Optical Properties and the Electro-optic and Photoelastic Effects in Crystals Expressed in Tensor Form

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I. Introduction

THE electro-optic and photoelastic effects in crystals were first investigated by Pöckels,[1] who developed a phenomenological theory for these effects and measured the constants for a number of crystals. Since then not much work has been done on the subject till the very large electro-optic effects were discovered in two tetragonal crystals ammonium dihydrogen phosphate (ADP) and potassium dihydrogen phosphate (KDP). With these crystals light modulators can be obtained which work on voltages of 2000 volts or less. Their use has been suggested[2] in such equipment as light valves for sound on film recording and in television systems. Furthermore, since the electro-optic effect depends on a change in the dielectric constant with voltage, and the dielectric constant is known to follow the field up to 10^{10} cycles, it is obvious that this effect can be used to produce very short light pulses which may be of interest for physical investigations and for stroboscopic instruments of very high resolution. Hence these crystals renew an interest in the electro-optic effect.

In looking over the literature on the electro-optic effect and photoelastic effect in crystals, there do not seem to be any derivations that give them in terms of thermodynamic potentials, which allow one to investigate the condition under which equalities occur between the various electro-optic and photoelastic constants. Hence it is the purpose of this paper to give such a derivation. Another object is to give a derivation of Maxwell's equations in tensor form, and to apply them to the derivation of the Fresnel ellipsoid.

The first sections deal with the optics of crystals, and derive the Fresnel ellipsoid from Maxwell's equations. Other sections give a derivation of the two effects, discuss methods for measuring them by determining the birefrigence in various directions and give the constants for the two effects in terms of crystal symmetries. The final section discusses the application of the photoelastic effect for measuring strains in isotropic media.

[1] F. Pöckels, Lehrbuck Der Kristalloptic, B. Teubner, Leipzig, 1906.

[2] See *Patent* 2,467,325 issued to the writer; "Light Modulation by P type Crystals," George D. Gotschall, *Jour. Soc. Motion Picture Engineers*, July, 1948, pp. 13–20; B. H. Billings, *Jour. Opt. Soc. Am.*, 39, 797, 802 (1949).

II. Solution of Maxwell's Equations In Tensor Form

In tensor notation, Maxwell's equations for a nonmagnetic medium with no free charges take the form

$$\frac{1}{V}\frac{\partial D_i}{\partial t} = \epsilon_{ijk}\frac{\partial H_j}{\partial x_k}; \quad \frac{1}{V}\frac{\partial H_j}{\partial t} = -\epsilon_{jki}\frac{\partial E_k}{\partial x_i}; \quad \frac{\partial D_i}{\partial x_i} = 0; \quad \frac{\partial H_j}{\partial x_j} = 0 \quad (1)$$

where D_i is the electric displacement, H_j the magnetic field, E_k the electric field, V the velocity of light in vacuo and ϵ_{ijk} a tensor equal to zero when $i = j$ or k or $j = k$, but equal to 1 or -1 when all three numbers are different. If the numbers are in rotation, i.e. 1, 2, 3; 2, 3, 1; 3, 1, 2 the value is $+1$ while, if they are out of rotation, the value is -1.

We assume the electric vector to be representable by a plane wave whose planes of equal phase are taken normal to the unit vector n_i. Then

$$E_k = E_{0_k} e^{j\omega(t - x_i n_i/v)} \quad (2)$$

where E_{0_k} are constants representing the maximum values of the field along the three rectangular coordinates and $j = \sqrt{-1}$. Substituting (2) in the second of equations (1), noting that E_{0_k} are not functions of the space coordinates, we have

$$\frac{1}{V}\frac{\partial H_j}{\partial t} = \frac{j\omega}{v}[\epsilon_{jki} E_{0_k} n_i] e^{j\omega[t - x_i n_i/v]}. \quad (3)$$

Integrating with respect to the time

$$H_j = \frac{V}{v}[\epsilon_{jki} E_{0_k} n_i] e^{j\omega[t - x_i n_i/v]} = H_{0_j} e^{j\omega(t - x_i n_i/v)}. \quad (4)$$

Hence,

$$H_{0_j} = \frac{V}{v}[\epsilon_{jki} E_{0_k} n_i] \quad (5)$$

and therefore the magnetic vector is normal to the plane determined by E_{0_k} and n_i.

Next, using the first of equations (1),

$$\begin{aligned}\frac{\partial D_i}{\partial t} &= V\epsilon_{ijk}\frac{\partial H_j}{\partial x_k} = V\epsilon_{ijk} H_{0_j}\frac{\partial e^{j\omega[t - x_k n_k/v]}}{\partial x_k} \\ &= -\frac{j\omega V}{v}[\epsilon_{ijk} H_{0_j} n_k] e^{j\omega[t - x_k n_k/v]}.\end{aligned} \quad (6)$$

Integrating with respect to time,

$$D_i = -\frac{V}{v}[\epsilon_{ijk} H_{0_j} n_k] e^{j\omega[t - x_k n_k/v]}. \quad (7)$$

Inserting the value of H_{0_j} from (5), this equation takes the form

$$D_i = -\frac{V^2}{v^2}[\epsilon_{ijk}(\epsilon_{jki}E_{0_k}n_i)n_k]e^{j\omega[t-x_in_i/v]}$$

and, in general,

$$D_i = -\frac{V^2}{v^2}[\epsilon_{ijk}(\epsilon_{jki}E_kn_i)n_k]. \tag{9}$$

Expanding the inner parenthesis, we have the components

$$(E_2n_3 - E_3n_2)_1; \qquad (E_3n_1 - E_1n_3)_2; \qquad (E_1n_2 - E_2n_1)_3. \tag{10}$$

Then

$$\epsilon_{ijk}[(E_2n_3 - E_3n_2);\ (E_3n_1 - E_1n_3);\ (E_1n_2 - E_2n_1)]n_k \text{ gives}$$

$$\begin{aligned}
D_1 &= -\frac{V^2}{v^2}[(E_3n_1 - E_1n_3)n_3 - (E_1n_2 - E_2n_1)n_2] \\
&= [(E_3n_3 + E_2n_2 + E_1n_1)n_1 - E_1(n_1^2 + n_2^2 + n_3^2)] \\
D_2 &= -\frac{V^2}{v^2}[(E_1n_2 - E_2n_1)n_1 - (E_2n_3 - E_3n_2)n_3] \\
&= [(E_3n_3 + E_2n_2 + E_1n_1)n_2 - E_2(n_1^2 + n_2^2 + n_3^2)] \\
D_3 &= -\frac{V^2}{v^2}[(E_2n_3 - E_3n_2)n_2 - (E_3n_1 - E_1n_3)n_1] \\
&= [(E_3n_3 + E_2n_2 + E_1n_1)n_3 - E_3(n_1^2 + n_2^2 + n_3^2)].
\end{aligned} \tag{11}$$

Now, since $n_1^2 + n_2^2 + n_3^2 = 1$ because n is a unit vector, we have

$$D_i = \frac{V^2}{v^2}[E_i - (E_jn_j)n_i] \quad \text{or} \quad \frac{v^2}{V^2}D_i - E_i - (E_jn_j)n_i = 0. \tag{12}$$

This equation states that D_i, E_i and n_i are in the same plane, H_j being normal to the plane as shown by Fig. 1. The energy flow vector

$$S_i = \frac{V^2}{4\pi}\epsilon_{ijk}E_jH_k \tag{13}$$

also lies in the plane since it is perpendicular to E and H. It is at the same angle θ with n that E is with D. The velocity of energy flow is $v/\cos\theta$. The energy velocity is called the ray velocity and the energy path the ray path.

Next, from the relation for a material medium, that

$$D_i = K_{ij}E_j \text{ or conversely } E_j = \beta_{ji}D_i \tag{14}$$

where K_{ij} are the dielectric constants measured at optical frequencies and β_{ji} are the impermeability constants determined from the relations

$$\beta_{ji} = \Delta^{ji}/\Delta^{K} \tag{15}$$

where

$$\Delta^{K} = \begin{vmatrix} K_{11} & K_{12} & K_{13} \\ K_{12} & K_{22} & K_{23} \\ K_{13} & K_{23} & K_{33} \end{vmatrix}$$

and Δ^{ji} the determinant obtained by suppressing the j^{th} row and i^{th} column, we can eliminate E_i from equation (12) and obtain

$$\begin{aligned} \frac{v^2}{V^2} D_1 &= \beta_{11} D_1 + \beta_{12} D_2 + \beta_{13} D_3 - (E_j n_j) n_1 \\ \frac{v^2}{V^2} D_2 &= \beta_{12} D_1 + \beta_{22} D_2 + \beta_{23} D_3 - (E_j n_j) n_2 \\ \frac{v^2}{V^2} D_3 &= \beta_{13} D_1 + \beta_{23} D_2 + \beta_{33} D_3 - (E_j n_j) n_3. \end{aligned} \tag{16}$$

This can be put in the form

$$\begin{aligned} (E_j n_j) n_1 &= D_1[\beta_{11} - v^2/V^2] + \beta_{12} D_2 + \beta_{13} D_3 \\ (E_j n_j) n_2 &= \beta_{12} D_1 + (\beta_{22} - v^2/V^2) D_2 + \beta_{23} D_3 \\ (E_j n_j) n_3 &= \beta_{13} D_1 + \beta_{23} D_2 + (\beta_{33} - v^2/V^2) D_3. \end{aligned} \tag{17}$$

Solving for D_1, D_2 and D_3

$$\begin{aligned} D_1 &= [(\beta_{22} - v^2/V^2)(\beta_{33} - v^2/V^2) - \beta_{23}^2][E_j n_j] n_1 \\ D_2 &= [(\beta_{11} - v^2/V^2)(\beta_{33} - v^2/V^2) - \beta_{13}^2][E_j n_j] n_2 \\ D_3 &= [(\beta_{11} - v^2/V^2)(\beta_{22} - v^2/V^2) - \beta_{12}^2][E_j n_j] n_3. \end{aligned} \tag{18}$$

Now, since D and n are at right angles,

$$D_1 n_1 + D_2 n_2 + D_3 n_3 = 0. \tag{19}$$

Hence,

$$\begin{aligned} 0 = {} & [(\beta_{22} - v^2/V^2)(\beta_{33} - v^2/V^2) - \beta_{23}^2] n_1^2 \\ & + [(\beta_{11} - v^2/V^2)(\beta_{33} - v^2/V^2) - \beta_{13}^2] n_2^2 \\ & + [(\beta_{11} - v^2/V^2)(\beta_{22} - v^2/V^2) - \beta_{12}^2] n_3^2. \end{aligned} \tag{20}$$

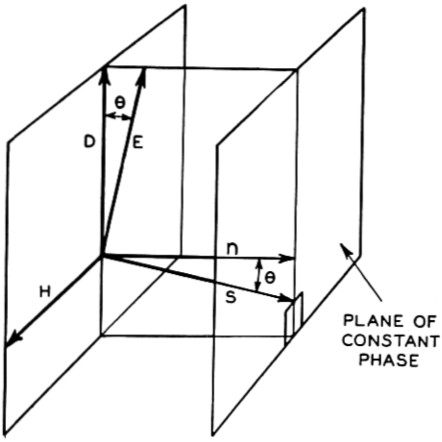

Fig. 1—Position of electric, magnetic and normal vectors for an electromagnetic plane wave in a crystal.

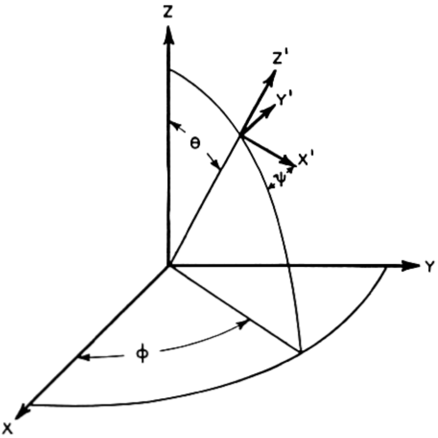

Fig. 2—Rotated axes and angles for relating them to unrotated axes.

By choosing the original x, y, z axes so that $\beta_{12} = \beta_{13} = \beta_{23} = 0$ and using the values $\beta_{11} = \beta_1$, $\beta_{22} = \beta_2$, $\beta_{33} = \beta_3$ this gives the equation

$$\frac{n_1^2}{\beta_1 - \frac{v^2}{V^2}} + \frac{n_2^2}{\beta_2 - \frac{v^2}{V^2}} + \frac{n_3^2}{\beta_3 - \frac{v^2}{V^2}} = 0. \tag{21}$$

For transmission along the X axis $n_1 = 1$, $n_2 = n_3 = 0$ and the two velocities are given by

$$v^2 = \beta_2 V^2 = b^2, \qquad v^2 = \beta_3 V^2 = c^2. \tag{22}$$

Similarly the third velocity $v^2 = \beta_1 V^2 = a^2$ can also be used and equation (21) reduces to

$$\frac{n_1^2}{a^2 - v^2} + \frac{n_2^2}{b^2 - v^2} + \frac{n_3^2}{c^2 - v^2} = 0. \tag{23}$$

This is a quadratic equation for the velocities v in terms of the principal velocities a, b and c which are usually taken so that $a > b > c$.

Solving for the velocities, we obtain the quadratic equation

$$\begin{aligned} v^4 - v^2[n_1^2(b^2 + c^2) + n_2^2(a^2 + c^2) + n_3^2(a^2 + b^2)] \\ + n_1^2 b^2 c^2 + n_2^2 a^2 c^2 + n_3^2 a^2 b^2 = 0. \end{aligned} \tag{24}$$

Letting $L = n_1^2(b^2 - c^2)$, $M = n_2^2(c^2 - a^2)$, $N = n_3^2(a^2 - b^2)$ the solutions for the velocities become

$$\begin{aligned} 2v^2 = n_1^2(b^2 + c^2) + n_2^2(c^2 + a^2) + n_3^2(a^2 + b^2) \\ \pm \sqrt{L^2 + M^2 + N^2 - 2LM - 2LN - 2MN}. \end{aligned} \tag{25}$$

This equation can be put into a simpler form if we change to the coordinate system shown by Fig. 2. Here the rotated system is related to the original system by three angles θ, φ, ψ. θ is the angle between the Z' axis and the Z axis, φ is the angle the plane containing Z and Z' makes with the X axis while ψ represents a rotation of the primed coordinate systems about the Z' axis. The direction cosines for the primed system with respect to the normal system are designated by the matrix

$$\begin{array}{c|ccc} & X & Y & Z \\ \hline X' & \ell_1 & m_1 & n_1 \\ Y' & \ell_2 & m_2 & n_2 \\ Z' & \ell_3 & m_3 & n_3 \end{array} \tag{26}$$

where, in terms of θ, φ and ψ, these direction cosines are,

$$\ell_1 = \cos\theta \cos\varphi \cos\psi - \sin\varphi \sin\psi,$$

$$m_1 = \cos\theta \sin\varphi \cos\psi + \cos\varphi \sin\psi, \qquad n_1 = -\sin\theta \cos\psi$$

$$\ell_2 = -\cos\theta \cos\varphi \sin\psi - \sin\varphi \cos\psi,$$

$$m_2 = \cos\varphi \cos\psi - \sin\varphi \sin\psi \cos\theta, \qquad n_2 = \sin\theta \sin\psi$$

$$\ell_3 = \cos\varphi \sin\theta, \qquad m_3 = \sin\varphi \sin\theta, \qquad n_3 = \cos\theta. \tag{27}$$

If we take Z' as the direction of the wave normal, then in equation (25)

$$n_1 = \ell_3, \qquad n_2 = m_3, \qquad n_3 = n_3$$

and the equation for the velocities becomes

$$2v^2 = a^2(\sin^2\varphi\sin^2\theta + \cos^2\theta) + b^2(\cos^2\varphi\sin^2\theta + \cos^2\theta) + c^2\sin^2\theta \pm \sqrt{\begin{matrix}(a^2 - b^2)^2(\cos^2\theta\cos^2\varphi + \sin^2\varphi)^2 + 2(a^2 - b^2)(c^2 - b^2) \\ \sin^2\theta(\cos^2\theta\cos^2\varphi - \sin^2\varphi) + (c^2 - b^2)^2\sin^4\theta\end{matrix}} \quad (28)$$

A very elegant construction for the wave-velocities and the directions of vibration is the Fresnel index ellipsoid. Consider the ellipsoid

$$a^2x^2 + b^2y^2 + c^2z^2 = 1 \quad (29)$$

Then Fresnel[3] showed that, for any diametral plane perpendicular to the wave normal, the two principal axes of the ellipse were the directions of the two permitted vibrations, while the wave velocities were the reciprocals of the principal semi-axes.

We wish to show now that the maximum and minimum values of the impermeability constants in a plane perpendicular to the direction of the wave normal determine the directions of vibration and the values of the two velocities. To show this we make use of the fact that β_{ij} is a second rank tensor and transforms according to the tensor transformation formula

$$\beta'_{ij} = \frac{\partial x'_i}{\partial x_k}\frac{\partial x'_j}{\partial x_\ell}\beta_{k\ell} \quad (30)$$

where the partial derivatives are the direction cosines

$$\frac{\partial x'_1}{\partial x_1} = \ell_1, \quad \frac{\partial x'_1}{\partial x_2} = m_1, \quad \frac{\partial x'_1}{\partial x_3} = n_1$$

$$\frac{\partial x'_2}{\partial x_1} = \ell_2, \quad \frac{\partial x'_2}{\partial x_2} = m_2, \quad \frac{\partial x'_2}{\partial x_3} = n_2$$

$$\frac{\partial x'_3}{\partial x_1} = \ell_3, \quad \frac{\partial x'_3}{\partial x_2} = m_3, \quad \frac{\partial x'_3}{\partial x_3} = n_3.$$

Expanding equation (30) the six transformation equations become

$$\beta'_{11} = \ell_1^2\beta_{11} + 2\ell_1m_1\beta_{12} + 2\ell_1n_1\beta_{13} + m_1^2\beta_{22} + 2m_1n_1\beta_{23} + n_1^2\beta_{33}$$

$$\beta'_{12} = \ell_1\ell_2\beta_{11} + (\ell_1m_2 + m_1\ell_2)\beta_{12} + (\ell_1n_2 + n_1\ell_2)\beta_{13} + m_1m_2\beta_{22} + (m_1n_2 + n_1m_2)\beta_{23} + n_1n_2\beta_{33}$$

$$\beta'_{13} = \ell_1\ell_3\beta_{11} + (\ell_1m_3 + m_1\ell_3)\beta_{12} + (\ell_1n_3 + n_1\ell_3)\beta_{13} + m_1m_3\beta_{22} + (n_1m_3 + m_1n_3)\beta_{23} + n_1n_3\beta_{33} \quad (31)$$

[3] See for example "Photoelasticity," Coker and Filon, Cambridge University Press, pages 17 and 18.

$$\beta'_{22} = \ell_2^2\beta_{11} + 2\ell_2 m_2\beta_{12} + 2\ell_2 n_2\beta_{13} + m_2^2\beta_{22} + 2m_2 n_2\beta_{23} + n_2^2\beta_{33}$$

$$\beta'_{23} = \ell_2\ell_3\beta_{11} + (\ell_2 m_3 + m_2\ell_3)\beta_{12} + (\ell_2 n_3 + n_2\ell_3)\beta_{13} + m_2 m_3\beta_{22} + (m_2 n_3 + n_2 m_3)\beta_{23} + n_2 n_3\beta_{33}$$

$$\beta'_{33} = \ell_3^2\beta_{11} + 2\ell_3 m_3\beta_{12} + 2\ell_3 n_3\beta_{13} + m_3^2\beta_{22} + 2m_3 n_3\beta_{23} + n_3^2\beta_{33}.$$

Now, if the axes refer to the axes of a Fresnel ellipsoid, $\beta_{12} = \beta_{13} = \beta_{23} = 0$ and one of the impermeability constants for any direction, say β'_{33}, can be expressed in the form

$$\beta'_{33} = \ell_3^2\beta_1 + m_3^2\beta_2 + n_3^2\beta_3 \tag{32}$$

If r, which lies along Z' of Fig. 2, is the radius vector of the Fresnel ellipsoid, then the direction cosines ℓ_3, m_3 and n_3 are

$$\ell_3 = \frac{x}{r}, \qquad m_3 = \frac{y}{r}, \qquad n_3 = \frac{z}{r}.$$

From equation (24) $\beta_1 = a^2/V^2$, $\beta_2 = b^2/V^2$, $\beta_3 = c^2/V^2$ and equation (32) becomes

$$r^2V^2\beta'_{33} = a^2x^2 + b^2y^2 + c^2z^2 = 1.$$

Hence the square of the radius vector of the Fresnel ellipsoid is $1/V^2\beta'_{33}$ and the radius vector of the impermeability ellipsoid agrees with that of the Fresnel ellipsoid. Hence, the directions of vibration can be determined from the principal axes of the impermeability ellipsoid for any diametral plane.

When light transmission occurs along Z', the direction for maximum and minimum impermeability can be obtained by evaluating β'_{11} and determining the angle ψ for which it has an extreme value. Inserting the direction cosines ℓ_1, m_1 and n_1 from equation (27), we find

$$\begin{aligned}\beta'_{11} = {} & \beta_1\left[\cos^2\theta\cos^2\varphi\cos^2\psi - \frac{\sin 2\varphi\sin 2\psi\cos\theta}{2} + \sin^2\varphi\sin^2\psi\right] \\ & + \beta_2\left[\cos^2\theta\sin^2\varphi\cos^2\psi + \frac{\sin 2\varphi\sin 2\psi\cos\theta}{2} + \cos^2\varphi\sin^2\psi\right] \\ & + \beta_3\sin^2\theta\cos^2\psi.\end{aligned} \tag{33}$$

Differentiating with respect to ψ and setting the resultant derivative equal to zero, the value of ψ that will satisfy the equation is given by

$$\begin{aligned}\tan 2\psi &= \frac{(\beta_2 - \beta_1)\sin 2\varphi\cos\theta}{(\beta_1 - \beta_2)(\cos^2\theta\cos^2\varphi - \sin^2\varphi) + (\beta_3 - \beta_2)\sin^2\theta} \\ &= \frac{(b^2 - a^2)\sin 2\varphi\cos\theta}{(a^2 - b^2)(\cos^2\theta\cos^2\varphi - \sin^2\varphi) + (c^2 - b^2)\sin^2\theta}.\end{aligned} \tag{34}$$

For a given value on the right-hand side there are two values of ψ, 90° apart, that will satisfy the equation and hence we have two directions of vibration at right angles to each other. Inserting (34) in (33) the values of β'_{11} and β''_{11} for these two directions are

$$2\beta'_{11} = \beta_1(\sin^2\varphi \sin^2\theta + \cos^2\theta) + \beta_2(\cos^2\varphi \sin^2\theta + \cos^2\theta) + \beta_3 \sin^2\theta$$
$$\pm \sqrt{(\beta_1 - \beta_2)^2 (\cos^2\theta \cos^2\varphi + \sin^2\varphi)^2 + 2(\beta_1 - \beta_2)(\beta_3 - \beta_2) \cdot \sin^2\theta (\cos^2\theta \cos^2\varphi - \sin^2\varphi) + (\beta_3 - \beta_2)^2 \sin^4\theta}.$$

Since β_1 corresponds to a^2, etc., this equation agrees with the two velocities given in equation (28) and shows that the directions of vibration correspond with the maximum and minimum values of β'_{11}.

It can also be shown that the two directions of electric displacement coincide with the two values of ψ given by equation (34). Transforming the electrical displacements to the X', Y', Z' set of axes we have

$$\begin{aligned}
D'_1 &= \frac{\partial x'_1}{\partial x_1} D_1 + \frac{\partial x'_1}{\partial x_2} D_2 + \frac{\partial x'_1}{\partial x_3} D_3 = \ell_1 D_1 + m_1 D_2 + n_1 D_3 \\
D'_2 &= \frac{\partial x'_2}{\partial x_1} D_1 + \frac{\partial x'_2}{\partial x_2} D_2 + \frac{\partial x'_2}{\partial x_3} D_3 = \ell_2 D_1 + m_2 D_2 + n_2 D_3 \\
D'_3 &= \frac{\partial x'_3}{\partial x_1} D_1 + \frac{\partial x'_3}{\partial x_2} D_2 + \frac{\partial x'_3}{\partial x_3} D_3 = \ell_3 D_1 + m_3 D_2 + n_3 D_3 .
\end{aligned} \tag{35}$$

Hence, inserting the values of D_1, D_2, D_3 from equation (18), we find

$$\begin{aligned}
D'_1 &= \ell_1\ell_3(\beta_2 - \beta'_{11})(\beta_3 - \beta'_{11}) + m_1m_3(\beta_1 - \beta'_{11})(\beta_3 - \beta'_{11}) \\
&\qquad + n_1n_3(\beta_1 - \beta'_{11})(\beta_2 - \beta'_{11}) \\
D'_2 &= \ell_2\ell_3(\beta_2 - \beta'_{11})(\beta_3 - \beta'_{11}) + m_2m_3(\beta_1 - \beta'_{11})(\beta_3 - \beta'_{11}) \\
&\qquad + n_2n_3(\beta_1 - \beta'_{11})(\beta_2 - \beta'_{11}) \\
D'_3 &= \ell_3^2(\beta_2 - \beta'_{11})(\beta_3 - \beta'_{11}) + m_3^2(\beta_1 - \beta'_{11})(\beta_3 - \beta'_{11}) \\
&\qquad + n_3^2(\beta_1 - \beta'_{11})(\beta_2 - \beta'_{11}).
\end{aligned} \tag{36}$$

From equation (20) with $\beta_{12} = \beta_{13} = \beta_{23} = 0$, it is evident that the D_3 component vanishes and hence the two values of electric displacement lie in a plane perpendicular to Z'. By inserting the values of β'_{11} and the value of ψ found from equation (34) we find that $D_2 = 0$ and hence the electric displacement lies along the directions of the greatest value of β'_{11}. Similarly, from the second value of β'_{11}, D_1 vanishes and hence the second wave is perpendicular to the first and in the direction of the smallest value of β'_{11}.

III. Location of Optic Axes in a Crystal

When the expression in the radical of equation (28) vanishes the two velocities are equal and an optic axis exists. Since the expression inside the radical can be written

$$[(a^2 - b^2)(\cos^2\theta\cos^2\varphi + \sin^2\varphi) - (b^2 - c^2)\sin^2\theta]^2 - 4(a^2 - b^2)(c^2 - b^2)\sin^2\theta\sin^2\varphi = 0 \tag{37}$$

then, since the square is always positive and since $(a^2 - b^2) > 0$ and $(b^2 - c^2) > 0$, the equation can vanish only if $\varphi = 0$. But $\varphi = 0$ indicates that the two optic axes always lie in a plane perpendicular to the intermediate velocity b. With $\varphi = 0$ then the square vanishes when

$$\tan^2\theta = \frac{(a^2 - b^2)}{(b^2 - c^2)} \quad \text{or} \quad \tan\theta = \pm\sqrt{\frac{a^2 - b^2}{b^2 - c^2}}. \tag{38}$$

If $(a^2 - b^2) < (b^2 - c^2)$ the value of the $\tan\theta$ is less than unity and the crystal is called a positive crystal. For this case the two axes approach more closely the Z axis having the velocity c than they do the X axis. If $(a^2 - b^2) > (b^2 - c^2)$ the crystal is negative.

If $a = b$ or $b = c$ the crystal has a single optic axis and is respectively a positive or negative uniaxial crystal. For the first case the two velocities are given by

$$v_1 = a = b, \qquad v_2 = \sqrt{a^2\cos^2\theta + c^2\sin^2\theta}. \tag{39}$$

The first velocity is that of the ordinary ray while that of the second is that of the extraordinary ray. Since $a > c$, the ordinary ray will have a velocity greater than the extraordinary ray except along the optic axis where they are equal. Since $c < a$, the maximum axis for any ellipse, formed by intersecting the Fresnel ellipsoid at an angle to the optic axis, will lie in the plane formed by the normal and the c axis and hence the direction of polarization of the extraordinary ray will lie in the c, n plane. The polarization of the ordinary ray will be perpendicular to this plane.

If $b = c$ the a axis is the optic axis and the velocities of the two rays are again

$$v_1 = c \text{ and } v_2^2 = a^2(1\text{-}\sin^2\theta\cos^2\varphi) + c^2(\sin^2\theta\cos^2\varphi) \tag{40}$$

Hence, when $\theta = 90°$, $\varphi = 0°$, the two velocities are equal and a is the optic axis. In this case the velocity of the extraordinary ray is greater than that of the ordinary ray except along the a axis, and the crystal is a negative uniaxial crystal. The polarization of the extraordinary ray lies again in the plane of the normal and the optic axis while the ordinary ray is perpendicular to it.

IV. Derivation of the Electro-optic and Photoelastic Effects

In a previous paper[4] and in the book "Piezoelectric Crystals and Their Application to Ultrasonics", D. Van Nostrand, 1950, it was shown that the electro-optic and photoelastic effects can be expressed as third derivatives of one of the thermodynamic potentials. Probably the most fundamental way of developing these properties is to express them in terms of the strains, electric displacements and the entropy. For viscoelastic substances it has been shown that the photoelastic effects are directly related to the strains. In terms of the electric displacements, the electro-optic constants do not vary much with temperature whereas, if they are expressed in terms of the fields, the constants of a ferroelectric type of crystal such as KDP increase many fold near the Curie temperature. The entropy is chosen as the fundamental heat variable, since most measurements are carried out so rapidly that the entropy does not vary.

The thermodynamic potential which has the strains, electric displacements and entropy as the independent variables is the internal energy U, given by

$$dU = T_{ij}\, dS_{ij} + E_m \frac{dD_m}{4\pi} + \Theta\, d\sigma \tag{41}$$

where S_{ij} are the strains, T_{ij} the stresses, E_m the fields, D_m the electric displacements, Θ the temperature and σ the entropy. In this equation the strains S_{ij} are defined in the tensor form

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{42}$$

where the u's are the displacements along the three axis. In the case of a shearing strain occurring when $i \neq j$, the strain is only half that usually used in engineering practice. In order to avoid writing the factor $1/4\pi$, we use the variable $\delta_m = D_m/4\pi$. Then, from (41),

$$T_{ij} = \frac{\partial U}{\partial S_{ij}}, \qquad E_m = \frac{\partial U}{\partial \delta_n}, \qquad \theta = \frac{\partial U}{\partial \sigma}. \tag{43}$$

Since, for most conditions of interest, adiabatic conditions prevail, we can set $d\sigma$ equal to zero and can develop the dependent variables, the fields and

[4] "First and Second Order Equations for Piezoelectric Crystals Expressed in Tensor Form," W. P. Mason, *B.S.T.J.*, Vol. 26, pp. 80–138, Jan., 1947.

the stresses in terms of the independent variables, the strains and the electric displacements. Up to the second derivatives, these are

$$E_m = \frac{\partial E_m}{\partial S_{ij}} S_{ij} + \frac{\partial E_m}{\partial \delta_n} \delta_n + \frac{1}{2!}\left[\frac{\partial^2 E_m}{\partial S_{ij}\,\partial S_{qr}} S_{ij} S_{qr} + \frac{2\partial^2 E_m}{\partial S_{ij}\,\partial \delta_n} S_{ij}\delta_n + \frac{\partial^2 E_m}{\partial \delta_n\,\partial \delta_o}\delta_n\delta_o\right] + \cdots$$

$$T_{k\ell} = \frac{\partial T_{k\ell}}{\partial S_{ij}} S_{ij} + \frac{\partial T_{k\ell}}{\partial \delta_n} \delta_n + \frac{1}{2!}\left[\frac{\partial^2 T_{k\ell}}{\partial S_{ij}\,\partial S_{qr}} S_{ij} S_{qr} + \frac{2\partial^2 T_{k\ell}}{\partial S_{ij}\,\partial \delta_n} S_{ij}\delta_n + \frac{\partial^2 T_{k\ell}}{\partial \delta_n\,\partial \delta_o}\delta_n\delta_o\right] + \cdots. \tag{44}$$

For the electro-optic and photoelastic cases, the two tensors of interest are

$$\frac{\partial^2 T_{k\ell}}{\partial \delta_n\,\partial \delta_o} = \frac{\partial^3 U}{\partial S_{k\ell}\,\partial \delta_n\,\partial \delta_o} = \frac{\partial^2 E_n}{\partial S_{k\ell}\,\partial \delta_o} = 4\pi m_{k\ell no}$$

$$\frac{\partial^2 E_m}{\partial \delta_n\,\partial \delta_o} = \frac{\partial^3 U}{\partial \delta_m\,\partial \delta_n\,\partial \delta_o} = (4\pi) r_{mno}. \tag{45}$$

For the first partial derivatives, we have the values

$$\frac{\partial T_{k\ell}}{\partial S_{ij}} = c^D_{ijk\ell}; \qquad \frac{\partial T_{k\ell}}{\partial \delta_n} = \frac{\partial^2 U}{\partial S_{k\ell}\,\partial \delta_n} = \frac{\partial E_n}{\partial S_{k\ell}} = -h_{nk\ell}$$

$$\frac{\partial E_m}{\partial \delta_n} = 4\pi \beta^S_{mn} \tag{46}$$

where $c^D_{ijk\ell}$ are the elastic stiffnesses measured at constant electric displacement, $h_{nk\ell}$ are the piezoelectric constants that relate the open circuit voltages to the strains, and β^S_{mn} are the impermeability constants measured for constant strain.

With these substitutions and neglecting the other second partial derivatives, we have, from (44),

$$E_m = -h_{mij} S_{ij} + D_n\left[\beta^S_{mn} + m_{ijmn} S_{ij} + \frac{r^S_{mno}}{2} D_o\right] + \cdots$$

$$T_{k\ell} = c^D_{ijk\ell} S_{ij} + D_o\left[-\frac{h_{ok\ell}}{4\pi} + \frac{m_{k\ell on} D_n}{2}\right]. \tag{47}$$

This equation shows that there is a relation between the change in the impermeability constant due to stress in the first equation, and the electrostrictive constant in the second equation through the tensor m_{ijmn}. These effects, however, have to be measured at the same frequency before equality exists.

To obtain the changes in the optical properties caused by the strain and the electric displacement we have to determine the fields and displacements occurring at the high frequencies of optics. Even for piezoelectric vibrations occurring at as high frequencies as they can be driven by the piezoelectric effect, these frequencies are small compared to the optic frequencies f and can be considered to be static displacements or strains. Hence, writing

$$E_m = E_m^0 + E_m e^{j\omega t}, \qquad D_n = D_n^0 + D_n e^{j\omega t},$$
$$D_o = D_o^0 + D_o e^{j\omega t}, \qquad S_{ij} = S_{ij}^0$$

where $\omega = 2\pi f$, the first of equation (47) can be written in the form

$$\begin{aligned} E_m^0 &= -h_{mij} S_{ij} + D_n^0 \left[\beta_{mn}^S + m_{ijmn} S_{ij} + \frac{r_{mno}}{2} D_o^0 \right] \\ E_m e^{j\omega t} &= D_n e^{j\omega t} \left[\beta_{mn}^S + m_{ijmn} S_{ij} + \frac{r_{mno}^S}{2} D_o^0 \right] + \frac{r_{mno}}{2} D_n^0 D_o e^{j\omega t}. \end{aligned} \tag{48}$$

If we develop one of the fields, say E_1, this can be written in the form

$$\begin{aligned} E_1 e^{j\omega t} &= [\beta_{11} + m_{ij11} S_{ij} + r_{111} D_1^0 + r_{112} D_2^0 + r_{113} D_3^0] D_1 e^{j\omega t} \\ &+ [\beta_{12} + M_{ij12} S_{ij} + r_{121} D_1^0 + r_{122} D_2^0 + r_{123} D_3^0] D_2 e^{j\omega t} \\ &+ [\beta_{13} + M_{ij13} S_{ij} + r_{131} D_1^0 + r_{132} D_2^0 + r_{133} D_3^0] D_3 e^{i\omega t} \end{aligned} \tag{49}$$

where the first number of r refers to the field, the second to the optical value of D and the third to the static value of D. Hence, for the general case,

$$E_m e^{j\omega t} = D_n e^{j\omega t} [\beta_{mn} + m_{ijmn} S_{ij} + r_{mno} D_o^0]. \tag{50}$$

From the definition of the two tensors m_{ijno} and r_{mno} given by equation (45), we can show that there are relations between the various components of the tensors. For the first tensor m_{ijno}, since $S_{ij} = S_{ji}$ is a symmetrical tensor, then

$$m_{ijno} = m_{jino} \tag{51}$$

From the definition of the tensor m_{ijno} in the form

$$4\pi m_{ijno} = \frac{\partial}{\partial S_{ij}} \left(\frac{\partial^2 U}{\partial \delta_n \, \partial \delta_o} \right) \tag{45}$$

it is obvious that we can interchange the order of δ_n and δ_o so that

$$m_{ijno} = m_{ijon}$$

Since ij and no are reversible, it has been customary to abbreviate the tensor by writing one number in place of the two in the following form:

$$11 = 1; 22 = 2; 33 = 3; 12 = 21 = 6; 13 = 31 = 5; 23 = 32 = 4 \tag{52}$$

Since the reduced tensor is associated with the engineering strains, it is necessary to investigate the numerical relationships between the four index symbols and the two index symbols. From equation (48), when $m \neq n$, the change in the impermeability constant β_{mn} is given by

$$m_{ijmn} S_{ij} + m_{jimn} S_{ji} = m_{rs} S_r \tag{53}$$

Since $S_r = 2S_{ij} = 2S_{ji}$ we have the relation that

$$m_{ijmn} = m_{rs}(i, j, m, n = 1 \text{ to } 3, r, s, = 1 \text{ to } 6) \tag{54}$$

In equation (45) we cannot in general interchange the order of ij and no since U does not contain product terms of strains and electric displacements and hence in general

$$m_{rs} \neq m_{sr}. \tag{55}$$

Hence in the most general case there are 36 photoelastic constants. Crystal symmetrics cut down the number of constants as shown in a later section.

The tensor r_{mno} defined in equation (45) as

$$(4\pi)^2 r_{mno} = \frac{\partial^3 U}{\partial \delta_m \partial \delta_n \partial \delta_o} \tag{56}$$

shows that we can interchange the order of m and n since U contains product terms of δ_m and δ_n. Hence

$$r_{mno} = r_{nmo} \tag{57}$$

and this is usually replaced by the two index symbols

$$r_{qo} = r_{mno}(m, n, o = 1 \text{ to } 3; q = 1 \text{ to } 6).$$

The so called "true" electro-optic constants are measured at constant strain and for this case the modifications in the impermeability constants are given by the equation

$$E_m = D_n [\beta^S_{mn} + r^S_{mno} D_o]. \tag{58}$$

Since m and n are interchangeable, the third rank tensor is usually replaced by the two index symbols

$$r^S_{mno} = r_{qo}(m, n, o = 1 \text{ to } 3; q = 1 \text{ to } 6). \tag{59}$$

As discussed in the next sections, these constants can be determined by applying an electric field of a frequency high enough so that the principal resonances and their harmonics cannot be excited by the applied field, and measuring the resulting birefringence along definite directions in the crystal. On the other hand if we apply a static field to the crystal, an additional effect occurs because the crystal is strained by the piezoelectric effect and this causes a photoelastic effect in addition to the "true" electro-optic effect. A better designation for these effects is the electro-optic effect at constant strain and stress.

This latter effect can be calculated from equation (47) by setting the stresses $T_{k\ell}$ equal to zero and eliminating the S_{ij} strains. After neglecting second order corrections,

$$E_m = D_n e^{j\omega t}\left[\beta^S_{mn} + \left(r^S_{mno} + \frac{m_{ijmn} h_{ok\ell}}{4\pi c^D_{ijk\ell}}\right) D^0_o\right]. \tag{60}$$

Since $h_{ok\ell}/c^D_{ijk\ell} = g_{oij}$, the other piezoelectric constant relating the open circuit voltage to the stress, the electro-optic effect at constant stress can be written in the form

$$r^T_{mno} = r^S_{mno} + \frac{m_{ijmn} g_{oij}}{4\pi}. \tag{61}$$

In terms of the two index symbols

$$r^T_{qo} = r^S_{qo} + \frac{m_{pq} g_{op}}{4\pi} \tag{62}$$

since it has been shown[4] that $g_{oij} = g_{op}/2$ when $i \neq j$, and the tensor in (61) has ij as common symbols which involves the summations of two terms.

The electro-optic effect is usually measured in terms of an applied field. The change in the impermeability constant β^S_{mn} for this case can be determined from the first equations (47), setting $T_{k\ell}$ equal to zero and neglecting second order terms. Multiplying through by the tensor K^T_{op} of the dielectric constants

$$D^0_p = E^0_0 K^T_{op} \tag{63}$$

since the product $K^T_{op}\beta^T_{op} = 1$. Introducing this equation into (58) we have

$$E_m = D_n[\beta^S_{mn} + r^S_{mnp} K^T_{op} E^0_o] = D_n[\beta^S_{mn} + z^S_{mno} E^0_o]. \tag{64}$$

where the new tensor z_{mno} is equal to

$$z^S_{mno} = r^S_{mnp} K^T_{op}. \tag{65}$$

In terms of the two index symbols

$$z^S_{qo} = r^S_{qp} K^T_{op}. \tag{66}$$

in which the repeated index indicates a summation. The difference between the electro-optical constant at constant stress expressed in terms of the field and the electro-optical constant at constant strain is

$$z^T_{mno} = z^S_{mno} + \frac{m_{ijmn} g_{oij}}{4\pi} K^T_{op} = z^S_{mno} + m_{ijmn} d_{pij} \tag{67}$$

since the piezoelectric constants d_{pij} are related to the g constants by the equation

$$d_{pij} = \frac{g_{oij} K^T_{op}}{4\pi}. \tag{68}$$

In terms of two index symbols

$$z_{qo}^{T} = z_{qo}^{S} + m_{pq}d_{op}(p,\ q = 1 \text{ to } 6;\ o = 1 \text{ to } 3) \tag{69}$$

where a repeated index means a summation with respect to this index.

Finally the photoelastic effect is sometimes expressed in terms of the stresses rather than the strains. As can be seen from equation (47), the new set of constants is

$$\pi_{pq} = m_{pr}s_{rq}^{D} \tag{70}$$

where the s_{rq}^{D} are the elastic compliances measured at constant electric displacement.

V. Birefringence Along Any Direction In the Crystal and Determination of the Electro-optic and Photoelastic Constants

If we take axes along the Fresnel ellipsoid when no stress or field is applied to the crystal, the result of the electro-optic and photoelastic effects is to change the impermeability constants by the values

$$\begin{aligned} \beta_{11} &= \beta_1 + \Delta_1; \quad & \beta_{22} &= \beta_2 + \Delta_2; \quad & \beta_{33} &= \beta_3 + \Delta_3 \\ \beta_{23} &= \Delta_4; & \beta_{13} &= \Delta_5; & \beta_{12} &= \Delta_6 \end{aligned} \tag{71}$$

where

$$\begin{aligned}
\Delta_1 &= z_{11}E_1 + z_{12}E_2 + z_{13}E_3 + m_{11}S_1 + m_{12}S_2 + m_{13}S_3 + m_{14}S_4 \\
&\qquad + m_{15}S_5 + m_{16}S_6 \\
\Delta_2 &= z_{21}E_1 + z_{22}E_2 + z_{23}E_3 + m_{21}S_1 + m_{22}S_2 + m_{23}S_3 + m_{24}S_4 \\
&\qquad + m_{25}S_5 + m_{26}S_6 \\
\Delta_3 &= z_{31}E_1 + z_{32}E_2 + z_{33}E_3 + m_{31}S_1 + m_{32}S_2 + m_{33}S_3 + m_{34}S_4 \\
&\qquad + m_{35}S_5 + m_{36}S_6 \\
\Delta_4 &= z_{41}E_1 + z_{42}E_2 + z_{43}E_3 + m_{41}S_1 + m_{42}S_2 + m_{43}S_3 + m_{44}S_4 \\
&\qquad + m_{45}S_5 + m_{46}S_6 \\
\Delta_5 &= z_{51}E_1 + z_{52}E_2 + z_{53}E_3 + m_{51}S_1 + m_{52}S_2 + m_{53}S_3 + m_{54}S_4 \\
&\qquad + m_{55}S_5 + m_{56}S_6 \\
\Delta_6 &= z_{61}E_1 + z_{62}E_2 + z_{63}E_3 + m_{61}S_1 + m_{62}S_2 + m_{63}S_3 + m_{64}S_4 \\
&\qquad + m_{65}S_5 + m_{66}S_6.
\end{aligned} \tag{72}$$

If we transmit light along the z' axis which, as shown by Fig. 2, makes an angle of θ degrees with the z axis in a plane making an angle φ with the xz plane, the birefringence can be calculated as follows: Keeping z' fixed and rotating the other two axes about z' by varying the angle ψ, one light vector will occur when β'_{11} is a maximum and the other when β'_{11} is a minimum. Using the transformation equations (31) and the direction cosines of (27), we find that β'_{11} is given by the equations

$$\begin{aligned}\beta'_{11} = \beta_{11}\left[\cos^2\theta\cos^2\varphi\cos^2\psi - \frac{\sin 2\varphi\sin 2\psi\cos\theta}{2} + \sin^2\varphi\sin^2\psi\right] \\ + \beta_{12}[\sin 2\varphi\cos 2\psi - \sin^2\theta\sin 2\varphi\cos^2\psi + \cos\theta\sin 2\psi\cos 2\varphi] \\ + \beta_{13}[-\sin 2\theta\cos\varphi\cos^2\psi + \sin\varphi\sin\theta\sin 2\psi] \\ + \beta_{22}\left[\cos^2\theta\sin^2\varphi\cos^2\psi + \frac{\cos\theta\sin 2\varphi\sin 2\psi}{2} + \cos^2\varphi\sin^2\psi\right] \\ + \beta_{23}[-\sin 2\theta\sin\varphi\cos^2\psi - \sin\theta\cos\varphi\sin 2\psi] + \beta_{33}\sin^2\theta\cos^2\psi\end{aligned} \tag{73}$$

Differentiating with respect to ψ and setting $\dfrac{\partial\beta'_{11}}{\partial\psi} = 0$, we find an expression for $\tan 2\psi$ in the form

$$\tan 2\psi = \frac{\begin{aligned}-\beta_{11}\sin 2\varphi\cos\theta + 2\beta_{12}\cos\theta\cos 2\varphi \\ + 2\beta_{13}\sin\varphi\sin\theta + \beta_{22}\cos\theta\sin 2\varphi - 2\beta_{23}\sin\theta\cos\varphi\end{aligned}}{\begin{aligned}\beta_{11}[\cos^2\theta\cos^2\varphi - \sin^2\varphi] + \beta_{12}[(1+\cos^2\theta)\sin 2\varphi] \\ - \beta_{13}\sin^2\theta\cos\varphi + \beta_{22}(\cos^2\theta\sin^2\varphi - \cos^2\varphi) \\ - \beta_{23}\sin 2\theta\sin\varphi + \beta_{33}\sin^2\theta\end{aligned}} \tag{74}$$

Inserting this value back in equation (73) we find that the two extreme values of β'_{11} are given by the equation

$$\begin{aligned}2\beta'_{11} = 2\beta_{22} + (\beta_{11} - \beta_{22})(\cos^2\theta\cos^2\varphi + \sin^2\varphi) + (\beta_{33} - \beta_{22})\sin^2\theta \\ - \beta_{12}\sin^2\theta\sin 2\varphi - \beta_{13}\sin 2\theta\cos\varphi - \beta_{23}\sin 2\theta\sin\varphi\end{aligned}$$

$$\pm\sqrt{\begin{aligned}(\beta_{11} - \beta_{22})^2(\cos^2\theta\cos^2\varphi + \sin^2\varphi)^2 + 2(\beta_{11} - \beta_{22})(\beta_{33} - \beta_{22})\sin^2\theta\times \\ (\cos^2\theta\cos^2\varphi - \sin^2\varphi) + (\beta_{33} - \beta_{22})^2\sin^4\theta - 2(\beta_{11} - \beta_{22})\times \\ [\beta_{12}(\sin 2\varphi\sin^2\theta(\cos^2\theta\cos^2\varphi + \sin^2\varphi) + \beta_{13}\sin 2\theta\cos\varphi\times \\ (\cos^2\theta\cos^2\varphi + \sin^2\varphi) - \beta_{23}\sin 2\theta\sin\varphi(1 + \cos^2\varphi\sin^2\theta))] \\ + 2(\beta_{33} - \beta_{22})\sin^2\theta[\beta_{12}\sin 2\varphi(1 + \cos^2\theta) - \beta_{13}\sin 2\theta\cos\varphi \\ - \beta_{23}\sin 2\theta\sin\varphi] + (2\beta_{12})^2[\sin^4\theta\sin^2\varphi\cos^2\varphi + \cos^2\theta] \\ - 4\beta_{12}\beta_{13}\sin^2\theta\sin\varphi[\cos^2\theta\cos^2\varphi + \sin^2\varphi] - 4(\beta_{12}\beta_{23}) \\ [\sin 2\theta\cos\varphi(\sin^2\varphi\cos^2\theta + \cos^2\varphi)] + (2\beta_{13})^2\sin^2\theta\times \\ (\cos^2\theta\cos^2\varphi + \sin^2\varphi) - 4\beta_{13}\beta_{23}\sin 2\varphi\sin^4\theta \\ + (2\beta_{23})^2\sin^2\theta(\cos^2\theta\sin^2\varphi + \cos^2\varphi)\end{aligned}} \tag{75}$$

The birefringence in any direction can be calculated from equation (75); since $\beta'_{11} = v_1^2/V^2$, it equals $1/\mu_1^2$ where μ_1 is the index of refraction corresponding to a light wave with its electric displacement in the β'_{11} direction. Similarly, for the second solution at right angle to the first,

$$\beta''_{11} = \frac{v_2^2}{V^2} = \frac{1}{\mu_2^2} \tag{76}$$

Hence if we designate the expression under the radical by K_2 and half the expression on the right outside the radical by K_1, we have

$$\frac{1}{\mu_1^2} + \frac{1}{\mu_2^2} = K_1; \qquad \frac{1}{\mu_1^2} - \frac{1}{\mu_2^2} = \sqrt{K_2}. \tag{77}$$

Since μ_1 and μ_2 are very nearly equal even in the most birefringent crystal, we have nearly

$$\mu_2 - \mu_1 = B = \frac{\mu^3}{2}\sqrt{K_2}. \tag{78}$$

For special directions in the crystal, the expression for K_2 simplifies very considerably. Along the x, y and z axes, the values are

$$\begin{aligned}
&X, (\varphi = 0°, \theta = 90°); && B_x = \frac{\mu^3}{2}\sqrt{(\beta_{33} - \beta_{22})^2 + (2\beta_{23})^2} \\
&Y, (\varphi = 90°, \theta = 90°); && B_y = \frac{\mu^3}{2}\sqrt{(\beta_{11} - \beta_{33})^2 + (2\beta_{13})^2} \\
&Z, (\varphi = 0°, \theta = 0°); && B_z = \frac{\mu^3}{2}\sqrt{(\beta_{11} - \beta_{22})^2 + (2\beta_{12})^2}.
\end{aligned} \tag{79}$$

If any natural birefringence exists along these axes, $(2\beta_{23})^2$ will be very small compared to this and

$$\begin{aligned}
B_x &= \frac{\mu^3}{2}(\beta_3 - \beta_2 + \Delta_3 - \Delta_2) = \frac{\mu^3}{2}\left(\frac{1}{\mu_c^2} - \frac{1}{\mu_b^2} + \Delta_3 - \Delta_2\right) \\
B_y &= \frac{\mu^3}{2}(\beta_1 - \beta_3 + \Delta_1 - \Delta_3) = \frac{\mu^3}{2}\left(\frac{1}{\mu_a^2} - \frac{1}{\mu_c^2} + \Delta_1 - \Delta_3\right) \\
B_z &= \frac{\mu^3}{2}(\beta_1 - \beta_2 + \Delta_1 - \Delta_2) = \frac{\mu^3}{2}\left(\frac{1}{\mu_a^2} - \frac{1}{\mu_b^2} + \Delta_1 - \Delta_2\right).
\end{aligned} \tag{80}$$

Hence, for this case, measurements along the three axes will tell the difference between the three effects Δ_1, Δ_2 and Δ_3. To get absolute values requires a direct measurement of the index of refraction along one of the axes and its change with fields or stresses. This is a considerably more difficult measurement than a birefringence measurement and requires the use of an accurate interferometer.

If, however, the Z axis is an optic axis as it is in ADP, for example, and $\Delta_1 = \Delta_2 = 0$, a birefringence occurs due to the term β_{12}. As shown in the next section, the electro-optic constants for ADP (tetragonal $\bar{4}2m$) are z_{41} and z_{63}. z_{63} occurs in the expression for $\beta_{12} = \Delta_6$, as can be seen from equations (72), and hence the birefringence along the Z axis is

$$B_z = \frac{\mu_a^3}{2} x 2\beta_{12} = \mu_a^3 z_{63} E_3. \tag{81}$$

The constants z_{63} and z_{41} have been measured independently by W. L. Bond, Robert O'B. Carpenter, and Hans Jaffe. Probably the most accurate measurements, and the only one published, are those of Carpenter,[5] who finds that the indices of refraction and the z_{63} and z_{41} constants for ADP and KDP are in cgs units

	μ_a	μ_c	$r_{63} \times 10^7$	$r_{41} \times 10^7$
ADP	1.5254	1.4798	2.54 ± 0.05	6.25 ± 0.1
KDP	1.5100	1.4684	3.15 ± 0.07	2.58 ± 0.05

An even larger constant has been found for heavy hydrogen KDP by Zwicker and Scherrer.[6] They find at 20°C that $r_{63} = 6 \times 10^{-7}$. Using this constant, a half wave retardation for a $\lambda = 5461$ A° mercury line occurs for a voltage of 4000 volts.

For tetragonal crystals of these types the only photoeleastic constant for the z axis is m_{66}, and the birefringence for this case is given by

$$B_z = \mu_a^3 m_{66} S_6 \tag{82}$$

When a natural birefringence exists for the crystal, measurements of the other three effects Δ_4, Δ_5 and Δ_6 can be made by determining the birefringence along other directions than the Fresnel ellipsoid axes. In a direction of Z' lying in the XZ plane $\varphi = 0$, $\theta =$ variable and

$$B_{xz} = \frac{\mu^3}{2} \sqrt{\begin{aligned}&[(\beta_{11} - \beta_{22}) \cos^2\theta + (\beta_{33} - \beta_{22}) \sin^2\theta - \beta_{13} \sin^2\theta]^2 \\ &\quad + [2\beta_{12} \cos\theta + 2\beta_{23} \sin\theta]^2.\end{aligned}} \tag{83}$$

When a natural birefringence exists, this reduces to

$$\begin{aligned} B_{xy} \doteq \frac{\mu^3}{2} \Bigg[&\left(\frac{1}{\mu_a^2} - \frac{1}{\mu_b^2} + \Delta_1 - \Delta_2\right) \cos^2\theta \\ &+ \left(\frac{1}{\mu_c^2} - \frac{1}{\mu_b^2} + \Delta_3 - \Delta_2\right) \sin^2\theta - \Delta_5 \sin 2\theta \Bigg] \end{aligned} \tag{84}$$

[5] "The Electro-optic Effect in Uniaxial Crystals of the Type XH_2PO_4," Robert O'B. Carpenter, *Jour. Opt. Soc. Am.*, in course of publication.

[6] Zwicker and Scherrer, *Helv. Phys. Acta.*, 17, 346 (1944).

and hence, by measuring at 45° between the two axes, one can evaluate the Δ_5 term.

Similarly, for the YZ plane, $\varphi = 90°$, $\theta =$ variable and

$$B_{yz} = \frac{\mu^3}{2} \sqrt{\begin{aligned}&[-(\beta_{11} - \beta_{22}) + (\beta_{33} - \beta_{22}) \sin^2 \theta - \beta_{23} \sin 2\theta]^2 \\ &\qquad + [2\beta_{12} \cos \theta - 2\beta_{13} \sin \theta]^2.\end{aligned}} \tag{85}$$

Hence, when a natural birefringence exists, we have

$$B_{yz} \doteq \frac{\mu^3}{2} \left[-\left(\frac{1}{\mu_a^2} - \frac{1}{\mu_b^2} + \Delta_1 - \Delta_2 \right) + \left(\frac{1}{\mu_c^2} - \frac{1}{\mu_b^2} + \Delta_3 - \Delta_2 \right) \sin^2 \theta - \Delta_4 \sin 2\theta \right]. \tag{86}$$

In the XY plane $\theta = 90°$, $\varphi =$ variable and

$$B_{xy} = \frac{\mu^3}{2} \sqrt{\begin{aligned}&[(\beta_{11} - \beta_{12}) \sin^2 \varphi - (\beta_{33} - \beta_{22}) - \beta_{12} \sin 2\varphi]^2 \\ &\qquad + [2\beta_{13} \sin \varphi - \beta_{23} \cos \varphi]^2.\end{aligned}} \tag{87}$$

Then, for natural birefringence,

$$B_{xy} \doteq \frac{\mu^3}{2} \left[\left(\frac{1}{\mu_a^2} - \frac{1}{\mu_b^2} + \Delta_1 - \Delta_2 \right) \sin^2 \varphi - \left(\frac{1}{\mu_c^2} - \frac{1}{\mu_b^2} + \Delta_3 - \Delta_2 \right) - \Delta_6 \sin 2\varphi \right]. \tag{88}$$

Hence, with measurements at 45° between the axes and with suitably applied fields and strains, the three effects Δ_4, Δ_5 and Δ_6 can be measured. Since the axes of the test specimen are turned with respect to the X, Y and Z axes, suitable transformations of the effects Δ_1 to Δ_6 with respect to the new axes will have to be made. These can be done as shown in reference (4) by means of tensor transformation formulae.

Another method for measuring the constants in Δ_4, Δ_5, Δ_6 is to measure the amount they rotate the axes of the Fresnel ellipsoid. As an example consider the z_{41} constant of ADP. For example, if we look along the X axis and apply a field in the same direction, then, in equation (74), $\theta = 90°$, $\varphi = 0$ and

$$\tan 2\psi = \frac{-2\beta_{23}}{\beta_{33} - \beta_{22}} = \frac{-2z_{41} E_1}{\dfrac{1}{\mu_c^2} - \dfrac{1}{\mu_b^2}} = \frac{-2\mu_b^2 \mu_c^2 z_{41} E_1}{(\mu_b + \mu_c)(\mu_b - \mu_c)}. \tag{89}$$

According to Carpenter, the z_{41} electro-optic constant of ADP is 6.25×10^{-7} in cgs units. $\mu_a = \mu_b = 1.5254$; $\mu_c = 1.4798$; hence the angle of rotation for a field of 30,000 volts per centimeter = 100 stat volts cm is

$$\psi = -2.25 \times 10^{-3} \text{ radians} = 7.7 \text{ minutes of arc.} \tag{90}$$

VI. Electro-optic and Photoelastic Tensors for Various Crystal Classes

Since $r_{mno} = r_{nmo}$ and $z_{mno} = z_{nmo}$ are third rank tensors similar to the h_{mij} piezoelectric tensor, they will have the same components for the various crystal classes. For the twenty crystal classes that show the electro-optic effect these tensors are given below. They are given with the crystal system they belong to, and the symmetry is designated by the Hermann-Mauguin symbol. The last number of the subscript of z designates the direction of the applied static field.

(91)

Triclinic; 1

$$\begin{vmatrix} z_{11} & z_{21} & z_{31} & z_{41} & z_{51} & z_{61} \\ z_{12} & z_{22} & z_{32} & z_{42} & z_{52} & z_{62} \\ z_{13} & z_{23} & z_{33} & z_{43} & z_{53} & z_{63} \end{vmatrix}$$

Monoclinic; 2

$$\begin{vmatrix} 0 & 0 & 0 & z_{41} & 0 & z_{61} \\ z_{12} & z_{22} & z_{32} & 0 & z_{52} & 0 \\ 0 & 0 & 0 & z_{43} & 0 & z_{63} \end{vmatrix}$$

Monoclinic; $\bar{2} = m$

$$\begin{vmatrix} z_{11} & z_{21} & z_{31} & 0 & z_{51} & 0 \\ 0 & 0 & 0 & z_{42} & 0 & z_{62} \\ z_{13} & z_{23} & z_{33} & 0 & z_{53} & 0 \end{vmatrix}$$

Orthorhombic; 222

$$\begin{vmatrix} 0 & 0 & 0 & z_{41} & 0 & 0 \\ 0 & 0 & 0 & 0 & z_{52} & 0 \\ 0 & 0 & 0 & 0 & 0 & z_{63} \end{vmatrix}$$

Orthorhomic; 2mm

$$\begin{vmatrix} 0 & 0 & 0 & 0 & z_{51} & 0 \\ 0 & 0 & 0 & z_{42} & 0 & 0 \\ z_{13} & z_{23} & z_{33} & 0 & 0 & 0 \end{vmatrix}$$

Tetragonal; $\bar{4}$

$$\begin{vmatrix} 0 & 0 & 0 & z_{41} & z_{51} & 0 \\ 0 & 0 & 0 & -z_{51} & z_{41} & 0 \\ z_{13} & -z_{13} & 0 & 0 & 0 & z_{63} \end{vmatrix}$$

Tetragonal; 4

$$\begin{vmatrix} 0 & 0 & 0 & z_{41} & z_{51} & 0 \\ 0 & 0 & 0 & z_{51} & -z_{41} & 0 \\ z_{13} & z_{13} & z_{33} & 0 & 0 & 0 \end{vmatrix}$$

Tetragonal; $\bar{4}$2m

$$\begin{vmatrix} 0 & 0 & 0 & z_{41} & 0 & 0 \\ 0 & 0 & 0 & 0 & z_{41} & 0 \\ 0 & 0 & 0 & 0 & 0 & z_{63} \end{vmatrix}$$

Tetragonal; 422

$$\begin{vmatrix} 0 & 0 & 0 & z_{41} & 0 & 0 \\ 0 & 0 & 0 & 0 & -z_{41} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

Tetragonal; 4mm

$$\begin{vmatrix} 0 & 0 & 0 & 0 & z_{51} & 0 \\ 0 & 0 & 0 & z_{51} & 0 & 0 \\ z_{13} & z_{13} & z_{33} & 0 & 0 & 0 \end{vmatrix}$$

Trigonal; 3

$$\begin{vmatrix} z_{11} & -z_{11} & 0 & z_{41} & z_{51} & -z_{22} \\ -z_{22} & z_{22} & 0 & z_{51} & -z_{41} & -z_{11} \\ z_{13} & z_{13} & z_{33} & 0 & 0 & 0 \end{vmatrix}$$

Trigonal; 32

$$\begin{vmatrix} z_{11} & -z_{11} & 0 & z_{41} & 0 & 0 \\ 0 & 0 & 0 & 0 & -z_{41} & -z_{11} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

Trigonal; 3m

$$\begin{vmatrix} 0 & 0 & 0 & 0 & z_{51} & -z_{22} \\ -z_{22} & z_{22} & 0 & z_{51} & 0 & 0 \\ z_{13} & z_{13} & z_{33} & 0 & 0 & 0 \end{vmatrix}$$

Hexagonal; $\bar{6}$

$$\begin{vmatrix} z_{11} & -z_{11} & 0 & 0 & 0 & -z_{22} \\ -z_{22} & z_{22} & 0 & 0 & 0 & -z_{11} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

Hexagonal; $\bar{6}$m2

$$\begin{vmatrix} z_{11} & -z_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -z_{11} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

Hexagonal; 6

$$\begin{vmatrix} 0 & 0 & 0 & z_{41} & z_{51} & 0 \\ 0 & 0 & 0 & z_{51} & -z_{41} & 0 \\ z_{13} & z_{13} & z_{33} & 0 & 0 & 0 \end{vmatrix}$$

Hexagonal; 622

$$\begin{vmatrix} 0 & 0 & 0 & z_{41} & 0 & 0 \\ 0 & 0 & 0 & 0 & -z_{41} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

Hexagonal; 6mm

$$\begin{vmatrix} 0 & 0 & 0 & 0 & z_{51} & 0 \\ 0 & 0 & 0 & z_{51} & 0 & 0 \\ z_{13} & z_{13} & z_{33} & 0 & 0 & 0 \end{vmatrix}$$

Cubic; 23 and $\bar{4}$3m

$$\begin{vmatrix} 0 & 0 & 0 & z_{41} & 0 & 0 \\ 0 & 0 & 0 & 0 & z_{41} & 0 \\ 0 & 0 & 0 & 0 & 0 & z_{41} \end{vmatrix}$$

The r tensor has similar terms.

The photoelastic constants are similar to the elastic constant tensors except that $m_{rs} \neq m_{sr}$ in general. However, for the tetragonal, trigonal, hexagonal and cubic systems, Pockels found that $m_{12} = m_{21}$. This follows from the transformation equations about the Z axis which is the n fold axes for these groups. For a rotation of an angle θ about Z, the direction cosines are

$$\begin{vmatrix} \ell_1 = \dfrac{\partial x_1'}{\partial x_1} = \cos\theta & m_1 = \dfrac{\partial x_1'}{\partial x_2} = \sin\theta & n_1 = \dfrac{\partial x_1'}{\partial x_3} = 0 \\ \ell_2 = \dfrac{\partial x_2'}{\partial x_1} = -\sin\theta & m_2 = \dfrac{\partial x_2'}{\partial x_2} = \cos\theta & n_2 = \dfrac{\partial x_2'}{\partial x_3} = 0 \\ \ell_3 = \dfrac{\partial x_3'}{\partial x_1} = 0 & m_3 = \dfrac{\partial x_3'}{\partial x_2} = 0 & n_3 = \dfrac{\partial x_3'}{\partial x_3} = 1 \end{vmatrix} \quad (92)$$

Transforming the two terms $m'_{1122} = m'_{12}$ and $m'_{2211} = m'_{21}$ by the tensor transformation equation

$$m_{ijk\ell} = \frac{\partial x'_i}{\partial x_m} \frac{\partial x'_j}{\partial x_n} \frac{\partial x'_k}{\partial x_o} \frac{\partial x'_\ell}{\partial x_p} m_{mnop} \tag{93}$$

we find, for these two coefficients,

$$\begin{aligned} m'_{12} &= (m_{11} + m_{22} - 4m_{66}) \sin^2\theta \cos^2\theta + 2(m_{62} - m_{16}) \\ &\quad \sin\theta \cos^3\theta + 2(m_{61} - m_{16}) \sin^3\theta \cos\theta + m_{12} \cos^4\theta + m_{21} \sin^4\theta \\ m'_{21} &= (m_{11} + m_{22} - 4m_{66}) \sin^2\theta \cos^2\theta + 2(m_{16} - m_{62}) \\ &\quad \sin^3\theta \cos\theta + 2(m_{26} - m_{61}) \sin\theta \cos^3\theta + m_{21} \cos^4\theta + m_{12} \sin^4\theta \end{aligned} \tag{94}$$

If $m'_{12} = m'_{21}$ for all angles of rotation we must have

$$m_{16} + m_{26} = m_{61} + m_{62}$$

For all the classes that $m_{12} = m_{21}$, either $m_{26} = -m_{16}$ and $m_{62} = -m_{61}$ or else $m_{16} = m_{26} = m_{61} = m_{62} = 0$.

Now, if Z is a four-fold axis, as it is in the tetragonal and cubic systems, then, for a 90° rotation, the value of m'_{12} or m'_{21} must repeat. From the first of (92) this means that

$$m_{12} = m_{21} \text{ and } m_{21} = m_{12}$$

For a trigonal or hexagonal system additional relations are obtained between m_{66} and m_{11}, m_{22} and m_{12} in the usual manner. Hence the photoelastic matrices become, for the various crystal classes,

(95)

Triclinic 36 Constant	m_{11}	m_{12}	m_{13}	m_{14}	m_{15}	m_{16}	The π tensor is entirely analogous
	m_{21}	m_{22}	m_{23}	m_{24}	m_{25}	m_{26}	
	m_{31}	m_{32}	m_{33}	m_{34}	m_{35}	m_{36}	
	m_{41}	m_{42}	m_{43}	m_{44}	m_{45}	m_{46}	
	m_{51}	m_{52}	m_{53}	m_{54}	m_{55}	m_{56}	
	m_{61}	m_{62}	m_{63}	m_{64}	m_{65}	m_{66}	
Monoclinic 20 Constants	m_{11}	m_{12}	m_{13}	0	m_{15}	0	The π tensor is entirely analogous
	m_{21}	m_{22}	m_{23}	0	m_{25}	0	
	m_{31}	m_{32}	m_{33}	0	m_{35}	0	
	0	0	0	m_{44}	0	m_{46}	
	m_{51}	m_{52}	m_{53}	0	m_{55}	0	
	0	0	0	m_{64}	0	m_{66}	

Orthorhombic 12 Constants	$\begin{vmatrix} m_{11} & m_{12} & m_{13} & 0 & 0 & 0 \\ m_{21} & m_{22} & m_{23} & 0 & 0 & 0 \\ m_{31} & m_{32} & m_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & m_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & m_{66} \end{vmatrix}$	The π tensor is entirely analogous
Tetragonal $4, \bar{4}, 4/m$ 9 Constants	$\begin{vmatrix} m_{11} & m_{12} & m_{13} & 0 & 0 & m_{16} \\ m_{12} & m_{11} & m_{13} & 0 & 0 & -m_{16} \\ m_{31} & m_{31} & m_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & m_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{44} & 0 \\ m_{61} & -m_{61} & 0 & 0 & 0 & m_{66} \end{vmatrix}$	The π tensor is entirely analogous
Tetragonal $\bar{4}2m$, 422 $4mm$, $(4/m)mm$ 7 Constants	$\begin{vmatrix} m_{11} & m_{12} & m_{13} & 0 & 0 & 0 \\ m_{12} & m_{11} & m_{13} & 0 & 0 & 0 \\ m_{31} & m_{31} & m_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & m_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & m_{66} \end{vmatrix}$	The π tensor is entirely analogous
Trigonal $3, \bar{3}$ 11 Constants	$\begin{vmatrix} m_{11} & m_{12} & m_{13} & m_{14} & -m_{25} & 0 \\ m_{12} & m_{11} & m_{13} & -m_{14} & m_{25} & 0 \\ m_{31} & m_{31} & m_{33} & 0 & 0 & 0 \\ m_{41} & -m_{41} & 0 & m_{44} & m_{45} & m_{52} \\ -m_{52} & m_{52} & 0 & -m_{45} & m_{44} & m_{41} \\ 0 & 0 & 0 & m_{25} & m_{14} & \frac{m_{11}-m_{12}}{2} \end{vmatrix}$	The π tensor is analogous except that $\pi_{46} = 2\pi_{52}$ $\pi_{56} = 2\pi_{41}$ $\pi_{66} = (\pi_{11} - \pi_{12})$
Trigonal $32, 3m$ $\bar{3}(2/m)$ 8 Constants	$\begin{vmatrix} m_{11} & m_{12} & m_{13} & m_{14} & 0 & 0 \\ m_{12} & m_{11} & m_{13} & -m_{14} & 0 & 0 \\ m_{31} & m_{31} & m_{33} & 0 & 0 & 0 \\ m_{41} & -m_{41} & 0 & m_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{44} & m_{41} \\ 0 & 0 & 0 & 0 & m_{14} & \frac{m_{11}-m_{12}}{2} \end{vmatrix}$	The π tensor is analogous except that $\pi_{56} = 2\pi_{41}$ $\pi_{66} = \pi_{11} - \pi_{12}$

Hexagonal $6, \bar{6}m2, 6$ $622, 6/m$; $6mm, \frac{6}{m}mm$ 6 Constants	m_{11}	m_{12}	m_{13}	0	0	0	The π tensor is analogous except that $\pi_{66} = \pi_{11} - \pi_{12}$
	m_{12}	m_{11}	m_{13}	0	0	0	
	m_{31}	m_{31}	m_{33}	0	0	0	
	0	0	0	m_{44}	0	0	
	0	0	0	0	m_{44}	0	
	0	0	0	0	0	$\frac{m_{11}-m_{12}}{2}$	

Cubic System $23, 432$ $\frac{2}{m}3, \bar{4}3m, \frac{4}{m}\bar{3}\frac{2}{m}$ 3 Constants	m_{11}	m_{12}	m_{12}	0	0	0	The π tensor is entirely analogous (95)
	m_{12}	m_{11}	m_{12}	0	0	0	
	m_{12}	m_{12}	m_{11}	0	0	0	
	0	0	0	m_{44}	0	0	
	0	0	0	0	m_{44}	0	
	0	0	0	0	0	m_{44}	

Isotropic Systems 2 Constants	m_{11}	m_{12}	m_{12}	0	0	0	The π tensor is analogous except that $\pi_{66} = \pi_{11} - \pi_{12}$
	m_{12}	m_{11}	m_{12}	0	0	0	
	m_{12}	m_{12}	m_{11}	0	0	0	
	0	0	0	$\frac{m_{11}-m_{12}}{2}$	0	0	
	0	0	0	0	$\frac{m_{11}-m_{12}}{2}$	0	
	0	0	0	0	0	$\frac{m_{11}-m_{12}}{2}$	

From measurement[7] on the photoelastic effects at high pressure for cubic crystals, it has become apparent that the second derivatives of equation (44) are not sufficient to represent the experimental results and derivatives up to the fourth power should be included. This extension, however, is not considered in the present paper.

VII. Photoelasticity in Isotropic Media

The photoelastic effect in isotropic solids has been used extensively in studying the stresses existing in machine parts and other pieces. For this purpose a plastic model cut in the shape of the original is used and is loaded in a similar manner to that of the machine part to be studied. Since stresses are applied, the π_i photoelastic constants are most useful. If we look along

[7] H. B. Maris, *Jour. Optical Society of Amer.*, Vol. 15, pp. 194–200, 1927.

the Z axis, the last of equations (79) shows that the birefringence is equal to

$$B_z = \frac{\mu^3}{2} \sqrt{(\beta_1 + \Delta_1 - \beta_2 - \Delta_2)^2 + 4(\Delta_6)^2} \tag{96}$$

Since, for an isotropic substance $\beta_1 = \beta_2$, we have, after substituting the value of Δ_1 and Δ_2, with the appropriate photoelastic constants from equation (95), (last tensor):

$$B_z = \frac{\mu^3}{2} (\pi_{11} - \pi_{12}) \sqrt{(T_1 - T_2)^2 + 4T_6^2} \tag{97}$$

If we transform to axes rotated by an angle θ about Z, the values of T'_{11} and T'_{22} are given by

$$\begin{aligned} T'_{11} &= \cos^2\theta T_1 + 2 \sin\theta \cos\theta T_6 + \sin^2\theta T_2 \\ T'_{22} &= \sin^2\theta T_1 - 2 \sin\theta \cos\theta T_6 + \cos^2\theta T_2 \end{aligned} \tag{98}$$

If, now, we choose the angle θ so that T'_{11} is a maximum, we find

$$\tan 2\theta = \frac{+2T_6}{T_1 - T_2} \tag{99}$$

Inserting this value of $\tan 2\theta$ in (98) we find

$$\begin{aligned} T'_1 &= \frac{T_1 + T_2}{2} + \tfrac{1}{2}\sqrt{(T_1 - T_2)^2 + 4T_6^2} \\ T'_2 &= \frac{T_1 + T_2}{2} - \tfrac{1}{2}\sqrt{(T_1 - T_2)^2 + 4T_6^2} \end{aligned} \tag{100}$$

and, hence,

$$T'_1 - T'_2 = \sqrt{(T_1 - T_2)^2 + 4T_6^2} \tag{101}$$

Hence the birefringence obtained in stressing a material is proportional to the difference in the principal stresses. By observing the isoclinic lines of a photoelastic picture, methods[8] are available for determining the stresses in a model. A photograph[9] of a stressed disk is shown by Fig. 3. The high concentration of lines near the surface shows that the shearing stress is very high at these points. By counting the number of lines from the edge and knowing the stress optical constant, the stress can be calculated at any point.

If we apply a single stress T_1, the birefringence is given by the equation

$$B_z = \frac{\mu^3}{2} (\pi_{11} - \pi_{12}) T_1 \tag{102}$$

[8] See Photoelasticity, Coker and Filon, Cambridge University Press, 1931.
[9] This photograph was taken by T. F. Osmer.

Instead of using the constants π_{11} and π_{12} it is customary to use a single constant C given by

$$B = \mu_e - \mu_o = r = CT \tag{103}$$

where the constant C is called the relative stress optical constant and r the retardation. The dimensions of C are the reciprocal of a stress and are

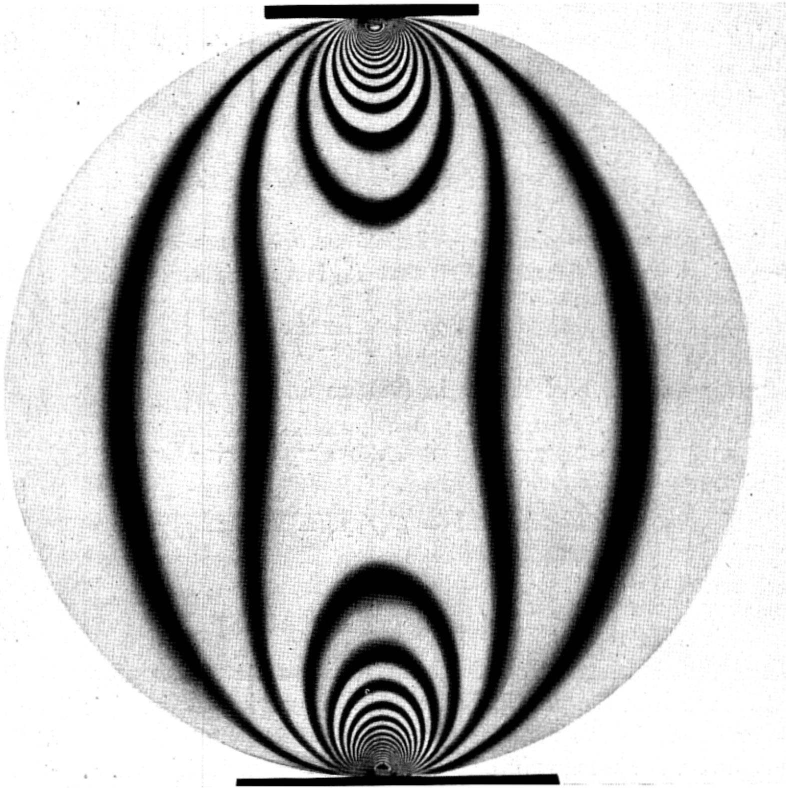

Fig. 3—Photoelastic picture of a disk in compression.

measured in cm^2 per dyne. A convenient unit for most purposes is one of 10^{-13} cm^2/dyne; if this is used, the stress optical coefficients of most glasses are from 1 to 10 and most plastics are from 10 to 100. This unit so defined has been called the "Brewster". In terms of the Brewster, the retardation is

$$r = CTd \tag{104}$$

If C is measured in Brewsters, d in millimeters and T in bars (10^6 dynes/cm^2) then r, as given by the formula, is expressed in angstrom units.