Traveling-Wave Tubes

By J. R. PIERCE

[THIRD INSTALLMENT]

CHAPTER VII

EQUATIONS FOR TRAVELING-WAVE TUBE

SYNOPSIS OF CHAPTER

IN CHAPTER VI we have expressed the properties of a circuit in terms of its normal modes of propagation rather than its physical dimensions. In this chapter we shall use this representation in justifying the circuit equation of Chapter II and in adding to it a term to take into account the local fields produced by a-c space charge. Then, a combined circuit and ballistical equation will be obtained, which will be used in the following chapters in deducing various properties of traveling-wave tubes.

In doing this, the first thing to observe is that when the propagation constant Γ of the impressed current is near the propagation constant Γ_1 of a particular active mode, the excitation of that mode is great and the excitation varies rapidly as Γ is changed, while, for passive modes or for active modes for which Γ is not near to the propagation constant Γ_n, the excitation varies more slowly as Γ is changed. It will be assumed that Γ is nearly equal to the propagation constant Γ_1 of one active mode, is not near to the propagation constant of any other mode and varies over a small fractional range only. Then the sum of terms due to all other modes will be regarded as a constant over the range of Γ considered. It will also be assumed that the phase velocities corresponding to Γ and Γ_1 are small compared with the speed of light. Thus, (6.47) and (6.47a) are replaced by (7.1), where the first term represents the excitation of the Γ_1 mode and the second term represents the excitation of passive and "non-synchronous" modes. In another sense, this second term gives the field produced by the electrons in the absence of a wave propagating on the circuit, or, the field due to the "space charge" of the bunched electron stream. Equation (7.1) is the equation for the distributed circuit of Fig. 7.1. This is like the circuit of Fig. 2.3 save for the addition of the capacitances C_1 between the transmission circuit and the electron beam. We see that, because of the presence of these capacitances, the charge of a bunched electron beam will produce a field in addition to the field of a wave traveling down the circuit. This circuit is intuitively so appealing that it was originally thought of by guess and justified later.

Equation (7.1), or rather its alternative form, (7.7), which gives the voltage in terms of the impressed charge density, can be combined with the ballistical equation (2.22), which gives the charge density in terms of the voltage, to give (7.9), which is an equation for the propagation constant. The attenuation, the difference between the electron velocity and the phase velocity of the wave on the circuit in the absence of electrons and the difference between the propagation constant and that for a wave traveling with the electron speed are specified by means of the gain parameter C and the parameters d, b and δ. It is then assumed that d, b and δ are around unity or smaller and that C is much smaller than unity. This makes it possible to neglect certain terms without serious error, and one obtains an equation (7.13) for δ.

In connection with (7.7) and Fig. 7.1, it is important to distinguish between the *circuit voltage* V_c, corresponding to the first term of (7.7), and the total voltage V acting on the electrons. These quantities are related by (7.14). The a-c velocity v and the convection current i are given within the approximation made ($C \ll 1$) by (7.15) and (7.16).

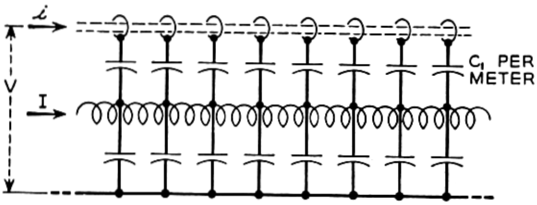

Fig. 7.1

7.1 Approximate Circuit Equation

From (6.47) we can write for a current $J = i$ and a summation over n modes

$$E_z = (1/2)(\Gamma^2 + \beta_0^2)i \sum_n \frac{(E^2/\beta^2 P)_n \Gamma_n^3}{(\Gamma_n^2 + \beta_0^2)(\Gamma_n^2 - \Gamma^2)} \tag{6.47a}$$

This has a number of poles at $\Gamma = \Gamma_n$. We shall be interested in cases in which Γ is very near to a particular one of these, which we shall call Γ_1. Thus the term in the expansion involving Γ_1 will change rapidly with small variations in Γ. Moreover, even if $(E^2/\beta^2 P)_1$ and Γ_1 have very small real components, $\Gamma_1^2 - \Gamma^2$ can be almost or completely real for values of Γ which have only small real components. Thus, one term of the expansion, that involving Γ_1, can go through a wide range of phase angles and magnitudes for very small fractional variations in Γ, fractional variations, as it turns out, which are of the order of C over the range of interest.

The other modes are either passive modes, for which even in a lossy circuit $(E^2/\beta^2 P)_n$ is almost purely imaginary, and Γ_n almost purely real, or they are active modes which are considerably out of synchronism with the electron velocity. Unless one of these other active modes has a propagation constant Γ_n such that $|(\Gamma_1 - \Gamma_2)/\Gamma_1|$ is so small as to be of the order of C, the terms forming the summation will not vary very rapidly over the range of variation of Γ which is of interest.

We will thus write the circuit equation in the approximate form

$$E = \left[\frac{\Gamma^2\Gamma_1(E^2/\beta^2P)}{2(\Gamma_1^2 - \Gamma^2)} - \frac{j\Gamma^2}{\omega C_1}\right] i \tag{7.1}$$

Here there has been a simplification of notation. E is the z component of electric field, as in Chapter II, and is assumed to vary as $\exp(-\Gamma z)$. (E^2/β^2P) is taken to mean the value for the Γ_1 mode. It has been assumed that β_0^2 is small compared with $|\Gamma_1^2|$ and $|\Gamma^2|$, and β_0^2 has been neglected in comparison with these quantities.

Further, it has been pointed out that for slightly lossy circuits, (E^2/β^2P) will have only a small imaginary component, and we will assume as a valid approximation that (E^2/β^2P) is purely real. We cannot, however, safely assume that Γ_1 is purely imaginary, for a small real component of Γ_1 can affect the value of $\Gamma_1^2 - \Gamma^2$ greatly when Γ is nearly equal to Γ_1.

The first term on the right of (7.1) represents fields associated with the active mode of the circuit, which is nearly in synchronism with the electrons. We can think of these fields as summing up the effect of the electrons on the circuit over a long distance, propagated to the point under consideration.

The term $(-j\Gamma^2/\omega C_1)$ in (7.1) sums up the effect of all passive modes and of any active modes which are far out of synchronism with the electrons. It has been written in this form for a special purpose; the term will be regarded as constant over the range of Γ considered, and C_1 will be given a simple physical meaning.

This second term represents the field resulting from the local charge density, as opposed to that of the circuit wave which travels to the region from remote points. Let us rewrite (7.1) in terms of voltage and charge density

$$E = -\frac{\partial V}{\partial z} = \Gamma V \tag{7.2}$$

From the continuity equation

$$i = (j\omega/\Gamma)\rho \tag{2.18}$$

$$V = \left[\frac{j\omega\Gamma_1(E^2/\beta^2P)}{2(\Gamma_1^2 - \Gamma^2)} + \frac{1}{C_1}\right]\rho \tag{7.3}$$

We see that C_1 has the form of a capacitance per unit length. We can, for instance, redraw the transmission-line analogue of Fig. 2.3 as shown in Fig. 7.1. Here, the current I is still the line current; but the voltage V acting on the beam is the line voltage plus the drop across a capacitance of C_1 farads per meter.

Consider as an illustration the case of unattenuated waves for which

$$\Gamma_1 = j\beta_1 \tag{7.5}$$

$$\Gamma = j\beta \tag{7.6}$$

where β_1 and β are real. Then

$$V = \left[\frac{\omega\beta_1(E^2/\beta^2P)}{2(\beta_1^2 - \beta^2)} + \frac{1}{C_1}\right]\rho \tag{7.7}$$

In (7.7), the first term in the brackets represents the impedance presented to the beam by the "circuit"; that is, the ladder network of Figs. 2.3 and 7.1. The second term represents the additional impedance due to the capacitance C_1, which stands for the impedance of the nonsynchronous modes. We note that if $\beta < \beta_1$, that is, for a wave faster than the natural phase velocity of the circuit, the two terms on the right are of the same sign. This must mean that the "circuit" part of the impedance is capacitive. However, for $\beta > \beta_1$, that is, for a wave slower than the natural phase velocity, the first term is negative and the "circuit" part of the impedance is inductive. This is easily explained. For small values of β the wavelength of the impressed current is long, so that it flows into and out of the circuit at widely separated points. Between such points the long section of series inductance has a higher impedance than the shunt capacitance to ground; the capacitive effect predominates and the circuit impedance is capacitive. However, for large values of β the current flows into and out of the circuit at points close together. The short section of series inductance between such points provides a lower impedance path than does the shunt capacitance to ground; the inductive impedance predominates and the circuit impedance is inductive. Thus, for *fast* waves the circuit appears *capacitive* and for *slow* waves the circuit appears *inductive*.

Since we have justified the use of the methods of Chapter II within the limitations of certain assumptions, there is no reason why we should not proceed to use the same notation in the light of our fuller understanding. We can now, however, regard V not as a potential but merely as a convenient variable related to the field by (7.2).

From (2.18) and (7.3) we obtain

$$V = \left[\frac{\Gamma\Gamma_1(E^2/\beta^2P}{2(\Gamma_1^2 - \Gamma^2)} - \frac{j\Gamma}{\omega C_1}\right]i \tag{7.8}$$

We use this together with (2.22)

$$i = \frac{jI_0\beta_e\Gamma V}{2V_0(j\beta_e - \Gamma)^2} \tag{2.22}$$

We obtain the overall equation

$$1 = \frac{jI_0\beta_e\Gamma}{2V_0(j\beta_e - \Gamma)}\left[\frac{\Gamma\Gamma_1(E^2/\beta^2P)}{2(\Gamma_1 - \Gamma)} - \frac{j\Gamma}{\omega C_1}\right] \tag{7.9}$$

In terms of the gain parameter C, which was defined in Chapter II,

$$C^3 = (E^2/\beta^2P)(I_0/8V_0) \tag{2.43}$$

we can rewrite (7.8)

$$(j\beta_e - \Gamma)^2 = \frac{j2\beta_e\Gamma^2\Gamma_1C^3}{(\Gamma_1^2 - \Gamma^2)} + \frac{4\beta_e\Gamma^2C^3}{\omega C_1(E^2/\beta^2P)} \tag{7.10}$$

We will be interested in cases in which Γ and Γ_1 differ from β_e by a small amount only. Accordingly, we will write

$$-\Gamma = -j\beta_e + \beta_eC \tag{7.11}$$

$$-\Gamma_1 = -j\beta_e - j\beta_eCb - \beta_eCd \tag{7.12}$$

The propagation constant Γ describes propagation in the presence of electrons. A positive real value of δ means an increasing wave. A positive imaginary part means a wave traveling faster than the electrons.

The propagation constant Γ_1 refers to propagation in the circuit in the absence of electrons. A positive value of b means the electrons go faster than the undisturbed wave. A positive value d means that the wave is an attenuated wave which decreases as it travels.

If we use (7.11) and (7.12) in connection with (7.10) we obtain

$$\delta = \frac{[1 + C(2j\delta - C\delta^2)][1 + C(b - jd)]}{[-b + jd + j\delta + C(jbd - b^2/2 + d^2/2 + \delta^2/2)]} - \frac{4\beta_e[(1 + C(2j\delta - C\delta^2)]C}{\omega C_1(E^2/\beta^2P)} \tag{7.13}$$

We will now assume that $|\delta|$ is of the order of unity, that $|b|$ and $|d|$ range from zero to unity or a little larger, and that $C \ll 1$. We will then neglect the parentheses multiplied by C, obtaining

$$\delta = \frac{1}{(-b + jd + j\delta)} - 4QC \tag{7.14}$$

$$Q = \frac{\beta_e}{\omega C_1(E^2/\beta^2P)} \tag{7.15}$$

The quantity ωC_1 has the dimensions of admittance per unit length, β_e has the dimensions of (length)$^{-1}$ and $(E^2/\beta^2 P)$ has the dimensions of impedance. Thus, Q is a dimensionless parameter (the space-charge parameter) which may be thought of as relating to the impedance parameter $(E^2/\beta^2 P)$ associated with the synchronous mode the impedance $(\beta_e/\omega C_1)$, attributable to all modes but the synchronous mode.

At this point it is important to remember that there are not only two impedances, but two voltage components as well. Thus, in (7.8), the first term in the brackets times the current represents the "circuit voltage", which we may call V_c. The second term in the brackets represents the voltage due to space charge, the voltage across the capacitances C_1. The two terms in the brackets are in the same ratio as the two terms on the right of (7.14), which came from them. Thus, we can express the circuit component of voltage V_c in terms of the total voltage V acting on the beam either from (7.8) as

$$V_c = \left[1 - \frac{j2(\Gamma_1^2 - \Gamma^2)}{\omega C_1 \Gamma_1 (E^2/\beta^2 P)}\right]^{-1} V \tag{7.16}$$

or, alternatively, from (7.14) as

$$V_c = [1 - 4QC(-b + jd + j\delta)]^{-1} V \tag{7.17}$$

From Chapter II we have relations for the electron velocity (2.15) and electron convection current (2.22). If we make the same approximations which were made in obtaining (7.14), we have

$$(ju_0 C/\eta)v = \frac{V}{\delta} \tag{7.18}$$

$$(-2V_0 C^2/I)i = \frac{V}{\delta^2} \tag{7.19}$$

We should remember also that the variation of all quantities with z is as

$$e^{-j\beta_e z} e^{\beta_e C \delta z} \tag{7.20}$$

The relations (7.18)–(7.19) together with (2.36), which tells us that the characteristic impedance of the circuit changes little in the presence of electrons if C is small, sum up in terms of the more important parameters the linear operation of traveling-wave tubes in which C is small. The parameters are: the gain parameter C, relative electron velocity parameter b, circuit attenuation parameter d and space-charge parameter Q. In following chapters, the practical importance of these parameters in the operation of traveling-wave tubes will be discussed.

There are other effects not encompassed by these equations. The effect of transverse electron motions is small in most tubes because of the high focusing fields employed; it will be discussed in a later chapter. The differences between a field theory in which different fields act on different electrons and the theory leading to (7.14)–(7.20), which apply accurately only when all electrons at a given z-position are acted on by the same field, will also be discussed.

CHAPTER VIII

THE NATURE OF THE WAVES

Synopsis of Chapter

IN THIS CHAPTER we shall discuss the effect of the various parameters on the rate of increase and velocity of propagation of the three forward waves. Problems involving boundary conditions will be deferred to later chapters.

The three parameters in which we are interested are those of (7.13), that is, b, the velocity parameter, d, the attenuation parameter and QC, the space-charge parameter. The fraction by which the electron velocity is greater than the phase velocity for the circuit in the absence of electrons is bC. The circuit attenuation is 54.6 dC db/wavelength. Q is a factor depending on the circuit impedance and geometry and on the beam diameter. For a helically conducting sheet of radius a and a hollow beam of radius a_1, Q can be obtained from Fig. 8.12.

The three forward waves vary with distance as

$$e^{-j\beta_e(1-yC)z} e^{\beta_e x C z}$$

$$\beta_e = \frac{\omega}{u_0}$$

Thus, a positive value of y means a wave which travels faster than the electrons, and a positive value of x means an increasing wave. The gain in db per wavelength of the increasing waves is BC, and B is defined by (8.9).

Figure 8.1 shows x and y for the three forward waves for a lossless circuit ($d = 0$). The increasing wave is described by x_1, y_1. The gain is a maximum when the electron velocity is equal to the velocity of the undisturbed wave, or, when $b = 0$. For large positive values of b (electrons much faster than undisturbed wave), there is no increasing wave. However, there is an increasing wave for all negative values of b (all low velocities). For the increasing wave, y_1 is negative; thus, the increasing wave travels more slowly than the electrons, *even when the electrons travel more slowly than the circuit wave in the absence of electrons*. For the range of b for which there is an increasing wave, there is also an attenuated wave, described by $x_2 = -x_1$ and $y_2 = y_1$. There is also an unattenuated wave described by $y_3(x_3 = 0)$.

For very large positive and negative values of b, the velocity of two of the waves approaches the electron velocity (y approaches zero) and the velocity of the third wave approaches the velocity of the circuit wave in the absence of electrons (y approaches minus b). For large negative values of b, x_1, y_1 and x_2, y_2 become the "electron" waves and y_3 becomes the "circuit" wave. For large values of b, y_1 and y_3 become the "electron" waves and y_2 becomes the "circuit" wave. The "circuit" wave is essentially the wave in the absence of electrons, modified slightly by the presence of a non-synchronous electron stream. The "electron waves" represent the motion of "bunches" along the electron stream, slightly affected by the presence of the circuit.

Figures 8.2 and 8.3 indicate the effect of loss. Loss decreases the gain of the increasing wave, adds to the attenuation of the decreasing wave and adds attenuation to the wave which was unattenuated in the lossless case. For large positive and negative values of b, the attenuation of the circuit wave (given by x_3 for negative values of b and x_2 for positive values of b) approaches the attenuation in the absence of electrons.

Figure 8.4 shows B, the gain of the increasing wave in db per wavelength per unit C. Figure 8.5 shows, for $b = 0$, how B varies with d. The dashed line shows a common approximation: that the gain of the increasing wave is reduced by $\frac{1}{3}$ of the circuit loss. Figure 8.6 shows how, for $b = 0$, x_1, x_2 and x_3 vary with d. We see that, for large values of d, the wave described by x_2 has almost the same attenuation as the wave on the circuit in the absence of electrons.

Figures 8.7–8.9 show x, y for the three waves with no loss ($d = 0$) but with a-c space charge taken into account ($QC \neq 0$). The immediately striking feature is that there is now a minimum value of b below which there is no increasing wave.

We further note that, for large negative and positive values of b, y for the electron waves approaches $\pm 2\sqrt{QC}$. In these ranges of b the electron waves are dependent on the electron inertia and the field produced by a-c space charge, and have nothing to do with the active mode of the circuit.

As QC is made larger, the value of b for which the gain of the increasing wave is a maximum increases. Now, C is proportional to the cube root of current. Thus, as current is increased, the voltage for maximum gain of the increasing wave increases. An increase in optimum operating voltage with an increase in current is observed in some tubes, and this is at least partly explained by these curves.* There is also some decrease in the maximum value of x_1 and hence of B as QC is increased. This is shown more clearly in Fig. 8.10.

If x and B remained constant when the current is varied, then the gain per wavelength would rise as C, or, as the $\frac{1}{3}$ power of current. However,

* Other factors include a possible lowering of electron speed because of d-c space charge, and boundary condition effects.

we see from Fig. 8.10 that B falls as QC is increased. The gain per wavelength varies as BC and, because Q is constant for a given tube, it varies as BQC. In Fig. 8.11, BQC, which is proportional to the gain per wavelength of the increasing wave, is plotted vs QC, which is proportional to the $\frac{1}{3}$ power of current. For very small values of current (small values of QC), the gain per wavelength is proportional to the $\frac{1}{3}$ power of current. For larger values of QC, the gain per wavelength becomes proportional to the $\frac{1}{4}$ power of current.

It would be difficult to present curves covering the simultaneous effect of loss (d) and space charge (QC). As a sort of substitute, Figs. 8.13 and 8.14 show $\partial x_1/\partial d$ for $d = 0$ and b chosen to maximize x_1, and $\partial x_1/\partial(QC)$ for $QC = 0$ and $b = 0$. We see from 8.13 that, while for small values of QC the gain of the increasing wave is reduced by $\frac{1}{3}$ of the circuit loss, for large values of QC the gain of the increasing wave is reduced by $\frac{1}{2}$ of the circuit loss.

8.1 Effect of Varying the Electron Velocity

Consider equation (7.13) in case $d = 0$ (no attenuation) and $Q = 0$ (neglect of space-charge). We then have

$$\delta^2(\delta + jb) = -j \tag{8.1}$$

Here we will remember that

$$\beta_e = \omega/u_0 \tag{8.2}$$

$$-\Gamma_1 = -j\beta_e(1 + Cb) = -j\omega/v_1 \tag{8.3}$$

Here v_1 is the phase velocity of the wave in the absence of electrons, and u_0 is the electron speed. We see that

$$u_0 = (1 + Cb)v_1 \tag{8.4}$$

Thus, $(1 + Cb)$ is the ratio of the electron velocity to the velocity of the *undisturbed wave*, that is, the wave in the absence of electrons. Hence, b is a measure of velocity difference between electrons and undisturbed wave. For $b > 0$, the electrons go faster than the undisturbed wave; for $b < 0$ the electrons go slower than the undisturbed wave. For $b = 0$ the electrons have the same speed as the undisturbed wave.

If $b = 0$, (8.1) becomes

$$\delta^3 = -j \tag{8.5}$$

which we obtained in Chapter II.

In dealing with (8.1), let

$$\delta = x + jy$$

The meaning of this will be clear when we remember that, in the presence of electrons, quantities vary with z as (from (7.10))

$$\begin{aligned} &e^{-j\beta_e(1+jC\delta)z} \\ &= e^{-j\beta_e(1-Cy)z} e^{\beta_e Cxz} \end{aligned} \tag{8.6}$$

If v is the phase velocity in the presence of electrons, we have

$$\omega/v = (\omega/u_0)(1 - Cy) \tag{8.7}$$

If $Cy \ll 1$, very nearly

$$v = u_0(1 + Cy) \tag{8.8}$$

In other words, if $y > 0$, the wave travels faster than the electrons; if $y < 0$ the wave travels more slowly than the electrons.

From (8.6) we see that, if $x > 0$, the wave increases as it travels and if $x < 0$ the wave decreases as it travels. In Chapter II we expressed the gain of the increasing wave as

$$BCN \text{ db}$$

where N is the number of wavelengths. We see that

$$\begin{aligned} B &= 20(2\pi)(\log_{10}e)x \\ B &= 54.5x \end{aligned} \tag{8.9}$$

In terms of x and y, (8.1) becomes

$$(x^2 - y^2)(y + b) + 2x^2y + 1 = 0 \tag{8.10}$$

$$x(x^2 - 3y^2 - 2yb) = 0 \tag{8.11}$$

We see that (8.11) yields two kinds of roots: those corresponding to unattenuated waves, for which $x = 0$ and those for which

$$x^2 = 3y^2 + 2yb \tag{8.12}$$

If $x = 0$, from (8.10)

$$\begin{aligned} y^2(y + b) &= 1 \\ b &= -y + 1/y^2 \end{aligned} \tag{8.13}$$

If we assume values of y ranging from perhaps $+4$ to -4 we can find the corresponding values of b from (8.13), and plot out y vs b for these unattenuated waves.

For the other waves, we substitute (8.12) into (8.10) and obtain

$$2yb^2 + 8y^2b + 8y^3 + 1 = 0 \tag{8.14}$$

This equation is a quadratic in b, and, by assigning various values of y, we can solve for b. We can then obtain x from (8.12).

In this fashion we can construct curves of x and y vs b. Such curves are shown in Fig. 8.1.

We see that for

$$b < (3/2)(2)^{1/3}$$

there are two waves for which $x \neq 0$ and one unattenuated wave. The increasing and decreasing waves ($x \neq 0$) have equal and opposite values of x, and since for them $y < 1$, they travel more slowly than the electrons, *even when the electrons travel more slowly than the undisturbed wave.* It can be

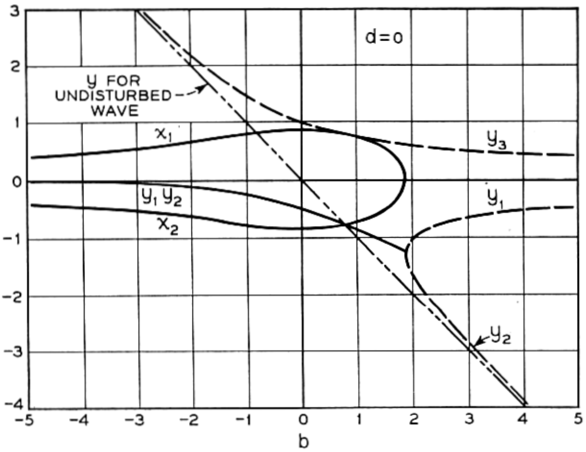

Fig. 8.1—The three waves vary with distance as exp $(-j\beta_e + j\beta_e Cy + \beta_e Cx)z$. Here the x's and y's for the three waves are shown vs the velocity parameter b for no attenuation ($d = 0$) and no space charge ($QC = 0$).

shown that the electrons must travel faster than the increasing wave in order to give energy to it.

For $b > (3/2)(2)^{1/3}$, there are 3 unattenuated waves: two travel faster than the electrons and one more slowly.

For large positive or negative values of b, two waves have nearly the electron speed ($|y|$ small) and one wave travels with the speed of the undisturbed wave. We measure velocity with respect to electron velocity. Thus, if we assigned a parameter y to describe the velocity of the undisturbed wave relative to the electron velocity, it would vary as the 45° line in Fig. 8.1.

The data expressed in Fig. 8.1 give the variation of gain per wavelength of the undisturbed wave with electron velocity, and are also useful in fitting boundary conditions; for this we need to know the three x's and the three y's.

In a tube in which the total gain is large, a change in b of $\pm$ 1 about $b =$ 0 can make a change of several db in gain. Such a change means a difference between phase velocity of the undisturbed wave, v_1, and electron velocity u_0 by a fraction approximately $\pm C$. Hence, the allowable difference between phase velocity v_1 of the undisturbed wave, which is a function of frequency, and electron velocity, which is not, is of the order of C.

8.2 Effect of Attenuation

If we say that $d \neq 0$ but has some small positive value, we mean that the circuit is lossy, and in the absence of electrons the voltage decays with distance as

$$e^{-\beta_e C d}$$

Hence, the loss L in db/wavelength is

$$\begin{aligned} L &= 20(2\pi)(\log_{10} e)Cd \\ L &= 54.5Cd \text{ db/wavelength} \end{aligned} \tag{8.15}$$

or

$$d = .01836\ (L/C) \tag{8.16}$$

For instance, for $C = .025$, $d = 1$ means a loss of 1.36 db/wavelength.

If we assume $d \neq 0$ we obtain the equations

$$(x^2 - y^2)(y + b) + 2xy(x + d) + 1 = 0 \tag{8.17}$$

$$(x^2 - y^2)(x + d) - 2xy(y + b) = 0 \tag{8.18}$$

The equations have been solved numerically for $d = .5$ and $d = 1$, and the curves which were obtained are shown in Figs. 8.2 and 8.3. We see that for a circuit with attenuation there is an increasing wave for all values of b (electron velocity). The velocity parameters y_1 and y_2 are now distinct for all values of b.

We see that the maximum value of x_1 decreases as loss is increased. This can be brought out more clearly by showing x_1 vs b on an expanded scale. It is perhaps more convenient to plot B, the db gain per wavelength per unit C, vs b, and this has been done for various values of d in Fig. 8.4.

We see that for small values of d the maximum value of x_1 occurs very near to $b = 0$. If we let $b = 0$ in (8.17) and (8.18) we obtain

$$y(x^2 - y^2) + 2xy(x + d) + 1 = 0 \tag{8.19}$$

$$(x^2 - y^2)(x + d) - 2xy^2 = 0 \tag{8.20}$$

We can rewrite (8.20) in the form

$$y = \pm x \left(\frac{1 + d/x}{3 + d/x}\right)^{1/2} \tag{8.21}$$

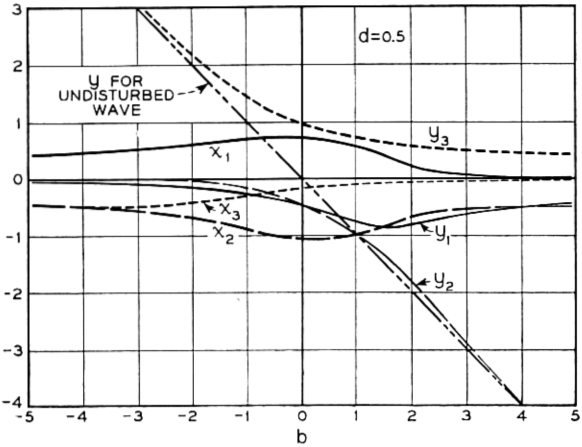

Fig. 8.2—The x's and y's for a circuit with attenuation ($d = .5$).

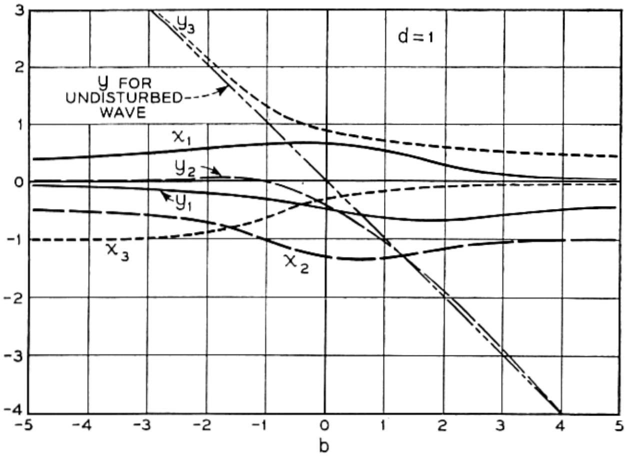

Fig. 8.3—The x's and y's for a circuit with attenuation ($d = 1$).

If we substitute this into (8.19) we can solve for x in terms of the parameter d/x

$$x = \mp \left[\frac{\left(\dfrac{3 + d/x}{1 + d/x}\right)^{1/2}}{2\left(\dfrac{1}{3 + d/x} + 1 + d/x\right)}\right]^{1/3} \tag{8.22}$$

Here we take both upper signs or both lower signs in (8.21) and (8.22).

If we assume $d/x \ll 1$ and expand, keeping no powers of d/x higher than the first, we obtain

$$x = \mp (\sqrt{3}/2)(1 - (1/3(d/x)) \tag{8.23}$$

The plus sign will give x_1, which is the x for the increasing wave. Let x_{10} be the value of x_1 for $d = 0$ (no loss).

$$x_{10} = \sqrt{3}/2 \tag{8.24}$$

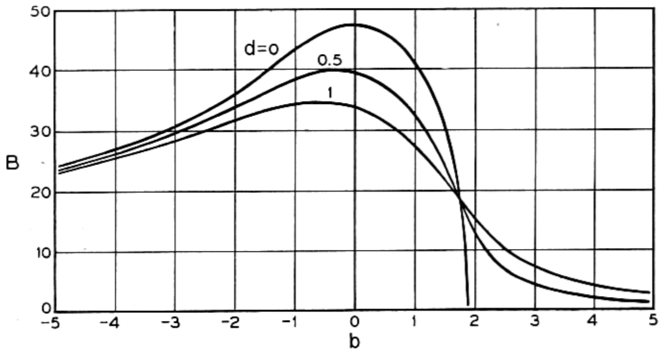

Fig. 8.4—The gain of the increasing wave is BCN db, where N is the number of wavelengths.

Then for small values of d

$$x_1 = x_{10}(1 - (1/3)(d/x_{10}))$$
$$x_1 = x_{10} - 1/3d \tag{8.25}$$

This says that, for small losses, the reduction of gain *of the increasing wave* from the gain in db for zero loss is $\frac{1}{3}$ of the circuit attenuation in db. The reduction of *net gain*, which will be greater, can be obtained only by matching boundary conditions in the presence of loss (see Chapter IX).

In Fig. 8.5, $B = 54.6\ x_1$ has been plotted vs d from (8.22). The straight line is for $x_{10} = d/3$.

In Fig. 8.6, $-x_1$, x_2 and x_3 have been plotted vs d for a large range in d. As the circuit is made very lossy, the waves which for no loss are unattenuated and increasing turn into a pair of waves with equal and opposite small attenuations. These waves will be essentially disturbances in the electron stream, or space-charge waves. The original decreasing wave turns into a wave which has the attenuation of the circuit, and is accompanied by small disturbances in the electron stream.

8.3 Space-Charge Effects

Suppose that we let d, the attenuation parameter, be zero, but consider cases in which the space-charge parameter QC is not zero. We then obtain

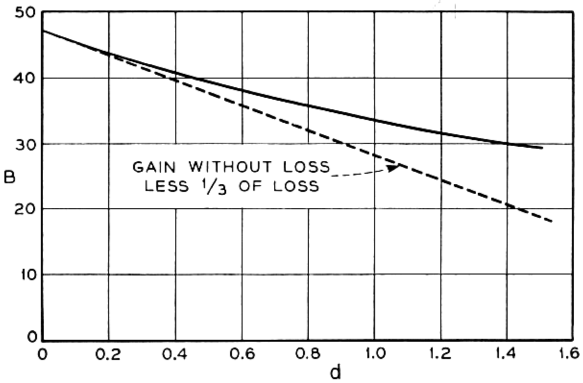

Fig. 8.5—For $b = 0$, that is, for electrons with a velocity equal to the circuit phase velocity, the gain factor B falls as the attenuation parameter d is increased. For small values of d, the gain is reduced by $\frac{1}{3}$ of the circuit loss.

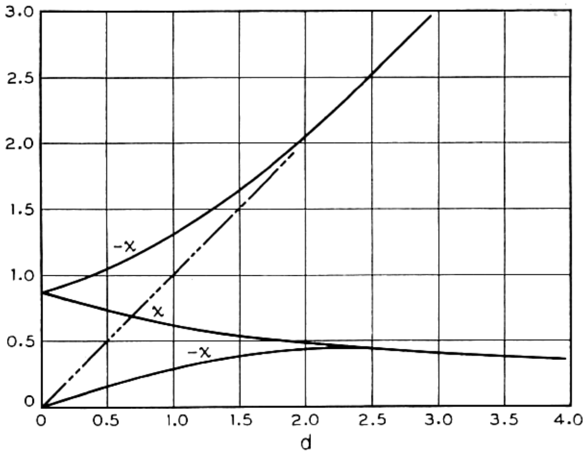

Fig. 8.6—How the three x's vary for $b = 0$ and for large losses.

the equations

$$(x^2 - y^2)(b + y) + 2x^2y + 4QC(b + y) + 1 = 0 \tag{8.26}$$

$$x[(x^2 - y^2) - 2y(y + b) + QC] = 0 \tag{8.27}$$

Solutions of this have been found by numerical methods for $QC = .25$, .5 and 1; these are shown in Figs. 8.7–8.9.

We see at once that the electron velocity for maximum gain shifts markedly as QC is increased. Hence, the region around $b = 0$ is not in this case worthy of a separate investigation.

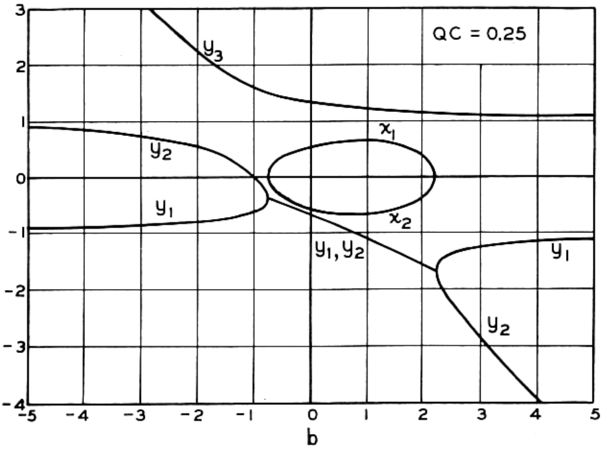

Fig. 8.7—The x's and y's for the three waves with zero loss ($d = 0$) but with space charge ($QC = .25$).

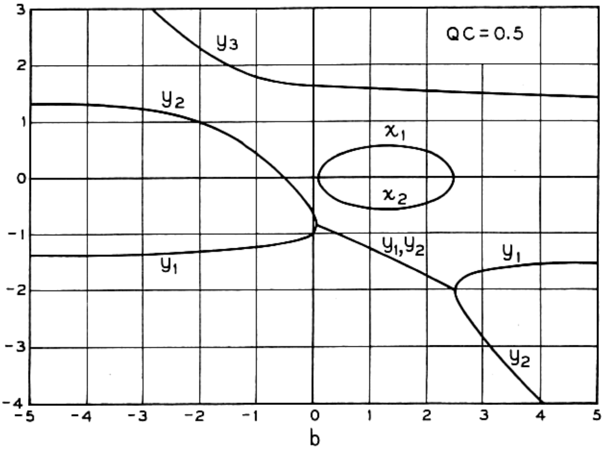

Fig. 8.8—The x's and y's with greater space charge ($QC = .5$).

It is interesting to plot the maximum value of x_1 vs. the parameter QC. This has, in effect, been done in Fig. 8.10, which shows B, the gain in db per wavelength per unit C, vs. QC.

We can obtain a curve proportional to db per wavelength by plotting BQC vs. QC. (Q is independent of current.) This has been done in Fig. 8.11. For $QC < 0.025$, the gain in db per wavelength varies linearly with

QC. Chu and Rydbeck found that under certain conditions gain varies approximately as the $\frac{1}{4}$ power of the current. This would mean a slope of $\frac{3}{4}$ on Fig. 8.11. A $\frac{3}{4}$ power dashed line is shown in Fig. 8.11; it fits the upper part of the curve approximately.

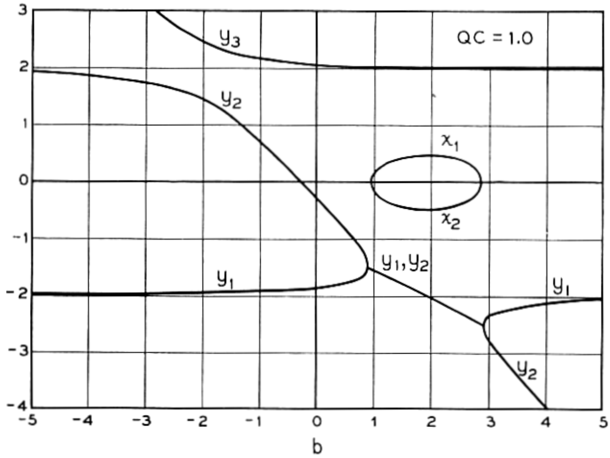

Fig. 8.9—The x's and y's with still greater space charge ($QC = 1$).

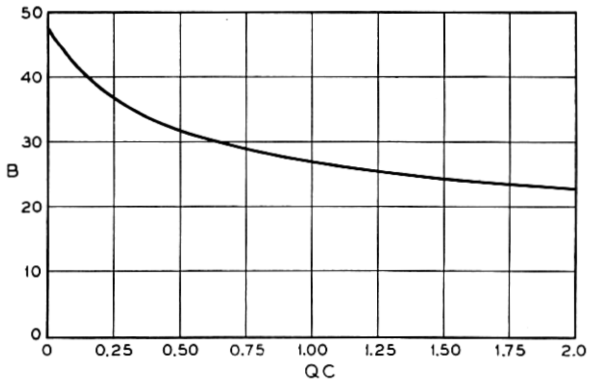

Fig. 8.10—How the gain factor B decreases as QC is increased, for the value of b which gives a maximum value of x_1.

If we examine Figs. 8.7–8.9 we find that for large and small values of b there are, as in other cases, a circuit wave, for which y is nearly equal to $-b$, and two space-charge waves. For these, however, y does not approach zero.

Let us consider equation (7.13). If b is large, the first term on the right becomes small, and we have approximately

$$\delta = \pm j2\sqrt{QC} \tag{8.28}$$

These waves correspond to the space-charge waves of Hahn and Ramo, and are quite independent of the circuit impedance, which appears in (8.28) merely as an arbitrary parameter defining the units in which δ is measured. Equation (8.28) also describes the disturbance we would get if we shorted out the circuit by some means, as by adding excessive loss.

Practically, we need an estimate of the value of Q for some typical circuit. In Appendix IV an estimate is made on the following basis: The helix

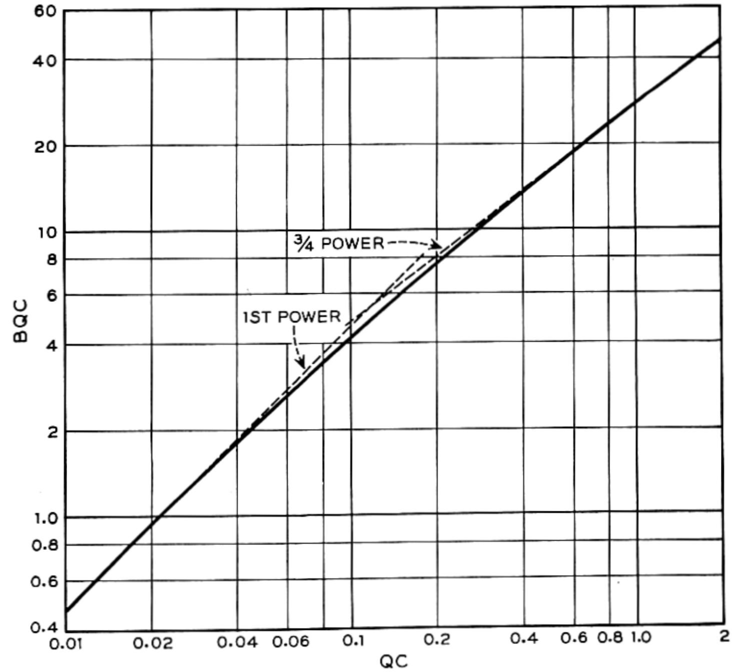

Fig. 8.11—The variation of a quantity proportional to the cube of the gain of the increasing wave (ordinate) with a quantity proportional to current (abscissa). For very small currents, the gain of the increasing wave is proportional to the $\frac{1}{3}$ power of current, for large currents to the $\frac{1}{4}$ power of current.

of radius a is replaced by a conducting cylinder of the same radius, a thin cylinder of convection current of radius a_1 and current of $i \exp(-j\beta z)$ is assumed, and the field is calculated and identified with the second term on the right of (7.1). R. C. Fletcher has obtained a more accurate value of Q by a rigorous method. His work is reproduced in Appendix VI, and in Fig. 1 of that appendix, Fletcher's value of Q is compared with the approximate value of Appendix IV.

In Fig. 8.12, the value $Q(\beta/\gamma)^2$ of Appendix IV is plotted vs. γa for $a_1/a = .9, .8, .7$. For $a_1/a = 1$, $Q = 0$. In a typical 4,000 mc traveling-wave tube, $\gamma a = 2.8$ and C is about .025. Thus, if we take the effective beam radius as .5 times the helix radius, $Q = 5.6$ and $QC = .14$.

We note from (7.14) that Q is the ratio of a capacitive impedance to $(E^2/\beta^2 P)$. In obtaining the curves of Fig. 8.12, the value of $(E^2/\beta^2 P)$ for a helically conducting sheet was assumed. This is given by (3.8) and (3.9). If $(E^2/\beta^2 P)$ is different for the circuit actually used, and it is somewhat different, even for an actual helix, Q from Fig. 8.12 should be multiplied by $(E^2/\beta^2 P)$ for the helically conducting sheet, from (3.8) and (3.9), and divided by the value of $(E^2/\beta^2 P)$ for the circuit used.

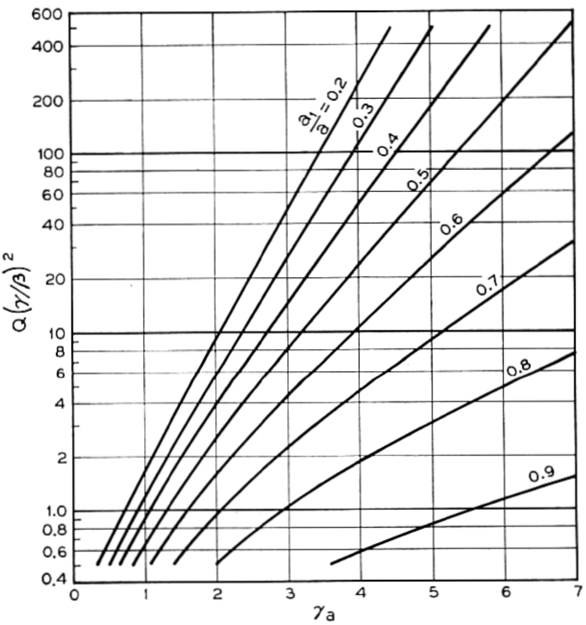

Fig. 8.12—Curves for obtaining Q for a helically conducting sheet and a hollow beam. The radius of the helically conducting sheet is a and that of the beam is a_1.

8.4 Differential Relations

It would be onerous to construct curves giving δ as a function of b for many values of attenuation and space charge. In some cases, however, useful information may be obtained by considering the effect of adding a small amount of attenuation when QC is large, or of seeing the effect of space charge when QC is small but the attenuation is large. We start with (7.13)

$$\delta^2 = \frac{1}{(-b + jd + j\delta)} - 4QC \tag{7.13}$$

Let us first differentiate (7.13) with respect to δ and d

$$2\delta\, d\delta = \frac{-j\, dd - j\, d\delta}{(-b + j\, d + j\delta)^2} \tag{8.29}$$

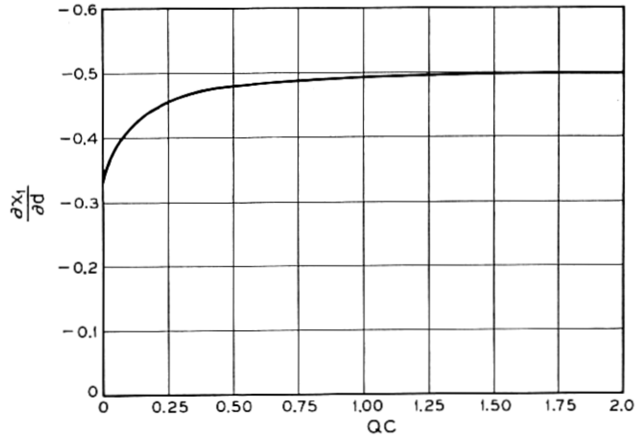

Fig. 8.13—A curve giving the rate of change of x_1 with attenuation parameter d for $d = 0$ and for various values of the space-charge parameter QC. For small values of QC the gain of the increasing wave is reduced by $\frac{1}{3}$ of the circuit loss; for large values of QC the gain of the increasing wave is reduced by $\frac{1}{2}$ of the circuit loss.

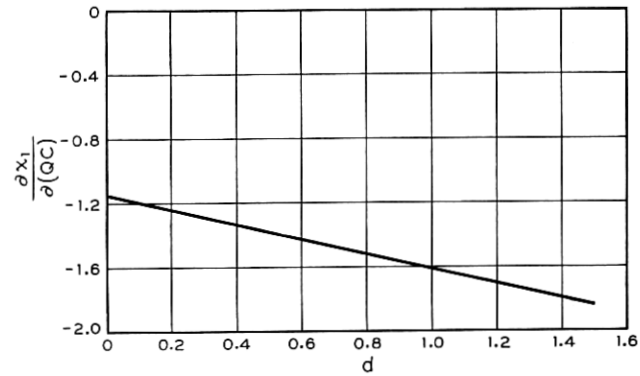

Fig. 8.14—A curve showing the variation of x_1 with QC for $QC = 0$ and for various values of the attenuation parameter d.

By using (7.13) we obtain

$$d\delta = \left(\frac{-j2\delta}{(\delta^2 + 4QC)^2} - 1\right)^{-1} dd \tag{8.30}$$

If we allow d to be small, we can use the values of δ of Figs. 8.7–8.9 to plot the quantity

$$\mathrm{Re}(d\delta_1/dd) = dx_1/dd \tag{8.31}$$

vs. QC. In Fig. 8.13, this has been done for b chosen to make x_1 a maximum. We see that a small loss dd causes more reduction of gain as QC is increased (more space charge).

Let us now differentiate (7.13) with respect to QC

$$2\delta\, d\delta = \frac{-j\, d\delta}{(-b + j\, d + j\delta)^2} - 4\, d(QC) \tag{8.32}$$

By using (7.13) with $QC = 0$ we obtain

$$d\delta = \left(\frac{-4}{2\delta + j\delta^4}\right) d(QC) \tag{8.33}$$

In Fig. 8.14, $dx/d(QC)$ has been plotted vs. d for $b = 0$.

We see that the reduction of gain for a small amount of space charge becomes greater, the greater the loss is increased (d increased).

Both Fig. 8.13 and Fig. 8.14 indicate that for large values of QC or d the gain will be overestimated if space charge (QC) and loss (d) are considered separately.

CHAPTER IX

DISCONTINUITIES

SYNOPSIS OF CHAPTER

WE WANT TO KNOW the overall gain of traveling-wave tubes. So far, we have evaluated only the gain of the increasing wave, and we must find out how strong an increasing wave is set up when a voltage is applied to the circuit.

Beyond this, we may wish for some reason to break the circuit up into several sections having different parameters. For instance, it is desirable that a traveling-wave tube have more loss in the backward direction than it has gain in the forward direction. If this is not so, small mismatches will result either in oscillation or at least in the gain fluctuating violently with frequency. We have already seen in Chapter VIII the effect of a uniform loss in reducing the gain of the increasing wave. We need to know also the overall effect of short sections of loss in order to know how loss may best be introduced.

Such problems are treated in this chapter by matching boundary conditions at the points of discontinuity. It is assumed that there is no reflected wave at the discontinuity. This will be very nearly so, because the characteristic impedances of the waves differ little over the range of loss and velocity considered. Thus, the total voltages, a-c convection currents and the a-c velocities on the two sides of the point of discontinuity are set equal.

For instance, at the beginning of the circuit, where the unmodulated electron stream enters, the total a-c velocity and the total a-c convection current—that is, the sums of the convection currents and the velocities for the three waves—are set equal to zero, and the sum of the voltages for the three waves is set equal to the applied voltage.

For the case of no loss ($d = 0$) and an electron velocity equal to circuit phase velocity ($b = 0$) we find that the three waves are set up with equal voltages, each $\frac{1}{3}$ of the applied voltage. The voltage along the circuit will then be the sum of the voltages of the three waves, and the way in which the magnitude of this sum varies with distance along the circuit is shown in Fig. 9.1. Here CN measures distance from the beginning of the circuit and the amplitude relative to the applied voltage is measured in db.

The dashed curve represents the voltage of the increasing wave alone.

For large values of CN corresponding to large gains, the increasing wave predominates and we can neglect the effect of the other waves. This leads to the gain expression

$$G = A + BCN \text{ db}$$

Here BCN is the gain in db of the increasing wave and A measures its initial level with respect to the applied voltage.

In Fig. 9.2, A is plotted vs. b for several values of the loss parameter d. The fact that A goes to ∞ for $d = 0$ as b approaches $(3/2)\ (2)^{1/3}$ does not imply an infinite gain for, at this value of b, the gain of the increasing wave approaches zero and the voltage of the decreasing wave approaches the negative of that for the increasing wave.

Figure 9.3 shows how A varies with d for $b = 0$. Figure 9.4 shows how A varies with QC for $d = 0$ and for b chosen to give a maximum value of B (the greatest gain of the increasing wave).

Suppose that for $b = QC = 0$ the loss parameter is suddenly changed from zero to some finite value d. Suppose also that the increasing wave is very large compared with the other waves reaching the discontinuity. We can then calculate the ratio of the increasing wave just beyond the discontinuity to the increasing wave reaching the discontinuity. The solid line of Fig. 9.5 shows this ratio expressed in decibels. We see that the voltage of the increasing wave excited in the lossy section is less than the voltage of the incident increasing wave.

Now, suppose the waves travel on in the lossy section until the increasing wave again predominates. If the circuit is then made suddenly lossless, we find that the increasing wave excited in this lossless section will have a greater voltage than the increasing wave incident from the lossy section, as shown by the dashed curve of Fig. 9.5. This increase is almost as great as the loss in entering the lossy section. Imagine a tube with a long lossless section, a long lossy section and another long lossless section. We see that the gain of this tube will be less than that of a lossless tube of the same total length by about the reduction of the gain of the increasing wave in lossy section.

Suppose that the electromagnetic energy of the circuit is suddenly absorbed at a distance beyond the input measured by CN. This might be done by severing a helix and terminating the ends. The a-c velocity and convection current will be unaffected in passing the discontinuity, but the circuit voltage drops to zero. For $d = b = QC = 0$, Fig. 9.6 shows the ratio of V_1, the amplitude of the increasing wave beyond the break, to V, the amplitude the increasing wave would have had if there were no break. We see that for CN greater than about 0.2 the loss due to the break is not serious. For CN large (the break far from the input) the loss approaches 3.52 db.

Beyond such a break, the total voltage increases with CN as shown in Fig. 9.7, and from $CN = 0.2$ the circuit voltage is very nearly equal to the voltage of the increasing wave.

Often, for practical reasons loss is introduced over a considerable distance, sometimes by putting lossy material near to a helix. Suppose we use CN computed as if for a lossless section of circuit as a measure of length of the lossy section, and assume that the loss is great enough so that the circuit voltage (as opposed to that produced by space charge) can be taken as zero. Such a lossy section acts as a drift space. Suppose that an increasing wave only reaches this lossy section. The amplitude of the increasing wave excited beyond the lossy section in db with respect to the amplitude of the increasing wave reaching the lossy section is shown vs. CN, which measures the length of the lossy section, in Fig. 9.8.

9.1 General Boundary Conditions

We have already assumed that C is small, and when this is so the characteristic impedance of the various waves is near to the circuit characteristic impedance K. We will neglect any reflections caused by differences among the characteristic impedances of the various waves.

We will consider cases in which the circuit is terminated in the $+z$ direction, so as to give no backward wave. We will then be concerned with the 3 forward waves, for which δ has the values δ_1, δ_2, δ_3 and the waves represented by these values of δ have voltages V_1, V_2, V_3, electron velocities v_1, v_2, v_3 and convection currents i_1, i_2, i_3.

Let V, v, i be the total voltage, velocity and convection current at $z = 0$. Then we have

$$V_1 + V_2 + V_3 = V \tag{9.1}$$

and from (7.15) and (7.16),

$$\frac{V_1}{\delta_1} + \frac{V_2}{\delta_2} + \frac{V_3}{\delta_3} = (ju_0C/\eta)v \tag{9.2}$$

$$\frac{V_1}{\delta_1^2} + \frac{V_2}{\delta_2^2} + \frac{V_3}{\delta_3^2} = (-2V_0C^2/I_0)i \tag{9.3}$$

These equations yield, when solved,

$$V_1 = [V - (\delta_2 + \delta_3)(ju_0C/\eta)v + \delta_2\delta_3(-2V_0C^2/I_0)i] \cdot [(1 - \delta_2/\delta_1)(1 - \delta_3/\delta_1)]^{-1} \tag{9.4}$$

We can obtain the corresponding expressions for V_2 and V_3 simply by interchanging subscripts; to obtain V_2, for instance, we substitute subscript 2 for 1 and subscript 1 for 2 in (9.4).

9.2 Lossless Helix, Synchronous Velocity, No Space Change

Suppose we consider the case in which $b = d = Q = 0$, so that we have the values of δ obtained in Chapter II

$$\begin{aligned} \delta_1 &= e^{-j\pi/6} = \sqrt{3}/2 - j1/2 \\ \delta_2 &= e^{-j5\pi/6} = -\sqrt{3}/2 - j1/2 \\ \delta_3 &= e^{j\pi/2} = j \end{aligned} \tag{9.5}$$

Suppose we inject an unmodulated electron stream into the helix and apply a voltage V. The obvious thing is to say that, at $z = 0$, $v = i = 0$. It is not quite clear, however, that $v = 0$ at $z = 0$ (the beginning of the circuit). Whether or not there is a stray field, which will give an initial velocity modulation, depends on the type of circuit. Two things are true, however. For the small values of C usually encountered such a velocity modulation constitutes a small effect. Also, the fields of the first part of the helix act essentially to velocity modulate the electron stream, and hence a neglect of any small initial velocity modulation will be about equivalent to a small displacement of the origin.

If, then, we let $v = i = 0$ and use (9.4) we obtain

$$V_1 = V[(1 - \delta_2/\delta_1)(1 - \delta_3/\delta_1)]^{-1} \tag{9.6}$$

$$V_1 = V/3 \tag{9.7}$$

Similarly, we find that

$$V_2 = V_3 = V/3 \tag{9.8}$$

We have used V to denote the voltage at $z = 0$. Let V_z be the voltage at z. We have

$$\begin{aligned} V_z &= (V/3)e^{-j\beta_e z}(e^{j(1/2)\beta_e Cz+(\sqrt{3}/2)\beta_e Cz} + e^{j(1/2)\beta_e Cz-(\sqrt{3}/2)\beta_e Cz} + e^{j\beta_e Cz}) \\ V_z &= (V/3)e^{-j\beta_e(1-C)z}(1 + 2\cosh((\sqrt{3}/2)\beta eCz)e^{-j(3/2)\beta_e Cz}) \end{aligned} \tag{9.9}$$

From this we obtain

$$\begin{aligned} |V_z/V|^2 = (1/9)[1 &+ 4\cosh^2(\sqrt{3}/2)\beta_e Cz \\ &+ 4\cos(3/2)\beta_e Cz \cosh(\sqrt{3}/2)\beta_e Cz] \end{aligned} \tag{9.10}$$

We can express gain in db as $10 \log_{10} |V_z/V|^2$, and, in Fig. 9.1, gain in db is plotted vs CN, where N is the number of cycles.

We see that initially the voltage does not change with distance. This is natural, because the electron stream initially has no convection current, and hence cannot act on the circuit until it becomes bunched. Finally, of course, the increasing wave must predominate over the other two, and the slope of the line must be

$$B = 47.3/CN \tag{9.11}$$

The dashed line represents the increasing wave, which starts at $V_z/V = \frac{1}{3}$ (-9.54 db) and has the slope specified by (9.11). Thus, if we write for the increasing wave that gain G is

$$G = A + BCN \text{ db} \tag{9.12}$$

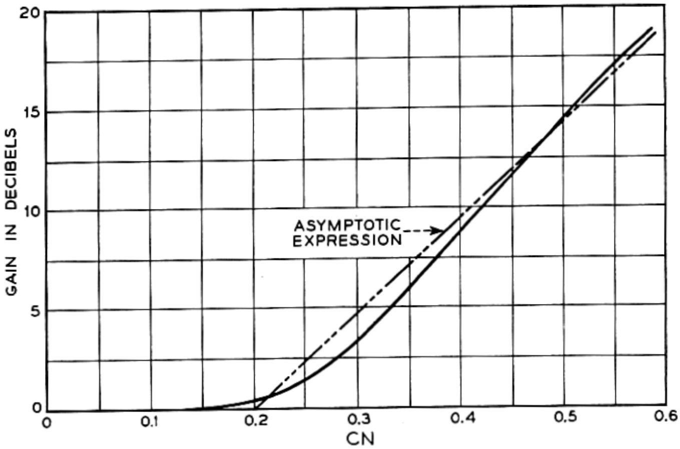

Fig. 9.1—How the signal level varies along a traveling-wave tube for the special case of zero loss and space charge and an electron velocity equal to the circuit phase velocity (solid curve). The dashed curve is the level of the increasing wave alone, which starts off with $\frac{1}{3}$ of the applied voltage, or at -9.54 *db.*

This is an asymptotic expression for the total voltage at large values of z, where $|V_1| \gg |V_2|, |V_3|$, and for $b = d = Q = 0$

$$\begin{aligned} A &= -\ 9.54 \text{ db} \\ B &= 47.3 \end{aligned} \tag{9.13}$$

We see that (9.11) is pretty good for $CN > .4$, and not too bad for $CN > .2$.

9.3 Loss in Helix

In Chapter VIII, curves were given for δ_1, δ_2, δ_3 vs. b for $QC = 0$ and for d, the loss parameter, equal to 0, 0.5 and 1. From the data from which these curves were derived one can calculate the initial loss parameter by means of (9.6)

$$A = 20 \log_{10} |V_1/V| \tag{9.14}$$

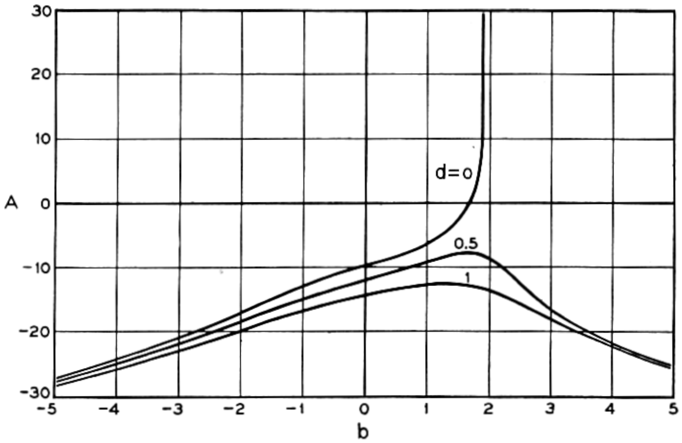

Fig. 9.2—When the gain is large we need consider the increasing wave only. Using this approximation, the gain in db is $A + BCN$ db. Here A is shown vs the velocity parameter b, several values of the attenuation parameter d, for no space charge ($QC = 0$).

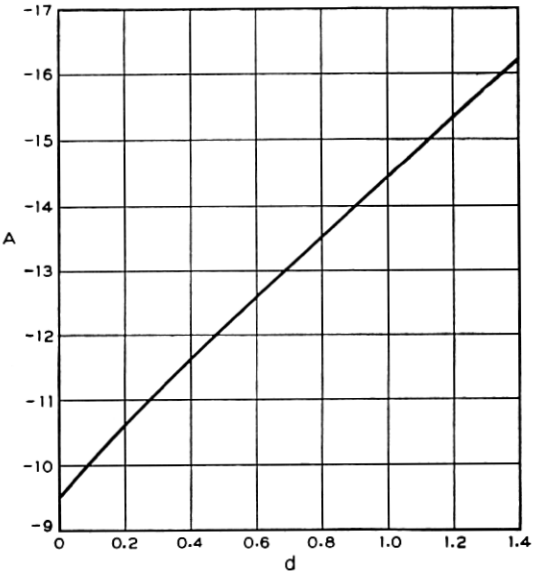

Fig. 9.3—A vs d for $b = 0$ and $QC = 0$.

In Fig. 9.2, A is plotted vs b for these three values of d.

It is perhaps of some interest to plot A vs d for $b = 0$ (the electron velocity equal to the phase velocity of the undisturbed wave). Such a plot is shown in Fig. 9.3.

9.4. Space Charge

We will now consider the case in which $QC \neq 0$. We will deal with this case only for $d = 0$, and for b adjusted for maximum gain per wavelength.

There is a peculiarity about this case in that a certain voltage V is applied to the *circuit* at $z = 0$, and we want to evaluate the *circuit* voltage associated with the increasing wave, V_{c1}, in order to know the gain.

At $z = 0$, $i = 0$. Now, the term which multiplies i to give the space-charge component of voltage (the second term on the right in (7.11)) is the same for all three waves and hence at $z = 0$ the circuit voltage is the total voltage. Thus, (9.1)–(9.3) hold. However, after V_1 has been obtained from (9.4), with

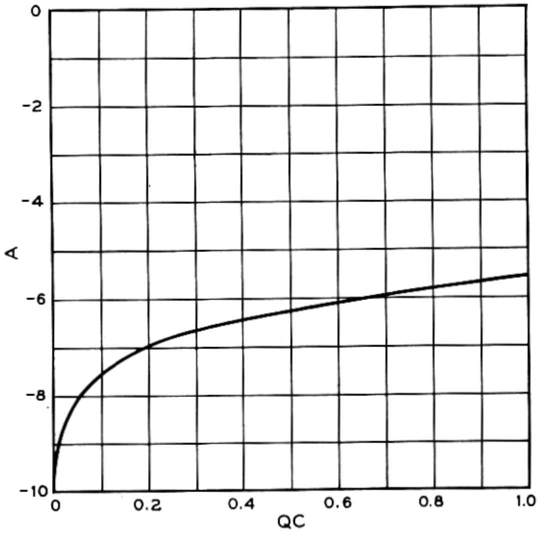

Fig. 9.4—A vs QC for $d = 0$ and b chosen for maximum gain of the increasing wave.

$V = V_1$, $v = i = 0$, then the circuit voltage V_{c1} must be obtained through the use of (7.14), and the initial loss parameter is

$$A = 20 \log_{10} | V_{c1}/V | \tag{9.15}$$

By using the appropriate values of δ, the same used in plotting Figs. 8.1 and 8.7–8.9, the loss parameter A was obtained from (9.15) and plotted vs QC in Fig. 9.4.

9.5 Change in Loss

We might think it undesirable in introducing loss to make the whole length of the helix lossy. For instance, we might expect the power output to be higher if the last part of the helix had low loss. Also, from Figs. 8.2

and 8.3 we see that the initial loss A becomes higher as d is increased. This is natural, because the electron stream can act to cause gain only after it is bunched, and if the initial section of the circuit is lossy, the signal decays before the stream becomes strongly bunched.

Let us consider a section of a lossless helix which is far enough from the input so that the increasing wave predominates and the total voltage V can be taken as that corresponding to the increasing wave

$$V = V_1 \tag{9.16}$$

Then, at this point

$$(ju_0C/\eta)v = V_1/\delta_1 \tag{9.17}$$

$$(-2V_0C^2/I_0)i = V_1/\delta_1^2\,. \tag{9.18}$$

Here δ_1 is the value for $d = 0$ (and, we assume, $b = 0$). If we substitute the values from (9.16) in (9.4), and use in (9.4) the values of δ corresponding to $b = Q = 0$, $d \neq 0$, and call the value of V_1 we obtain V_1', we obtain the ratio of the initial amplitude of the increasing wave in the lossy section to the value of the increasing wave just to the left of the lossy section. Thus, the loss in the amplitude of the increasing wave in going from a lossless to a lossy section is $20 \log_{10} | V_1'/V_1 |$. This loss is plotted vs d in Fig. 8.5.

This loss is accounted for by the fact that $| i_1/V_1 |$ becomes larger as the loss parameter d is increased. Thus, the convection current injected into the lossy section is insufficient to go with the voltage, and the voltage must fall.

If we go from a lossy section ($d \neq 0$, $b = 0$) to a lossless section ($d = 0$, $b = 0$) we start with an excess of convection current and $| V_1' |$, the initial amplitude of the increasing wave to the right of the discontinuity is greater than the amplitude $| V_1 |$ of the increasing wave to the left. In Fig. 9.5, $20 \log_{10} | V_1'/V_1 |$ is plotted vs d for this case also.

We see that if we go from a lossless section to a lossy section, and if the lossy section is long enough so that the increasing wave predominates at the end of it, and if we go back to a lossless section at the end of it, the net loss and gain at the discontinuities almost compensate, and even for $d = 3$ the net discontinuity loss is less than 1 db. This does not consider the reduction of gain of the increasing wave in the lossy section.

9.6 Severed Helix

If the loss introduced is distributed over the length of the helix, the gain will decrease as the loss is increased (Fig. 8.5). If, however, the loss is distributed over a very short section, we easily see that as the loss is increased more and more, the gain must approach a constant value. The circuit will be in effect severed as far as the electromagnetic wave is concerned, and any excitation in the output will be due to the a-c velocity and convection current of the electron stream which crosses the lossy section.

We will first idealize the situation and assume that the helix is severed and by some means terminated looking in each direction, so that the voltage falls from a value V to a value 0 in zero distance, while v and i remain unchanged.

We will consider a case in which $b = d = Q = 0$, and in which a voltage

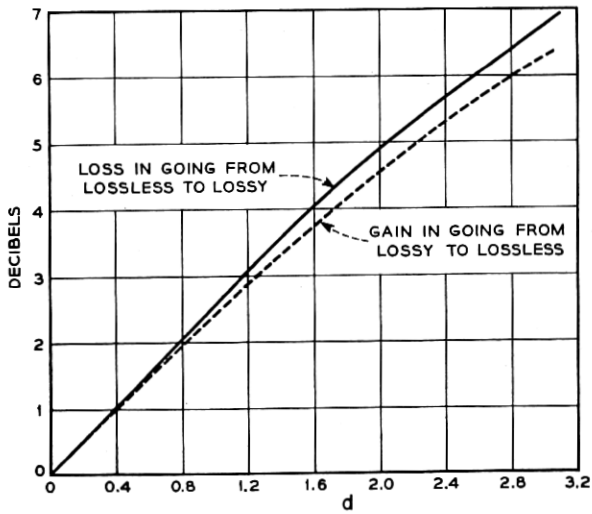

Fig. 9.5—Suppose that the circuit loss parameter changes suddenly with distance from 0 to d or from d to 0. Suppose there is an increasing wave only incident at the point of change. How large will the increasing wave beyond the point of change be? These curves tell ($b = QC = 0$).

V is applied to the helix N wavelengths before the cut. Then, just before the cut,

$$\begin{aligned} V_1 &= (V/3)e^{-j2\pi N}\, e^{2\pi NC\delta_1} \\ V_2 &= (V/3)e^{-j2\pi N}\, e^{2\pi NC\delta_2} \\ V_3 &= (V/3)e^{-j2\pi N}\, e^{2\pi NC\delta_3} \end{aligned} \tag{9.19}$$

and

$$\begin{aligned} (ju_0C/\eta)v_1 &= V_1/\delta_1 \\ (-2V_0C^3/I_0)i_1 &= V_1/\delta_1^2 \end{aligned} \tag{9.20}$$

etc.

Whence, just beyond the break which makes $V = 0$, V, v and i are

$$V = 0$$

$$(ju_0C/\eta)v = V_1/\delta_1 + V_2/\delta_2 + V_3/\delta_3 \tag{9.21}$$

$$(-2V_0C^3/I_0)i = V_1/\delta_1^2 + V_2/\delta_2^2 + V_3/\delta_3^2$$

Putting these values in (9.4), we can find V_1', the value of the increasing wave to the right of the break. The ratio of the magnitude of the increasing wave to the magnitude it would have if it were not for the break is then $|V_1'/V_1|$, and this ratio is plotted vs CN in Fig. 9.6, where N is the number of wavelengths in the first section.

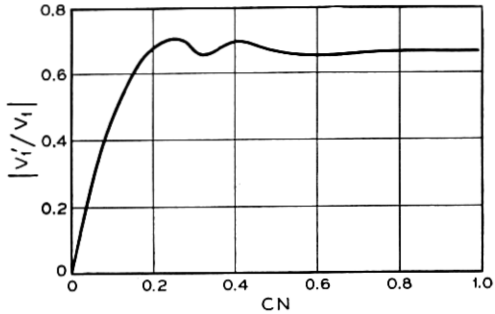

Fig. 9.6—Suppose the circuit is severed a distance measured by CN beyond the input, so that the voltage just beyond the break is zero. The ordinate is the ratio of the amplitude of the increasing wave beyond the break to that it would have had with an unbroken circuit ($b = QC = 0$).

We see that there will be least loss in severing the helix for CN equal to approximately $\frac{1}{4}$. From Fig. 9.1, we see that at $CN = \frac{1}{4}$ the voltage is just beginning to rise. In a typical 4,000 megacycle traveling-wave tube, CN is approximately unity for a 10 inch helix, so the loss should be put at least 2.5″ beyond the input. Putting the loss further on changes things little; asymptotically, $|V_1'/V|$ approaches $\frac{2}{3}$, or 3.52 db loss, for large values of CN (loss for from input).

It is of some interest to know how the voltage rises to the right of the cut. It was assumed that the cut was far from the point of excitation, so that only increasing wave of magnitude V_1 was present just to the left of the cut. The initial amplitudes of the three waves, V_1', V_2', V_3' to the right of the cut were computed and the magnitude of their sum plotted vs CN as it varies with distance to the right of the cut. The resulting curve, expressed in db with respect to the magnitude of the increasing wave V_1 just to the left of the cut, is shown in Fig. 9.7. Again, we see that at a distance $CN = \frac{1}{4}$ to the right of the cut the increasing wave (dashed straight line) predominates.

9.7 Severed Helix With Drift Space

In actually putting concentrated loss in a helix, the loss cannot be concentrated in a section of zero length for two reasons. In the first place, this is physically difficult if not impossible; in the second place it is desirable that the two halves of the helix be terminated in a reflectionless manner at the cut, and it is easiest to do this by tapering the loss. For instance, if the loss is put in by spraying aquadag (graphite in water) on ceramic rods supporting the helix, it is desirable to taper the loss coating at the ends of the lossy section.

Perhaps the best reasonably simple approximation we can make to such a lossy section is one in which the section starts far enough from the input

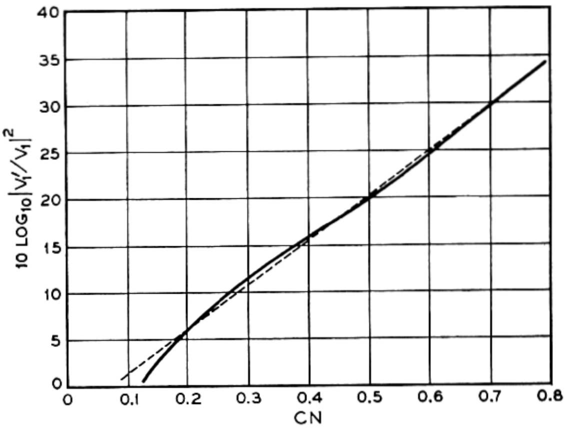

Fig. 9.7—Suppose that the circuit is severed and an increasing wave only is incident at the break. How does the signal build up beyond the break? The solid curve shows ($b = QC = 0$). 0 db is the level of the incident increasing wave.

so that at the beginning of the lossy section only an increasing wave is present. In the lossy section CN long we will consider that the loss completely shorts out the circuit, so that (8.28) holds. Thus, in the lossy section we will have only two values of δ, which we will call δ_I and δ_{II}.

$$\delta_I = jk \tag{9.21}$$

$$\delta_{II} = -jk \tag{9.22}$$

$$k = 2\sqrt{QC} \tag{9.23}$$

Let V_I and V_{II} be the voltages of the waves corresponding to δ_I and δ_{II} at the beginning of the lossy section. Let δ_1, δ_2, δ_3 be the values of δ to the left and right of the lossy section. Let V_1 be the amplitude of the increasing wave just to the left of the lossy section. Then, by equating velocities and convection currents at the start of the lossy section, we obtain

$$V_1/\delta_1 = V_I/\delta_I + V_{II}/\delta_{II} \tag{9.24}$$

and, from (9.21) and (9.22)

$$V_1/\delta_1 = (-j/k)(V_I - V_{II}) \tag{9.25}$$

Similarly

$$\begin{aligned} V_1/\delta_1^2 &= V_I/\delta_{II}^2 + V_{II}\,\delta_{II}^2 \\ V_1/\delta_1^2 &= -(1/k^2)(V_I + V_{II}) \end{aligned} \tag{9.26}$$

So that

$$V_I = j(V_1/2)(k/\delta_1)(jk/\delta_1 + 1) \tag{9.27}$$

$$V_{II} = j(V_1/2)(k/\delta_1)(jk/\delta_1 - 1) \tag{9.28}$$

At the output of the lossy section we have the voltages V_I' and V_{II}'

$$V_I' = V_I e^{-j2\pi N} e^{-j2\pi kCN} \tag{9.29}$$

$$V_{II}' = V_{II} e^{-j2\pi N} e^{-j2\pi kCN} \tag{9.30}$$

Thus, at the end of the lossy section we have

$$V = V_I' + V_{II}' \tag{9.31}$$

$$\begin{aligned} (ju_0C/\eta)v &= V_I'/\delta_I + V_{II}'/\delta_{II} \\ (ju_0C/\eta)v &= (-j/k)(V_I' - V_{II}') \end{aligned} \tag{9.32}$$

and similarly

$$(-2V_0C^2/I_0)i = (-1/k^2)(V_I' + V_{II}') \tag{9.33}$$

From (9.27) and (9.28) we see that

$$V_I' + V_{II}' = -(k/\delta_1)[+(k/\delta_1)\cos 2\pi kCN + \sin 2\pi kCN]V_1 e^{-j2\pi N} \tag{9.34}$$

$$V_I' - V_{II}' = j(k/\delta_1)[-(k/\delta_1)\sin 2\pi kCN + \cos 2\pi kCN]V_1 e^{-j2\pi N} \tag{9.35}$$

Whence

$$V = -(k/\delta_1)[+(k/\delta_1)\cos 2\pi kCN + \sin 2\pi kCN]V_1 e^{-j2\pi N} \tag{9.36}$$

$$(ju_0C/\eta)v = (1/\delta_1)[-(k/\delta_1)\sin 2\pi kCN + \cos 2\pi kCN]V_1 e^{-j2\pi N} \tag{9.37}$$

$$(-2V_0C^2/I_0)i = (1/\delta_1)[(1/\delta_1)\cos 2\pi kCN + (1/k)\sin 2\pi kCN]V_1 e^{-j2\pi N} \tag{9.38}$$

These can be used in connection with (9.4) in obtaining V_1', the value of V_1 just beyond the lossy section; that is, the amplitude of the component of increasing wave just beyond the lossy section.

In typical traveling-wave tubes the lossy section usually has a length such that CN is $\frac{1}{4}$ or less. In Fig. 9.8 the loss in db in going through the lossy section, $20 \log_{10} | V_1'/V_1 |$, has been plotted vs. CN for $QC = 0, .25, .5$ for the range $CN = 0$ to $CN = .5$.

We see that, for low space charge, increasing the length of a drift space increases the loss. For higher space charge it may either increase or decrease the loss. It is not clear that the periodic behavior characteristic of the curves for $QC = 0.5$ and 1, for instance, will obtain for a drift space with tapered loss at each end. The calculations may also be considerably in error for broad electron beams (γa large). The electric field pattern in the helix differs

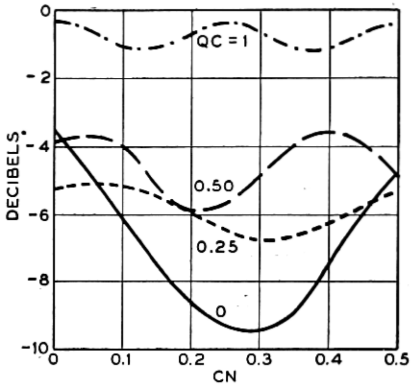

Fig. 9.8—Suppose that we break the circuit and insert a drift tube of length measured by CN in terms of the traveling-wave tube C and N. Assume an increasing wave only before the drift tube. The increasing wave beyond the drift tube will have a level with respect to the incident increasing wave as shown by the ordinate. Here $d = 0$ and b is chosen to maximize x_1.

from that in the drift space. In the case of broad electron beams this may result in the excitation in the drift space of several different space charge waves having different field patterns and different propagation constants.

A suggestion has been made that the introduction of loss itself has a bad effect. The only thing that affects the electrons is an electric field. Unpublished measurements made by Cutler mode by moving a probe along a helix indicate that in typical short high-loss sections the electric field of the helix is essentially zero. Hence, except for a short distance at the ends, such lossy sections should act simply as drift spaces.

9.8 Overall Behavior of Tubes

The material of Chapters VIII and IX is useful in designing traveling-wave tubes. Prediction of the performance of a given tube over a wide range of voltage and current is quite a different matter. For instance, in order to predict gain for voltage or current ranges for which the gain is small, the three waves must be taken into account. As current is varied, the loss parameter d varies, and this means different x's and y's must be computed for different currents. Finally, at high currents, the space-charge parameter Q must be taken into account. In all, a computation of tube behavior under a variety of conditions is an extensive job.

Fortunately, for useful tubes operating as intended, the gain is high. When this is so, the gain can be calculated quite accurately by asymptotic relations. Such an overall calculation of the gain of a helix-type tube with distributed loss is summarized in Appendix VII.

CHAPTER X

NOISE FIGURE

Synopsis of Chapter

BECAUSE THERE IS no treatment of the behavior at high frequencies of an electron flow with a Maxwellian distribution of velocities, one might think there could be no very satisfactory calculation of the noise figure of traveling-wave tubes. Various approximate calculations can be made, and two of these will be discussed here. Experience indicates that the second and more elaborate of these is fairly well founded. In each case, an approximation is made in which the actual multi-velocity electron current is replaced by a current of electrons having a single velocity at a given point but having a mean square fluctuation of velocity or current equal to a mean square fluctuation characteristic of the multi-velocity flow.

In one sort of calculation, it is assumed that the noise is due to a current fluctuation equal to that of shot noise (equation (10.1)) in the current entering the circuit. For zero loss, an electron velocity equal to the phase velocity of the circuit and no space charge, this leads to an expression for noise figure (10.5), which contains a term proportional to beam voltage V_0 times the gain parameter C. One can, if he wishes, add a space-charge noise reduction factor multiplying the term 80 V_0C. This approach indicates that the voltage and the gain per wavelength should be reduced in order to improve the noise figure.

In another approach, equations applying to single-valued-velocity flow between parallel planes are assumed to apply from the cathode to the circuit, and the fluctuations in the actual multi-velocity stream are represented by fluctuations in current and velocity at the cathode surface. It is found that for space-charge-limited emission the current fluctuation has no effect, and so all the noise can be expressed in terms of fluctuations in the velocity of emission of electrons.

For a special case, that of a gun with an anode at circuit potential V_0, a cathode-anode transit angle θ_1, and an anode-circuit transit angle θ_2, an expression for noise figure (10.28) is obtained. This expression can be rewritten in terms of a parameter L which is a function of P

$$F = 1 + (\tfrac{1}{2})(4 - \pi)(T_c/T)(1/C)L$$

$$P = (\theta_1 - \theta_2)C$$

Formally, F can be minimized by choosing the proper value of P. In Fig. 10.3, the minimum value of L, L_m, is plotted vs. the velocity parameter b for zero loss and zero space charge ($d = QC = 0$). The corresponding value of P, P_m, is also shown.

P is a function of the cathode-anode transit angle θ_1, which cannot be varied without changing the current density and hence C, and of anode-circuit transit angle θ_2, which can be given any value. Thus, P can be made very small if one wishes, but it cannot be made indefinitely large, and it is not clear that P can always be made equal to P_m. On the other hand, these expressions have been worked out for a rather limited case: an anode potential equal to circuit potential, and no a-c space charge. It is possible that an optimization with respect to gun anode potential and space charge parameter QC would predict even lower noise figures, and perhaps at attainable values of the parameters.

In an actual tube there are, of course, sources of noise which have been neglected. Experimental work indicates that partition noise is very important and must be taken into account.

10.1 Shot Noise in the Injected Current

A stream of electrons emitted from a temperature-limited cathode has a mean square fluctuation in convection current $\overline{i_s^2}$

$$\overline{i_s^2} = 2eI_0B_0 \tag{10.1}$$

Here e is the charge on an electron, I_0 is the average or d-c current and B is the bandwidth in which the frequencies of the current components whose mean square value is $\overline{i_s^2}$ lie. Suppose this fluctuation in the beam current of a traveling-wave tube were the sole cause of an increasing wave ($V = v = 0$). Then, from (9.4) the mean square value of that increasing wave,, $\overline{V_{1s}^2}$, would be

$$\overline{V_{1s}^2} = (8eBV_0^2C^4/I_0)\ |\ \delta_2\delta_3\ |^2\ |\ (1 - \delta_2/\delta_1)(1 - \delta_3/\delta_1)\ |^{-2} \tag{10.2}$$

Now, suppose we have an additional noise source: thermal noise voltage applied to the circuit. If the helix is matched to a source of temperature T, the thermal noise power P_t drawn from the source is

$$P_t = kTB \tag{10.3}$$

Here k is Boltzman's constant, T is temperature in degrees Kelvin and, as before, B is bandwidth in cycles. If K_t is the longitudinal impedance of the circuit the mean square noise voltage $\overline{V_t^2}$ associated with the circuit will be

$$\overline{V_t^2} = kTBK_t \tag{10.4}$$

and the component of increasing wave excited by this voltage, $\overline{V_{1t}^2}$, will be, from (9.4),

$$\overline{V_{1t}^2} = kTBK_t \mid (1 - \delta_2/\delta_1)(1 - \delta_3/\delta_1) \mid^{-2} \tag{10.5}$$

The noise figure of an amplifier is defined as the ratio of the total noise output power to the noise output power attributable to thermal noise at the input alone. We will regard the mean-square value of the initial voltage V_1 of the increasing wave as a measure of noise output. This will be substantially true if the signal becomes large prior to the introduction of further noise. For example, it will be substantially true in a tube with a severed helix if the helix is cut at a point where the increasing wave has grown large compared with the original fluctuations in the electron stream which set it up.

Under these circumstances, the noise figure F will be given by

$$F = (\overline{V_{1s}^2} + \overline{V_{1t}^2})/(\overline{V_{1t}^2})$$

$$F = 1 + (e/kT)(8V_0^2C^4/I_0K_t) \mid \delta_2\delta_3 \mid^2 \tag{10.3}$$

Now we have from Chapter II that

$$C^3 = I_0K_t/4V_0$$

whence

$$F = 1 + 2(eV_0/kT)C \mid \delta_2\delta_3 \mid^2 \tag{10.4}$$

The standard reference temperature is 290°K. Let us assume $b = d = QC = 0$. For this case we have found $\mid \delta_2 \mid = \mid \delta_3 \mid = 1$. Thus, for these assumptions we find

$$F = 1 + 80V_0C \tag{10.5}$$

A typical value of V_0 is 1,600 volts; a typical value of C is .025. For these values

$$F = 3,201$$

In db this is a noise of 35 db.

This is not far from the noise figure of traveling-wave tubes when the cathode temperature is lowered so as to give temperature-limited emission. The noise figure of traveling-wave tubes in which the cathode is at normal operating temperature and is active, so that emission is limited by space-charge, can be considerably lower. In endeavoring to calculate the noise figure for space-charge-limited electron flow from the cathode we must proceed in a somewhat different manner.

10.2 The Diode Equations

Llewellyn and Peterson[1] have published a set of equations governing the behavior of parallel plane diodes with a single-valued electron velocity. They sum up the behavior of such a diode in terms of nine coefficients A^*–I^*, in the following equations

$$V_b - V_a = A^* I + B^* q_a + C^* v_a \tag{10.6}$$

$$q_b = D^* I + E^* q_a + F^* v_a \tag{10.7}$$

$$v_b = G^* I + H^* q_a + I^* v_a \tag{10.8}$$

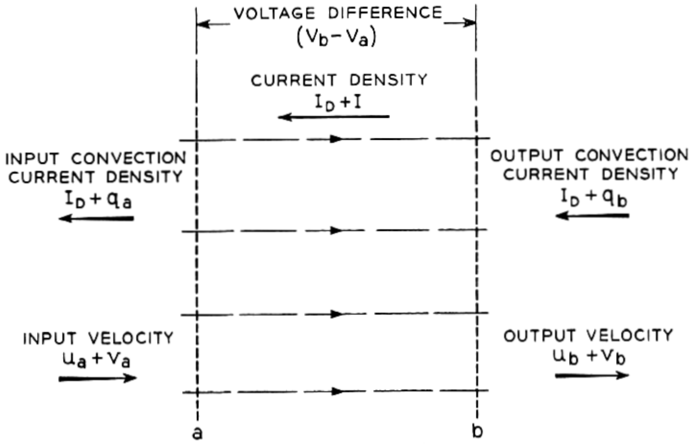

Fig. 10.1—Parallel electron flow between two planes *a* and *b* normal to the flow, showing the currents, velocities and voltages.

These equations and the values of the various coefficients in terms of current, electron velocity and transit angle are given in Appendix V. The diode structure to which they apply is indicated in Fig. 10.1. Electrons enter normal to the left plane and pass out at the right plane. The various quantities involved are transit angle between the two planes and:

I_0	d-c current density to left
I	a-c current density to left
q_a	a-c convection current density to left at input plane *a*
q_b	a-c convection current density to left at output plane *b*
u_a	d-c velocity to right at plane *a*
u_b	d-c velocity to right at plane *b*
v_a	a-c velocity to right at plane *a*
v_b	a-c velocity to right at plane *b*
V_b-V_a	a-c potential difference between plane *b* and plane *a*

[1] F. B. Llewellyn and L. C. Peterson, "Vacuum Tube Networks," *Proc. I.R.E.*, Vol. 32, pp. 144–166, March, 1944.

We will notice that I and the q's are current *densities* and that, contrary to the convention we have used, they are taken as positive to the left. Thus, if the area is σ, we would write the output convection current; as

$$i = -\sigma q_b$$

where q_b is the convection current density used in (10.6)–(10.8).

Peterson has used (10.6)–(10.8) in calculating noise figure by replacing the actual multi-velocity flow from the cathode by a single-velocity flow with the same mean square fluctuation in velocity, namely,[2]

$$\overline{v_t^2} = (4 - \pi)\eta\,(kT_c/I_0)B \tag{10.9}$$

Here T_c is the cathode temperature in degrees Kelvin and I_0 is the cathode current.

Whatever the justification for such a procedure, Rack[3] has shown that it gives a satisfactory result at low frequencies, and unpublished work by Cutler and Quate indicates surprisingly good quantitative agreement under conditions of long transit angle at 4,000 mc.

We must remember, however, that the available values of the coefficients of (10.6)–(10.8) are for a broad electron beam in which there are a-c fields in the z direction only. Now, the electron beam in the gun of a traveling-wave tube is ordinarily rather narrow. While the a-c fields may be substantially in the z-direction near the cathode, this is certainly not true throughout the whole cathode-anode space. Thus, the coefficients used in (10.6)–(10.8) are certainly somewhat in error when applied to traveling-wave tube guns.

Various plausible efforts can be made to amend this situation, as, by saying that the latter part of the beam in the gun acts as a drift region in which the electron velocities are not changed by space-charge fields. However, when one starts such patching, he does not know where to stop. In the light of available knowledge, it seems best to use the coefficients as they stand for the cathode-anode region of the gun.

Let us then consider the electron gun of the traveling-wave tube to form a space-charge limited diode which is short-circuited at high frequencies.

If we assume complete space charge (space-charge limited emission) and take the electron velocity at the cathode to be zero, we find that the quantities multiplying q_a in (10.6)–(10.8) are zero.

$$B^* = E^* = H^* = 0^* \tag{10.10}$$

[2] L. C. Peterson, "Space-Charge and Transit-Time Effects on Signal and Noise in Microwave Tetrodes," *Proc. I.R.E.*, Vol. 35, pp. 1264–1272, November, 1947.

[3] A. J. Rack, "Effect of Space Charge and Transit Time on the Shot Noise in Diodes," *Bell System Technical Journal*, Vol. 17, pp. 592–619, October, 1938.

Accordingly, the magnitude of the noise convection current at the cathode does not matter. If we assume that the gun is a short-circuited diode as far as r-f goes

$$V_b - V_a = 0 \tag{10.11}$$

Then from (10.6), (10.10) and (10.11) we obtain

$$I = -\frac{C^*}{A^*} v_a \tag{10.12}$$

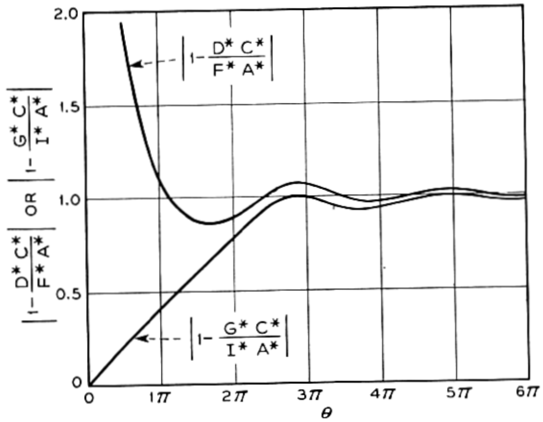

Fig. 10.2—Some expressions useful in noise calculations, showing how they approach unity at large transit angles.

Accordingly, from (10.7) and (10.8) we obtain

$$q_b = \left(1 - \frac{D^*C^*}{F^*A^*}\right) F^* v_a \tag{10.13}$$

$$v_b = \left(1 - \frac{G^*C^*}{I^*A^*}\right) I^* v_a \tag{10.14}$$

In Fig. 10.2, $|1 - D^*C^*/F^*A^*|$ and $|1 - G^*C^*/I^*A^*|$ are plotted vs θ, the transit angle. We see that for transit angles greater than about 3π these quantities differ negligibly from unity, and we may write

$$q_b = F^* v_a \tag{10.15}$$

$$v_b = I^* v_a \tag{10.16}$$

More specifically, we find

$$q_b = \frac{v_a I_0 \beta_1 e^{-\beta_1}}{u_b} \tag{10.17}$$

$$v_b = -v_a e^{-\beta_1} \tag{10.18}$$

Here β_1 is j times the transit angle in radians from cathode to anode. For v_a we use a velocity fluctuation with the mean-square value given by (10.9).

Suppose now that there is a constant-potential drift space following the diode anode, of length β_2/j in radians. If we apply (10.6)–(10.8) and assume that the space-charge is small and the transit angle long, we find that q_b', the value of q_b at the end of this drift space, is given in terms of q_a' and v_a', the values at the beginning of this drift space, by

$$q_b' = (q_a' + (I_0/u_b)\beta_2 v_a')e^{-\beta_2} \tag{10.19}$$

The case of v_b', the velocity at the end of this drift space, is a little different. The first term on the right of (10.8) can be shown to be negligible for long transit angles and small space charge. The last term on the right represents the purely kinematic bunching. For the assumption of small space charge the middle term gives not zero but a first approximation of a space-charge effect, assuming that all the space-charge field acts longitudinally. Thus, this middle term gives an overestimate of the effect of space-charge in a narrow, high-velocity beam. If we include both terms, we obtain

$$v_b' = H_2^* q_a' + e^{-\beta_2} v_a' \tag{10.20}$$

Here the term on the right is the purely kinematic term.*

Now, the current from the gun is assumed to go into the drift space, so that q_a' is q_b from (10.17) and v_a' is v_a from (10.18). The *d-c* velocity at the gun anode and throughout the drift space are both given by u_b. If we make these substitutions in (10.19) and (10.20) we obtain

$$q_b' = (I_0/u_b)(\beta_1 - \beta_2)e^{-(\beta_1+\beta_2)} v_a \tag{10.21}$$

$$v_b' = -\left(2\frac{\beta_1}{\beta_2} + 1\right) e^{-(\beta_1+\beta_2)}\, v_a \tag{10.22}$$

The term $2\beta_1/\beta_2$ in (10.22) is the "space-charge" term. We will in the following analysis omit this, making the same sort of error we do in neglecting space charge in the traveling-wave section of the tube. If space charge in the drift space is to be taken into account, it is much better to proceed as in 9.7.

From the drift-space the current goes into the helix. It is now necessary to change to the notation we have used in connection with the traveling-wave tube. The chief difference is that we have taken currents as positive to the right, but allowed I_0 to be the d-c current to the left. If i and v are

* The first term has been written as shown because it is easiest to use the small space-charge value of H^* for the drift region (H_2^*) in connection with the space-charge limited value of F^* for the cathode-anode region rather than in connection with (10.17).

our a-c convection current and velocity at the beginning of the helix, and I_0 and u_0 the d-c beam current and velocity, and σ the area of the beam,

$$\begin{aligned} i &= -\sigma q_b' \\ v &= v_b \\ I_0 &= \sigma I_0 \\ u_0 &= u_0 \end{aligned} \tag{10.23}$$

In addition, we will use transit angles θ_1 and θ_2 in place of β_1 and β_2

$$\begin{aligned} \beta_1 &= j\theta_1 \\ \beta_2 &= j\theta_2 \end{aligned} \tag{10.24}$$

We then obtain from (10.21) and (10.22)

$$q = -j(I_0/u_0)(\theta_1 - \theta_2)e^{-j(\theta_1+\theta_2)}v_a \tag{10.25}$$

$$v = -e^{-j(\theta_1+\theta_2)}v_a \tag{10.26}$$

10.3 Overall Noise Figure

We are now in a position to use (9.4) in obtaining the overall noise figure. We have already assumed that the space-charge is small in the drift space between the gun anode and the helix ($QC = 0$). If we continue to assume this in connection with (9.4), the only voltage is the helix voltage and for the noise caused by the velocity fluctuation at the cathode, v_a, $V = 0$ at the beginning of the helix. Thus, the mean square initial noise voltage of the increasing wave, $\overline{V_{1s}^2}$, will be, from (10.21), (10.22), (9.4) and (10.9),

$$\begin{aligned} \overline{V_{1s}^2} = (2(4 - \pi)kT_cCBV_0/I_0)| \, \delta_2\delta_3(\theta_1 - \theta_2)C + (\delta_2 + \delta_3) \,|^2 \\ |\, (1 - \delta_2/\delta_1)(1 - \delta_3/\delta_1) \,|^{-2} \end{aligned} \tag{10.27}$$

As before, we have, from the thermal noise input to the helix

$$\overline{V_{1t}^2} = kTBK_t| \, (1 - \delta_2/\delta_1)(1 - \delta_3/\delta_1) \,|^{-2} \tag{10.5}$$

and the noise figure becomes

$$F = 1 + \overline{V_{1s}^2}/\overline{V_{1t}^2}$$

$$F = 1 + (1/2)(4 - \pi)(T_c/T)(1/C)| \, \delta_2\delta_3(\theta_1 - \theta_2)C + (\delta_2 + \delta_3) \,|^2 \tag{10.28}$$

Here use has been made of the fact that

$$C = K_tI/4V_0$$

Let us investigate this for the case $b = d = 0$ (we have already assumed $QC = 0$). In this case

$$\delta_2 = \sqrt{3}/2 - j1/2$$

$$\delta_3 = j$$

and we obtain

$$F = 1 + (1/2)(4 - \pi)(T_c/T)(1/C)|\,(P/2 - \sqrt{3}/2) - j(\sqrt{3}P/2 - 1/2)\,|^2 \tag{10.29}$$

$$P = (\theta_1 - \theta_2)C \tag{10.30}$$

For a given gun transit-angle θ_1, the parameter P can be given values ranging from $\theta_1 C$ to large negative values by increasing the drift angle θ_2 between the gun anode and the beginning of the helix.

We see that

$$F = 1 + (1/2)(4 - \pi)(T_c/T)(1/C)(P^2 - \sqrt{3}P + 1) \tag{10.31}$$

The minimum value of $(P^2 - \sqrt{3}P + 1)$ occurs when

$$P = \sqrt{3}/2 \tag{10.32}$$

if the product of the gun transit angle and C is large enough, this can be attained. The corresponding value of $(P^2 - \sqrt{3}P + 1)$ is $\frac{1}{4}$, and the corresponding noise figure is

$$F = 1 + (1/2)(1 - \pi/4)(T_c/T)(1/C) \tag{10.33}$$

A typical value for T_c is 1020°K, and for a reference temperature of 290°K,

$$T_c/T = 3.5$$

A typical value of C is .025. For these values

$$F = 17$$

or a noise figure of 12 db.

Let us consider cases for no attenuation or space-charge but for other electron velocities. In this case we write, as before

$$\delta_2 = x_2 + jy_2$$

$$\delta_3 = x_3 + jy_3$$

Let us write, for convenience,

$$L = |\,\delta_2\delta_3 P + \delta_1 + \delta_2\,|^2 \tag{10.34}$$

Then we find that

$$L = [(x_2x_3)^2 + (y_2y_3)^2 + (x_2y_3)^2 + (x_3y_2)^2]P^2$$
$$+ 2[x_3(y_2^2 + x_2^2) + x_2(x_3^2 + y_3^2)]P \quad (10.35)$$
$$+ (x_2 + x_3)^2 + (y_2 + y_3)^2$$

This has a minimum value for $P = P_m$

$$P_m = \frac{-[x_3(x_2^2 + y_2^2) + x_2(x_3^2 + y_3^2)]}{(x_2 x_3)^2 + (y_2 y_3)^2 + (x_2 y_3)^2 + (x_3 y_2)^2} \quad (10.36)$$

We note that, as we are not dealing with the increasing wave, x_2 and x_3

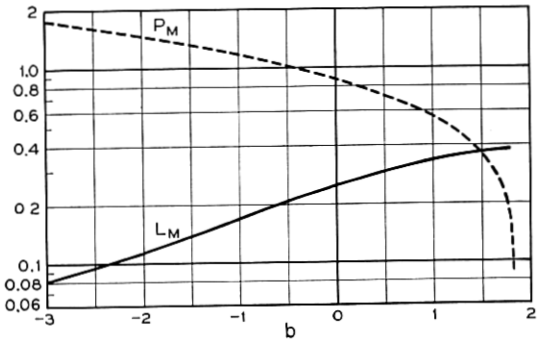

Fig. 10.3—According to the theory presented, the overall noise figure of a tube with a lossless helix and no space charge is proportional to L. Here we have a minimum value of L_m, minimized with respect to P, which is dependent on gun transit angle, and also the corresponding value of P, P_m. According to this curve, the optimum noise figure should be lowest for low electron velocities (low values of b). It may, however, be impossible to make P equal to P_m.

must be either negative or zero, and hence P_m is always positive. For no space-charge and no attenuation, x_3 is zero for all values of b and

$$P_m = \frac{-x_2}{y_2^2 + x_2^2} \quad (10.37)$$

From (10.36) and (10.35), the minimum value of L, L_m, is

$$L_m = (x_2 + x_3)^2 + (y_2 + y_3)^2 - \frac{[x_3(y_2^2 + x_2^2) + x_2(x_3^2 + y_3^2)]^2}{(x_2 x_3)^2 + (y_2 y_3)^2 + (x_2 y_3)^2 + (x_3 y_2)^2} \quad (10.38)$$

When $x_3 = 0$, as in (10.37)

$$L_m = x_2^2 + y_2^2 + 2y_2 y_3 + \frac{y_2^2 y_3^2}{x_2^2 + y_2^2} \quad (10.39)$$

In Fig. 10.3, P_m and L_m are plotted vs b for no attenuation ($d = 0$). We see that P_m becomes very small as b approaches $(3/2)2^{1/3}$, the value at which the increasing wave disappears.

If space charge is to be taken into account, it should be taken into account both in the drift space between anode and helix and in the helix itself. In the helix we can express the effect of space-charge by means of the parameter QC and boundary conditions can be fitted as in Chapter IX. The drift space can be dealt with as in Section 9.7 of Chapter IX. The inclusion of the effect of space-charge by this means will of course considerably complicate the analysis, especially if $b \neq 0$.

While working with Field at Stanford, Dr. C. F. Quate extended the theory presented here to include the effect of all three waves in the case of low gain, and to include the effect of a fractional component of beam current having pure shot noise, which might arise through failure of space-charge reduction of noise toward the edge of the cathode. His extended theory agreed to an encouraging extent with his experimental results. Subsequent unpublished work carried out at these Laboratories by Cutler and Quate indicates a surprisingly good agreement between calculations of this sort and observed noise current, and emphasizes the importance of properly including both partition noise and space charge in predicting noise figure.

10.4 Other Noise Considerations

Space-charge reduction of noise is a cooperative phenomenon of the whole electron beam. If some electrons are eliminated, as by a grid, additional "partition" noise is introduced. Peterson shows how to take this into account.[2]

An electron may be ineffective in a traveling-wave tube not only by being lost but by entering the circuit near the axis where the r-f field is weak rather than near the edge where the r-f field is high. Partition noise arises because sidewise components of thermal velocity cause a fluctuation in the amount of current striking a grid or other intercepting circuit. If such sidewise components of velocity appreciably alter electron position in the helix, a noise analogous to partition noise may arise even if no electrons actually strike the helix. Such a noise will also occur if the "counteracting pulses" of low-charge density which are assumed to smooth out the electron flow are broad transverse to the beam.

These considerations lead to some maxims in connection with low-noise traveling-wave tubes: (1) do not allow electrons to be intercepted by various electrodes (2) if practical, make sure that $I_0(\beta r)$ is reasonably constant over the beam, and/or (3) provide a very strong magnetic focusing field, so that electrons cannot move appreciably transversely.

10.5 Noise in Transverse-Field Tubes

Traveling-wave tubes can be made in which there is no longitudinal field component at the nominal beam position. One can argue that, if a narrow, well-collimated beam is used in such a tube, the noise current in the beam can induce little noise signal in the circuit (none at all for a beam of zero thickness with no sidewise motion). Thus, the idea of using a transverse-field tube as a low-noise tube is attractive. So far, no experimental results on such tubes have been announced.

A brief analysis of transverse-field tubes is given in Chapter XIII.

CHAPTER XI

BACKWARD WAVES

WE NOTED IN CHAPTER IV that, in filter-type circuits, there is an infinite number of spatial harmonics which travel in both directions. Usually, in a tube which is designed to make use of a given forward component the velocity of other forward components is enough different from that of the component chosen to avoid any appreciable interaction with the electron stream. It may well be, however, that a backward-traveling component has almost the same speed as a forward-traveling component.

Suppose, for instance, that a tube is designed to make use of a given forward-traveling component of a forward wave. Suppose that there is a forward-traveling component of a backward wave, and this forward-traveling component is also near synchronism with the electrons. Does this mean that under these circumstances both the backward-traveling and the forward-traveling waves will be amplified?

The question is essentially that of the interaction of an electron stream with a circuit in which the phase velocity is in step with the electrons but the group velocity and the energy flow are in a direction contrary to that of electron motion.

We can most easily evaluate such a situation by considering a distributed circuit for which this is true. Such a circuit is shown in Fig. 11.1. Here the series reactance X per unit length is negative as compared with the more usual circuit of Fig. 11.2. In the circuit of Fig. 11.2, the phase shift is 0° per section at zero frequency and assumes positive values as the frequency is increased. In the circuit of Fig. 11.1 the phase shift is −180° per section at a lower cutoff frequency and approaches 0° per section as the frequency approaches infinity.

Suppose we consider the equations of Chapter II. In (2.9) we chose the sign of X in such a manner as to make the series reactance positive, as in Fig. 11.2, rather than negative, as in Fig. 11.1. All the other equations apply equally well to either circuit. Thus, for the circuit of Fig. 11.1, we have, instead of (2.10),

$$V = \frac{+\Gamma\Gamma_1 i}{(\Gamma^2 - \Gamma_1^2)} \tag{11.1}$$

The sign is changed in the circuit equation relating the convection current and the voltage. Similarly, we can modify the equations of Chapter VII, (7.9) and (7.12), by changing the sign of the left-hand side. From Chapter VIII, the equation for a lossless circuit with no space charge is

$$\delta^2(\delta + jb) = -j \tag{8.1}$$

The corresponding modification is to change the sign preceding δ^2, giving

$$\delta^2(\delta + jb) = +j \tag{11.2}$$

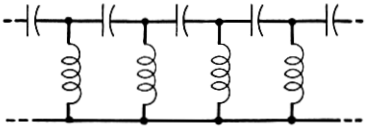

Fig. 11.1

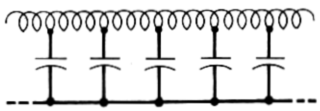

Fig. 11.2

Fig. 11.1—A circuit with a negative phase velocity. The electrons can be in synchronism with the field only if they travel in a direction opposite to that of electromagnetic energy flow.

Fig. 11.2—A circuit with a positive phase velocity.

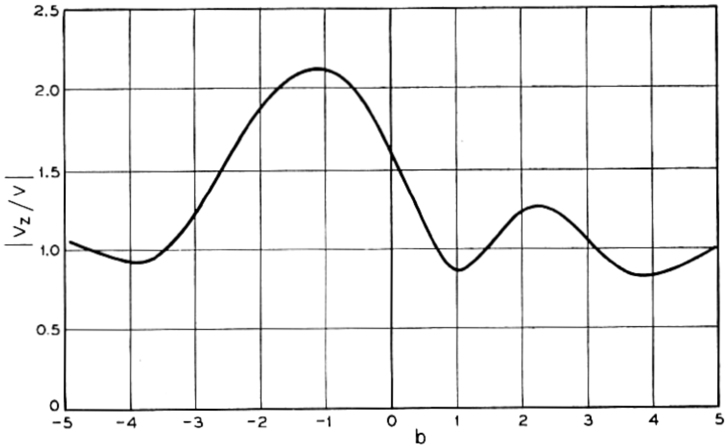

Fig. 11.3—Suppose we have a tube with a circuit such as that of Fig. 11.1, in which the circuit energy is really flowing in the opposite direction from the electron motion. Here, for $QC = d = 0$, we have the ratio of the magnitude of the voltage V_z a distance z from the point of injection of electrons to the magnitude of the voltage V at the point of injection of electrons. V_z is really the input voltage, and there will be gain at values of b for which $|V_z/V| < 1$.

In (11.2), b and δ have the usual meaning in terms of electron velocity and propagation constant.

Now consider the equation

$$\delta^2(\delta - jk) = j \tag{11.3}$$

Equations (11.2) and (8.1) apply to different systems. We have solutions of (8.1) and we want solutions of (11.2). We see that a solution of (11.2) is a solution of (11.3) for $k = -b$. We see that a solution of (11.3) is the conjugate of a solution of (8.1) if we put b in (8.1) equal to k in (11.3). Thus, a solution of (11.2) is the conjugate of a solution of (8.1) in which b in (8.1) is made the negative of the value of b for which it is desired to solve (11.2).

We can use the solutions of Fig. 8.1 in connection with the circuit of Fig. 11.1 in the following way: wherever in Fig. 8.1 we see b, we write in instead $-b$, and wherever we see y_1, y_2 or y_3 we write in instead $-y_1$, $-y_2$ or $-y_3$.

Thus, for synchronous velocity, we have

$$\delta_1 = \sqrt{3}/2 + j\tfrac{1}{2}$$

$$\delta_2 = -\sqrt{3}/2 + j\tfrac{1}{2}$$

$$\delta_3 = -j$$

We can determine what will happen in a physical case only by fitting boundary conditions so that at $z = 0$ the electron stream, as it must, enters unmodulated.

Let us, for convenience, write Φ for the quantity βCz

$$\beta Cz = \Phi \tag{11.4}$$

We will have for the total voltage V_z at z in terms of the voltage V at $z = 0$

$$\begin{aligned} V_z = Ve^{-j\beta z}([(1 - \delta_2/\delta_1)(1 - \delta_3/\delta_1)]^{-1}e^{-j\Phi y_1}e^{\Phi x_1} \\ + [(1 - \delta_3/\delta_2)(1 - \delta_1/\delta_2)]^{-1}e^{-j\Phi y_2}e^{\Phi x_2} \\ + [(1 - \delta_1/\delta_3)(1 - \delta_2/\delta_3)]^{-1}e^{-j\Phi y_3}e^{\Phi x_3}) \end{aligned} \tag{11.5}$$

We must remember that in using values from an unaltered Fig. 8.1 we use in the δ's and as the y's the negative of the y's shown in the figure (the sign of the x's is unchanged), and for a given value of b we enter Fig. 8.1 at $-b$.

In Fig. 11.3, $|V_z/V|$ has been plotted vs b for $\Phi = 2$. We see that, for several values of b, $|V_z|$ (the input voltage) is less than $|V|$ (the output voltage) and hence there can be "backward" gain.

We note that as Φ is made very large, the wave which increases with increasing Φ will eventually predominate, and $|V_z|$ will be greater than $|V|$. "Backward gain" occurs not through a "growing wave" but rather through a sort of interference between wave components, as exhibited in Fig. 11.3.

Fig. 11.3 is for a lossless circuit; the presence of circuit attenuation would alter the situation somewhat.

APPENDIX IV

EVALUATION OF SPACE—CHARGE PARAMETER Q

Consider the system consisting of a conducting cylinder of radius a and an internal cylinder of current of radius a_1 with a current

$$ie^{j\omega t}e^{-\Gamma z}. \tag{1}$$

Let subscript 1 refer to inside and 2 to outside. We will assume magnetic fields of the form

$$H_{\varphi 1} = AI_1(\gamma r) \tag{2}$$

$$H_{\varphi 2} = BI_1(\gamma r) + CK_1(\gamma r) \tag{3}$$

From Maxwell's equations we have,

$$\frac{\partial}{\partial r}(rH_\varphi) = j\omega\epsilon rE_z + rJ_z \tag{4}$$

Now

$$\frac{\partial}{\partial z}(zI_1(z)) = zI_0(z) \tag{5}$$

$$\frac{\partial}{\partial z}(zK_1(z)) = -zK_0(z) \tag{6}$$

Hence

$$E_{z1} = \frac{-j\gamma}{\omega\epsilon}AI_0(\gamma r) \tag{7}$$

$$E_{z2} = \frac{-j\gamma}{\omega\epsilon}(BI_0(\gamma r) - CK_0(\gamma r)) \tag{8}$$

at $r = a$, $E_{z2} = 0$

$$C = B\frac{I_0(\gamma a)}{K_0(\gamma a)} \tag{9}$$

at $r = a_1$, $E_{z1} = E_{z2}$

$$\begin{aligned} AI_0(\gamma a_1) &= B\left(I_0(\gamma a_1) - \frac{I_0(\gamma a)}{K_0(\gamma a)}K_0(\gamma a_1)\right) \\ A &= B\left(1 - \frac{I_0(\gamma a)}{K_0(\gamma a)}\frac{K_0(\gamma a_1)}{I_0(\gamma a_1)}\right) \end{aligned} \tag{10}$$

In going across boundary, we integrate (4) over the infinitesimal radial distance which the current is assumed to occupy

$$\begin{aligned} r dH_\varphi &= rJdr \\ 2\pi rJdr &= i \\ rjdr &= \frac{i}{2\pi} \end{aligned} \tag{11}$$

Thus

$$dH_\varphi = \frac{i}{2\pi r} = \frac{i}{2\pi a_1} = (H_{\varphi 2} - H_{\varphi 1})_{a_1} \tag{12}$$

$$B\left[I_1(\gamma a_1) + \frac{I_0(\gamma a)}{K_0(\gamma a)} K_1(\gamma a_1) - I_1(\gamma a_1)\left(1 - \frac{I_0(\gamma a)K_0(\gamma a_1)}{K_0(\gamma a)I_0(\gamma a_1)}\right)\right] = \frac{i}{2\pi a_1}$$

$$B = \frac{i}{2\pi a_1}\left[\frac{I_0(\gamma a)}{K_0(\gamma a)} K_1(\gamma a_1) + \frac{I_0(\gamma a)}{K_0(\gamma a)} \frac{K_0(\gamma a_1)}{I_0(\gamma a_1)} I_1(\gamma a_1)\right]^{-1}$$

$$B = \frac{i}{2\pi a_1} \frac{K_0(\gamma a)}{I_0(\gamma a) I_1(\gamma a_1)}\left[\frac{K_1(\gamma a_1)}{I_1(\gamma a_1)} + \frac{K_0(\gamma a_1)}{I_0(\gamma a_1)}\right]^{-1} \tag{13}$$

at $r = a_1$

$$\begin{aligned} E_{z1} = E_{z2} = \left(\frac{-j\gamma}{\omega\epsilon}\right)\left(\frac{i}{2\pi a_1}\right) \frac{K_0(\gamma a)}{I_0(\gamma a)} \frac{I_0(\gamma a_1)}{I_1(\gamma a_1)} \\ \left(1 - \frac{I_0(\gamma a)}{K_0(\gamma a)} \frac{K_0(\gamma a_1)}{I_0(\gamma a_1)}\right)\left[\frac{K_1(\gamma a_1)}{I_1(\gamma a_1)} + \frac{K_0(\gamma a_1)}{I_0(\gamma a_1)}\right]^{-1} \end{aligned} \tag{14}$$

Now

$$\frac{1}{\omega\epsilon} = \frac{\sqrt{\mu/\epsilon}}{\beta_0} = \frac{377}{\beta_0} \tag{15}$$

Hence

$$\begin{aligned} i\beta V = E_z = j\frac{\gamma^2}{\beta_0} I_0^2(\gamma a_1) G(\gamma a, \gamma a_1) i \\ V = \left(\frac{\gamma}{\beta_0}\right)\left(\frac{\gamma}{\beta}\right) I_0^2(\gamma a_1) G(\gamma a, \gamma a_1) q \end{aligned} \tag{16}$$

$$G(\gamma a, \gamma a_1) = 60\left[\frac{K_0(\gamma a_1)}{I_0(\gamma a_1)} - \frac{K_0(\gamma a)}{I_0(\lambda a)}\right] \tag{17}$$

In obtaining this form, use was made of the fact that

$$K_1(z)I_0(z) + K_0(z)I_1(z) = \frac{1}{z}$$

Now

$$Q = \frac{\beta}{\omega C_1(E^2/\beta^2 P)} \tag{18}$$

where $(E^2/\beta^2 P)$ is the value of this quantity at $r = a_1$. In order to evaluate Q we note that

$$V = -\frac{j\Gamma}{\omega C_1} i = \frac{-j(j\beta)}{\omega C_1} i$$

$$V = \frac{\beta}{\omega C_1} i$$

$$\frac{\beta}{\omega C_1} = \frac{V}{i} = \left(\frac{\gamma}{\beta_0}\right)\left(\frac{\gamma}{\beta}\right) I_0^2(\gamma a_1) G(\gamma a, \gamma a_1) \tag{20}$$

$$\frac{\beta}{\omega C_1} = \left(\frac{\beta}{\beta_0}\right)\left(\frac{\gamma}{\beta}\right)^2 I_0^2(\gamma a_1) G(\gamma a, \gamma a_1)$$

On the axis, $(E^2/\beta^2 P)$ has a value $(E^2/\beta^2 P)_0$

$$(E^2/\beta^2 P)_0 = \left(\frac{\beta}{\beta_0}\right)\left(\frac{\gamma}{\beta}\right)^4 F^3(a) \tag{21}$$

At a radius a_1

$$(E^2/\beta^2 P) = \left(\frac{\beta}{\beta_0}\right)\left(\frac{\gamma}{\beta}\right)^4 F^3(\gamma a) I_0^2(\gamma a_1) \tag{22}$$

Hence

$$Q(\gamma/\beta)^2 = \frac{G(\gamma a, \gamma a_1)}{F^3(\gamma a)} \tag{23}$$

APPENDIX V

DIODE EQUATIONS

FROM LLEWELLYN AND PETERSON

These apply to electrons injected into a space between two planes a and b normal to the x direction. Plan b is in the $+x$ direction from plane a. Current density I and convection current q are positive in the $-x$ direction. The d-c velocities u_a, u_b and the a-c velocities v_a, v_b are in the $+x$ direction. T is the transit time. The notation in this appendix should not be confused with that used in other parts of this book. It was felt that it would be confusing to change the notation in Llewellyn's and Peterson's[1] well-known equations.

TABLE I

ELECTRONICS EQUATIONS

Numerics Employed:

$$\eta = 10^7 \frac{e}{m} = 1.77 \times 10^{15}, \qquad \epsilon = 1/(36\pi \times 10^{11}) \; \frac{\eta}{\epsilon} \doteq 2 \times 10^{28}$$

Direct-Current Equations:

Potential-velocity: $\eta V_D = (1/2)u^2$ (1)

$$\left.\begin{aligned} &\text{Space-charge-factor definition: } \zeta = 3(1 - T_0/T) \\ &\text{Distance: } x = (1 - \zeta/3)(u_a + u_a)T/2 \\ &\text{Current density: } (\eta/\epsilon)I_D = (u_a + u_b)2\zeta/T^2 \end{aligned}\right\} \quad (2)$$

Space-charge ratio: $I_D/I_m = (9/4)\zeta(1 - \zeta/3)^2$ (3)

Limiting-current density:

$$I_m = \frac{2.33}{10^6} \frac{(\sqrt{V_{Da}} + \sqrt{V_{Db}})^3}{x^2} \quad (4)$$

Alternating-Current Equations:

Symbols employed:

$$\beta = i\theta, \qquad \theta = \omega T, \qquad i = \sqrt{-1}$$

[1] F. B. Llewellyn and L. C. Peterson "Vacuum Tube Networks," *Proc. I.R.E.*, vol. 32, pp. 144–166, March, 1944.

$$P = 1 - e^{-\beta} - \beta e^{-\beta} \doteq \frac{\beta^2}{2} - \frac{\beta^3}{3} + \frac{\beta^4}{8} \cdots$$

$$Q = 1 - e^{-\beta} \doteq \beta - \frac{\beta^2}{2} + \frac{\beta^3}{6} - \frac{\beta^4}{24} \cdots$$

$$S = 2 - 2e^{-\beta} - \beta - \beta e^{-\beta} \doteq -\frac{\beta^3}{6} + \frac{\beta^4}{12} - \frac{\beta^5}{40} + \frac{\beta^6}{180}$$

General equations for alternating current

q = alternating conduction-current density

v = alternating velocity

$$\left.\begin{aligned} V_b - V_a &= A^*I + B^*q_a + C^*v_a \\ q_b &= D^*I + E^*q_a + F^*v_a \\ v_b &= G^*I + H^*q_a + I^*v_a \end{aligned}\right\} \tag{5}$$

TABLE II

VALUES OF ALTERNATING-CURRENT COEFFICIENTS

$$A^* = \frac{1}{\epsilon} u_a + u_b \frac{T^2}{2} \frac{1}{\beta} \left[1 - \frac{\zeta}{3}\left(1 - \frac{12S}{\beta^3}\right)\right]$$

$$B^* = \frac{1}{\epsilon} \frac{T^2}{\beta^3} [u_a(P - \beta Q) - u_b P + \zeta(u_a + u_b)P]$$

$$C^* = -\frac{1}{\eta} 2\zeta(u_a + u_b) \frac{P}{\beta^2}$$

$$D^* = 2\zeta \frac{(u_a + u_b)}{u_b} \frac{P}{\beta^2}$$

$$E^* = \frac{1}{u_b} [u_b - \zeta(u_a + u_b)]e^{-\beta}$$

$$F^* = \frac{\epsilon}{\eta} \frac{2\zeta}{T^2} \frac{(u_a + u_b)}{u_b} \beta e^{-\beta}$$

$$G^* = -\frac{\eta}{\epsilon} \frac{T^2}{\beta^3} \frac{1}{u_b} [u_b(P - \beta Q) - u_a P + \zeta(u_a + u_b)P]$$

$$H^* = -\frac{\eta}{\epsilon} \frac{T^2}{2} \frac{(u_a + u_b)}{u_b} (1 - \zeta) \frac{e^{-\beta}}{\beta}$$

$$I^* = \frac{1}{u_b} [u_a - \zeta(u_a + u_b)]e^{-\beta}$$

Complete space-charge, $\zeta = 1$.

$$A^* = \frac{1}{\epsilon} (u_a + u_b) \frac{T^2}{3\beta} \left(1 + \frac{6S}{\beta^3}\right)$$

$$B^* = \frac{1}{\epsilon} \frac{T^2}{\beta^3} u_a(2P - \beta Q)$$

$$C^* = -\frac{2}{\eta}(u_a + u_b)\frac{P}{\beta^2}$$

$$D^* = 2\,\frac{(u_a + u_b)}{(u_b)}\,\frac{P}{\beta^2}$$

$$E^* = -\frac{u_a}{u_b}\,e^{-\beta}$$

$$F^* = \frac{\epsilon}{\eta}\,\frac{2}{T^2}\,\frac{(u_a + u_b)}{(u_b)}\,\beta e^{-\beta}$$

$$G^* = -\frac{\eta}{\epsilon}\,\frac{T^2}{\beta^3}\,(2P - \beta Q)$$

$$H^* = 0$$

$$I^* = -e^{-\beta}$$

APPENDIX VI

EVALUATION OF IMPEDANCE AND Q FOR THIN AND SOLID BEAMS[1]

Let us first consider a thin beam whose breadth is small enough so that the field acting on the electrons is essentially constant. The normal mode solutions obtained in Chapters VI and VII apply only to this case. The more practical situation of a thick beam will be considered later. The normal mode method consists of simultaneously solving two equations, one relating the r-f field produced on the circuit by an impressed r-f current from the electron stream and the other relating r-f current produced in the electron stream by an impressed r-f field from the circuit.

We have the circuit equation

$$E = -\left[\frac{\Gamma^2\Gamma_0 K}{\Gamma^2 - \Gamma_0^2} + \frac{2jQK\Gamma^2}{\beta_e}\right] i \tag{1}$$

and the electronic equation

$$i = \frac{j\beta_e}{(j\beta_e - \Gamma)^2}\frac{I_0}{2V_0}E. \tag{2}$$

The solution of these two equations gives Γ in terms of Γ_0, K, and Q, which must be evaluated separately for the particular circuit being considered.

The field solution is obtained by solving the field equations in various regions and appropriately matching at the boundaries. For a hollow beam of electrons of radius b traveling in the z direction inside a helix of radius a and pitch angle ψ, the matching consists of finding the admittances $\left(\frac{H_\varphi}{E_z}\right)$ inside and outside the beam and setting the difference equal to the admittance of the beam. Thus the admittance just outside the beam for an idealized helix will be[2]

$$Y_0 = \frac{H_{\varphi 0}}{E_{z0}} = j\frac{\omega\epsilon}{\gamma}\frac{I_1(\gamma b) - \delta K_1(\gamma b)}{I_0(\gamma b) + \delta K_0(\gamma b)}, \tag{3}$$

[1] This appendix is taken from R. C. Fletcher, "Helix Parameters in Traveling-Wave Tube Theory," *Proc. I.R.E.*, Vol. 38, pp. 413–417 (1950).

[2] L. J. Chu and J. D. Jackson, "Field Theory of Traveling-Wave Tubes," *I.R.E., Proc.*, Vol. 36, pp. 853–863, July, 1948.

O. E. H. Rydbeck, "Theory of the Traveling-Wave Tube," *Ericsson Technics*, No. 46 pp. 3–18, 1948.

where

$$\delta = \frac{1}{K_0^2(\gamma a)} \left(\left(\frac{\beta_0 a \cot \Psi}{\gamma a} \right)^2 I_1(\gamma a) K_1(\gamma a) - I_0(\gamma a) K_0(\gamma a) \right),$$

$$\beta_0^2 = \omega^2 \mu \epsilon,$$

and

$$\gamma^2 = -\Gamma^2 - \beta_0^2 .$$

(The I's and K's are modified Bessel functions). The admittance inside the beam is

$$Y_i = \frac{H_{\varphi i}}{E_{zi}} = \frac{j\omega\epsilon}{\gamma} \frac{I_1(\gamma b)}{I_0(\gamma b)} . \tag{4}$$

Boundary conditions require that $E_{z0} = E_{zi} = E_z$ and $H_{z0} - H_{zi} = \dfrac{i}{2\pi b}$. Combining the boundary conditions, we see that

$$Y_0 - Y_i = \frac{1}{2\pi b} \frac{i}{E_z}, \tag{5}$$

where the ratio of $\dfrac{i}{E_z}$ is given by (2). Thus the field method gives two equations which are equivalent to the circuit and electronic equations of the normal mode method.

A6.1 Normal Mode Parameters for Thin Beam

The constants appearing in eq. (1) can be evaluated by equating the circuit equation (1) to the circuit equation (5). Thus if $Y_c = Y_0 - Y_i$,

$$-\frac{\Gamma^2 \Gamma_0 K}{\Gamma^2 - \Gamma_0^2} - \frac{2jQK\Gamma^2}{\beta_e} = +\frac{1}{2\pi b Y_c} . \tag{6}$$

The constants can be obtained by expanding each side of eq. (6) in terms of the zero and pole occurring in the vicinity of Γ_0. Thus if γ_0 and γ_p are the zero and pole of Y_c, respectively,

$$Y_c \simeq -(\gamma_p - \gamma_0) \left(\frac{\partial Y_c}{\partial \gamma} \right)_{\gamma = \gamma_0} \left(\frac{\gamma - \gamma_0}{\gamma - \gamma_p} \right), \tag{7}$$

and the two sides of eq. (6) will be equivalent if

$$\Gamma_0^2 = -\gamma_0^2 - \beta_0^2 , \tag{8}$$

$$\frac{2Q}{\beta_e} = \left(1 + \frac{\beta_0^2}{\gamma_0^2} \right)^{-1/2} \frac{\gamma_0}{\gamma_p^2 - \gamma_0^2}, \tag{9}$$

and

$$\frac{1}{K} = -j\pi b\gamma_0^2\left(1 + \frac{\beta_0^2}{\gamma_0^2}\right)^{3/2}\left(\frac{\partial Y_c}{\partial Y}\right)_{\gamma=\gamma_0}. \tag{10}$$

γ_0 and γ_p can be obtained from eqs. (3) and (4) through the implicit equations

$$(\beta\, a \cot \Psi)^2 = (\gamma_0 a)^2 \frac{I_0(\gamma_0 a)K_0(\gamma_0 a)}{I_1(\gamma_0 a)K_1(\gamma_0 a)}, \tag{11}$$

$$\begin{aligned} \frac{I_0(\gamma_p b)}{K_0(\gamma_p b)} &= -\frac{1}{K_0^2(\gamma_p a)} \\ &\cdot \left[\left(\frac{\beta_0\, a \cot \Psi}{\gamma_p a}\right)^2 I_1(\gamma_p a)K_1(\gamma_p a) - I_0(\gamma_p a)K_0(\gamma_p a)\right], \end{aligned} \tag{12}$$

and $1/K$ is found to be

$$\begin{aligned} \frac{1}{K} = \pi\sqrt{\frac{\epsilon}{\mu}}\left(1 + \frac{\beta_0^2}{\gamma_0^2}\right)^{3/2} \frac{\beta_0^2}{I_0^2(\gamma_0 b)} \frac{I_0(\gamma_0 a)}{K_0(\gamma_0 a)}\left[\frac{I_1(\gamma_0 a)}{I_0(\gamma_0 a)} - \frac{I_0(\gamma_0 a)}{I_1(\gamma_0 a)}\right. \\ \left. + \frac{K_0(\gamma_0 a)}{K_1(\gamma_0 a)} - \frac{K_1(\gamma_0 a)}{K_0(\gamma_0 a)} + \frac{4}{\gamma_0 a}\right]. \end{aligned} \tag{13}$$

The equations for γ_0 and K are the same as those given by Appendix II, evaluated by solving the field equations for the helix without electrons present. The evaluation of γ_p, and thus Q, represents a new contribution. Values of $Q\frac{\gamma_0}{\beta_e}\left(1 + \frac{\beta_0^2}{\gamma_0^2}\right)^{-1/2}$ are plotted in Fig. A6.1 as a function of $\gamma_0 a$ for various ratios of b/a. (It should be noted that for most practical applications the factor $\frac{\gamma_0}{\beta_e}\left(1 + \frac{\beta_0^2}{\gamma_0^2}\right)^{-1/2}$ is very close to unity, so that the ordinate is practically the value of Q itself.)

Appendix IV gives a method for estimating Q based on the solution of the field equations for a conductor replacing the helix and considering the resultant field to be $-\frac{2jKQ\Gamma^2}{\beta_e}\,i$. This estimate of Q is plotted as the dashed lines of Fig. A6.1.

A6.2 Thick Beam Case

For an electron beam which entirely fills the space out to the radius b, the electronic equations of both the normal mode method and the field method are altered in such a way as to considerably complicate the solution. In order to find a solution for this case some simplifying assumptions must be made. A convenient type of assumption is to replace the thick beam by an "equivalent" thin beam, for which the solutions have already been worked out.

Two beams will be equivalent if the value of $\frac{H_\varphi}{E_z}$ is the same outside the beams, since the matching to the circuit depends only on this admittance.

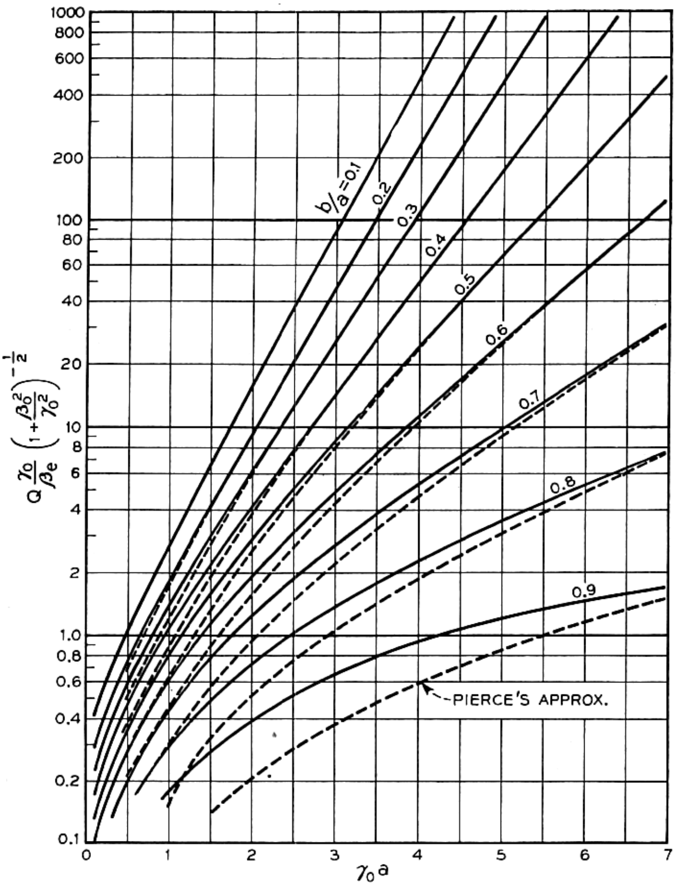

Fig. A6.1—Passive mode parameter Q for a hollow beam of electrons of radius b inside a helix of radius a and natural propagation constant γ_0. The solid line was obtained by equating the circuit equation of the normal mode method, which defines Q, with a corresponding circuit equation found from the field theory method. The dashed line was obtained in Appendix IV from a solution of the field equations for a conductor replacing the helix.

The problem, then, of making a thin beam the equivalent of a thick beam is the problem of arranging the position and current of a thin beam to give the same admittance at the radius b of the thick beam. This is of course impossible for all values of γ. It is desirable therefore that the admittances be the same close to the complex values of γ which will eventually solve the equations.

The solution of the field equations for the solid beam yields the value for $\frac{H_\varphi^{(1)}}{E_z}$ at the radius b as

$$\frac{H_\varphi}{E_z} = \frac{j\omega\epsilon}{\gamma}\frac{nI_1(n\gamma b)}{I_0(n\gamma b)}, \tag{14}$$

where

$$n^2 = 1 + \frac{1}{\beta_0}\sqrt{\frac{\mu}{\epsilon}}\frac{\beta_e I_0}{2\pi b^2 V_0}\frac{1}{(j\beta_e - \Gamma)^2}. \tag{15}$$

Thus the electronic equation for the solid beam which must be solved simultaneously with the circuit equation (given above by either the normal mode approximation or the field solution) must be

$$Y_e = \frac{H_\varphi}{E_z} - Y_i = \frac{j\omega\epsilon b}{\gamma b}\left[\frac{nI_1(n\gamma b)}{I_0(n\gamma b)} - \frac{I_1(\gamma b)}{I_0(\gamma b)}\right]. \tag{16}$$

Complex roots for γ will be expected in the vicinity of real values of γ for which $Y_e \approx Y_c$ and $\frac{dY_e}{d\gamma} \approx \frac{dY_c}{d\gamma}$. By plotting Y_e and Y_c vs. real values of γ, it is found that the two curves become tangent close to the value of γ for which $n = 0$, using typical operating conditions (Fig. A6.2). Our procedure for choosing a hollow beam equivalent of the solid beam, then, will be to equate the values of Y_e and $\frac{dY_e}{d\gamma}$ at $n = 0$. This will give us two equations from which to solve for the electron beam diameter and d-c current for the equivalent hollow beam.

If the hollow beam is placed at the radius sb with a current of tI_0, the value of $\frac{H_\varphi}{E_z}$ at the radius b gives the value for Y_{eH} as

$$\begin{aligned} Y_{eH} = \left(\frac{H_\varphi}{E_z}\right)_b - Y_i = -j\omega\epsilon b\,\frac{t}{2}\,(1 - n^2)\,\frac{I_0^2(s\gamma b)}{I_0^2(\gamma b)} \\ \cdot\left(1 - \gamma^2 b^2 I_0^2(s\gamma b)\,\frac{t}{2}\,(1 - n^2)\left[\frac{K_0(s\gamma b)}{I_0(s\gamma b)} - \frac{K_0(\gamma b)}{I_0(\gamma b)}\right]\right)^{-1}. \end{aligned} \tag{17}$$

Equating this with eq. (16) at $n = 0$ yields the equation

$$\frac{1}{t} = \frac{1}{2}\theta^2 I_0^2(s\theta)\left[\frac{K_0(s\theta)}{I_0(s\theta)} + \frac{K_1(\theta)}{I_1(\theta)}\right], \tag{18}$$

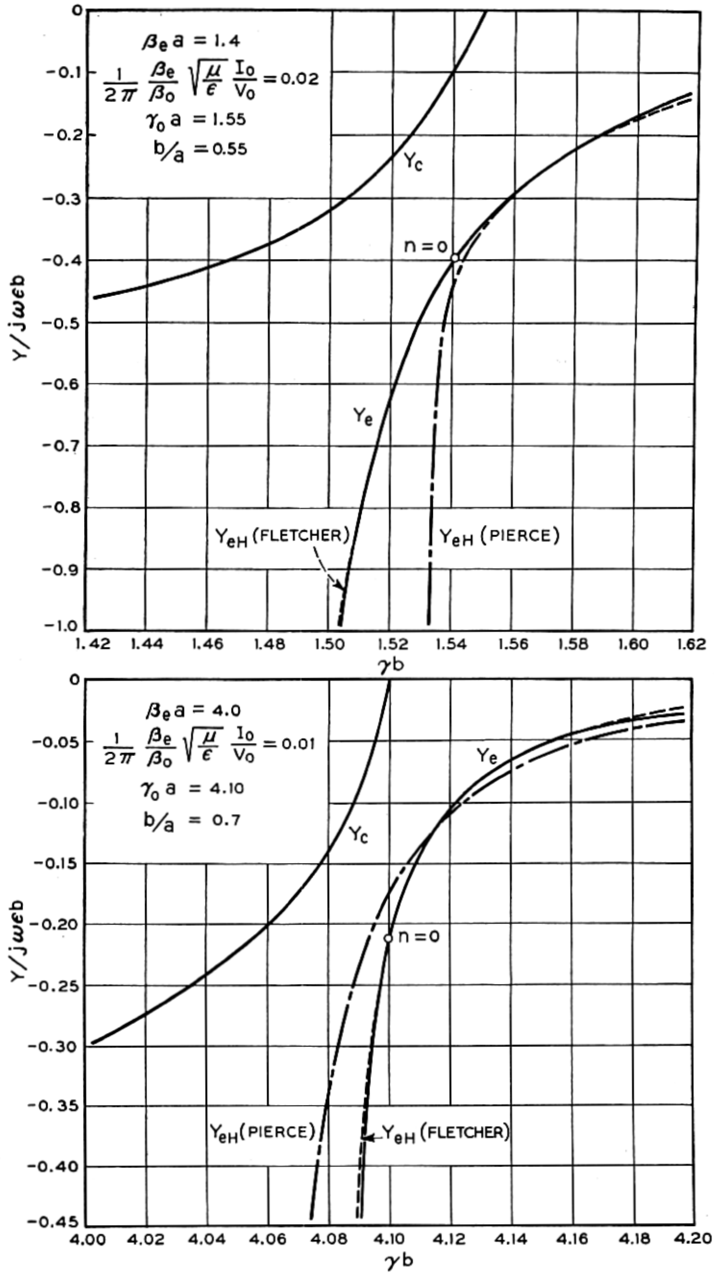

Fig. A6.2—Electronic admittance Y_e of a solid electron beam of radius b and circuit admittance Y_c of a helix of radius a plotted vs. real values of the propagation constant γ in the vicinity of where $\frac{dY_e}{d\gamma} = \frac{dY_c}{d\gamma}$ where complex solutions for γ are expected, for two typical sets of operating conditions. Plotted on the same graph is the electron admittance Y_{eH} for two equivalent hollow electron beams: the dashed curve (Fletcher) is matched to Y_e at $n = 0$, while the dot-dashed curve (Pierce, Appendix IV) is matched at $n = 1$ (off the graph).

where $\theta = \gamma_e b$ and γ_e is the value of γ at $n = 0$; i.e. for $\gamma_e \gg \beta_0$

$$\gamma_e = \beta_e + \sqrt{\frac{1}{\beta_0}\sqrt{\frac{\mu}{\epsilon}}\frac{\beta_e I_0}{2\pi b^2 V_0}} \approx \beta_e . \tag{19}$$

In the vicinity of $n = 0$, n varies very rapidly with γ, and hence matching $\left(\frac{\partial Y_e}{\partial n}\right)_\gamma$ is practically the same as matching $\frac{dY_e}{d\gamma}$. With this approximation eqs. (16) and (17) can be differentiated with respect to n and set equal at

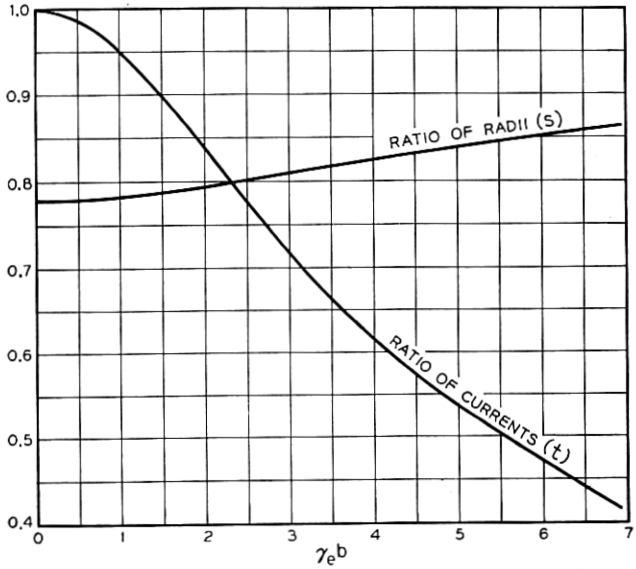

Fig. A6.3—Parameters of the hollow electron beam which is matched to the solid electron beam of radius b and current I_0 at $\gamma = \gamma_e \simeq \beta_e$, where $n = 0$. sb is the radius and tI_0 is the current of the equivalent hollow beam.

$n = 0$ to yield the second relation

$$\frac{1}{t} = \theta^2 I_0^2(\theta) I_0^2(s\theta) \left[\frac{K_0(s\theta)}{I_0(s\theta)} + \frac{K_1(\theta)}{I_1(\theta)}\right]^2 \tag{20}$$

Equations (18) and (20) can then be solved to give the implicit equation for s as

$$\frac{K_0(s\theta)}{I_0(s\theta)} = -\frac{K_1(\theta)}{I_1(\theta)} + \frac{1}{2I_1^2(\theta)} \tag{21}$$

and the simpler equation for t

$$t = \frac{4}{\theta^2}\frac{I_1^2(\theta)}{I_0^2(s\theta)} . \tag{22}$$

s and t are plotted as a function of θ in Fig. A6.3. The value of Y_{eH} using these values of s and t is compared in Fig. A6.2 with Y_e in the vicinity of where Y_c is almost tangent to Y_e for two typical sets of operating conditions.

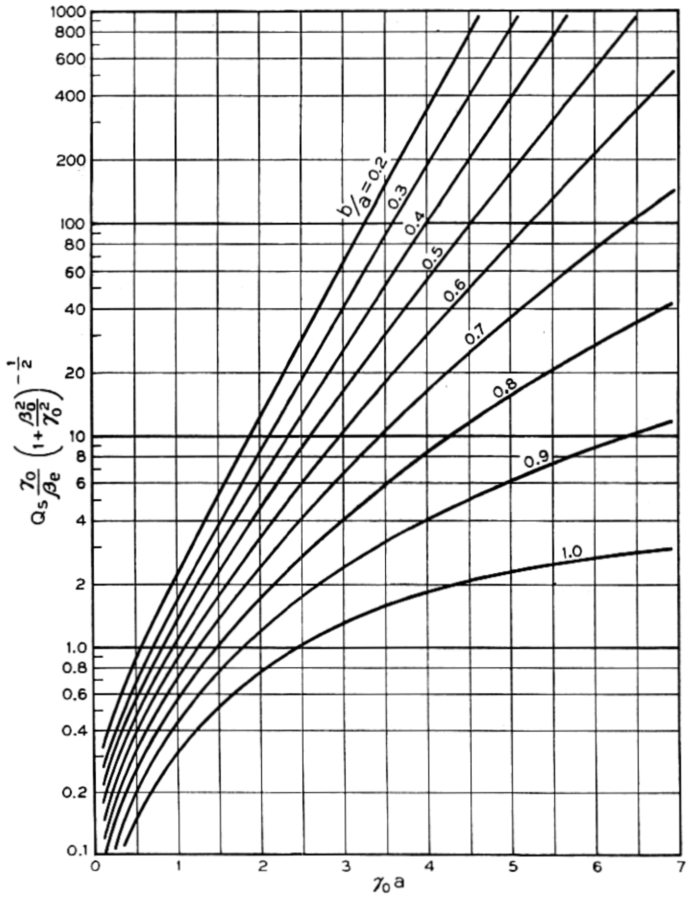

Fig. A6.4—Passive mode parameter Q_s for a solid beam of electrons of radius b inside a helix of radius a and natural propagation constant γ_0, obtained from the equivalent hollow beam parameters of Fig. 3 taken at $\gamma_e = \gamma_0$. All the normal mode solutions which have been found(2), (3) for a hollow beam will be approximately valid for a solid beam if Q is replaced by Q_s and K is replaced by K_s (Fig. 5).

It is of course possible to pick other criteria for determining an "equivalent" hollow beam. In Chapter XIV, in essence, Y_c and Y_{eH} were expanded in terms of $(1 - n^2)$ and the coefficients of the first two terms were equated. This has been done for the cylindrical beams, and the values of s and t found by this method determine values of Y_{eH} shown in Fig. A6.2. The greater departure from the true curve of Y_e would indicate that this approximation is not as good as that described above.

It is now possible to find the values of Q_s and K_s appropriate to the solid

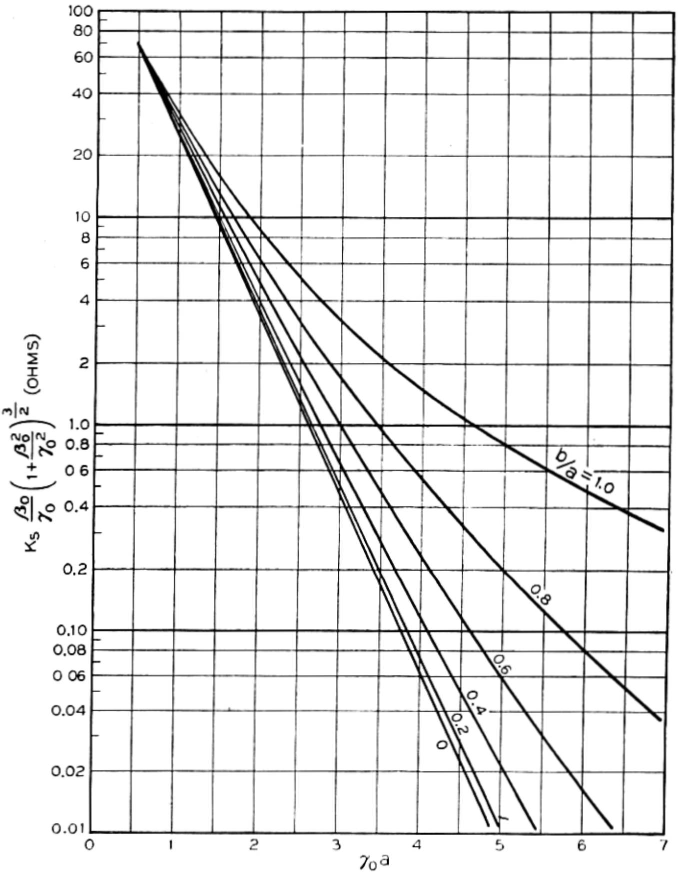

Fig. A6.5—Circuit impedance K_s for a solid beam of electrons of radius b inside a helix of radius a and natural propagation constant γ_0, obtained from the equivalent hollow beam parameters of Fig. 3 taken at $\gamma_e = \gamma_0$. K_s should replace $K = \frac{E^2\ {}^{(2),\ (3)}}{2\beta^2{}_p}$ in order for the normal mode solutions for a hollow beam to be applicable to a solid beam.

beam. Thus if $Q\left(\gamma_0 a, \frac{b}{a}\right)$ and $K\left(\gamma_0 a, \frac{b}{a}\right)$ are the values for the hollow beam calculated from eqs. (9), (12) and (13),

$$Q_s = Q\left(\gamma_0 a, s\frac{b}{a}\right), \tag{23}$$

and

$$K_s = tK\left(\gamma_0 a, s\frac{b}{a}\right). \tag{24}$$

The t is placed in front of K in eq. (24) because tI_0 and K appear in the thin beam solutions only in the combination tI_0K. Using tK instead of K allows us to use I_0, the actual value of the current in the solid beam in the solutions instead of tI_0, the equivalent current. Values of $Q_s\frac{\gamma_0}{\beta_e}\left(1+\frac{\beta_0^2}{\gamma_0^2}\right)^{-1/2}$ and $K_s\frac{\beta_0}{\gamma_0}\cdot\left(1+\frac{\beta_0^2}{\gamma_0^2}\right)^{+3/2}$ are plotted vs. $\gamma_0 a$ in Figs. A6.4 and A6.5 for different values of b/a and for values of t and s taken at $\gamma_e = \gamma_0$. All the solutions obtained for the hollow beam will be valid for the solid beam if Q_s and K_s are substituted for Q and K.

APPENDIX VII

HOW TO CALCULATE THE GAIN OF A TRAVELING-WAVE TUBE

The gain calculation presented here neglects the effect at the output of all waves except the increasing wave. Thus, it can be expected to be accurate only for tubes with a considerable net gain. The gain is expressed in db as

$$G = A + BCN \tag{1}$$

Here A represents an initial loss in setting up the increasing wave and BCN represents the gain of the increasing wave.

We will modify (1) to take into account approximately the effect of the cold loss of L db in reducing the gain of the increasing wave by writing

$$G = A + [BCN - \alpha L] \tag{2}$$

Here α is the fraction of the cold loss which should be subtracted from the gain of the increasing wave. This expression should hold even for moderately non-uniform loss (see Fig. 9.5).

Thus, what we need to know to calculate the gain are the quantities

$$A,\ B,\ C,\ N,\ \alpha,\ L$$

A7.1 Cold Loss L db

The best way to get the cold loss L is to measure it. One must be sure that the loss measured is the loss of a wave traveling in the circuit and not loss at the input and output couplings.

A7.2 Length of Circuit in Wavelengths, N

We can arrive at this in several ways. The ratio of the speed of light c to the speed of an electron u_0 is

$$\frac{c}{u_0} = \frac{505}{\sqrt{V_0}} \tag{3}$$

where V_0 is the accelerating voltage. Thus, if ℓ is the length of the circuit and λ is the free-space wavelength and λ_g is the wavelength along the axis of the helix

$$\lambda_g = \lambda \frac{u_0}{c} \tag{4}$$

$$N = \frac{\ell}{\lambda_g} = \frac{\ell}{\gamma} \frac{c}{u_0} \tag{5}$$

Also, if $\mathfrak{L}_w$ is the total length of wire in the helix, approximately

$$N = \frac{L_w}{\lambda} \tag{6}$$

A7.3 The Gain Parameter C

The gain parameter can be expressed

$$C = \left(\frac{E^2}{\beta^2 P} \frac{I_0}{8V_0}\right)^{1/3} = \left(\frac{KI_0}{4V_0}\right)^{1/3} \tag{7}$$

Here K is the helix impedance properly defined. I_0 is the beam current in amperes and V_0 is the beam voltage.

A7.4 Helix Impedance K

In Fig. 5 of Appendix VI, $K\left(\frac{\beta_0}{\gamma_0}\right)\left(1 + \left(\frac{\beta_0}{\gamma_0}\right)^2\right)^{3/2}$ is plotted vs. γ_0a for values of b/a. K_s is the effective value of K for a solid beam of radius b, and a is the radius of the helix. γ_0 is to be identified with γ for present purposes, and is given by

$$\gamma_0 = \frac{2\pi}{\lambda_g}\left[1 - \left(\frac{\gamma_g}{\lambda}\right)^2\right]^{1/2} \tag{8}$$

where λ_g is given in terms of λ by (4). We see that in most cases (for voltages up to several thousand)

$$(\lambda_g/\lambda)^2 \ll 1 \tag{9}$$

and we may usually use as a valid approximation

$$\gamma_0 = \frac{2\pi}{\lambda_g} \tag{10}$$

and

$$\gamma_0 a = \frac{2\pi a}{\lambda_g} \tag{11}$$

As $\beta_0 = 2\pi/\lambda$, this approximation gives

$$1 + \left(\frac{\beta_0}{\gamma_0}\right)^2 = 1 + \left(\frac{\lambda_g}{\lambda}\right)^2$$

and we may assume

$$\left(1 + \left(\frac{\beta_0}{\gamma_0}\right)^2\right)^{3/2} = 1 \tag{12}$$

Thus, we may take K_s as the ordinate of Fig. 5 multiplied by c/u_0, from (3), for instance.

The true impedance may be somewhat less than the impedance for a helically conducting sheet. If the ratio of the circuit impedance to that of a helically conducting sheet is known (see Sections 3 and 4.1 of Chapter III, and Fig. 3.13, for instance), the value of K_s from Fig. 5 can be multiplied by this ratio.

A7.5 The Space-Charge Parameter Q

The ordinate of Fig. 4 of Appendix VI shows $Q_s \dfrac{\gamma_0}{\beta_e}\left(1 + \left(\dfrac{\beta_0}{\gamma_0}\right)^2\right)^{-1/2}$ vs. γa for several values of b/a. Here Q_s is the effective value of Q for a solid beam of radius b. As before, for beam voltages of a few thousand or lower, we may take

$$\left(1 + \left(\frac{\beta_0}{\gamma_0}\right)^2\right)^{-1/2} = 1$$

The quantity β_e is just

$$\beta_e = \frac{2\pi a}{\lambda_g} \tag{13}$$

and from (8) we see that for low beam voltages we can take

$$\beta_e = \gamma = \gamma_0$$

so that the ordinate in Fig. 4 can usually be taken as simply Q_s.

A7.6 The Increasing Wave Parameter B

In Fig. 8.10, B is plotted vs. QC. C can be obtained by means of Sections 3 and 4, and Q by means of Section 5. Hence we can obtain B.

A7.7 The Gain Reduction Parameter α

From (2) we see that we should subtract from the gain of the increasing wave in db α times the cold loss L in db. In Fig. 8.13 a quantity $\partial x_1/\partial d$, which we can identity as α, is plotted vs. QC.

A7.8 The Loss Parameter d

The loss parameter d can be expressed in terms of the cold loss, L in db, the length of the circuit in wavelengths, N, and C

$$d = \left(\frac{2.3L}{20}\right)\left(\frac{1}{2\pi NC}\right) \tag{14}$$

$$d = 0.0183 \frac{L}{NC} \tag{15}$$

A7.9 The Initial Loss A

The quantity A of (2) is plotted vs. d in Fig. 9.3. This plot assumes $QC = 0$, and may be somewhat in error. Perhaps Fig. 9.4 can be used in estimating a correction; it looks as if the initial loss should be less with $QC \neq 0$ even when $d \neq 0$. In any event, an error in A means only a few db, and is likely to make less error in the computed gain than does an error in B, for instance.