduced form, with a fraction f_ϵ of this current carried by holes. Denote by C^- and by C^+ the reduced total currents for $X < 0$ and for $X > 0$, respectively. Since the injection of holes requires that C_ϵ be positive, at least one of C^- and C^+ must be positive, since total current is conserved. Let f^- and f^+ denote, respectively, the ratio of the hole current at the origin to the left, C_p^-, to the total current C^-, and the ratio of the hole current at the origin to the right, C_p^+, to the total current, C^+. It might be noted that, for a flow of holes to the left, say, against the field, C^- and C^+ are positive and f^- is negative, and that, if C^- is (plus) zero, f^- is (negatively) infinite, corresponding to the flow of holes under zero applied field. Now, general boundary-condition equations of the form of (33) or (34) hold with the sign conventions here employed, as indicated in Section 2.4. One may be written for the flow to the left, another for the flow to the right, making use of the condition that the relative concentration P is everywhere continuous; G exhibits a discontinuity of the first kind at the source, with a change in sign. Writing G^- for the limiting value of the reduced concentration gradient as the origin is approached from the left, and G^+ the limiting value as the origin is approached from the right, the boundary-condition equations are, for the n-type semiconductor,

$$(56)\quad \begin{cases} G^- = -\dfrac{b + (b+1)P^0}{1 + 2P^0}\left[f^- - \dfrac{P^0}{b + (b+1)P^0}\right]C^- \\[2ex] G^+ = -\dfrac{b + (b+1)P^0}{1 + 2P^0}\left[f^+ - \dfrac{P^0}{b + (b+1)P^0}\right]C^+. \end{cases}$$

For the intrinsic semiconductor, they are

$$(57)\quad \begin{cases} G^- = -\dfrac{(b+1)^2}{2b}\left[f^- - \dfrac{1}{b+1}\right]C^- \\[2ex] G^+ = -\dfrac{(b+1)^2}{2b}\left[f^+ - \dfrac{1}{b+1}\right]C^+. \end{cases}$$

There are, in addition, an equation which expresses the conservation of hole flow, and one which expresses the conservation of total current, as follows:

$$(58)\quad \begin{cases} f^+ C^+ - f^- C^- = f_\epsilon C_\epsilon \\ C^+ - C^+ = C_\epsilon\ . \end{cases}$$

The solution of the problem is determined by f_ϵ and the three parameters which specify the total currents: With these four quantities known, then, from equations (56) or (57) in conjunction with (58) and the known general solutions in the $(\Delta P, G)$-plane which apply to the left and to the right of the origin, all of the quantities P^0, G^-, G^+, f^- and f^+ can be found and the problem completely solved.

The technique of obtaining the solution depends on a simple fundamental result which may be expressed as follows:

For fixed f_ϵ and C_ϵ, consider the sum of the magnitudes of the concentration gradients at a single common source from which holes flow into a number of similar filaments in parallel, for any consistent distribution among the filaments of total currents, some of which may be produced by opposing fields. This sum is equal to the magnitude of the concentration gradient at the source if the entire flow, under the appropriate aiding field, were confined to a single filament.

The total magnitude of the concentration gradient, in this sense, is an invariant for fixed f_ϵ and C_ϵ. Specifically, for the n-type semiconductor, it follows from equations (56) and (58) that

$$G^+ - G^- = -\frac{b + (b+1)P^0}{1 + 2P^0}\left[f_\epsilon - \frac{P^0}{b + (b+1)P^0}\right]C_\epsilon. \tag{59}$$

Similarly, for the intrinsic semiconductor,

$$G^+ - G^- = -\frac{(b+1)^2}{2b}\left[f_\epsilon - \frac{1}{b+1}\right]C_\epsilon. \tag{60}$$

The left-hand sides of these equations are the negative of the sum of the magnitudes of the reduced concentration gradients, since G^- is always positive and G^+ always negative, and their right-hand sides are similar in form to those of equations (56) and (57), with the quantities f_ϵ and C_ϵ, characteristic of the source, replacing f^- and C^-, or f^+ and C^+.

The particular utility of these equations arises from their independence of the unknowns f^- and f^+. By means of equation (59) for the n-type semiconductor the evaluation of the five unknown quantities can now be effected as follows: With the current parameters known, the solutions in the (P, G)-plane to the left and right of the X-origin are determined; either both solutions are for field aiding, or else one is for field aiding and the other for field opposing. From them, the sum of the magnitudes of the reduced concentration gradients can be found as a function of P. It is also given, for the origin, as a function of the unknown P^0, by equation (59). The values of the sum for the origin and of P^0 are accordingly found as those which satisfy both relationships. The value of P^0 thus found determines both G^- and G^+ from the respective solutions

in the (P, G)-plane, and f^- and f^+ may be obtained by solving for them in equations (56).

For the intrinsic semiconductor, this method can be applied analytically, and the solution so obtained serves at the same time as an approximation for large relative hole concentrations in the n-type semiconductor, for which the method is otherwise essentially graphical or numerical in the general case. Making use of the symmetry of the solutions for the intrinsic semiconductor about a source, it follows from (57) and (58) that

$$\begin{aligned} G^+ &= -G^- = -\frac{(b+1)^2}{4b}\left[f_\epsilon - \frac{1}{b+1}\right]C_\epsilon \\ &= -\frac{(b+1)^2}{2b}\left[f^+ - \frac{1}{b+1}\right]C^+ = \frac{(b+1)^2}{2b}\left[f^- - \frac{1}{b+1}\right]C^-, \end{aligned} \tag{61}$$

whence

$$\begin{cases} f^- = \dfrac{1}{b+1} - \dfrac{1}{2}\left[f_\epsilon - \dfrac{1}{b+1}\right]\dfrac{C_\epsilon}{C^-} \\ f^+ = \dfrac{1}{b+1} + \dfrac{1}{2}\left[f_\epsilon - \dfrac{1}{b+1}\right]\dfrac{C_\epsilon}{C^+}. \end{cases} \tag{62}$$

It is easily verified that this result holds approximately for large relative concentrations in the n-type semiconductor. Three simple special cases of (62) might be considered: The first is

$$\begin{cases} C^- = -C^+ = -\tfrac{1}{2}C_\epsilon \\ f^- = f^+ = f_\epsilon . \end{cases} \tag{63}$$

This is the rather trivial case of symmetrical flows from a source which supplies all currents. A second special case is that for which C^- and C^+ are both positive, say, and such that there is no hole flow to the left against the field. It is readily found that, for this case,

$$\begin{cases} C^- = \dfrac{b+1}{2}\left[f_\epsilon - \dfrac{1}{b+1}\right]C_\epsilon; \quad C^+ = \dfrac{b+1}{2}\left[f_\epsilon + \dfrac{1}{b+1}\right]C_\epsilon \\ f^- = 0; \quad f^+ = \dfrac{2}{b+1}\dfrac{f_\epsilon}{f_\epsilon + 1/(b+1)}. \end{cases} \tag{64}$$

Here, the drift from the left under the applied field of holes normally present in the intrinsic semiconductor just cancels the diffusion from the source to the left.[33] A third special case is that in which the total current

[33] Using the numerically obtained solutions, the validity of (64) as an approximation for large concentrations in n-type germanium may be seen as follows: For f_ϵ equal to unity and C_ϵ, C^- and C^+ equal to 2, 1.5, and 3.5 respectively, P^0 is about 0.6 and the fraction of injected holes which flows against the field is nearly one-half; doubling these current densities increases P^0 to 1.45 and decreases the fraction to about one-fourth, and the fraction is less than about one-tenth if the current densities are increased so that C^+ exceeds 15

to the left of the source is zero, the left-hand side of the filament being open-circuited. For this case, equations (62) are better written in the form obtained by multiplying through by C^- or C^+, and the special case in question is then found to be given by

$$(65)\quad \begin{cases} C^- = 0; \qquad C^+ = C_\epsilon \\ C_p^- = -\dfrac{1}{2}\left[f_\epsilon - \dfrac{1}{b+1}\right]C_\epsilon; \quad C_p^+ = \dfrac{1}{2}\left[f_\epsilon + \dfrac{1}{b+1}\right]C_\epsilon, \end{cases}$$

according to which, if f_ϵ is equal to unity, the magnitude of the hole flow to the left into the open-circuit end is $b/(b + 2)$ times that into the circuit end, to the right; or a fraction $b/2(b + 1)$ of the holes flows to the left, and a fraction $(b + 2)/2(b + 1)$ to the right. Thus, for germanium, the hole flow into the open-circuit end is 0.43 as large as that into the circuit end, a fraction 0.30 flowing to the left, and 0.70 to the right. It might be observed that the fractions of the injected holes which flow to the left and right are, in this case, proportional to the total currents C^- and C^+ of the preceding case, for which there is zero hole flow to the left.

Another general limiting case for the n-type semiconductor is that for P_0 small, so that the exponential approximations of Section 3.31 apply. The restriction on the magnitude of P is $P << \frac{1}{2}$. This restriction obtains if C_ϵ is sufficiently small that C^- and C^+ do not differ appreciably. Equation (59) then gives

$$(66)\qquad G^+ - G^- = -bf_\epsilon C_\epsilon.$$

Writing C for C^- and C^+, equations (30) and (46) result in

$$(67)\quad \begin{cases} G^- = \frac{1}{2}[\sqrt{C^2 + 4} + C]\,P^0 = bC_p^+ \\ G^+ = -\frac{1}{2}[\sqrt{C^2 + 4} - C]\,P^0 = bC_p^-, \end{cases}$$

whence, solving for $G^+ - G^-$ and comparing with equation (66),

$$(68)\qquad P^0 = bf_\epsilon C_\epsilon/\sqrt{C^2 + 4}.$$

In accordance with (67), then,

$$(69)\quad \begin{cases} C_p^- = -\frac{1}{2}[1 - C/\sqrt{C^2 + 4}]\,f_\epsilon C_\epsilon \\ C_p^+ = \frac{1}{2}[1 + C/\sqrt{C^2 + 4}]\,f_\epsilon C_\epsilon. \end{cases}$$

These are the reduced hole flows to the left and right of the source. While it has been assumed that C_ϵ is small compared with C, no restriction has been placed on C itself. For C small compared with unity, the equations indicate that the hole flows to the left and right are the

same in magnitude, while for C large compared with unity,

$$\begin{cases} C_p^- \sim -\frac{1}{C^2} f_\epsilon C_\epsilon \\ C_p^+ \backsim f_\epsilon C_\epsilon . \end{cases} \tag{70}$$

Thus, according to this approximation, C should exceed about 10 if no more than one per cent of the holes are to flow against the field. From (75) in the Appendix, a value of 10 for C corresponds to a current density of about 1.2 amp cm^{-2} in germanium of 10 ohm cm resistivity, with τ equal to 10 μsec. This current density is moderately large among those which have been employed in experiments with germanium filaments.

Experimentally, the ideal one-dimensional geometry postulated in the present treatment of the problem of the single source in an infinite filament cannot easily be realized, hole injection generally being accomplished through a point contact or a side arm on one side of the actual filament. If suitable averages are employed, non-uniformity in P at the injection cross-section does not, however, vitiate the approximate results for ΔP large and ΔP small, since their applicability depends largely on the validity over the injection cross-section of the approximation assumed.

ACKNOWLEDGMENT

The author is indebted to a number of his colleagues for their stimulating interest and encouragement; to J. Bardeen and W. Shockley for a number of valuable and helpful comments, as well as to W. H. Brattain, J. R. Haynes, C. Herring, L. A. MacColl, G. L. Pearson, and R. C. Prim. J. Bardeen also suggested the numerical analysis for n-type germanium which constituted one of the initial points of attack, and aided materially in its inception. The rather difficult numerical integrations and associated problems were ably handled by R. W. Hamming, Mrs. G. V. Smith and J. W. Tukey.

5. APPENDIX

5.1 *The concentrations of ionized donors and acceptors*

While the donor and acceptor concentrations need not, of course, be considered for the intrinsic semiconductor, for the extrinsic semiconductor the fundamental equations, as they have been written, are in principle incomplete: Two additional equations in the variables D^+ and A^- are required. One of the required equations is trivial, since changes in the concentration of ionized centers which are compensated by those which determine the conductivity type of the extrinsic semiconductor

can certainly be neglected. For an n-type semiconductor, for example, the term $(A^- - A_0^-)$ in Poisson's equation may be suppressed. This procedure is strictly consistent with the neglect of p_0 and g_0, but undoubtedly holds to an even better approximation. If D is the total donor concentration in the n-type semiconductor, the concentration of ionized donors may be considered to satisfy the equation,

$$\frac{\partial D^+}{\partial t} = H(D - D^+) - KD^+n, \tag{71}$$

which applies to the homogeneous semiconductor, with H and K constants which characterize, respectively, the rate of ionization of unionized donors, and the rate of recombination of an ionized donor with an electron. If, as a result of a small thermal ionization energy, most of the donors are ionized, so that $KD/H << 1$, the change in ionized-donor concentration for the steady state is given by (71) as

$$D^+ - D_0^+ \sim -\frac{KD}{H}(n - n_0), \tag{72}$$

which is small compared with the corresponding change in electron concentration. In other cases, the use of the general expression obtainable from (71) for the steady-state concentration of ionized donors in terms of the electron concentration, or the expression for the other limiting case of relatively few ionized donors, might provide a more precise description provided the conditions under which solutions are sought do not involve unduly rapid changes with time.

5.2 *The carrier concentrations at thermal equilibrium*

The ratio of the thermal-equilibrium values of the hole and electron concentrations may be evaluated for n-type germanium from[5]

$$\begin{cases} np = 3 \cdot 10^{32} T^3 \exp\left(-\dfrac{8700}{T}\right) \equiv n_i^2 \\ n - p = n_s \sim n_0 = 1/b\mu e\rho_0 = 2.40 \cdot 10^{15}/\rho_0, \end{cases} \tag{73}$$

where the electron concentration excess n_s corresponds to complete ionization of the donors, and is approximately n_0 at the highest temperature at which P_0 is still negligible, which may be taken as room temperature[29]. The resistivity ρ_0 is that which determines n_0. Thus,

$$P_0 = \tfrac{1}{2}\left[\sqrt{1 + 4(n_i/n_0)^2} - 1\right], \tag{74}$$

[29] loc. cit.
[5] loc. cit.

with n_i, the concentration of holes or electrons in intrinsic germanium at T deg abs, given in (73). It may be estimated that temperature rises of less than 100 deg C will make 10 ohm cm n-type germanium substantially intrinsic in its behavior.

The range of values of the parameter C for which the numerical solutions are given corresponds, for example, to current densities up to the order of 10 amp cm^{-2} in germanium filaments of about 10 ohm cm resistivity, for the mean lifetime τ about 10 μsec; for this mean lifetime, the distance unit L_p is approximately $2 \cdot 10^{-2}$ cm. Current densities corresponding to the larger values of C will ordinarily produce appreciable joule heating in filaments some 10^{-3} cm^2 in area of cross-section, cemented to a backing, with temperature rises of the order of 100 deg C.

The effect of joule heating on L_p and C may be evaluated from

$$(75) \qquad \begin{cases} L_p = 6.6 \left[\dfrac{T}{300} \right]^{-\frac{1}{4}} \tau^{\frac{1}{2}} \\ C = 2.6 \cdot 10^2 \left[\dfrac{T}{300} \right]^{-\frac{5}{4}} \tau^{\frac{1}{2}} \rho\, I, \end{cases}$$

where τ is expressed in sec, I in amp cm^{-2}, and ρ is the normal resistivity in ohm cm of the germanium at T deg abs. These are obtained from the definitions (8), taking the hole mobility in the thermal scattering range to be proportional to $T^{-\frac{3}{2}}$, with the value 1700 cm^2 volt^{-1} sec^{-1} at 300 deg abs.[7]

5.3 *Series solutions for the extrinsic semiconductor in the steady state*

Maclaurin's series for G in the relative concentration P are of the form

$$(76) \qquad G = a_1 P + a_2 P^2 + a_3 P^3 + \ldots$$

for the cases of field opposing and field aiding, the solutions passing through the (P, G)-origin. Substituting the series (76) for G in the differential equation (27) for the n-type semiconductor in the steady state, it is found, in accordance with (30), that

$$(77) \qquad a_1 = \tfrac{1}{2} \left[C \pm \sqrt{C^2 + 4} \right],$$

the sign of C being taken before the radical for field opposing, the other sign for field aiding. The other coefficients are given in terms of a_1 and

[7] loc. cit.

also the b, C, and the constant, a, of the recombination function:

$$(78)\quad \begin{cases} a_2 = \dfrac{4a_1^2 - \left[2\,\dfrac{b+1}{b} + a\right]}{C - 3a_1} \\[2ex] a_3 = \dfrac{2a_2^2 + \dfrac{11b+1}{b}\,a_1 a_2 + 2\,\dfrac{b+1}{b}\,a_1^2 - \dfrac{b+1}{b}\left[\dfrac{b+1}{b} + 2a\right]}{C - 4a_1} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

The series in the current parameter are series in ascending powers of the reciprocal of C. Writing, for convenience,

$$(79)\qquad \gamma \equiv 1/C,$$

the differential equation (27) may be put in the form,

$$(80)\qquad \gamma\,[1 + 2P]\left[1 + \frac{b+1}{b}\,P\right]GG' + \gamma\,\frac{b-1}{b}\,G^2 - G - \gamma P\left[1 + \frac{b+1}{b}\,P\right]^2 R = 0,$$

using the prime to denote differentiation with respect to P. Consider expansions of the form,

$$(81)\qquad G = \sum_{j=j_0}^{\infty} A_j \gamma^j,$$

in which the A's are functions of P to be determined. Substituting in the differential equation, there results

$$(82)\qquad \sum_{j=j_0}^{\infty}\sum_{m=j_0}^{\infty}\left[[1 + 2P]\left[1 + \frac{b+1}{b}\,P\right]A_j A_m' + \frac{b-1}{b}\,A_j A_m\right]\gamma^{j+m+1} - \sum_{j=j_0}^{\infty} A_j \gamma^j - P\left[1 + \frac{b+1}{b}\,P\right]^2 R\gamma = 0.$$

Since the expansions are to hold for arbitrary values of γ, the A's must, for the cases of field opposing and field aiding, for which the solutions pass through the (P, G)-origin, vanish identically for P equal to zero, and be determined by equating to zero the coefficients of given powers of γ in (82). It can, without loss of generality, be assumed that the coefficient of the leading term in the expansion, A_{j_0}, is not identically zero. Then, from (82), it is found that there is no expansion for $j_0 = 0$, that is, no expansion starting with a term independent of γ. Formal expansions can be obtained, however, for $j_0 = -1$ and for $j_0 = +1$. These may be identified,

respectively, with the solutions for field opposing and field aiding, as will be seen.

For $j_0 = -1$, or field opposing, (82) leads to differential equations of the first order for the determination of the A's. The condition that these functions vanish identically for $P = 0$ suppresses all A's of even order. The first term of the expansion is found by solving

$$A'_{-1} + \frac{b-1}{b} \frac{A_{-1}}{[1+2P]\left[1 + \frac{b+1}{b} P\right]} = \frac{1}{[1+2P]\left[1 + \frac{b+1}{b} P\right]} \tag{83}$$

whence

$$A_{-1} = \frac{P}{1+2P}. \tag{84}$$

The second term is found from

$$A'_1 + \frac{b-1}{b} \frac{A_1}{[1+2P]\left[1 + \frac{b+1}{b} P\right]} = \left[1 + \frac{b+1}{b} P\right] R, \tag{85}$$

whence, with R equal to unity and $(1 + P)$, respectively,

$$\begin{cases} A_1 = \dfrac{P[1+P]\left[1 + \frac{b+1}{b} P\right]}{1+2P} & \text{for constant mean lifetime} \\[2ex] A_1 = \dfrac{P\left[1 + \frac{3}{2}P + \frac{2}{3}P^2\right]\left[1 + \frac{b+1}{b} P\right]}{1+2P} & \text{for mass-action recombination.} \end{cases} \tag{86}$$

For the third term, making use of (84), (85) and (86),

$$\begin{cases} A'_3 + \dfrac{b-1}{b} \dfrac{A_3}{[1+2P]\left[1 + \frac{b+1}{b} P\right]} \\ \qquad\qquad = -[1+P]\left[1 + \frac{b+1}{b} P\right]^2 & \text{for constant mean lifetime} \\[2ex] A'_3 + \dfrac{b-1}{b} \dfrac{A_3}{[1+2P]\left[1 + \frac{b+1}{b} P\right]} \\ = -[1+P]\left[1 + \frac{3}{2}P + \frac{2}{3}P^2\right]\left[1 + \frac{b+1}{b} P\right]^2 & \text{for mass-action recombination} \end{cases} \tag{87}$$

whence

$$
(88)\quad \begin{cases}
A_3 = \dfrac{-P\left[1+\dfrac{4b+1}{2b}P+\dfrac{5b+3}{3b}P^2+\dfrac{b+1}{2}P^3\right]\left[1+\dfrac{b+1}{b}P\right]}{1+2P} & \text{for constant mean lifetime} \\[2ex]
A_3 = \dfrac{-P\left[1+\dfrac{33b+6}{12b}P+\dfrac{70b+27}{18b}P^2+\dfrac{73b+43}{24b}P^3 + \dfrac{38b+30}{30b}P^4+\dfrac{2b+2}{9b}P^5\right]\left[1+\dfrac{b+1}{b}P\right]}{1+2P} & \text{for mass-action recombination.}
\end{cases}
$$

For $j_0 = +1$, or field aiding, the A's are determined somewhat more simply, recursive relationships obtaining. The results are:

$$
(89)\qquad A_1 = -P\left[1+\frac{b+1}{b}P\right]^2 R,
$$

and

$$
(90)\quad \begin{cases}
A_3 = [1+2P]\left[1+\dfrac{b+1}{b}P\right]A_1A_1' + \dfrac{b-1}{b}A_1^2 \\[1ex]
A_5 = [1+2P]\left[1+\dfrac{b+1}{b}P\right][A_1A_3]' + 2\dfrac{b-1}{b}A_1A_3 \\[1ex]
A_7 = [1+2P]\left[1+\dfrac{b+1}{b}P\right][[A_1A_5]' + A_3A_3'] \\
\qquad\qquad + 2\dfrac{b-1}{b}\left[A_1A_5+\dfrac{1}{2}A_3^2\right] \\[1ex]
A_9 = [1+2P]\left[1+\dfrac{b+1}{b}P\right][[A_1A_7]' + [A_3A_5]'] \\
\qquad\qquad + 2\dfrac{b-1}{b}[A_1A_7+A_3A_5] \\
\cdots\cdots\cdots\cdots\cdots\cdots
\end{cases}
$$

The identification of the series in the parameter γ as series for field opposing and field aiding is accomplished by evaluating them for small P and then comparing them with the first terms of the corresponding Maclaurin's series in P, expanded in powers of γ. Further agreement is

obtained by comparing the first terms of the series in γ with the functions of P which result from evaluating the Maclaurin's series for γ small.

5.4 *Symbols for Quantities*

$a \equiv \tau/\tau_v$, constant in recombination function.
$a_j \equiv$ coefficients in the Maclaurin's expansion of G in powers of P; j an integer.
$A_j \equiv$ coefficients in the expansion of G in powers of γ; j an integer.
$A^- \equiv$ concentration of ionized acceptors.
$A_0^- \equiv$ thermal-equilibrium concentration of ionized acceptors.
$b \equiv$ ratio of electron mobility to hole mobility.
$C \equiv I/I_0$, reduced total current density.
$C_\epsilon \equiv$ reduced emitter current.
$C^- \equiv$ reduced total current to the origin from the left.
$C^+ \equiv$ reduced total current from the origin to the right.
$C_n \equiv -I_n/I_0$, reduced electron flow density.
$C_p \equiv I_p/I_0$, reduced hole flow density.
$\gamma \equiv I/C$.
$\Gamma \equiv \epsilon/4\pi\sigma_0\tau$, reduced time for the dielectric relaxation of charge.
$D \equiv$ total donor concentration.
$D^+ \equiv$ concentration of ionized donors.
$D_0^+ \equiv$ thermal-equilibrium concentration of ionized donors.
$D_n \equiv kT\mu_n/e$, diffusion constant for electrons.
$D_p \equiv kT\mu_p/e$, diffusion constant for holes.
$e \equiv$ magnitude of the electronic charge.
$E \equiv$ electrostatic field.
$E_a \equiv$ applied or asymptotic field.
$E_0 \equiv kT/eL_p$, characteristic field.
$\epsilon \equiv$ dielectric constant.
$f \equiv$ fraction of total current carried by holes.
$f_\epsilon \equiv$ fraction of total current carried by holes at an emitter.
$f^- \equiv$ fraction of total current carried by holes at a source, to the left.
$f^+ \equiv$ fraction of total current carried by holes at a source, to the right.
$F \equiv E/E_0$, reduced electrostatic field.
$g_0 \equiv$ thermal rate of generation of hole-electron pairs, per unit volume.
$G \equiv dP/dX$, reduced concentration gradient.
$G^0 \equiv$ value of G for $X = 0$.
$G^- \equiv$ limiting value of G at a source, approached from the left.
$G^+ \equiv$ limiting value of G at a source, approached from the right.
$H \equiv$ probability of thermal ionization of an unionized donor, per unit time.

I ≡ total current density.
I_n ≡ current density of electrons.
I_0 ≡ σE_0, characteristic current.
I_p ≡ current density of holes.
J ≡ $\frac{1}{e} I$, total carrier flow density.
J_n ≡ $-\frac{1}{e} I_n$, electron flow density.
J_p ≡ $\frac{1}{e} I_p$, hole flow density.
k ≡ Boltzmann's constant.
K ≡ probability per unit time of electron capture by an ionized donor, per unit electron concentration.
L_d ≡ $(kT\epsilon/8\pi n_i e^2)^{\frac{1}{2}}$, characteristic length associated with space charge in the steady state.
L_p ≡ $(kT\mu\tau/e)^{\frac{1}{2}}$, diffusion length for holes for time τ.
M ≡ $1 + \frac{b+1}{b} P$.
M_0 ≡ $1 + \frac{b+1}{b} P_0$.
μ ≡ μ_p ≡ mobility for holes.
μ_n ≡ mobility for electrons.
n ≡ concentration of electrons.
n_i ≡ thermal-equilibrium concentration of electrons (or holes) in the intrinsic semiconductor.
n_0 ≡ thermal-equilibrium concentration of electrons.
n_s ≡ saturation concentration excess of electrons, corresponding to complete ionization of donors.
N ≡ $n/(n_0 - p_0)$, reduced electron concentration for an n-type semiconductor.
p ≡ concentration of holes.
p_0 ≡ thermal-equilibrium concentration of holes.
P ≡ $p/(n_0 - p_0)$, reduced hole concentration for an n-type semiconductor.
ΔP ≡ $(p - p_0)/(n_0 - p_0)$, reduced concentration of added holes.
P_0 ≡ $p_0/(n_0 - p_0)$, reduced hole concentration at thermal equilibrium.
P^0 ≡ value of P for $X = 0$.
Q ≡ τ/τ_p, lifetime ratio.
R ≡ general recombination function, equal to $1 + aP/(1 + P_0)$ for mass-action and constant-mean-lifetime mechanisms combined.
ρ ≡ volume resistivity in ohm cm.
s ≡ differential transport velocity.
S ≡ $s/(D_p/\tau)^{\frac{1}{2}}$, reduced differential transport velocity.

σ ≡ conductivity of semiconductor.
σ_0 ≡ normal conductivity of semiconductor, with no added carriers.
Σ ≡ $\sigma/\sigma_0 = M/M_0$, reduced conductivity of semiconductor.
t ≡ time variable.
T ≡ temperature in degrees absolute.
τ ≡ mean lifetime for holes for small added concentrations, in an n-type or in an intrinsic semiconductor.
τ_n ≡ mean lifetime for electrons (concentration-dependent).
τ_p ≡ mean lifetime for holes (concentration-dependent).
τ_v ≡ mean lifetime for holes, for small added concentrations in an n-type semiconductor, due to mass-action recombination alone.
U ≡ t/τ ≡ reduced time variable.
W ≡ eV/kT, reduced electrostatic potential.
x ≡ distance variable.
X ≡ x/L_p, reduced distance variable.
V ≡ electrostatic potential.