Transverse Fields in Traveling-Wave Tubes

By J. R. PIERCE

Traveling-wave tubes will have gain even if the r-f field at the mean position of the electron stream is purely transverse. The addition of a longitudinal magnetic focusing field reduces the gain due to transverse fields and increases the electron velocity for optimum gain.

ALL slow electromagnetic waves have both longitudinal and transverse electric field components. Sometimes either the longitudinal or the transverse field may go to zero along a line or plane parallel to the direction of propagation. For instance, for the slow mode of propagation there is no transverse field on the axis of a helically-conducting sheet. Still, over any plane normal to the direction of propagation there are bound to be both longitudinal and transverse field components.

If a very strong longitudinal magnetic field is used in connection with a traveling-wave tube, the transverse motions of electrons may be so restricted as to be of little importance. With weak focusing fields, however, the transverse motion of electrons may be important in producing gain. The transverse fields can force the electrons sidewise, and thus change the longitudinal fields acting on them in such a way as to abstract energy from the electron stream.[1] This is closely analagous to the action of the longitudinal fields in displacing electrons forward or backward into regions of greater or lesser longitudinal field.

The purpose of this paper is to analyze the behavior of traveling-wave tubes in which transverse fields are important. The attack will be similar to that used previously.[2]

1. Circuit Theory

In this paper we shall consider only the electric field associated with the slow mode of propagation along the circuit having a speed close to the electron speed, and we shall neglect other field components attributable to local space charge. The writer believes the results so obtained to be valid at low currents but in error at high currents, and an acceptable guide at currents usually encountered.

In an earlier paper[2] a relation was found between the longitudinal field E_z excited in a mode of propagation of a transmission system and the longitud-

[1] See, for instance, J. R. Pierce and W. G. Shepherd, "Reflex Oscillators," *B. S. T. J.*, Vol. 26, No. 3, pp. 666–670 (July, 1947).

[2] J. R. Pierce, "Theory of the Beam-Type Traveling-Wave Tube," *Proc. I. R. E.*, Vol. 35, pp. 111–123, Feb. 1947.

inal exciting current q. Both E_z and q vary as $(\exp j\omega t)(\exp - \Gamma z)$. The relation is

$$E_z = q \frac{\Gamma_0}{\psi_0^*(\Gamma^2 - \Gamma_0^2)}. \tag{1}$$

Here Γ_0 is the propagation constant of the transmission mode considered and is defined in such a sense that for unattenuated propagation, $\Gamma_0 = j\beta_0$ where β_0 is a positive number. The quantity ψ_0 is defined as

$$\psi_0 = \frac{2P}{E_z E_z^*}. \tag{2}$$

Here P is complex power transmitted by the mode and E_z is the field associated with the mode.

In generalizing (1), let us consider the combination of equations (1) and (2)

$$P^* = \tfrac{1}{2} q E_z^* \frac{\Gamma_0}{\Gamma^2 - \Gamma_0^2}. \tag{3}$$

Now, suppose there is motion of the electrons not only in the z direction but in a direction normal to the z direction, which we will call the y direction. We shall have two extra first-order terms of the same general nature as qE_z^*, which contribute to the power, giving

$$P^* = \frac{1}{2}\left(q_z E_z^* + (-I_0)y \frac{\partial E_z^*}{\partial y} + q_y E_y^*\right) \frac{\Gamma_0}{\Gamma^2 - \Gamma_0^2}. \tag{4}$$

Here q_z is the a-c convection current in the z direction, $-I_0$ is the d-c convection current in the z direction (assumed to be the only d-c convection current), y is a small displacement, q_y is the convection current in the y direction and E_y is the field in the y direction.

We will now specialize this expression. Suppose we consider a two-dimensional transverse magnetic wave propagating in the z direction with a phase velocity v such that $v^2 \ll c^2$. Then over a restricted region the electric field can be represented quite accurately as the gradient of a scaler potential of

$$V = \exp(-\Gamma z)(A \exp(j\Gamma y) + B \exp(-j\Gamma y)). \tag{5}$$

Here A and B are constants. Using our notation, in which the field is understood to include the factor $\exp(-\Gamma z)$, we obtain

$$E_z = \Gamma(A \exp(j\Gamma y) + B \exp(-j\Gamma y)) \tag{6}$$

$$\frac{\partial E_z}{\partial y} = j\Gamma^2(A \exp(j\Gamma y) - B \exp(-i\Gamma y)) \tag{7}$$

$$E_y = -j\Gamma(A \exp(j\Gamma y) - B \exp(-j\Gamma y)). \tag{8}$$

In other words

$$\frac{\partial E_z}{\partial y} = -\Gamma E_y . \tag{9}$$

This relation will also be approximately correct remote from the axis in an axially symmetrical tube. Here we let y represent a displacement in the r direction.

We may also define a quantity α so that

$$E_y = j\alpha E_z \tag{10}$$

$$\alpha = \frac{-(A \exp(j\Gamma y) - B \exp(-j\Gamma y))}{Ay \exp(j\Gamma_y) + B \exp(-j\Gamma y)} . \tag{11}$$

For an active mode, such as the one we consider, the chief component of $j\Gamma$ is a positive real number. Hence, for large positive values of y, the quantity α approaches a value

$$\alpha = 1 . \tag{12}$$

This is characteristic of a plane symmetrical field far from the axis and also of an axially symmetrical field far from the axis.

Using (9) and (12) we rewrite (4)

$$P^* = \tfrac{1}{2}E_z^*[q_z - j\alpha^*(\Gamma^* I_0 y + q_y)] \tag{13}$$

We see from this that, according to our assumptions, for the mode considered,

$$E_z = (q_z - j\alpha^*(\Gamma^* I_0 y + q_y)) \frac{\Gamma_0}{\psi_0^*(\Gamma^2 - \Gamma_0^2)} . \tag{14}$$

We will henceforward assume that α and ψ are so nearly real that we can regard them as real quantities, giving

$$E_z = [q_z - j\alpha(\Gamma^* I_0 y + q_y)] \frac{\Gamma_0}{\psi_0(\Gamma^2 - \Gamma_0^2)} . \tag{15}$$

This is, then, the circu't equation which we will use.

2. Electronics Equations

We will assume an unperturbed motion of velocity u_0 in the z direction, parallel to a uniform magnetic field of strength B. Products of a-c quantities will be neglected.

In the x direction, perpendicular to the y and z direction

$$\frac{d\dot{x}}{dt} = -\eta B\dot{y}. \tag{16}$$

Assume that $\dot{x} = 0$ at $y = 0$. Then

$$\dot{x} = -\eta B y. \tag{17}$$

In the y direction we have

$$\frac{d\dot{y}}{dt} = \eta(B\dot{x} - j\alpha E_z). \tag{18}$$

Now

$$\frac{d\dot{y}}{dt} = \frac{\partial \dot{y}}{\partial t} + \frac{\partial \dot{y}}{\partial z}\frac{dz}{dt} \tag{19}$$

$$\frac{d\dot{y}}{dt} = u_0(j\beta - \Gamma)\dot{y} \tag{20}$$

$$\beta = \frac{\omega}{u_0}. \tag{21}$$

We obtain from (20) and (18), and (17)

$$(j\beta - \Gamma)\dot{y} = -u_0\beta_0^2 y - \frac{j\eta\alpha E_z}{u_0} \tag{22}$$

$$\beta_0 = \frac{\eta B}{u_0}. \tag{23}$$

We may note that ηB is the electron cyclotron frequency. Now,

$$\dot{y} = \frac{\partial y}{\partial t} - \frac{\partial y}{\partial z}\frac{dz}{dt}$$
$$\dot{y} = u_0(j\beta - \Gamma)y. \tag{24}$$

From (24) and (22) we obtain

$$y = \frac{-j\eta\alpha E_z}{u_0^2[(j\beta - \Gamma)^2 + \beta_0^2]} \tag{25}$$

$$\dot{y} = \frac{-j\eta\alpha(j\beta - \Gamma)E_z}{u_0[(j\beta - \Gamma)^2 + \beta_0^2]}. \tag{26}$$

We will have for q_y

$$q_y = -I_0\frac{\dot{y}}{u_0} \tag{27}$$

$$q_y = \frac{j\eta\alpha I_0(j\beta - \Gamma)E_z}{u_0^2[(j\beta - \Gamma)^2 + \beta_0^2]}. \tag{28}$$

It is easily shown that

$$\dot{z} = \frac{\eta E_z}{u_0(j\beta - \Gamma)}. \tag{29}$$

If ρ_0 is the d-c linear charge density and ρ the a-c linear charge density

$$\rho_0 = -\frac{I_0}{u_0}. \tag{30}$$

If q_z is the z component of convention current, we have

$$\begin{aligned} q_z &= \rho_0 \dot{z} + u_0 \rho \\ &= -\frac{I_0 \dot{z}}{u_0} + u_0 \rho. \end{aligned} \tag{31}$$

We also have

$$\begin{aligned} \frac{\partial q_z}{\partial z} &= -\frac{\partial \rho}{\partial t} \\ \Gamma q_z &= j\omega\rho. \end{aligned} \tag{32}$$

From (31) and (32) we obtain

$$q_z = \frac{-j\beta I_0 \dot{z}}{(j\beta - \Gamma)}. \tag{33}$$

Thus

$$q_z = \frac{j\eta\beta I_0 E_z}{u_0^2(j\beta - \Gamma)^2}. \tag{34}$$

3. Combined Equation

Combining (34), (28) and (25) with (15), we obtain

$$1 = \frac{\eta I_0}{u_0^2} \frac{\Gamma_0}{\psi_0(\Gamma^2 - \Gamma_0^2)} \left[\frac{j\beta}{(j\beta - \Gamma)_0^2} - \frac{\alpha^2(\Gamma^* - (j\beta - \Gamma))}{[(j\beta - \Gamma) + \beta_0^2]} \right]. \tag{35}$$

We now introduce new parameters

$$K = \frac{1}{\beta^2 \psi_0} = \frac{E_z E_z^*}{2\beta^2 P} \tag{36}$$

$$C^3 = \left(\frac{K I_0}{4 V_0}\right). \tag{37}$$

Here P_0 is the power transmitted by the circuit for a field strength E_z. K has the dimensions of impedance. V_0 is the voltage specifying the electron velocity u_0

$$u_0^2 = 2\eta V_0. \tag{38}$$

From (36)–(38) and (35) we see

$$1 = \frac{2\beta^2 C^3 \Gamma_0}{(\Gamma^2 - \Gamma_0^2)} \left[\frac{j\beta}{(j\beta - \Gamma)^2} - \frac{\alpha^2(\Gamma^* - (j\beta - \Gamma))}{[(j\beta - \Gamma)^2 + \beta_0^2]} \right]. \tag{39}$$

We now make the approximation that

$$-\Gamma = -j\beta + \delta. \tag{40}$$

Where $|\delta| \ll |\beta|$. Neglecting higher order terms,

$$\frac{\Gamma^2 - \Gamma_0^2}{\Gamma_0} = 2j\beta^3 C^3 \left[\frac{1}{\delta^2} + \frac{\alpha^2}{\delta^2 + \beta_0^2}\right]. \tag{41}$$

4. Purely Transverse Field Along the Path

We can imagine a case in which α approaches infinity and the quantity

$$D^3 = \alpha^2 C^3 = \frac{E_y E_y^*}{\beta^2 P} \frac{I_0}{8V_0} \tag{42}$$

remains finite. In this case we have

$$\frac{\Gamma^2 - \Gamma_0^2}{\Gamma_0} = \frac{2j\beta^3 D^3}{\delta^2 + \beta_0^2}. \tag{43}$$

We will let

$$-\Gamma_0 = -j\beta - j\beta D b - \beta D d. \tag{44}$$

Here b is a parameter describing the difference in speed between the electrons and the unperturbed wave and d is a loss parameter.

Assuming $bD \ll 1$ and $dD \ll 1$, and letting $\beta D(x + jy) = \delta$, (45)

we find

$$(x^2 - y^2 + f^2)(y + b) + 2xy(x + d) = -1 \tag{46}$$

$$(x^2 - y^2 + f^2)(x + d) - 2xy(y + b) = 0 \tag{47}$$

where

$$f^2 = \frac{\beta_0^2}{\beta^2 D^2}. \tag{48}$$

If would be difficult to work with all of the parameters b, f and d. However, it scarcely seems that the attenuation parameter d should enter into any unusual phenomena due to the presence of the magnetic field. Accordingly, let us investigate (46) and (47) for $d = 0$. We then obtain

$$x^2(3y + b) + (f^2 - y^2)(y + b) = -1 \tag{46a}$$

$$x[x^2 + (f^2 - y^2) - 2y(y + b)] = 0. \tag{47a}$$

From the $x = 0$ solution of (47a) we obtain

$$x = 0 \tag{49}$$

$$b = \frac{1}{y^2 - f^2} - y. \tag{50}$$

If is found that this solution obtains for large and small values of b. For very large and very small values of b, either $y \doteq -b$ (51) or $y \doteq \pm f$ (52) The wave given by (50) is a *circuit* wave; that given by (51) represents the travel down the tube of electrons oscillating in the magnetic field with cyclotron frequency.

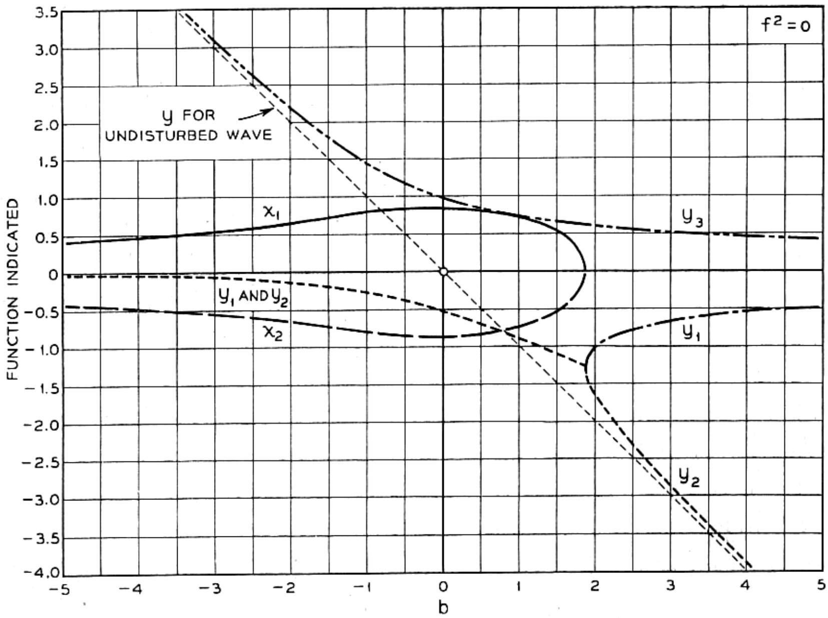

Fig. 1—Plot of parameters giving velocity and attenuation of the three forward waves vs. *a* parameter b proportional to electron speed with respect to the undisturbed wave. A positive value of x means an increasing wave; a positive value of y means a wave traveling faster than the electrons. This plot is for $f^2 = 0$ (no magnetic field).

In an intermediate range of b, we have from (47a)

$$x = \pm\sqrt{2y(y + b) - (f^2 - y^2)} \tag{53}$$

and

$$b = -2y \pm \sqrt{f^2 - 1/2y}. \tag{54}$$

For a given value of f^2 we can assume values of y and obtain values of b. Then, x can be obtained from (52) or (53). In Figs. 1–6, x and y are plotted vs. b for $f^2 = 0, .5, 1, 2, 4$ and 10. It should be noted that x_1, the parameter expressing the rate of increase of the increasing wave, has a maximum at larger values of b as f is increased (as the magnetic focusing field is increased). Thus, for higher magnetic focusing fields the electrons must be shot into the

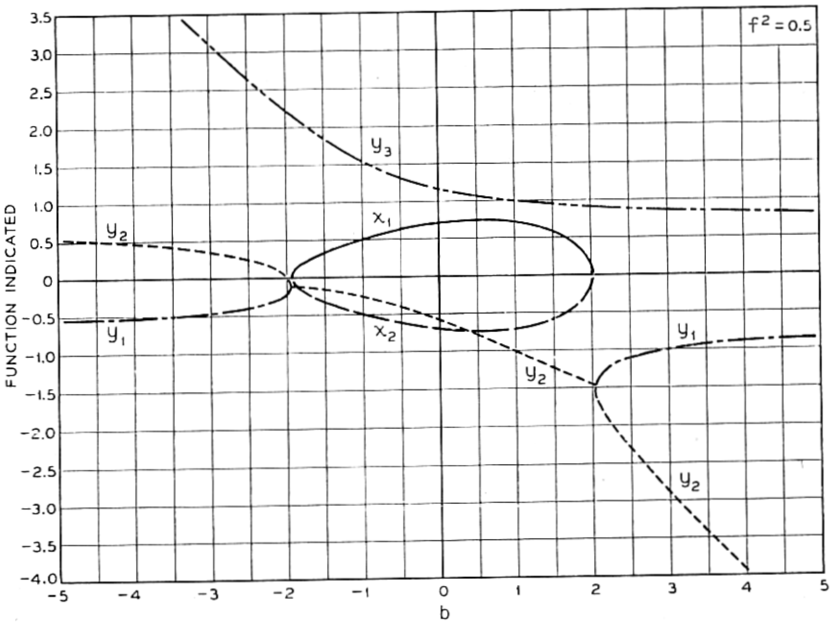

Fig. 2—Propagation parameters for $f^2 = 0.5$.

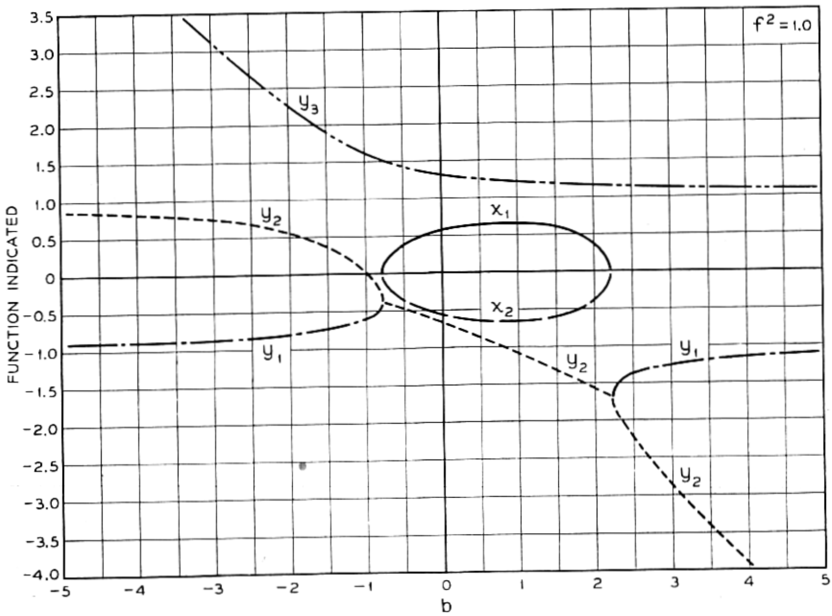

Fig. 3—Propagation parameters for $f^2 = 1.0$.

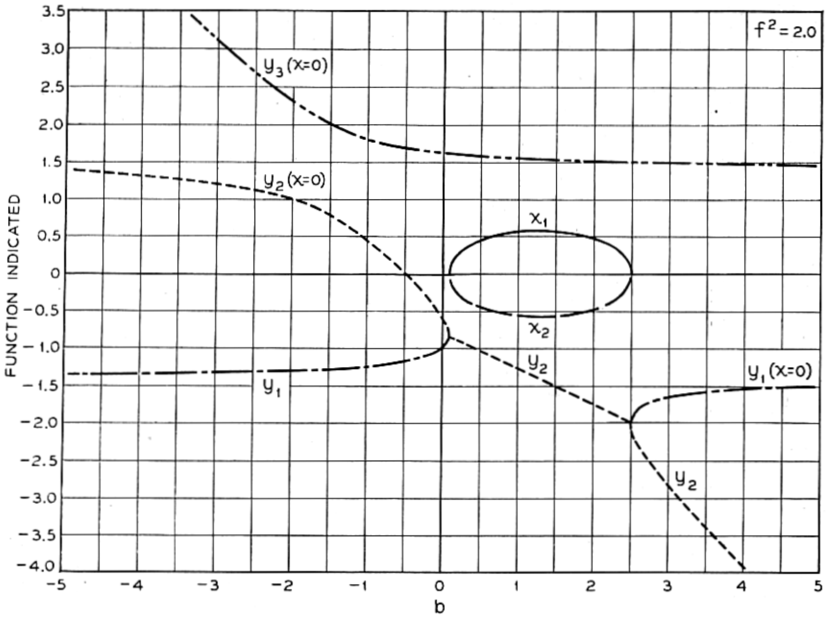

Fig. 4—Propagation parameters for $f^2 = 2.0$.

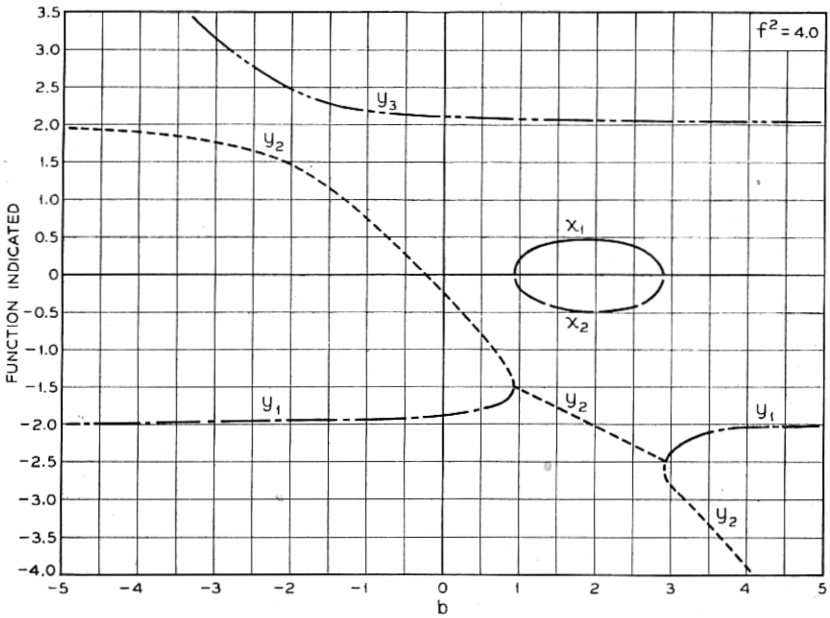

Fig. 5—Propagation parameters for $f^2 = 4.0$.

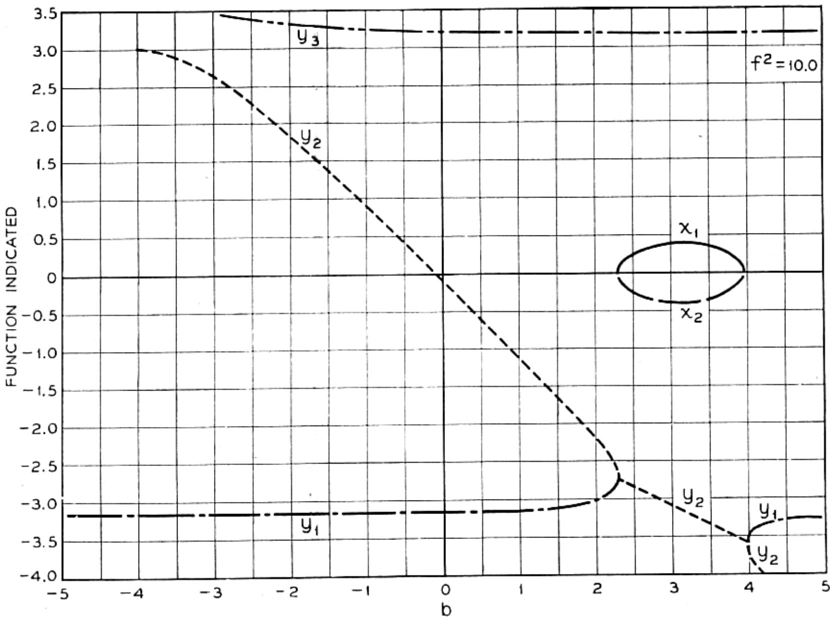

Fig. 6—Propagation parameters for $f = 10.0$.

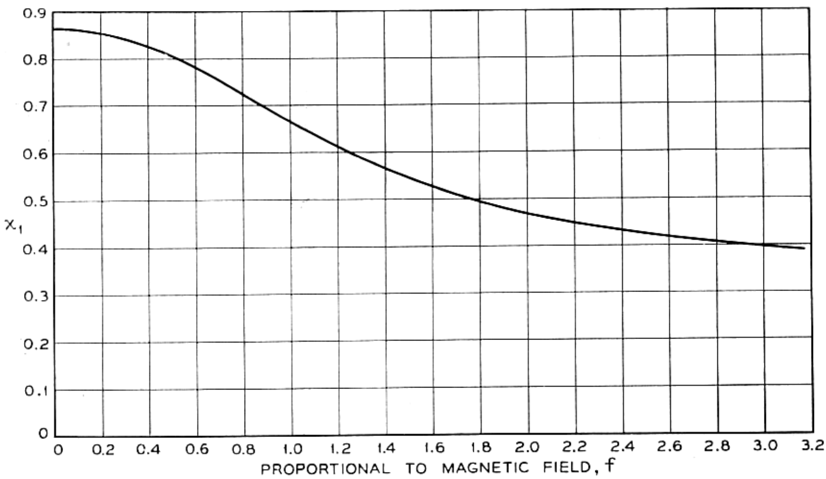

Fig. 7—Parameter x giving rate of increase of increasing wave vs. f, which is proportional to magnetic field strength.

circuit faster to get optimum results than for low fields. In Fig. 7, the maximum positive value of x is plotted vs. f. The plot serves to illustrate the effect on gain of increasing the magnetic field.

Let us consider an example. Suppose

$$\lambda = 7.5 \text{ cm}$$

$$D = .03$$

These values are chosen because there is a longitudinal field tube which operates at 7.5 cm with a value of C (which corresponds to D) of about .03. The table below shows the ratio of the maximum value of x_1 to the maximum value of x_1 for no magnetic field.

Magnetic Field in Gauss	f	x_1/x_{10}
0	0	1
50	1.17	.71
100	2.34	.50

A field of 50 to 100 gauss should be sufficient to give useful focusing action. Thus, it may be desirable to use magnetic focusing fields in deflection traveling wave tubes. This will be more especially true in low-voltage tubes, for which D may be expected to be higher than .03.

5. Mixed Fields

In tubes designed for use with longitudinal fields, the transverse fields far off the axis approach in strength the longitudinal fields. The same is true of transverse field tubes far off the axis. Thus, it is of interest to consider equation (41) for cases in which α is neither very small nor very large, but rather is of the order of unity.

If the magnetic field is very intense so that β_0^2 is large, then the term containing α^2, which represents the effect of transverse fields, will be very small and the tube will behave much as if the transverse fields were absent.

Consideration of both terms presents considerable difficulty as (41) leads to 5 waves (5 values of δ) instead of 3. The writer has attacked the problem only for the special case of $b = d = 0$. In this case we obtain from (41)

$$\delta = -j\beta^3 C^3 \left[\frac{1}{\delta^2} + \frac{\alpha^2}{\delta^2 + \beta_0^2} \right]. \tag{55}$$

In work which is given as an appendix, Dr. L. A. MacColl has shown that the two "new" waves (waves introduced when $\alpha = 0$) are unattenuated and thus unimportant and uninteresting (unless, as an off-chance, they have some drastic effect in fitting the boundary conditions).

Proceeding from this information, we will find the change in δ as β_0^2 is increased from zero. From (51) we obtain

$$d\delta = -j\beta^3 C^3 \left[-\frac{2d\delta}{\delta^3} - \frac{2\alpha^2 \delta d\delta}{(\delta^2 + \beta_0^2)^2} - \frac{2\alpha^2 d\beta_0^2}{(\delta^2 + \beta_0^2)^2} \right]. \tag{56}$$

Now, if $\beta_0 = 0$

$$-j\beta^3 C^3 = \frac{\delta^3}{(1 + \alpha^2)}. \tag{57}$$

Using this in connection with (56) for $\beta_0 = 0$ we obtain

$$d\delta = -\frac{2\alpha^2}{3\delta}\, d\beta_0^2. \tag{58}$$

For the increasing wave

$$\delta_1 = \beta(.866 - j.5). \tag{59}$$

Hence, for this wave

$$d\delta_1 = \tfrac{2}{9}(-.866 + j.5)\,\frac{d\beta_0^2}{\beta}. \tag{60}$$

This shows that applying a small magnetic field tends to decrease the gain. This does not mean, however, that the gain with a longitudinal and a transverse field and a magnetic field is less than the gain with the longitudinal field alone. To see this, we can assume that not β_0 but α^2 is small. Differentiating (55) we obtain

$$d\delta = -j\beta^3 C^3 \left[\frac{-2d\delta}{\delta^3} - \frac{2\alpha^2\delta d\delta}{(\delta^2 + \beta_0^2)^2} - \frac{d\alpha^2}{(\delta^2 + \beta_0^2)}\right]. \tag{61}$$

If $\alpha = 0$

$$-j\beta^3 C^3 = \delta^3 \tag{62}$$

$$d\delta = \frac{1}{3}\,\frac{\delta^3\, d\alpha^2}{(\delta^2 + \beta_0^2)}. \tag{63}$$

If β_0^2 is zero, a small transverse field (small increase in α^2) increases the magnitude of δ without changing the phase angle. If $\beta_0^2 \gg |\delta|$, then

$$d\delta = \frac{-j\beta^3 C^3}{\beta_0^2}\, d\alpha^2 \tag{64}$$

and the change in δ is purely imaginary. For the increasing wave, the change in δ as a transverse field is added will range from an increase in the real part for small magnetic fields to no change in the real part for large magnetic fields.

APPENDIX

Study of The Algebraic Equation

$$\delta = -j\beta^3 C^3 \left[\frac{1}{\delta^2} + \frac{\alpha^2}{\delta^2 + \beta_0^2}\right] \tag{1A}$$

$$\delta^3(\delta^2 + \beta_0^2) + j\beta^3 C^3(\delta^2 + \beta_0^2 + \alpha^2\delta^2) = 0$$

$$\delta^5 + \beta_0^2\delta^3 + j\beta^3 C^3(1 + \alpha^2)\delta^2 + j\beta_0^2\beta^3 C^3 = 0$$

$$\left(\frac{\delta}{\beta_0}\right)^5 + \left(\frac{\delta}{\beta_0}\right)^3 + j\left(\frac{\beta}{\beta_0}\right)^3 C^3(1 + \alpha^2)\left(\frac{\delta}{\beta_0}\right)^2 + j\left(\frac{\beta}{\beta_0}\right)^3 C^3 = 0 \quad (2A)$$

Write

$$\frac{\delta}{\beta_0} = z$$

$$\left(\frac{\beta}{\beta_0}\right)^3 C^3 = a \quad (3A)$$

$$\left(\frac{\beta}{\beta_0}\right)^3 C^3 a^2 = b$$

Then

$$z^5 + z^3 + j(a + b)z^2 + ja = 0 \quad (4A)$$

a is assumed to be positive, and b is assumed to be real and non-negative. For $b = 0$ we have

$$(z^3 + ja)(z^2 + 1) = 0 \quad (5A)$$

$$z = j,\ -j,\ ja^{1/3},\ ja^{1/3}e^{2\pi j/3},\ ja^{1/3}e^{4\pi j/3} \quad (6A)$$

We have

$$[5z^4 + 3^2z + 2j(a + b)z]\frac{\partial z}{\partial b} + jz^2 = 0$$

$$\frac{\partial z}{\partial b} = \frac{-jz^2}{5z^4 + 3z^2 + 2j(a + b)z} \quad (7A)$$

From this we draw the following conclusion. Suppose that for a certain value of b the five roots are distinct, and that among them there is a purely imaginary root. Then as b varies, in the neighborhood of its initial value, that root remains purely imaginary.

In particular, consider b as increasing from the initial value 0. As long as the five roots remain distinct, there are exactly three purely imaginary roots.

In order to have a real root $z = x$, we would have to have simultaneously

$$x^5 + x^3 = 0$$

$$(a + b)x^2 + a = 0 \quad (8A)$$

This is impossible (with $a > 0$). Hence there is never a real root.

In particular, as b increases from 0, no root can cross the real axis. Hence, as b increases from 0, as long as the roots remain distinct, there are two purely imaginary roots above the real axis, one purely imaginary root below the real axis, and two complex roots below the real axis.

Since there is no term in z^4 in the equation, the sum of all the roots is 0. Hence the two complex roots must be located symmetrically with respect to the imaginary axis.

First order variations of the roots with b can be calculated at once by means of the equation

$$\frac{\partial z}{\partial b} = \frac{-jz^2}{5z^4 + 3z^2 + 2j(a + b)z} \tag{9A}$$

In principle, higher-order variations can be calculated by carrying the differentiation to higher orders. However, the formulae get wonderfully complicated.

A very practical way of solving the equation is the following:

The three imaginary roots can be found by plotting a curve. If we let $z = jy$, (4A) becomes

$$y^5 - y^3 + (a + b)y^2 - a = 0 \tag{10A}$$

For the imaginary roots y is real and we have merely to plot the left-hand side of (10A) vs. y to find the roots. Denote them by z_1, z_2, z_3, which are now regarded as known numbers. These roots satisfy the equation

$$(z - z_1)(z - z_2)(z - z_3) \equiv z^3 - (z_1 + z_2 + z_3)z^2 + (z_1z_2 + z_1z_3 + z_2z_3)z$$
$$- z_1z_2z_3 \equiv z^3 + \alpha_1z^2 + \alpha_2z + \alpha_3 = 0 \tag{11A}$$

The two complex roots satisfy some equation

$$z^2 + \beta_1z + \beta_2 = 0 \tag{12A}$$

The β's are at present unknown. When we find them we can at once calculate the complex roots. We must have

$$(z^3 + \alpha_1z^2 + \alpha_2z + \alpha_3)(z^2 + \beta_1z + \beta_2) \equiv z^5 + z^3 + j(a + b)z^2 + ja \tag{13A}$$

Comparing the coefficients of z^4 and z^0, we get the equations

$$\alpha_1 + \beta_1 = 0$$
$$\alpha_3\beta_2 = ja \tag{14A}$$

which give us the β's.

Suppose that the magnetic field is very small, so that $\beta_0 \ll \beta$. Then unless α is very small, both a and b in (10A) will be very large numbers, and we find that two of the imaginary roots are given approximately by

$$y = \pm\left(\frac{a}{a + b}\right)^{1/2} \tag{15A}$$

As $\beta_0 \to 0$ the other three roots are given by

$$z = [-j(a + b)]^{1/3} \tag{16A}$$

These three roots correspond to the waves found for traveling-wave tubes with a purely longitudinal field. The roots according to (15A) represent such a combination of deflection and bunching as to produce no induced current in the circuit. The roots of (15A) are "extra" roots attributable to the consideration of transverse fields and transverse electron motion.

For the roots given by (15A), $\delta/\beta \to 0$ as $\beta_0 \to 0$. Thus in this case it is convenient to form the solution of two parts, one varying as

$$e^{-j\beta z} \sin\left(\frac{a\beta_0}{a + b}\right) z$$

and the other varying as

$$e^{-j\beta z} \cos\left(\frac{a\beta_0}{a + b}\right) z$$

As $\beta_0 \to 0$, the first of these approaches the form

$$ze^{-j\beta z}$$

and the second approaches the form

$$e^{-j\beta z}$$

Again, these "extra" waves produce no induced current in the circuit.

Two additional pieces of information:

As $a \to 0$, b a remaining fixed, the roots approach the limiting values

$$0, 0, 0, j, -j.$$

As $a \to \infty$, ba remaining fixed, two of the roots approach the limiting values

$$\pm j \sqrt{\frac{a}{a + b}},$$

the other roots behave as

$$j(a + b)^{1/3}, j(a + b)^{1/3} e^{2\pi j/3}, j(a + b)^{1/3} e^{4\pi j/3}$$

Much of the preceding discussion depends upon the roots remaining distinct. The condition that two or more of the roots coincide, which is a relation between a and b, can be written out, but it has not as yet been reduced to a compact and intelligible form.