

Some Circuit Aspects of the Transistor

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INTRODUCTION

THE purpose of this note is to discuss in a general way some circuit aspects of the transistor. It is rather interesting that in order to discuss its circuit aspects, little direct reference to the transistor is necessary. One needs only certain properties of the transistor which are empirically obtainable by measurement; these properties then determine behavior in the manner prescribed by the methods of general network theory. In principle, one needs no knowledge of the physics of the transistor in order to treat it circuitwise; any "black box" with the same electrical behavior at its terminals would act the same way.

It is rather fortunate for our purposes that the problem does separate nicely in this way. The operation of the transistor is reasonably well understood; but, for calculations of performance from physical properties, the numerical parameters needed are somewhat inaccessible, numerous and complicated. The paper by Shockley¹ gives some calculations of this kind which are illuminating for theoretical understanding. However, just as with electron tubes, practical engineering calculations often do not need to go back to the ultimate physics. Starting from the electrical properties of the transistor as empirically determined by measurements on its terminals, we need go only to the literature of electrical engineering to find much practically useful information on properties of circuits which could be built around the unit.

This method of characterizing the electrical performance of a device more or less independently of its physical construction has come into wide use in recent years. A considerable amount of work has been done with applications to both electron tubes and transistors at the Bell Telephone Laboratories by L. C. Peterson.² The purpose of the present note, however, is not to go deeply into the subject but rather to review it in a general way, indicating applications to some of the simpler transistor circuits and comparisons with electron tubes. For more profound analyses one may refer to Peterson's work.

¹ "The Theory of p-n Junctions in Semiconductors and p-n Junction Transistors," W. Shockley, this issue of *The Bell System Technical Journal*.

² "Equivalent Circuits of Linear Active Four-Terminal Networks," L. C. Peterson, *Bell System Technical Journal*, Oct. 1948, pp. 593-622.

The method used for circuit analysis may be grouped under the following headings:

1. Linear problems, like low-level amplifiers or the question of onset of oscillations. Such problems visualize the transistor as making only small excursions from an assumed operating point and are best treated by the method of small-signal analysis. The unit is assigned an equivalent circuit or, in mathematical terms, is dealt with by means of linear equations.
2. Slightly non-linear problems, like Class A power amplifiers. Here the excursions about the operating point are large enough to bring in higher-order effects like harmonic generation or intermodulation, but still small enough so that these effects can be treated by adding to the equivalent circuit certain distortion generators. Mathematically, some terms need to be added to the linear equations but these terms are of the nature of corrections, not big changes.
3. Highly non-linear problems, such as Class B or C amplifiers, oscillators, switches, harmonic generators. Here the excursions about the characteristic are so large as to reduce the linear approximation to the status of a qualitative guide or perhaps to invalidate it entirely; mathematically, the small signal series either require many terms for accuracy or else do not converge at all. These large-signal problems usually have to be treated by methods special for each problem. Frequently one uses graphical constructions from the static characteristics, or analytical methods starting from reasonable approximations to the static characteristics.
4. Finally, in certain highly non-linear problems the non-linear features are in a sense subsidiary; one is really interested in the behavior of a superposed small signal subject to a linear analysis. The non-linear part of the problem may appear in the form of circuit parameters or frequency shifts which may be left for empirical determination. Such problems are exemplified by mixers, modulators, or switches.

The subsequent discussion will emphasize mainly the linear problems where the methods of circuit analysis are most effective, but will touch on some of the other fields occasionally.

THE TYPE A TRANSISTOR

Perhaps at this point is the place to pay our respects to the physics of the transistor. A view of the Type A transistor³, currently being made in small quantities, is shown in Fig. 1. It is about $\frac{1}{2}$ inch long and $\frac{3}{16}$ inch in diameter. Two small phosphor bronze "cat-whiskers" make point contacts close together to a block of germanium. A large area ohmic contact to the

³ "Type A Transistor," R. M. Ryder, *Bell Laboratories Record*, March 1949, pp. 89-93.

germanium constitutes the third electrode, called the base. How it works is shown in a purely descriptive way in Fig. 2. One point, called the collector,

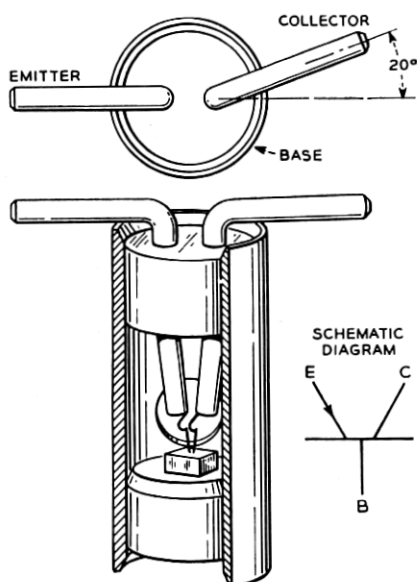


Fig. 1—Cutaway view of transistor.

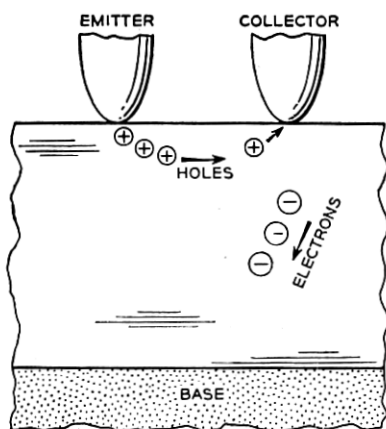


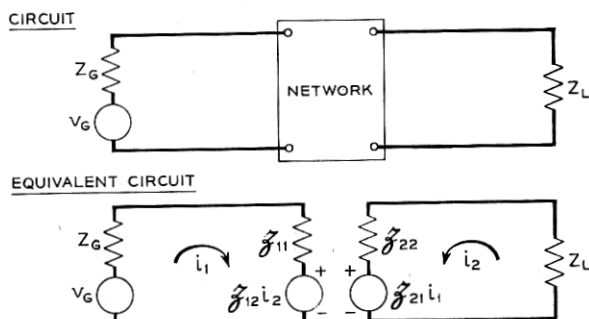
Fig. 2—Transistor mechanism.

is a rectifier biased strongly in the low-conducting direction. It therefore has a rectifying barrier in the germanium near it, which causes the collector impedance to be high. However, the collector can be influenced by the

emitter if the latter is arranged to emit anomalous charge carriers, that is, carriers of the sign not normally present in the interior of the material.

EQUIVALENT CIRCUITS

As has been explained by Bardeen, Brattain, and Shockley, many features of the transistor are nicely explained by this picture of its action; but, for present purposes of circuit analysis, we shall now take the purely empirical



Equations

$$i_1(Z_G + g_{11}) + i_2 g_{12} = v_G$$

$$i_1 g_{21} + i_2(g_{22} + Z_L) = 0$$

Circuit determinant $\Delta = (g_{11} + Z_G)(g_{22} + Z_L) - g_{12}g_{21}$

Input impedance $Z_{11} = g_{11} - \frac{g_{12}g_{21}}{g_{22} + Z_L}$

Output impedance $Z_{22} = g_{22} - \frac{g_{12}g_{21}}{g_{11} + Z_G}$

Operating power gain $G_0 = 4R_G R_L \left| \frac{-g_{21}}{\Delta} \right|^2$

Insertion power gain $G_1 = \left| \frac{(Z_G + Z_L)g_{21}}{\Delta} \right|^2$

Fig. 3—Synopsis of general four-pole—impedance analysis.

view and regard the transistor as a *black box* whose performance is to be determined by electrical measurements on its terminals.

A picture of a black box is shown in Fig. 3 along with the equations describing it. The performance is completely characterized if one knows the voltage and current at each of the two pairs of terminals. Now, of these four variables, only two are independent since, if any two are fixed, the other two are determined. One can therefore describe the network in terms of any two variables and, since there are six possible ways to choose a pair of variables from a set of four, there are six ways of describing the network.

To recall what is done for electron tubes is helpful. In the case of a triode

the voltages on grid and plate are usually taken as independent variables; the grid and plate currents are taken as functions of the voltages. It becomes natural, then, to measure tubes with regulated power supplies having low impedances to keep the voltages constant, and one is then naturally led to describe tubes in terms of admittances. Now the trouble with this scheme for transistors is that many of them oscillate when connected to low impedances, that is, many transistors are short-circuit unstable. To avoid this difficulty it is convenient to measure with high impedances in the leads; the analytical counterpart is to regard the currents as independent variables, leading naturally to a description of the transistor in terms of impedances, as shown in the figure.

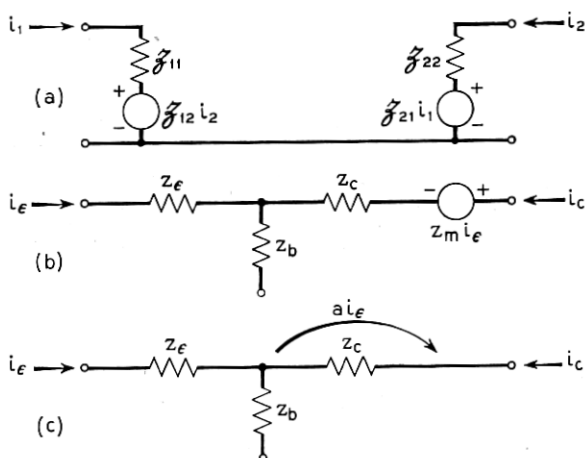
This description by open-circuit impedances happens to be a good one for many purposes, but there is nothing final or unique about it. In fact at high frequencies one of the other descriptions becomes more convenient.

By interpreting the \mathcal{Z} equations as circuit equations, one is led directly to the first equivalent circuit of Fig. 4. A little consideration shows why the \mathcal{Z} 's are called open-circuit impedances. For example, if the second mesh is open-circuited, then the equation says that \mathcal{Z}_{11} is the ratio of input voltage to input current, that is, the input open-circuit impedance; while \mathcal{Z}_{21} is the ratio of output voltage to input current, that is, the open-circuit forward transimpedance. Similarly \mathcal{Z}_{12} is the open-circuit feedback transimpedance and \mathcal{Z}_{22} is the open-circuit output impedance. Most of the subsequent discussion is concerned with low frequencies, where the impedances reduce to resistances.

This equivalent circuit for small signals is only one of many possibilities. Another, which is in fact more frequently used, is shown on Fig. 4. It consists of a T of resistors, each of which is associated with one of the transistor leads, and a voltage generator in series with the collector lead whose ratio to the emitter current is also of the dimensions of a resistance. The elements of this equivalent circuit are related to the former one by a simple subtraction. The other equivalent circuit on Fig. 4 is obtained by converting the series voltage generator to the equivalent shunt current generator, whose ratio to the emitter current is now a dimensionless constant which we shall call a .

These circuits, as well as all the other numerous possibilities, are equivalent in the sense that they all give exactly the same performance for any external connection of the unit. These three, however, are particularly well-behaved in that usually none of the circuit elements is negative; they are readily accessible to measurement; the association of the various circuit elements with corresponding regions within the transistor appears to have some physical significance; and, finally, the parameters are not too dreadfully dependent on the exact operating point used.

In the choice among various equivalent circuits, it appears that the optimum of convenience is also the one which most closely approaches the underlying physical situation. In agreeing to use the *black box* approach we have resolutely ignored the physical details, but here they are presenting themselves in a new way, having sneaked in the back door after we barred the front. Now, however, having chosen an equivalent circuit, we shall continue pursuing the circuit analysis in resolute ignorance of the physics. In what follows various equivalent circuits may be used, depending on the convenience of the moment.



Figs. 4—Some equivalent circuits.

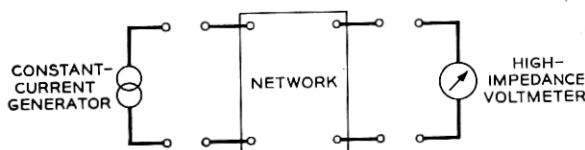


Fig. 5—Principle of measurement method.

The principle of a method used for rapid measurement of the transistor impedances is shown in Fig. 5, illustrating the measurement of forward transimpedance. A pair of terminals of the transistor is driven by a small alternating current of a few thousand cycles from a high impedance generator; the voltage developed is read by a high-impedance voltmeter. By calibrating the meter directly in ohms, one can read off the open circuit resistances of the unit as rapidly as one can switch and read meters.

Average values found by this method for the Type A transistor are shown on Fig. 6, together with data on the direct-current operating point. Since

development is still at an early stage, there are considerable variations between units.

SINGLE STAGE AMPLIFIERS. STABILITY, ELECTRON TUBE ANALOGY

An amplifier can be built in a straightforward manner by using the emitter as input electrode and collector as output electrode, the base being common to the two circuits. This amplifier is therefore called the grounded base amplifier. Figure 7 shows a schematic circuit using the average parameters just mentioned, working between 500 ohms and 20,000 ohms. The amplifier has an operating power gain of 17 db, power output Class A 10 milliwatts, noise figure at 1000 cycles 60 db with a variation inversely with frequency, and frequency response down 3 db at 5 megacycles.

Type A Transistor			
D.C. Operating Point:		$I_e = 0.6 \text{ ma}$	$V_e = 0.7V$
		$I_c = -2 \text{ ma}$	$V_c = -40V$
Circuit Parameters:		$r_e = 240 \text{ ohms}$	$r_b = 290 \text{ ohms}$
		$r_c = 19000 \text{ ohms}$	$r_m = 34000 \text{ ohms}$
		$Z_{11} = 530 \text{ ohms}$	$Z_{12} = 290 \text{ ohms}$
		$Z_{21} = 34000 \text{ ohms}$	$Z_{22} = 19000 \text{ ohms}$

Fig. 6—Equivalent circuit parameter values.

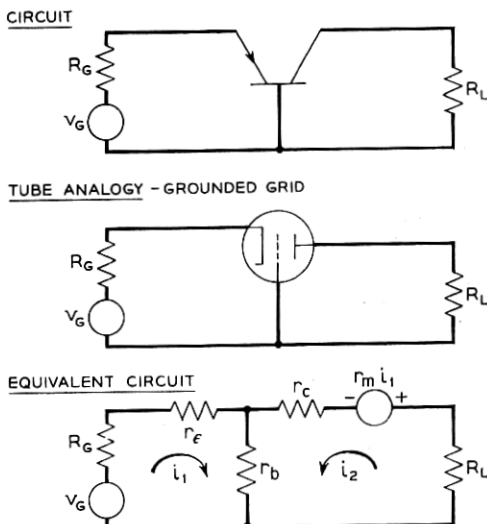
Some comments are in order on how this amplifier compares with an electron tube amplifier. First of all, the amplifying function and the manner of analyzing it from the circuit point of view are very similar, even though the internal mechanisms are markedly different. Secondly, there are qualitative differences in circuit behavior, which are set forth on Fig. 8. The base resistance r_b acts as a positive feedback element which, under adverse conditions, can cause the circuit to oscillate. A necessary condition for stability is that the circuit determinant shall be positive, and this can be written as follows:

$$\frac{r_m}{R_c} < 1 + \frac{R_E}{R_B} + \frac{R_E}{R_C} \quad (1)$$

Here the quantity r_m is the net mutual resistance of the transistor, and the capital R's are the total resistances in the corresponding leads, internal and external. One can see several features, as follows:

1. If $R_B = 0$, the circuit can be stable.
2. If $R_B > 0$, as usual, the circuit can be stable if the emitter and collector lead resistances are large enough or if r_m is not too large. In other words, resistance in the base lead tends toward instability if r_m is large; resistance in emitter or collector leads tends toward stability.

In the grounded base circuit the property of low base resistance is important, since the backward transmission depends directly on this property. In circuit terms, the base impedance is the feedback impedance in the grounded base circuit, and its value helps to set a limit on the stable gain which can be realized.



Equations:

$$\begin{aligned} i_1(R_G + r_e + r_b) + i_2 r_b &= v_G \\ i_1(r_b + r_m) + i_2(r_b + r_c + R_L) &= 0 \end{aligned}$$

Circuit determinant $\Delta = (R_G + r_e + r_b)(R_L + r_c + r_b) - r_b(r_b + r_m) > 0$ for stability

Input impedance $R_{11} = r_e + r_b - \frac{r_b(r_b + r_m)}{R_L + r_c + r_b}$

Output impedance $R_{22} = r_c + r_b - \frac{r_b(r_b + r_m)}{R_G + r_e + r_b}$

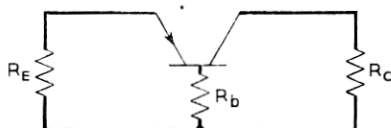
Operating power gain $G_0 = 4R_G R_L \left(\frac{-(r_b + r_m)}{\Delta} \right)^2$

Typical values: For $R_G = 500\Omega$, $R_L = 20,000\Omega$
Then $R_{11} = 280\Omega$, $R_{22} = 9600\Omega$
 $G_0 = 17\text{db}$

Fig. 7—Synopsis of grounded base amplifier.

The grounded base circuit has properties which are strongly reminiscent of the grounded grid electron triode amplifier in that both have low input impedance, high output impedance, and no change of signal polarity in transmission. The analogy was pointed out by Shockley. That this similarity is no coincidence can be seen by comparing the third equivalent circuit

above with the triode equivalent circuit of F. B. Llewellyn and L. C. Peterson⁴ in Fig. 9. Both circuits have the same topological form, and have similar impedance levels if the triode is considered to be operating in the frequency range of some tens of megacycles. The most important difference concerns the quantity a , a current amplification factor which, for the transistor, may be considerably greater than unity; while the analogous quantity



Can be stable if:

$$\frac{r_m}{R_c} < 1 + \frac{R_E}{R_b} + \frac{R_E}{R_c}$$

R 's include resistive elements both internal and external to the transistor.

Fig. 8—Stability

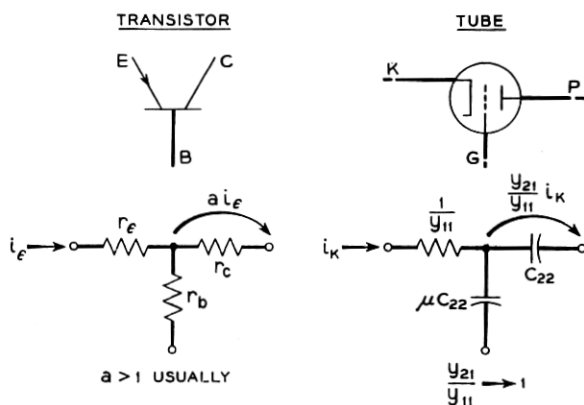


Fig. 9—Transistor-electron tube analogy.

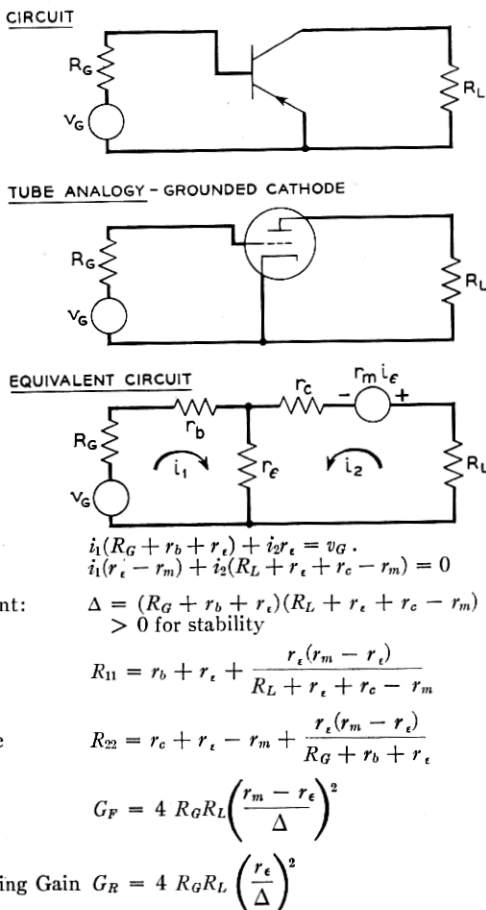
for the triode is close to unity for usual conditions. Another difference, of less importance, is the fact that the tube quantities analogous to r_c and r_b are capacitive reactances; their ratio, however, is like the ratio of r_c to r_b in magnitude.

One of the first consequences of this transistor-tube analogy is the suggestion that different transistor connections analogous to the different electron triode connections may be interesting.³ The analogy makes emitter analogous

⁴ "Vacuum Tube Networks," F. B. Llewellyn and L. C. Peterson, *Proc. I.R.E.*, March 1944, page 159, Fig. 13.

³ Loc. cit.

to cathode, base to grid, and collector to plate; the conventional or grounded cathode tube connection is therefore analogous to the grounded emitter connection of a transistor, shown on Fig. 10. It is found that when $a = 1$ the analogy is fairly close, in that the transistor has comparatively high-



Typical values: For $R_G = 500^\omega$, $R_L = 20000^\omega$. Then $R_{11} = 2100^\omega$, $R_{22} = -6900^\omega$, $G_F = 24^{db}$, $G_R = -19^{db}$

Fig. 10—Synopsis of grounded emitter amplifier.

input impedance, high-output impedance, and changes signal polarity in transmission. When $a > 1$, as is usual, the analogy becomes less close, and feedback effects tend to become large and obnoxious; the open-circuit output impedance is usually negative. This behavior is readily under-

standable from stability considerations, since the base lead is now one of the signal terminals and, as before mentioned, putting resistance in the base lead tends toward instability if a is enough greater than unity. The effect is so severe that often it is worth while to add resistance in the collector lead, thereby reducing a to the neighborhood of unity, and simultaneously reducing the amplifier to a state of greater tractability.

Another feature of the grounded emitter amplifier is that the base resistance r_b is usually negligible, in contrast to its pronounced effect on the reverse transmission of the grounded base amplifier. The role of feedback element is taken over here by the emitter resistance r_e . These considerations have important effects on the properties of cascaded amplifiers and will be reverted to later.

For numerical comparison we might work the grounded emitter amplifier between the same two terminations as the grounded base amplifier above, namely from 500 into 20,000 ohms. It would then have a gain of about 24 db, an improvement of 7 db over the grounded base, with about the same power output and noise figure. This improvement is obtained at greater risk of oscillation; in fact the output impedance of this amplifier is negative.

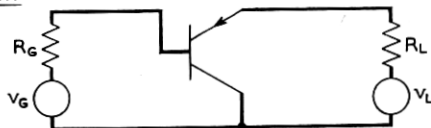
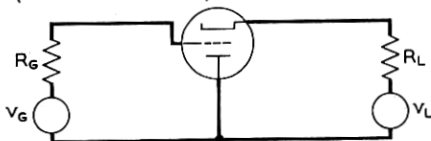
The remaining tube connection — the cathode follower or grounded plate — is analogous to the grounded collector connection (Fig. 11); again, when $a = 1$ the analogy is fairly close, in that the transistor has high-input impedance, low-output impedance, and no change of polarity in transmission. In fact when $a = 1$ the device is usable in very much the same manner as the cathode follower. The power output is lower than the other connections because the output electrode (the emitter) does not carry much direct current.

However, when we make a greater than 1 the effect is even more pronounced than it was in the grounded emitter case. As a increases from 1, the grounded collector amplifier rapidly loses its resemblance to the cathode follower and begins to transmit in both directions as a bilateral element. When $a = 2$, the operating gains in the two directions are the same; and for $a > 2$ the transmission is actually greater in the "backward" direction. Another curious feature is that, while the "forward" transmission is still without change in signal polarity, the "reverse" transmission inverts the signal polarity.

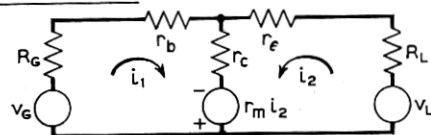
In any device which is supposed to give gain in both directions, naturally stability must be a controlling consideration. This amplifier is of course still subject to the aforementioned stability condition (1) and it is found that with care one can actually get power gains in both directions of transmission without instability, i.e. a simple bilateral amplifier is present. One numerical example may suffice. Assume a transistor having the properties

$r_e = 250$ ohms, $r_b = 250$ ohms, $r_o = 20,000$ ohms, $r_m = 40,000$ ohms, so that $a = 2$ and both base and emitter resistances r_e and r_b are negligible.

CIRCUIT

TUBE ANALOGY - GROUNDED PLATE
(CATHODE FOLLOWER)

EQUIVALENT CIRCUIT



Equations:

$$i_1(R_G + r_b + r_o) + i_2(r_e - r_m) = v_G$$

$$i_1 r_o + i_2(R_L + r_e + r_o - r_m) = v_L$$

Circuit determinant

$$\Delta = (R_G + r_b + r_o)(R_L + r_e + r_o - r_m) + r_o(r_m - r_e)$$

$$> 0 \text{ for stability}$$

Input impedance

$$R_{11} = r_b + r_o + \frac{r_o(r_m - r_e)}{R_L + r_e + r_o - r_m}$$

Output impedance

$$R_{22} = r_e + r_o - r_m + \frac{r_o(r_m - r_e)}{R_G + r_b + r_o}$$

Operating Gain

$$G_F = 4 R_G R_L \left(\frac{-r_o}{\Delta} \right)^2$$

Backward Operating Gain

$$G_R = 4 R_G R_L \left(\frac{-r_o + r_m}{\Delta} \right)^2 = (1 - a)^2 G_F$$

Typical values:

$$\text{For } R_G = 20000^\omega, R_L = 10000^\omega$$

$$\text{Then } R_{11} = -4100^\omega$$

$$R_{22} = -7600^\omega$$

$$G_F = 15^{db}$$

$$G_R = 13^{db}$$

Fig. 11—Synopsis of grounded collector amplifier

Working between 20,000-ohm terminations, such an amplifier should have 6 db power gain in both directions and should still be stable even if one of its terminations changes 50% in the unfavorable direction.

The grounded emitter connection can also exhibit bilateral properties.

Recapitulating these three single-stage amplifiers, we see that when $a = 1$ their properties are close enough to the analogous electron tube arrangements to be easily remembered; but that, when a is different from 1, their properties begin to diverge from their tube counterparts. Some of these circuits will perform in a simple manner functions which are impossible to the analogous tube connections, although of course the functions could be accomplished by using more tubes or more complicated circuits.

FREQUENCY RESPONSE

So far the analysis of transistors has been given only for the resistive case, appropriate at low frequencies. When the frequency is raised, reactive components appear and the situation becomes more complicated, although of course still subject to the same general methods of analysis.

One might expect that since semiconducting diodes work at microwave frequencies, so also would semiconducting triodes. For the Type A transistor, this hope is blasted because of the essentially different nature of the mechanism, involving as it does the physical transport of charge carriers over appreciable distances. For certain features of the transistor, however, the analogy does hold. For example, the emitter by itself is a diode; and, in keeping with this fact, its open-circuit impedance does not change much with frequency in the range in which we shall be interested. For most engineering purposes the open-circuit input impedance of a Type A transistor may be regarded as a resistance independent of frequency. Such deviations as occur are small and entirely similar to what take place in an analogous diode.

The same situation holds with respect to the base resistance r_b and the collector resistance r_c , that is, they act as one might expect of a diode. The base resistance is substantially constant with frequency; the collector resistance has associated with it a slight amount of capacitance, mostly due to the case, leads, and wiring external to the unit, which gives a variation of properties with frequency in high-impedance circuits. The analogous capacitance on the emitter side is negligible because of the lower value of emitter impedance. One has, therefore, the T of resistors in the equivalent circuit substantially constant with frequency.

The dominant factor governing frequency response of the transistor is therefore largely expressed as a variation of the net mutual impedance r_m or, one may say as well, in the factor a which is the ratio of r_m to r_c .

Measurements of r_m as a function of frequency encounter the practical difficulty that it is impossible to present to the transistor over a wide frequency range an impedance high compared to the collector impedance. It is, however, quite easy to present to the collector a relatively low impedance

(75 ohms), which is constant over the frequency range of interest. Concurrently it is relatively simple to present to the emitter a high impedance,

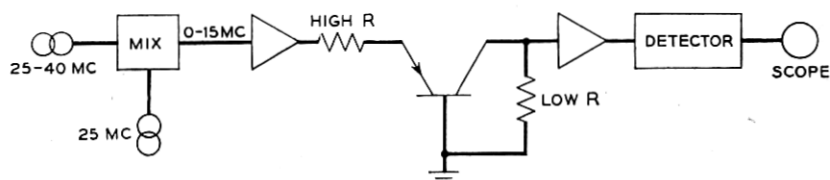


Fig. 12—Sweeper for measuring frequency response.

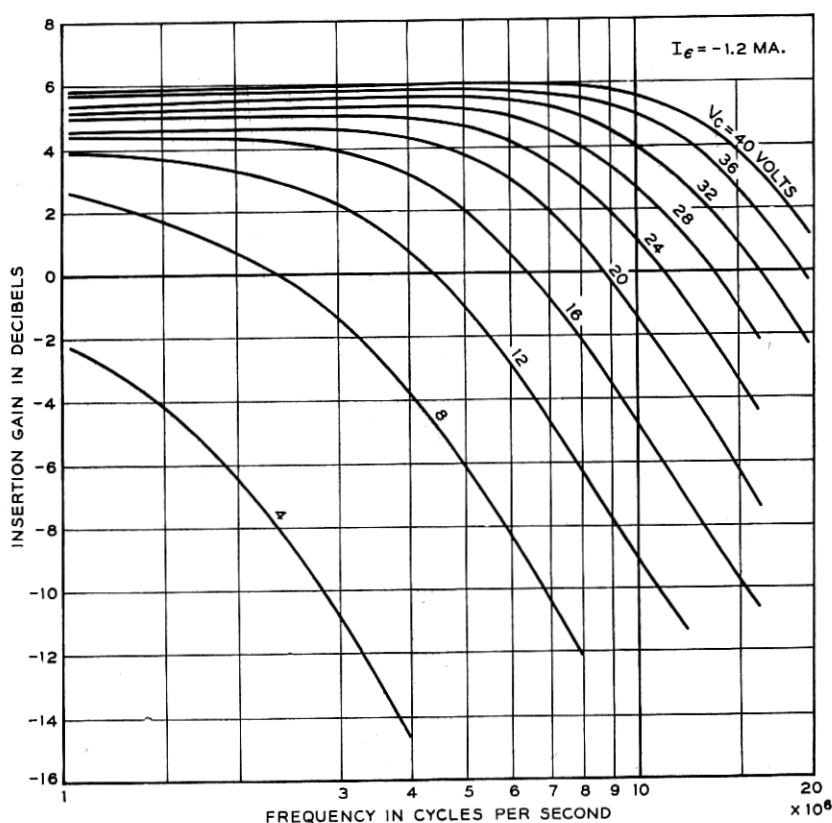


Fig. 13—Alpha versus frequency.

that is, to drive it with a constant current generator. Under these conditions the insertion power gain of the transistor is approximately α^2 , where the current amplification factor α is the ratio of increment in collector current to

increment in emitter current at constant collector voltage.⁵ The quantities α and a are usually nearly the same.

An oscilloscopic presentation of α versus frequency is possible and is a great convenience since many units can be measured quickly and variation with operating point observed directly. The sweep frequency generator built for this purpose is diagrammed in Fig. 12. It presents on an oscilloscope the magnitude of α as a function of frequency from 0 to 15 megacycles. Means are also available for making point-by-point plots which are more accurate, though much slower.

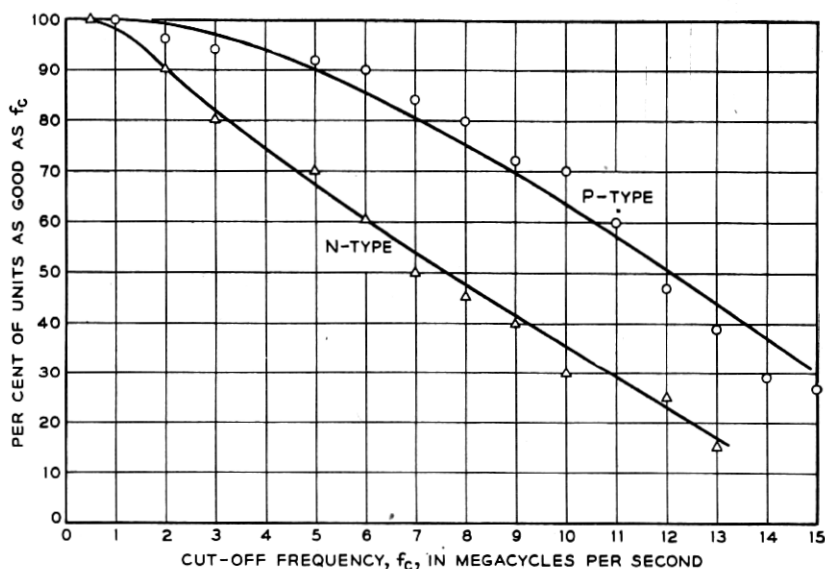


Fig. 14—Cut-off frequency statistics.

A set of curves of current amplification factor α versus frequency, as obtained with this apparatus, is shown in Fig. 13. The cutoff shape is a little sharper than that of a single R-C circuit but less so than that of a pair, one of which is shunt-peaked enough to make the combination flat. The apparent high-frequency asymptote varies in different units from 7 to 11 db per octave.

The phase shift associated with this curve has been found to be related to the amplitude in the same way as if the characteristic were that of a "mini-

⁵ Actually, $\alpha = (\partial I_c / \partial I_e)_{V_c}$ is only one of a set of four circuit parameters h_{ij} whose relationship to I_e and V_c is the same as that of the Z's to I_e and I_c , and which furnish an alternative circuit representation of the transistor. The other three h 's can be measured in a similar manner but are of less interest.

num phase" passive circuit.⁶ Accordingly the phase shift, like the amplitude variation, is also intermediate between a single R-C interstage and the flat compensated pair of interstages.

When variations between curve shapes are not too large, the shape can be characterized by a single parameter which we take as the cutoff frequency f_c . Cutoff is defined as the frequency where the magnitude of α^2 is halved. Some statistical data on cutoff frequency of different units made of N-type and P-type germanium are plotted in Fig. 14. The P-material is somewhat

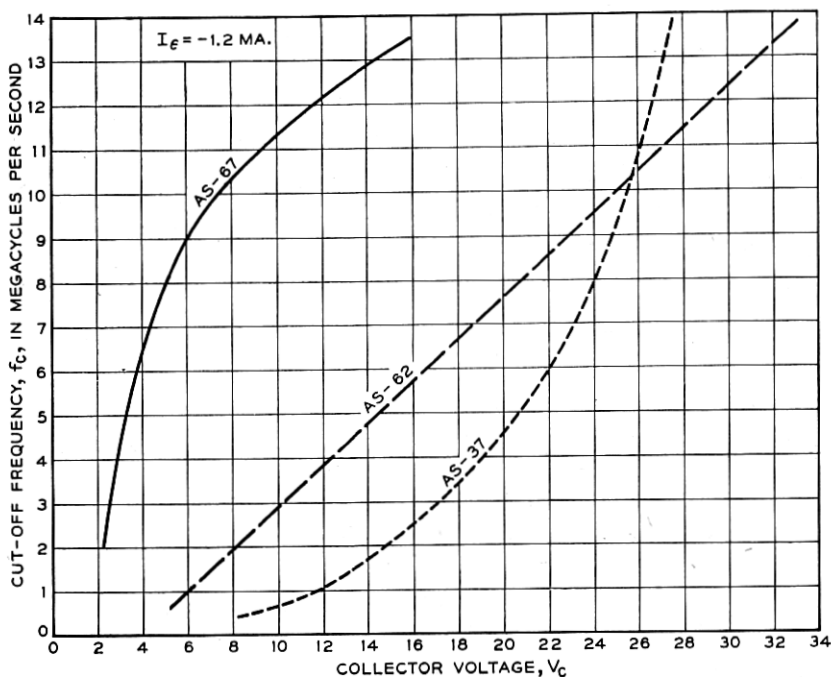


Fig. 15—Cut-off frequency versus collector voltage.

better, in keeping with the fact that the active charge carriers producing the transistor effect in it are electrons having greater mobility than the holes which are active in N-type germanium.

As one changes the operating point of the transistor the frequency response curve changes in such a way that the shape remains sensibly constant on a logarithmic frequency scale, but the scale changes. The cutoff frequency is usually roughly proportional to the collector voltage, with only minor dependence on the other operating parameter, as shown in Fig. 15 unit AS62.

⁶ "Network Analysis and Feedback Amplifier Design," H. W. Bode, D. Van Nostrand Publishing Co., 1945.

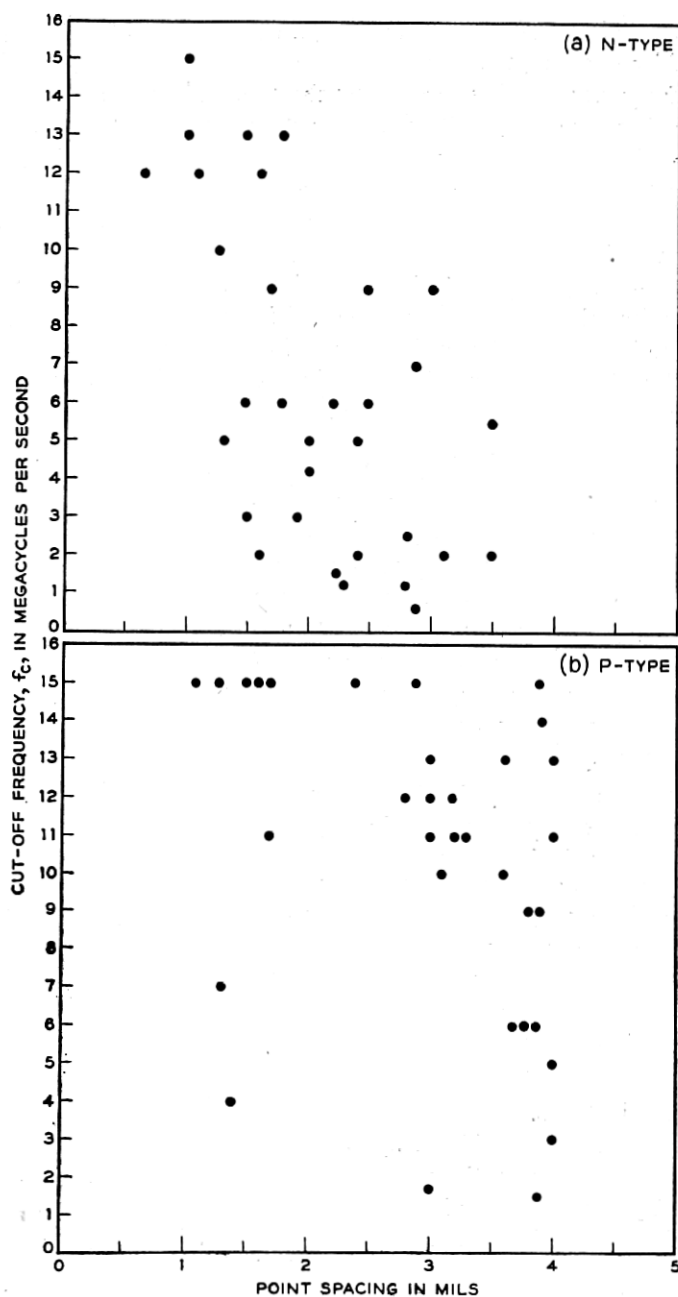


Fig. 16—Cut-off frequency versus point spacing.

Other types of variations of cutoff frequency with collector voltage are exhibited by some transistors.

That frequency cutoff is affected by the spacing between points of the transistor is shown in Fig. 16, which gives some support to the idea that the cutoff frequency might vary inversely as point spacing, other things being equal. However, one has only to look at the graph to see that other things are not equal for, at any given point spacing, the cutoff frequencies of different units vary by almost an order of magnitude. It is, however, clear that point spacing is one of the important factors.

In recapitulation of the measurements of frequency behavior, it appears possible to build Type A transistors with frequency cutoffs well above 10 megacycles. At the present time, the factors determining the frequency behavior are not yet under good control.

CASCADE AMPLIFIERS

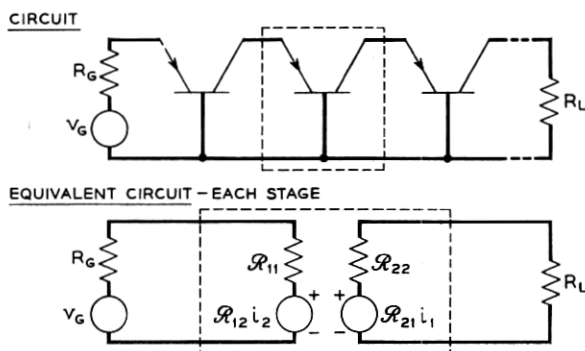
Many cascading possibilities exist, since any connection of the transistor might be used in combination with other connections, as well as involving all the parameter variations which might be made on each single stage. Some of the more elementary possibilities will be mentioned. Since feedback in each unit greatly complicates the situation, the essential features of the amplifiers may become clearer by discussing an idealized case where feedback is absent or greatly reduced. For similar reasons, the preliminary discussion is confined to frequencies low enough so that the equivalent circuits are purely resistive.

Perhaps the most straightforward cascade amplifier is the iterated grounded-base cascade, outlined in Fig. 17. Neglecting feedback, the insertion power gain is nearly equal to the current amplification factor α squared. For the Type A transistor this amounts to some 5 db per stage. For most uses this could be regarded as impractically low, but it might be pointed out that the tube analog (grounded grid cascade) is even worse; for when $\alpha = 1$ the maximum insertion gain is 0 db per stage. Both amplifiers of course can be made practical by interstage transformers (Fig. 18). For the Type A transistor, the matched gain without feedback rises to about 15 db per stage, which still compares favorably in magnitude with most grounded-grid tubes.

When feedback is considered by allowing r_b to return to its usual value of a few hundred ohms, the question of stability becomes important. The nominal Type A transistor is still stable when the cascade interstages are matched, the gain rising to about 21 db per stage. For many units having more than the usual amount of feedback, the interstages cannot be matched without violating the stability condition and therefore encountering os-

cillations; but one can normally count on stable gains of 15 to 20 db per stage, the transformers being perhaps somewhat mismatched.

Interesting possibilities for a good cascade amplifier with more gain than the grounded base cascade are offered by the grounded emitter connection. Incidentally, this gain advantage is also enjoyed by the grounded cathode or conventional tube connection, so that one would expect it to apply here from the electron tube analogy; but in transistors the feature that α may be greater than 1 brings in complications having no simple analogy for tubes.



Without feed back ($R_{12} = 0$):

Iterative impedance $R_G = R_{22}$, $R_L = R_{11}$

Circuit determinant $\Delta = (R_{11} + R_{22})^2$

Insertion Power Gain $G_I = \left| \frac{-R_{21}}{R_{11} + R_{22}} \right|^2$
 $= \left(\frac{\alpha}{1 + R_{11}/R_{22}} \right)^2$

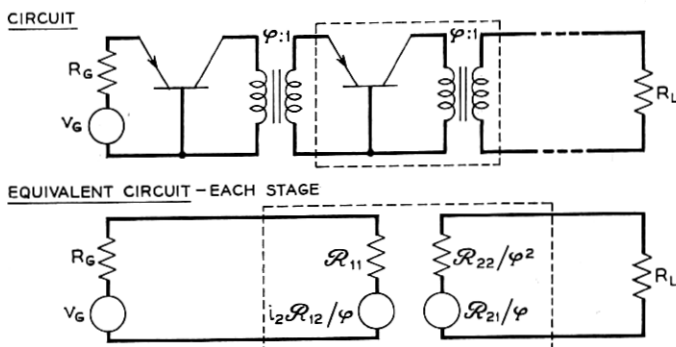
Nominal Type A Gain = 5^{db}

Fig. 17—Synopsis of grounded base cascade.

The iterated grounded emitter cascade without feedback (that is, emitter resistance $r_e = 0$) is unstable for the nominal Type A transistor, but can be stabilized in many ways of which we shall mention only one. The equivalent circuit of Fig. 19 shows an added resistor which may be thought of as adjusting the value of the collector resistance, and tends to make the unit more stable. When this resistor is adjusted to make the total collector resistance R_c about equal to the net mutual resistance r_m , thus reducing the effective value of α to the neighborhood of unity, then the cascade amplifier becomes stable, its gain being sensitive to the exact value chosen for the adjusting resistor. A numerical calculation for the grounded emitter

amplifier using the nominal Type A transistor adjusted in this way gives the following results:

Assuming an adjusted value of collector resistance of 36000 ohms to be satisfactory for stability, then the iterative input impedance is 2300 ohms, output impedance 4000 ohms, and insertion gain about 21 db per stage without transformers. Three-stage stable amplifiers having power gains of about 55 db have been operated.



Without feed back ($R_{12} = 0$):

Iterative impedance $R_G = R_{22}/\varphi^2$, $R_L = R_{11}$

Circuit determinant $\Delta = (R_{11} + R_{22}/\varphi^2)^2$

Insertion Power Gain $G_I = \left(\frac{-R_{21}/\varphi}{R_{11} + R_{22}/\varphi^2} \right)^2$

Maximum when $R_{11} = R_{22}/\varphi^2$

$$G_{I \text{ max.}} = \frac{R_{21}}{4 R_{11} R_{22}} = \frac{1}{4} \alpha^2 \frac{R_{22}}{R_{11}}$$

Nominal Type A Gain: without feed back = 15^{db}
with R_{12} normal = 17^{db}

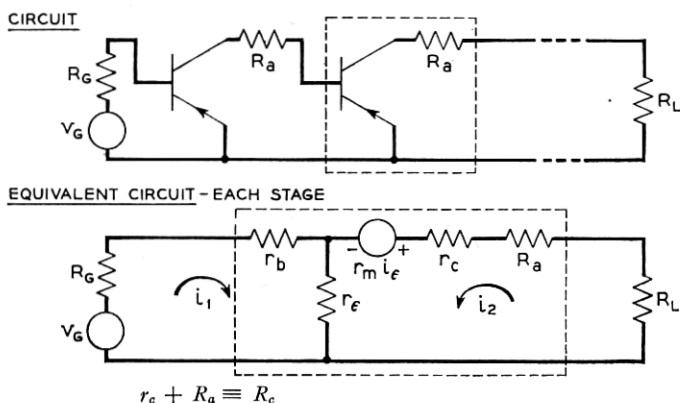
Fig. 18—Synopsis of grounded base cascade with transformers.

Another interesting feature of the grounded emitter amplifier is the ease with which negative feedback may be applied to it. A resistor inserted in the emitter lead gives local negative feedback analogous to the cathode feedback of tubes, while feedback involving several stages is also obtainable by common-lead methods analogous to common-cathode resistors familiar in the tube art. By such means, as is well known, distortion instability or gain variation may be reduced, or power output increased.

Theoretical study of these and other iterative amplifiers, particularly at higher frequencies, is conveniently carried on with the aid of the formulas

of Figs. 20 and 21 which give some of the iterative properties of a general fourpole and the effect thereon of an interstage matching transformer.

The iterative method of course does not exhaust the possibilities of cascade amplifiers. They can also be designed stage by stage. Even when feedback is large they can be cascaded together in the manner used for filter sections. A particular design of this sort is shown in Fig. 22. It is a grounded



Equations:

$$\begin{aligned} i_1(R_G + r_b + r_e) + i_2 r_e &= v_G \\ i_1(r_e - r_m) + i_2(R_L + r_e + R_c - r_m) &= 0 \end{aligned}$$

Circuit determinant $\Delta = (R_G + r_b + r_e)(R_L + r_e + R_c - r_m) - r_e(r_e - r_m) > 0$ for stability

Without feed back $(r_e = 0)$

Iterative impedance $R_G = R_c - r_m, R_L = r_b$

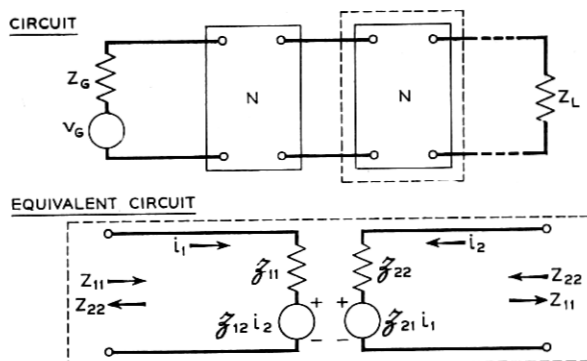
Circuit determinant $\Delta = (r_b + R_c - r_m)^2$

Insertion Power Gain $G_I = \left(\frac{r_m}{r_b + R_c - r_m} \right)^2$

Nominal Type A Gain with $R_c = 36000^\omega$
 without feed back 23^{db}
 with r_e normal 21^{db}

Fig. 19—Synopsis of grounded emitter cascade.

base stage followed by a grounded collector and accordingly has the tube analog grounded-grid, cathode follower, from which one would expect that the terminating impedances would be low and the interstage impedance high. This amplifier matched a 600-ohm line to better than 10% and had 16 db insertion gain, with a bandwidth of about a megacycle. An adaptation for video purposes was made to obtain over a band from 100 cycles to 3.5 megacycles, an insertion gain of 20 db in a 75-ohm coaxial line.



Equations:

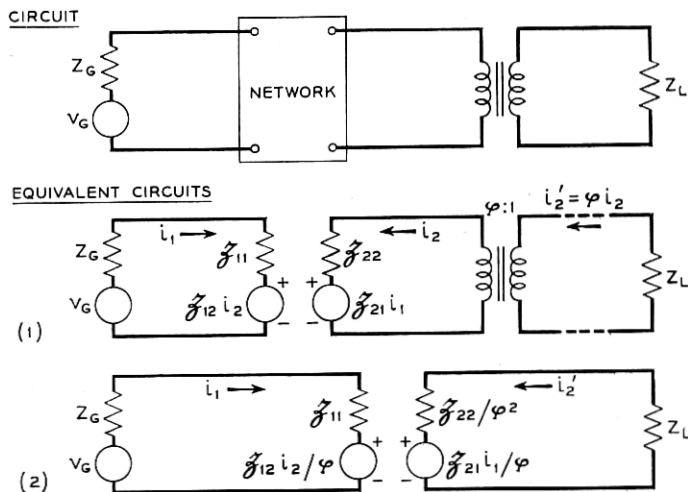
$$\begin{aligned} i_1(\mathfrak{G}_{11} + Z_{22}) + i_2 \mathfrak{G}_{12} &= v_G \\ i_1 \mathfrak{G}_{21} + i_2(\mathfrak{G}_{22} + Z_{11}) &= 0 \end{aligned}$$

Terminations:

$$\begin{aligned} Z_{11} &= Z_L = -\mathfrak{G}_{22} + \frac{1}{2}(\mathfrak{G}_{11} + \mathfrak{G}_{22})(1 + \sqrt{1-y}) \\ Z_{22} &= Z_G = -\mathfrak{G}_{11} + \frac{1}{2}(\mathfrak{G}_{11} + \mathfrak{G}_{22})(1 + \sqrt{1-y}) \\ y &= 4\mathfrak{G}_{12}\mathfrak{G}_{21}/(\mathfrak{G}_{11} + \mathfrak{G}_{22})^2 \end{aligned}$$

Circuit determinant $\Delta = \frac{1}{2}(\mathfrak{G}_{11} + \mathfrak{G}_{22})^2(1 - y + \sqrt{1 - y})$ Insertion Power Gain $G_I = \left| \frac{\mathfrak{G}_{21}}{\mathfrak{G}_{11} + \mathfrak{G}_{22}} \frac{2}{1 + \sqrt{1 - y}} \right|^2$

Fig. 20—Synopsis of iterated cascade of four-poles.

Equations: $i_1(\mathfrak{G}_{11} + Z_G) + i_2' \mathfrak{G}_{12}/\phi = v_G$

$$i_1 \mathfrak{G}_{21}/\phi + i_2' \left(\frac{\mathfrak{G}_{22}}{\phi^2} + Z_L \right) = 0$$

Fig. 21—Four-pole with ideal transformer.

The foregoing amplifiers both have rather low output powers because of the fact that the emitter, a low-current electrode, is the output electrode. A way of improving this situation has been suggested in the second amplifier schematic shown in Fig. 22. The first stage is a grounded emitter and the second a grounded collector transistor, the latter operating in what we have called the "backward" direction so that the output electrode is the base and the power level is improved. This amplifier can be stabilized by negative feedback obtainable by inserting a resistor in the first stage emitter lead.

These examples emphasize that one can cascade unlike stages and that feedback can be used to stabilize performance, just as with electron tubes. These amplifiers can be further cascaded to obtain more gain. Other possibilities worthy of mention include modifying the design of the first stage

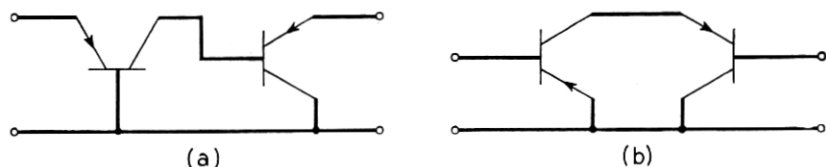


Fig. 22—Non-iterative cascade amplifiers.

of an iterative amplifier to obtain good noise figure, or of the last stage for greater power output.

BAND PASS AMPLIFIERS

Bandpass amplifiers require a few remarks before concluding the small-signal discussion. The design within the band may be carried out by the methods previously discussed; but frequently attention must also be paid to properties outside the band, to an extent unusual with tubes. The reason, of course, is connected with that Dr. Jekyll and Mr. Hyde of transistors, α (or a) greater than 1. When a transistor may be short-circuit unstable, then oscillations may result from the practise usual with electron tube amplifiers of letting the impedances outside the band fall to low values. For the same reason design of power leads requires more care than usual. The problems encountered are somewhat similar to those of tube amplifiers with feedback in that one must pay attention to characteristics far outside the useful band. In the case of transistors, one may have to exercise design care to avoid oscillations even when the gain of the amplifier is less than unity.

LARGE SIGNAL ANALYSIS

Large signals are those which involve considerable excursions over the electrical characteristics of the device and cannot be regarded as small

changes near an assumed operating point. For their general study a most convenient tool is provided by the set of static characteristics of the unit.

Since most analyses begin with the static characteristics, perhaps some excuse is needed for the unorthodox approach which has delayed them to this point. Two reasons may be cited: First, the small-signal behavior is in a sense simpler, being capable of discussion by the familiar linear methods of circuit theory. Second, the small-signal behavior has brought out some features, notably short-circuit instability, which have a bearing on certain features of the static characteristics, on the methods of measuring them, and on the particular manner of expressing them.

A set of characteristics representative of Type A transistor performance is shown in Fig. 23, consisting of four plots, one of each of the electrode voltages against each of the currents with the other current as parameter. Contrary to electron tube practise, rather than the voltages we take the currents as the independent variables. This choice avoids the experimental difficulty that the short-circuit unstable transistors might oscillate if we were to attempt to hold the electrode voltages constant, as well as the concomitant analytical trouble that in that case the voltage-dependent characteristics become double-valued.

The relationship of these characteristics to the open-circuit impedances is direct and quickly shown. Suppose the voltages are expressed formally as functions of the currents:

$$\begin{aligned} V_E &= f_1(I_E, I_C) \\ V_C &= f_2(I_E, I_C) \end{aligned} \quad (2)$$

Differentiating, and identifying the differentials as small-signal variables, we get immediately the equations for the open-circuit resistances:

$$\begin{aligned} v_E &= i_E \frac{\partial f_1}{\partial I_E} + i_C \frac{\partial f_1}{\partial I_C} \\ v_C &= i_E \frac{\partial f_2}{\partial I_E} + i_C \frac{\partial f_2}{\partial I_C} \end{aligned} \quad (3)$$

Accordingly, the open-circuit resistances are the slopes of these static characteristics. The reactive components do not appear because our assumptions (2) were not sufficiently general to take them into account or, in other words, the reactive information is not contained in the static characteristics.

Just as there are five other pairs of small signal parameters which could have been chosen, so there are five other ways in which the static characteristics could have been expressed. Often these other ways are convenient for special purposes or are closely connected with particular large signal circuits.

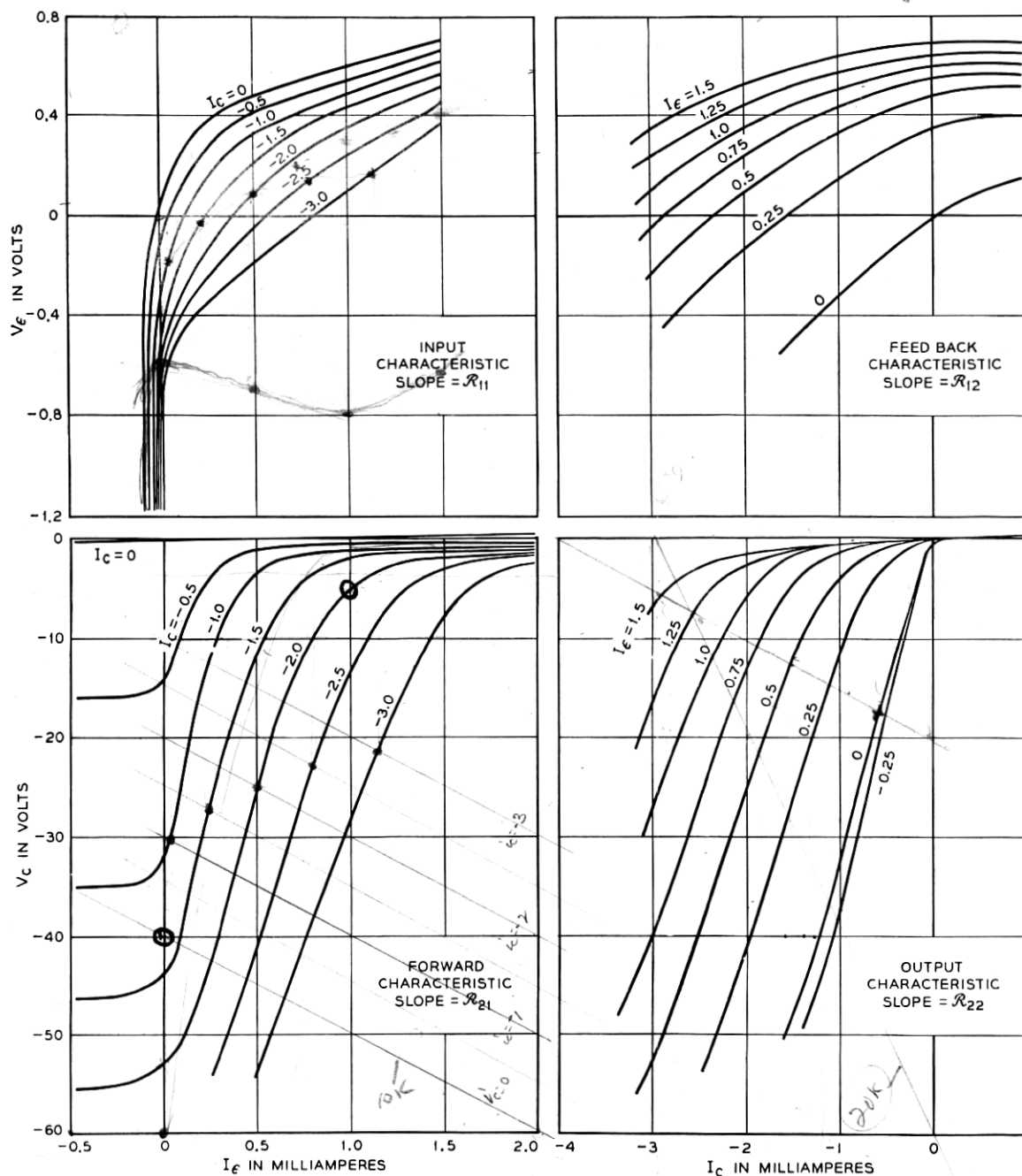


Fig. 23—Static characteristics.

Measurement of the characteristics can be by conventional point-by-point plots or by oscilloscope presentation. An oscilloscopic curve tracer has been built which can show any of the four characteristics for any of the six pairs of independent parameters of the Type A transistor, as well as any two-pole characteristic which might be of interest (such as a negative resistance characteristic).

Occasionally the static characteristics are affected by effects of a thermal nature such that an oscilloscope trace does not give the same results as a slow point-by-point plot. These thermal effects are small in the usual region of operation of the Type A transistor but may become appreciable if the unit is heated by excessive power dissipation in it.

POWER OUTPUT AND DISTORTION

The problem of obtaining good "undistorted" power output from a transistor at low frequencies is one which is conveniently discussed by means of the static characteristics. Analytically this question belongs to the class of slightly non-linear problems but, for descriptive purposes, it is illustrated by the curves of Fig. 24. The family of collector characteristics of a Type A transistor is shown. The region of linear operation is substantially that part of the plot where the curves are uniformly spaced, have constant slope, and lie within the permitted power dissipation of the unit.

In driving a Type A transistor harder and harder in an attempt to get greater power output, one may encounter four types of overload distortion, analogous to the types found in tubes.

1. One may drive the emitter negative into the cutoff region where the collector current fails to respond to changes in emitter potential, corresponding to grid cut-off in a tube.

2. One may drive the emitter positive into an emitter overload region where non-linear distortion may be encountered because the emitter impedance changes with its voltage. The corresponding tube phenomenon is positive grid distortion. For both tubes and transistors this effect is a minor one which may be actually beneficial in practical cases.

3. The collector may be driven down to low potential where it can no longer draw the current required to follow the impressed emitter current variations. This distortion corresponds to plate "bottoming" in electron tubes.

4. The collector may be driven up to high currents where it overloads because of the non-linear voltage response in that region arising from heating effects. This effect has practical consequences something like the overloading of electron tubes which may arise from insufficient cathode emission.

In other words, either emitter or collector may be driven into overload or cut-off and the problem of getting good power output reduces to choosing

an operating point and load impedance such as to avoid these non-linear effects as long as possible. Reverting to Fig. 24, since one wants as large a product of $\Delta V \cdot \Delta I$ as possible, the problem may be thought of in geometrical terms as approximately that of constructing the largest possible rectangle such that a load line extending diagonally across the corners of this rectangle

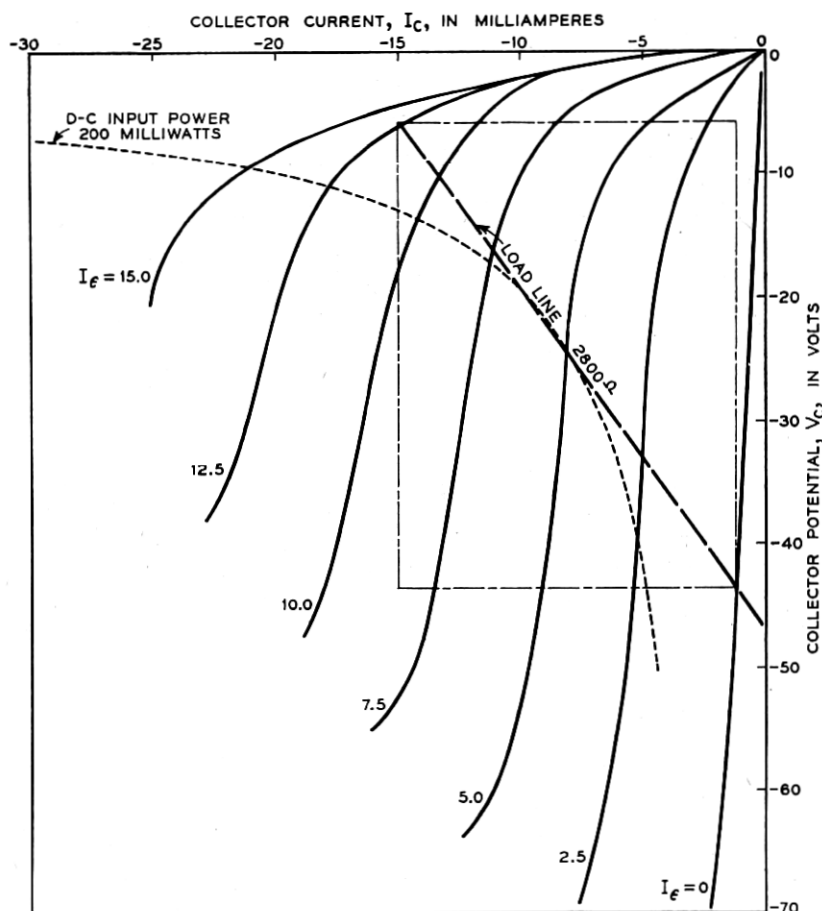


Fig. 24—Collector power output plot.

lies within the "linear" region of operation. The slope of this line gives the load impedance required, its intercept the collector supply voltage (for resistance coupling), and the sides of the rectangle give the extreme values of voltage and current. The center of the rectangle is approximately the quiescent or small-signal operating point.

Under optimum conditions of load impedance and operating point,

one obtains power efficiencies comparable to Class A electron tube operation, that is, 20 to 35% efficiency with a few percent harmonic distortion. As contrasted to recommendations for good low-level gain for the Type A transistor, the optimum conditions for power output have usually involved lower load impedances and higher currents. Representative values may be: load impedance, 5000 ohms; collector current, -8 milliamperes at -35 volts bias; emitter current, 3 milliamperes; power output, 60 milliwatts, with distortion less than ten percent.

One complication of the power transistor is that, when the optimum load impedance is low, the operating point gets nearer to the region where the transistor may tend to oscillate if it happens to be one of the kind which is short-circuit unstable. A saving circumstance here is available in that

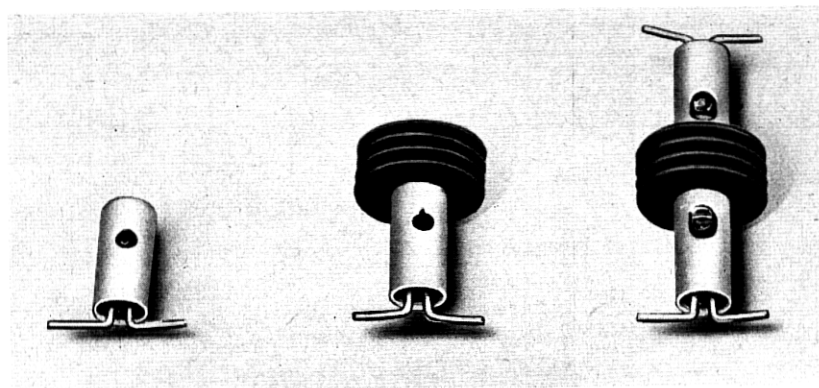


Fig. 25—Some power transistors.

added resistance in the emitter lead tends to promote stability, so that the transistor may be stabilized by operating out of a higher generator impedance, possibly at some cost in reduced gain. A corollary aspect of the same phenomenon is that the input impedance of a high-power transistor may become very low or even negative.

Higher power output from the transistor can also be obtained by increasing the permissible collector dissipation. This has been accomplished by using a thin wafer of germanium directly soldered to a copper base equipped with suitable fins to facilitate the removal of heat generated in the vicinity of the collector point. An increase in allowable dissipation from 200 to 600 milliwatts has been thereby obtained. Output powers of approximately 200 milliwatts at a conversion efficiency of 33% have been realized.

The photograph of Fig. 25 shows on the left the type A transistor, in the center the power version of this unit, and on the right is shown a double

ended type of power transistor using two germanium wafers with a common radiator for push-pull applications.

OTHER LARGE-SIGNAL APPLICATIONS

The static characteristics can be used for calculations of many large-signal circuits of which only a few examples can be given here. The first is a tickler feedback oscillator of Fig. 26, which uses the grounded-base circuit with a resonant circuit in the collector lead, transformer-coupled back to the emitter.

Other circuits making use of the special possibilities of the transistor include an oscillator with anti-resonant circuit in the base lead, or with a

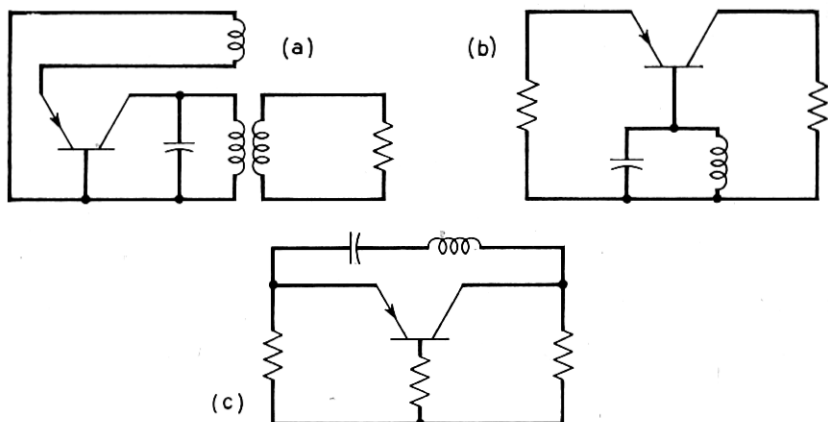


Fig. 26—Transistor oscillators.

series resonant circuit from collector to emitter. Some of these circuits make use of the short-circuit instability peculiar to the transistor and accordingly would not work with electron tubes.

NOISE

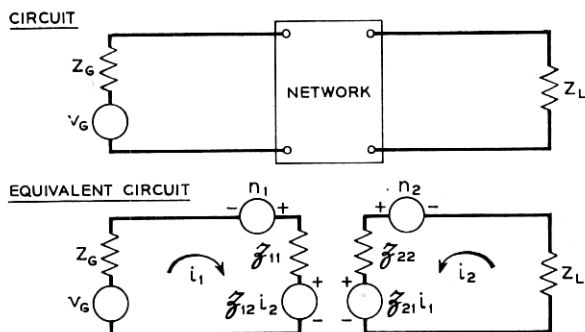
A discussion of small-signal amplifiers would be incomplete without some mention of the limiting factor of noise. The noise has been left to the last, however, because its discussion complicates the circuits slightly, and perhaps because it is not well to present too early an aspect of performance which is at the moment so much inferior to electron tubes.

On the circuit representation of noise as well as signal much work has been done by L. C. Peterson.⁷ It turns out that in the general four-terminal network in which we are interested, a complete noise representation for

⁷ "Signal and Noise in Microwave Tetrode," *Proc. I.R.E.*, Nov. 1947, pp. 1264-1272.

circuit purposes may be obtained by adding two noise generators to the equivalent circuit of four signal parameters, as shown in Fig. 27.

These noise representations are on an entirely similar basis to the signal representations. Just as four elements in any independent configuration suffice for signal description, so two noise generators in either series or shunt in any convenient independent locations can be added to account for the noise. All these representations give the same signal and noise behavior for any external connections. Still, some may be better than others in corresponding to the actual physics of the transistor; presumably the



$$\begin{aligned} \text{Equations: } i_1(Z_G + \mathcal{Z}_{11}) + i_2\mathcal{Z}_{12} &= v_G \oplus N_1 \\ i_1\mathcal{Z}_{21} + i_2(\mathcal{Z}_{22} + Z_L) &= \oplus N_2 \end{aligned}$$

Circled \oplus signs indicate addition with attention to any correlations which may exist between noise generators or mean square additions if no correlation exists.

$$\text{Noise Figure } F = 1 + \frac{1}{4 k T B R_G} \left\{ \overline{N_1^2} \oplus \overline{N_2^2} \left(\frac{\mathcal{Z}_{11} + Z_G}{\mathcal{Z}_{21}} \right)^2 \right\}$$

Fig. 27—Synopsis of general four-pole, including noise.

better representations will show particularly simple behavior, for example, in their dependence upon the d-c operating point of the transistor. The usual choice puts noise voltage generators in series with the emitter and collector leads, as shown.

If the two noise generators were truly independent physical sources of noise, their outputs would be expected to show no correlation and their noise power contributions would be simply additive. This independence is not usually the case for the Type A transistor. By adding the noise outputs and comparing the power in the sum to that in the separate components, correlation coefficients ranging from $-.8$ to $+.4$ have been found. From this the conclusion can be drawn that the physical sources of noise in the network do not act in series with the leads but at least to some extent arise elsewhere

in the transistor and contribute correlated noise output to both the generators of the circuit representation.

The transistor noise is of two types. One is a rushing sound somewhat similar qualitatively to thermal resistance noise; the other is a frying or rough sound which occurs erratically, usually in the noisier units. The noise

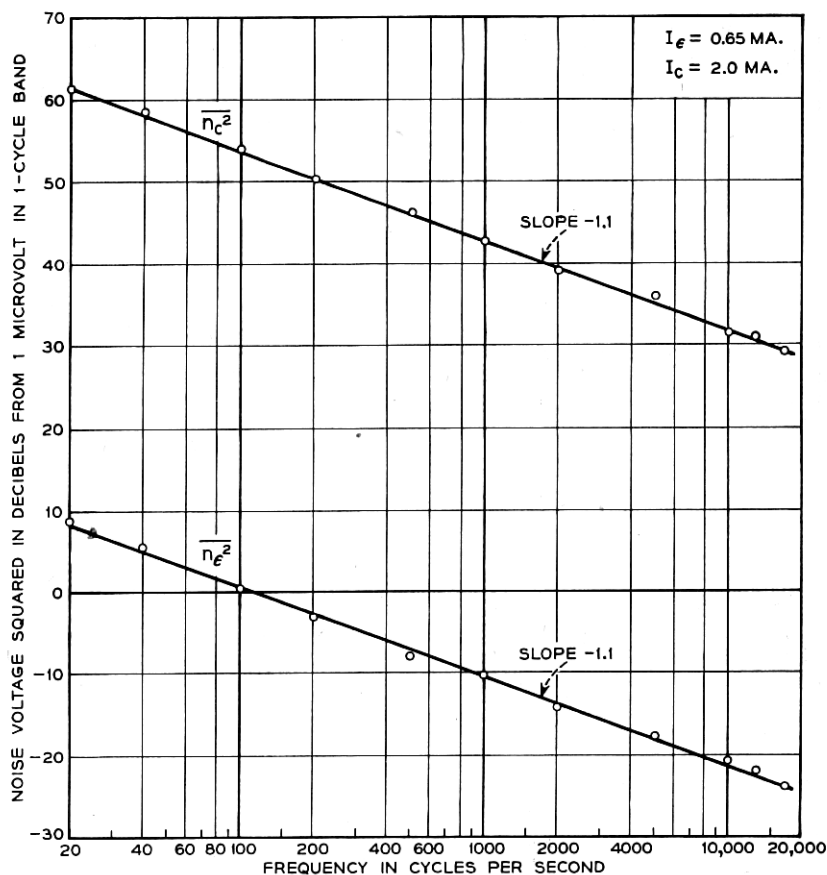


Fig. 28—Transistor noise versus frequency.

power per unit bandwidth varies almost exactly inversely with frequency as shown in Fig. 28, being in this respect reminiscent of contact noise.

Since the noise dependence on frequency is known, its level may be given as noise voltage per unit bandwidth at a reference frequency (1000 cycles). The collector noise usually dominates as far as practical effects on the output are concerned. Representative values are about 100 microvolts per cycle at 1000 cycles for the collector, and one or two microvolts for the emitter.

The noise voltages depend mainly on the collector direct voltage as shown in Fig. 29. While they do vary with the other operating parameter at constant collector voltage, such variations rarely exceed 10 db, which is much less than the variations with collector voltage.

More important than the actual level of the noise is its relation to thermal resistance noise, which is the ultimate limit to amplification. This relationship is conveniently expressed by means of the noise figure, or number of times noisier than amplified thermal noise in the output of the amplifier.

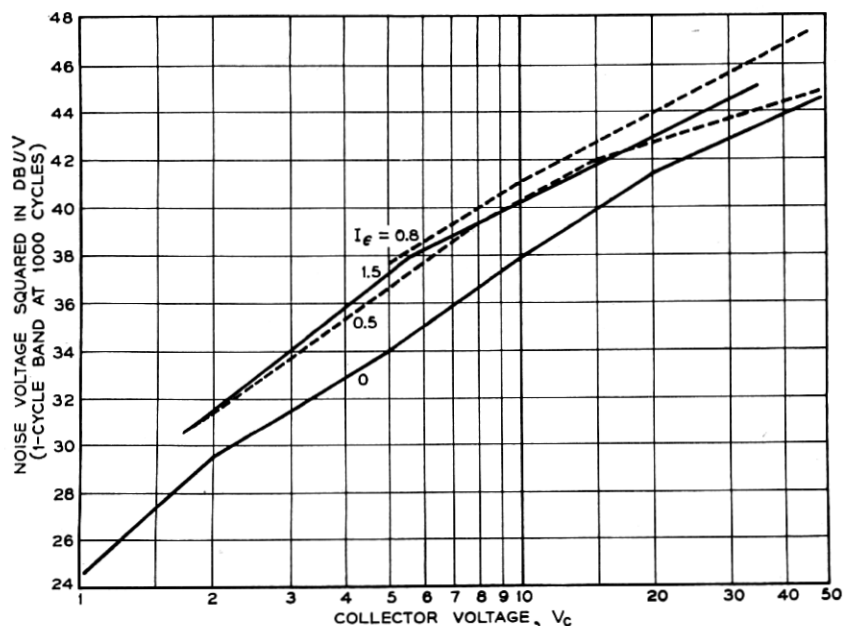


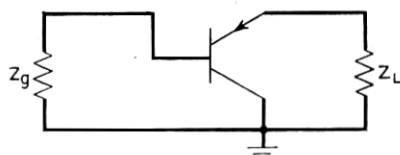
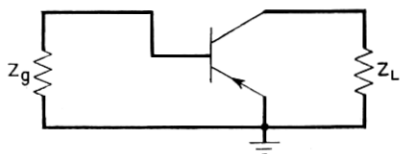
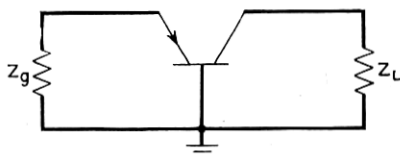
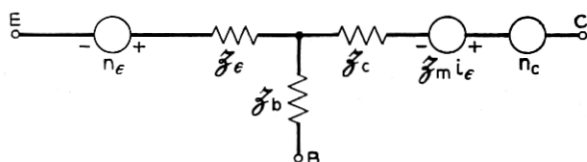
Fig. 29—Transistor noise versus operating point.

A representative noise figure for the Type A transistor at 1000 cycles is 60 db, with individual units ranging from 50 to 70 db.

Noise figure formulas for the three single-stage connections are given in Fig. 30. The noise performance of the three connections would usually not be very different if it were not for stability considerations, which may render unusable the generator impedance which would give optimum performance. Mainly, on account of stability, the grounded base connection may be said to give the best noise performance, with the grounded emitter running a close second.

The noise figure of any device depends upon the generator impedance out of which it works but does not depend upon the load. Accordingly, there exists an optimum generator impedance which gives the best noise

Equivalent Circuit



Grounded Base

$$F = 1 + \frac{1}{4kTBR_G} \left[\overline{N_\epsilon^2} \oplus \overline{N_c^2} \left(\frac{Z_G + z_\epsilon + z_b}{z_m + z_b} \right)^2 \right]$$

Grounded Emitter

$$F = 1 + \frac{1}{4kTBR_G} \left[\overline{N_\epsilon^2} \left(\frac{Z_G + z_m + z_b}{z_m - z_\epsilon} \right)^2 \oplus \overline{N_c^2} \left(\frac{Z_G + z_b + z_\epsilon}{z_m - z_\epsilon} \right)^2 \right]$$

Grounded Collector

$$\text{Forward } F = 1 + \frac{1}{4kTBR_G} \left[\overline{N_\epsilon^2} \left(\frac{Z_G + z_c + z_b}{z_c} \right)^2 \oplus \overline{N_c^2} \left(\frac{Z_G + z_b}{z_c} \right)^2 \right]$$

$$\text{Backward } F = 1 + \frac{1}{4kTBR_L} \left[\overline{N_\epsilon^2} \oplus \overline{N_c^2} \left(\frac{Z_L + z_\epsilon}{z_m - z_c} \right)^2 \right]$$

Fig. 30—Noise figure formulas.

figure of which the unit is capable. This optimum source impedance is best for signal-to-noise performance, not for signal performance alone; hence, as is well known for vacuum tubes, it is usually not a match for the unit, and in general both the resistive and reactive components of impedance may be mismatched to the unit.

For the transistor at low frequencies in the grounded-base connection, reactive effects are negligible and the emitter noise generator may usually be neglected. Under these conditions the optimum noise figure is obtained from a generator of impedance equal to the open-circuit input resistance of the transistor (not the actual working input resistance, which may be quite different).

The best operating point for low noise is usually obtained at a moderate collector voltage (20 volts) and a small emitter current (0.5 ma.).

SUMMARY

A tentative evaluation of the Type A transistor may be made on the basis of presently available information. Before making it, we should say that a comparison with the field of electron tubes is obviously unfair — there are many against one, and a little one at that. Furthermore the little one is a baby not only in size but in length of time under development. It is only natural that the full possibilities are not yet apparent. With these reservations, we can make the following statements about the present Type A transistor:

Gain: the transistor figure of about 17 db per stage is somewhat low compared to 30 or 40 db obtainable from audio tubes. When the bandwidth is taken into consideration the gain-band product of the transistor is good but, since the excess bandwidth cannot be exchanged for gain, this number is in this case illusory for narrow-band amplifiers. For video amplifiers the comparison is more favorable.

Stability considerations differ from the electron tube in such a way as to be likely to give more trouble at low frequencies. At video frequencies this difference is less marked if we play fair by comparing with a triode tube instead of a pentode. The latter is of course better shielded than the transistor.

Frequency response appears to be practical up to 10 megacycles or more.

Power output efficiency of around 30%, Class A, seems fully comparable to an electron tube, so that a comparison between the two can be based on input d-c power.

Noise figure of 60 db at 1000 cycles is much worse than that of a good electron tube, which can come close to 0 db. In view of the frequency dependence which brings the transistor noise figure down to 30 db at a megacycle, the comparison at video frequencies is less unfavorable, particularly if some developmental improvement can be made.

So far on most counts the comparison is not too favorable but, as we said before, it isn't fair to the baby. In addition there are a number of other considerations which are secondary from the point of view of pure technique but may be dominant from other points of view. Among favorable factors here are: small size; low power drain; no standby power, but instant response when needed; low heating effect when used in large numbers; and ruggedness.

The life of transistors should be fairly long on the basis of diode performance, but the device is too new to permit definite statement. The mechanical simplicity might well lead one to hope for low cost, but no production figures are as yet available.

In fine, even if Type A transistor performance does not excel all electron tubes, it is still good enough for many applications and will be considerably better in the future.

ACKNOWLEDGEMENT

This survey is based on the work of many people, only a few of whom have been mentioned in the text. The examples of circuits have not been numerous or exhaustive, but rather have been used to illustrate the methods adopted; these are general enough to be adapted to the solution of many particular problems.