Transient Response of an FM Receiver

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Introduction

THIS paper develops various formulas for the response of an FM receiver to signal or noise input voltages of arbitrary form. The principal object in view is to obtain a more complete understanding of how an FM receiver responds to transient voltages, such as those arising from ignition interference, but the more general aspects of the theory have other applications as well. In particular, general formulas are given for the response of a linear circuit to an applied voltage, or current, of variable frequency. The Fourier transforms, or frequency spectra, of the response, and the envelope thereof, are determined.

Two examples are given: (1) the audio response of an FM receiver to a very large impulse and (2) the response, including harmonic distortion, to a sinusoidal signal wave.

The element of an FM receiver that demands most discussion is the balanced frequency detector. The greater part of the paper accordingly deals with that important element. The general problem can be stated as follows: A limiter and frequency detector are transmitting a steady unmodulated carrier wave to an audio output circuit. At time, $t = 0$, frequency modulation of arbitrary form is applied to the carrier (either by signal modulation or a superposed noise transient). What is the audio output voltage that results?

Frequency Detector

Except for the greater bandwidth, the amplifiers and selective circuits between the antenna and the limiter of an FM receiver are similar to those of an AM receiver in their transmission features. If the selective circuits have a bandwidth ample to accommodate the maximum frequency swing of the FM transmitter, and if the transmission over the band is substantially "flat" and the phase shift nearly linear with frequency, the amplifiers will introduce little distortion. The limiter and frequency detector are therefore regarded as the distinctive elements of an FM receiver meriting theoretical discussion.

The literature contains descriptions of frequency detectors of several types together with adequate analyses of the action of the circuits based on the variable impedance concept.[1] The more generally used circuits can be

[1] See Items 3 to 6 in list of references attached.

reduced to the circuit shown schematically in Fig. 1, which can be taken to illustrate a generic form of frequency detector. Z_1 and Z_2 are two resonant impedances tuned to different frequencies, one above, the other below, the carrier frequency.[2] For example, the simplest version of Z_1 and Z_2 could be, for each, a parallel combination of R, L and C. Across each of these impedances is connected a rectifier with load circuit so proportioned that the rectification is substantially linear. The rectifiers are poled so that their low-frequency outputs are opposed, thereby obtaining cancellation of even-order demodulation products. With this arrangement, the low-frequency output voltage V_o, which is applied to the audio amplifier, is substantially proportional to the difference between the envelopes of the voltage drops across Z_1 and Z_2.

The resistance elements of the impedances, Z_1 and Z_2, each include a shunting resistance equal to half the load resistance of the associated rectifier, which therefore determines, to some extent, the Q of the tuned circuit. The output diode load, $R_o\,C_o$, has negligible impedance at the carrier frequency. Under these conditions, the low-frequency output voltage across the two-rectifier load impedances is

$$V_o = \eta\,([V_1] - [V_2])$$

where η = detection efficiency (nearly unity)
V_1, V_2 = high-frequency voltages across Z_1, Z_2, respectively (Fig. 1)
$[V]$ = envelope of V.

All this is in accord with the accepted understanding of the operation of a properly designed linear rectifier working at an efficiency approaching 100 per cent.

The amplitudes of the voltages, V_1 and V_2, across the resonant impedances, Z_1 and Z_2, of Fig. 1 are shown in Fig. 2. In the practical engineering analysis of this frequency detector circuit, employing the idea of impedance that varies in step with the instantaneous frequency, the two voltages of Fig. 2 are subtracted (owing to the opposed polarities of the rectifiers) to obtain the over-all voltage-frequency characteristic shown in Fig. 3. Then it is inferred, by physical intuition, that if the instantaneous frequency of the carrier is varied at the input, the output voltage wave will vary as indicated by the curve of Fig. 3. Strictly speaking, this is a false assumption, but where the rate of variation of the instantaneous frequency is at an audio signal frequency far below the carrier frequency, the error in the assumption is of no importance, whereas the simplification in thinking accomplished

[2] The term *carrier frequency* will be used to designate the value of the unmodulated received frequency after all heterodyne conversions. (This frequency is equal to the mid-band frequency of the last intermediate frequency amplifier ahead of the limiter, if tuning is perfect.)

by it is considerable. It is only where the rate of variation of the instantaneous frequency is high, as it can be in the case of a large noise transient caused by impulse excitation, that the error in the assumption in question

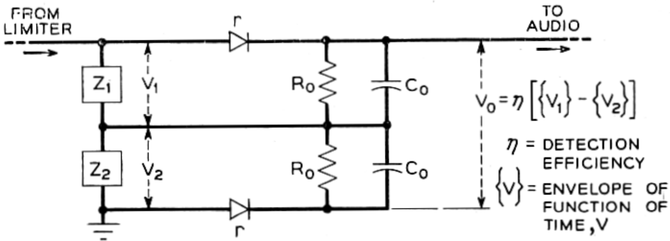

Fig. 1—Circuit of a balanced frequency detector.

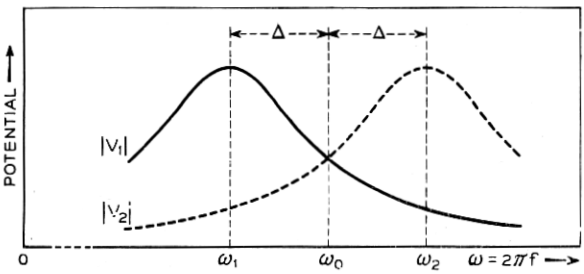

Fig. 2—Voltages across tuned circuits of frequency detector.

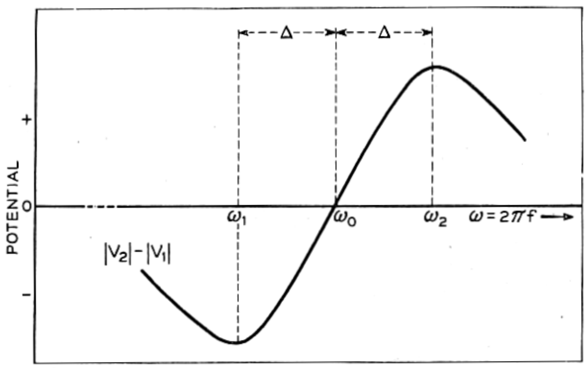

Fig. 3—Output voltage of frequency detector.

can become serious. A particularly subtle error that can arise from the assumption is to fall into the habit of regarding the characteristic curve of Fig. 3 as a frequency transmission curve of the sort obtained by measuring the ratio of output to input of a linear network over a range of frequencies.

The curve of Fig. 3 is not such a transmission curve, because the principle of superposition does not apply and a frequency conversion is involved.

Owing to the considerations discussed above, the analysis to follow avoids the assumption of variable impedance associated with the varying instantaneous frequency. This does not imply that the assumption, as employed by various writers, is considered seriously erroneous, but, rather, that it seems preferable to develop the theory without invoking the assumption, provided that this can be done without falling into unmanageable complications. Briefly, the procedure in the work to follow is to determine directly the envelopes of the voltages V_1 and V_2 as functions of time, one envelope then being subtracted from the other to obtain the output wave.

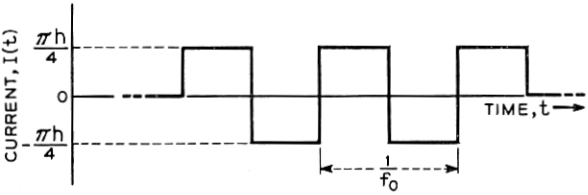

Fig. 4—Current wave out of limiter.

General Theory

When a carrier is being received, the limiter can be regarded as substantially a constant current source having an internal shunt admittance small compared to the admittance of the tuned impedance elements, Z_1 and Z_2, of the frequency detector. If the limiting is severe, as it should be for good operation, the current delivered by the limiter is a rectangular wave as illustrated in Fig. 4. When this current is driven through the impedances, Z_1 and Z_2, the voltage drops, V_1 and V_2, that arise across these elements are substantially sinusoidal in form, owing to the selectivity, which practically extinguishes all harmonics of the carrier frequency. We therefore take the current input to be sinusoidal in the first place, namely

$$I(t) = h \cos 2\pi f_0 t$$

This is the unmodulated current, $\pi h/4$ being the current cutoff point of the limiter and f_0 the frequency of the carrier. When the carrier is modulated in frequency, we write

$$I(t) = h \cos [2\pi f_0 t + \theta(t)] \tag{1}$$

where $\theta(t)$ is the phase angle varying with time. The instantaneous frequency then is

$$f(t) = \frac{1}{2\pi}\frac{d}{dt}[2\pi f_0 t + \theta(t)] = f_0 + \frac{1}{2\pi}\theta'(t). \tag{2}$$

In the transmission of signals by frequency modulation, the instantaneous radian frequency deviation, $\theta'(t)$, is made to vary in proportion to the signal amplitude, so that $\theta(t)$ then varies in proportion to the time integral of the signal amplitude.

As a preliminary step to the discussion of the frequency detector itself, we require a formula for the voltage drop across an impedance $Z(f)$ when the frequency-modulated current (1) flows through it. The point of view usually adopted is to regard the impedance as a composite function of time, viz., $Z[f(t)]$, and to say that the voltage across it is

$$V(t) = I(t)Z[f(t)] = I(t)Z\left[f_0 + \frac{1}{2\pi}\theta'(t)\right]. \quad (3)$$

This quasi-stationary viewpoint gives results that are nearly correct if the rate of change, $\theta''(t)/2\pi$, of the variable frequency is not too large. The magnitude of the error has been determined in a paper by Carson and Fry.[3] In the present paper, impedance is a function of frequency that is independent of time, as in the classic theory of linear systems. The frequent use of the term "instantaneous frequency," as defined by (2), does not imply a departure from this point of view.

In the following, $H(t)$ is, in general, the voltage response as a function of time, of a network to a unit impulse of current applied at time $t = 0$. In the case of a two-terminal impedance element, $H(t)$ is the voltage drop across the element when a unit impulse of current is sent through it. Then, if the frequency modulated current (1) flow through the impedance, the voltage drop is

$$V(t) = h\int_0^\infty \cos\left[\omega_0(t-\tau) + \theta(t-\tau)\right]H(\tau)\,d\tau \quad (4)$$

where $\omega_0 = 2\pi f_0$. $\theta(t)$ can have any form as a function of time. $V(t)$ can be written,

$$V(t) = \frac{h}{2}e^{i\omega_0 t}\int_0 e^{-i\omega_0\tau + i\theta(t-\tau)}H(\tau)\,d\tau + \frac{h}{2}e^{-i\omega_0 t}\int_0 e^{i\omega_0\tau - i\theta(t-\tau)}H(\tau)\,d\tau. \quad (5)$$

In the frequency detector problem, the result finally desired is the envelope of the voltage wave. It will clarify the discussion to explain first what is meant by an envelope. If the voltage is of the form

$$\begin{aligned} V(t) &= c(t)\cos\left[\omega_0 t + \phi(t)\right] \\ &= a(t)\cos\omega_0 t - b(t)\sin\omega_0 t \\ &= \tfrac{1}{2}[a(t) + ib(t)]\,e^{i\omega_0 t} + \tfrac{1}{2}[a(t) - ib(t)]\,e^{-i\omega_0 t} \end{aligned} \quad (6)$$

[3] Item 1 in the bibliography. See formula 21 in that paper.

the complex function,

$$[V(t)] = c(t)\, e^{i\phi(t)} = a(t) + ib(t) \tag{7}$$

is here called the "envelope function" of the voltage with respect to the radian frequency ω_0, $c(t)$ being a real amplitude modulation factor, which is the envelope[4] itself, as usually conceived, and exp $[i\phi(t)]$ a complex frequency modulation factor, in which $\phi'(t)$ is the instantaneous deviation of the radian frequency from the reference value, ω_0. If such a modulated voltage wave is applied to an ideal linear detector, the output voltage across the load circuit of the latter is the real envelope, $c(t) = [a^2(t) + b^2(t)]^{1/2}$. This concept of an envelope function provides a convenient generalization of modulation ideas. Both amplitude modulation and frequency modulation vary the envelope function, but in different ways. In amplitude modulation, the real magnitude, $c(t)$, is varied while the angle ϕ is constant, whereas, in frequency modulation, c is constant and it is the angle, $\phi(t)$, that is varied.

It will be seen that (5) is in precisely the same form as (6), so that we can write the envelope function of $V(t)$ immediately, as follows:

$$[V(t)] = a(t) + ib(t) = h \int_0^\infty e^{-i\omega_0\tau + i\theta(t-\tau)}\, H(\tau)\, d\tau. \tag{8}$$

The conjugate envelope function then is

$$[\overline{V(t)}] = a(t) - ib(t) = h \int_0^\infty e^{i\omega_0\tau - i\theta(t-\tau)}\, H(\tau)\, d\tau. \tag{9}$$

The spectrum of the envelope function is also of interest. To obtain the spectrum, which we shall call, $F_0(f)$, we find the Fourier transform (hereafter abbreviated, F.T.) of both sides of (8), viz.:

$$F_0(f) = \int_{-\infty}^{\infty} [V(t)]e^{-i\omega t}\, dt = h \int_{-\infty}^{\infty} e^{-i\omega t} \int_0^\infty e^{-i\omega_0\tau + i\theta(t-\tau)} \cdot H(\tau)\, d\tau\, dt. \tag{10}$$

It is permissible to reverse the order of integration of τ and t, obtaining

$$F_0(f) = h \int_0^\infty e^{-i\omega_0\tau} H(\tau) \int_{-\infty}^{\infty} e^{-i\omega t + i\theta(t-\tau)}\, dt\, d\tau. \tag{11}$$

The F.T. of h exp $[i\theta(t)]$ will be designated, $\Psi(f)$, i.e.

$$\Psi(f) = h \int_{-\infty}^{\infty} e^{i\theta(t) - i\omega t}\, dt. \tag{12}$$

[4] The "envelope", so defined, is an engineering concept and is not quite the same thing as the envelope of mathematics, which is always tangent to a curve or set of curves.

Putting $t - \tau$ in place of t in place of t as the variable of integration in (12) we have

$$\Psi(f) = he^{i\omega\tau} \int_{-\infty}^{\infty} e^{-i\omega t + i\theta(t-\tau)}\, dt. \tag{13}$$

Thus it is seen that the inner integral of (11) is equal to $e^{-i\omega\tau}\Psi(f)$ and the equation becomes

$$F_0(f) = \Psi(f) \int_0^{\infty} H(\tau) e^{-i(\omega+\omega_0)\tau}\, d\tau. \tag{14}$$

Now the F.T. of $H(t)$ is $Z(f)$, i.e.

$$Z(f) = \int_{-\infty}^{\infty} H(t) e^{-i\omega t}\, dt. \tag{15}$$

Therefore

$$Z(f + f_0) = \int_{-\infty}^{\infty} H(t) e^{-i(\omega+\omega_0)t}\, dt. \tag{16}$$

This differs from the integral in (14) only in the lower limit of integration. But since $H(t)$ is the response to an impulse applied at time $t = 0$, $H(t) = 0$ for $t < 0$ and the two integrals are therefore equal. Putting (16) in (14) we have, finally

$$F_0(f) = \int_{-\infty}^{\infty} [a(t) + ib(t)] e^{-i\omega t}\, dt = \Psi(f) Z(f + f_0). \tag{17}$$

The F.T. of the conjugate envelope function, $a - ib$, is

$$\bar{F}_0(-f) = \int_{-\infty}^{\infty} [a(t) - ib(t)] e^{-i\omega t}\, dt = \bar{\Psi}(-f)\bar{Z}(-f + f_0) \tag{18}$$

where symbols with the superbar denote the complex conjugates of unbarred symbols. Since $Z(f)$ is the F.T. of a real variable, $H(t)$, it must assume conjugate values for positive and negative values of f, i.e., $Z(f) = \bar{Z}(-f)$ and therefore $Z(f + f_0) = \bar{Z}(-f + f_0)$. Consequently, (18) could be written

$$\bar{F}_0(-f) = \bar{\Psi}(-f) Z(f + f_0) \tag{19}$$

(17) and (18) are the final solutions in frequency functions corresponding to the solutions (8) and (9) in time functions. The formulas in frequency functions have the advantage of compactness, which makes them easy to remember.

We require also the F.T. of the voltage itself, $V(t)$, which we shall call $F(f)$. From (6)

$$F(f) = \int_{-\infty}^{\infty} V(t) e^{-i\omega t}\, dt = \tfrac{1}{2} \int_{-\infty}^{\infty} (a + ib) e^{-i(\omega-\omega_0)t} dt + \tfrac{1}{2} \int_{-\infty}^{\infty} (a - ib) e^{-i(\omega+\omega_0)} d^t t \tag{20}$$

and from (17) and (18) this evidently is

$$F(f) = \tfrac{1}{2}\Psi(f - f_0)Z(f) + \tfrac{1}{2}\bar{\Psi}(-f - f_0)\bar{Z}(-f) \tag{21}$$

or, since $Z(f) = \bar{Z}(-f)$

$$F(f) = \tfrac{1}{2}Z(f)\,[\Psi(f - f_0) + \bar{\Psi}(-f - f_0)]. \tag{22}$$

Anyone familiar with the rules of Fourier transforms could write down this frequency function in the first place and then proceed to find the time functions by the reverse of the process just carried out. But the time functions are more closely related to the physics of the problem and therefore provide a more fundamental starting point for its solution.

It will be appreciated that, although the above discussion has been phrased to apply to the problem of finding the voltage drop across an impedance when a frequency modulated current flows through it, the formulas also give the current through an admittance when a frequency-modulated voltage is applied across it. They also give the output voltage or current of a four-terminal network when a frequency-modulated current or voltage is applied at the input. These various applications of the formulas obviously can be made by placing definitions on Z and H appropriate to the particular problem.

The next step is to assume suitable values of the impedance $Z(f)$ and various forms of the frequency modulation function θ and to employ these particular values in the general formulas 8, 9, 17 and 18.

Balanced Frequency Detector

The impedance can be defined, in general, as a rational algebraic function, viz.:

$$Z(i\omega) = \frac{(i\omega - a_1)(i\omega - a_2)\cdots(i\omega - a_m)}{(i\omega - p_1)(i\pi - p_2)\cdots(i\omega - p_n)} = \frac{P(i\omega)}{Q(i\omega)}. \tag{23}$$

Writing the polynominals P and Q in this way as the products of their factors exhibits the a's as the zeros of Z and the p's as the poles. The latter determine the frequencies of free vibration of the network. For the network to be stable, the p's must all have negative real parts. The a's and p's are either real or occur in conjugate complex pairs. By the partial fraction rule, the expression can be broken up into a series of simple fractions; thus

$$Z(i\omega) = \frac{A_1}{i\omega - p_1} + \frac{A_2}{i\omega - p_2} + \cdots. \tag{24}$$

If the poles are all simple, the A's are given by

$$A_j = \frac{P(p_j)}{Q'(p_j)}. \tag{25}$$

For the present purpose, only a pair of terms of (24) need be considered. This will provide a specific solution of the balanced frequency detector for the case, previously used for illustration, where the impedances are two simple resonant circuits of parallel R, L and C. At the same time, this solution can be extended to more complicated circuits by superposing a number of such elementary solutions, as is clearly possible with the type of impedance development indicated by (24).

The impedance of R, L and C in parallel, written in the form of (23) and (24), is

$$
\begin{aligned}
Z(i\omega) &= \frac{i\omega}{C}\,\frac{1}{(i\omega - p_1)(i\omega - p_2)} = \frac{k}{i\omega - p_1} + \frac{\bar{k}}{i\omega - p_2} \\
&= \frac{k}{i\omega + \gamma} + \frac{\bar{k}}{i\omega + \bar{\gamma}}
\end{aligned}
\tag{26}
$$

where

$$-p_1 = \gamma = \alpha + i\beta$$

$$-p_2 = \bar{\gamma} = \alpha - i\beta$$

$$k = \frac{1}{2C}\left(1 - \frac{i\alpha}{\beta}\right)$$

$$\bar{k} = \frac{1}{2C}\left(1 + \frac{i\alpha}{\beta}\right)$$

$$\alpha = \frac{1}{2RC}, \qquad \beta = \sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}} = \sqrt{\omega_c^2 - \alpha^2}, \qquad \omega_c = \frac{1}{\sqrt{LC}}.$$

The voltage response of the circuit to a unit impulse of current is then

$$H(t) = \int_{-\infty}^{\infty}\left(\frac{k}{i\omega + \gamma} + \frac{\bar{k}}{i\omega + \bar{\gamma}}\right)e^{i\omega t}\,df = ke^{-\gamma t} + \bar{k}e^{-\bar{\gamma}t}, \quad t > 0. \tag{27}$$

To find the envelope function of the voltage drop when the frequency modulated current

$$I(t) = \frac{h}{2}\left(e^{i\omega_0 t + i\theta(t)} + e^{-i\omega_0 t - i\theta(t)}\right) \tag{1}$$

is applied to the circuit, we make use of the general formula, (17), which states that the spectrum, or Fourier transform, of this envelope function is

$$F_0(f) = \int_{-\infty}^{\infty}[a(t) + ib(t)]e^{-i\omega t}\,dt = \Psi(f)Z(f + f_0) \tag{17}$$

where $\Psi(f)$ is the F.T. of $h \exp[i\theta(t)]$ and $Z(f)$ is the impedance, (26), in this case. The envelope function then is

$$
\begin{aligned}
a(t) + ib(t) &= \int_{-\infty}^{\infty} \Psi(f) Z(f + f_0) e^{i\omega t}\, df \\
&= hke^{-(i\omega_0+\gamma)t} \int_{-\infty}^{t} e^{(i\omega_0+\gamma)\tau + i\theta(\tau)}\, d\tau \qquad (28) \\
&\quad + h\bar{k}e^{-(i\omega_0+\bar{\gamma})t} \int_{-\infty}^{t} e^{(i\omega_0+\bar{\gamma})\tau + i\theta(\tau)}\, d\tau.
\end{aligned}
$$

This result is obtained by employing the convolution formula[5], which states that if F_1 and F_2 are F.T.'s of G_1 and G_2, respectively, then the F.T. of F_1F_2 is

$$\int_{-\infty}^{\infty} F_1(f)F_2(f)e^{i\omega t}\, df = \int_{-\infty}^{\infty} G_1(\tau)G_2(t - \tau)\, d\tau. \qquad (29)$$

(The upper limit of the integrals in (28) is t instead of ∞ for the reason that $H(t)$ is zero for $t < 0$.) The result could have been obtained equally well without using the Fourier transforms by substituting (27) in (8). When $\theta(t)$ is specified mathematically in the infinite interval $(-\infty, \infty)$ these formulas give the resultant of the steady state and transient oscillations. Various problems can be solved by specifying particular forms of variation for $\theta(t)$. Two examples follow: (1) where the instantaneous frequency, $\theta'(t)$, is an impulse and (2), where $\theta'(t)$ is a sinusoidal wave, as for elementary signal transmission.

Example 1: Impulse Modulation

Ignition interference comprises a sequence of sharp impulses, each of duration very brief compared to the interval between them, so that the transient in the receiver produced by one impulse dies away before the next one arrives. It is therefore sufficient to consider the disturbance caused by a single impulse.

If the receiver is perfectly tuned, an impulse produces, in the tuned circuits, a transient of the same nominal frequency[6] as the signal carrier. When superposed on the carrier, the interfering transient alters, or modulates, both the amplitude and phase of the carrier. The amplitude modulation is wiped out by the limiter, but the phase modulation remains to produce noise in the output. The phase shift caused by the transient is a random variable, because it depends upon the time of arrival of the impulse, and this is entirely fortuitous.

[5] See pair 202 of Item 10 in the bibliography.

[6] By "nominal frequency" is meant the frequency as determined by counting zeros of the wave. The transient actually comprises a spectrum of frequencies spread over the band of the tuned circuits, of course.

It is of engineering interest to determine the noise produced by a very large impulse, exceeding greatly the amplitude of the signal carrier. When such a large impulse arrives, it causes a sudden jump, or discontinuity, in the phase of the carrier. The excursion of the instantaneous frequency corresponding to the phase jump is indefinitely large and the problem accordingly cannot be solved satisfactorily by means of the usual assumption of quasi-stationary frequency. The problem of large impulsive interference provides, therefore, the principal justification for the more exact method of analysis here employed. In the paragraph following, the problem is restated in terms providing a suitable basis for mathematical analysis.

We assume, as before, that the limiter is delivering to the frequency detector a steady carrier current of constant amplitude h and frequency f_0. At time $t = 0$ a brief disturbance occurs specified by the statement that the instantaneous frequency, $\theta'(t)$, of the current suddenly executes an impulse of moment Θ. That is: $\theta'(t)$ is zero at all times except at $t = 0$, when it goes to infinity and back to zero again in such a way that the area of the impulse so formed is Θ. The carrier current amplitude then remains constant but the phase, $\theta(t)$, of the carrier takes a sudden jump of Θ radians at $t = 0$. What is the voltage output of the frequency detector?

The general formula (28) gives directly the envelope function of the voltage across the impedance (26) for a phase function $\theta(t)$. In this formula we have now to put $\theta(t) = 0$ before time $t = 0$ and Θ, after $t = 0$. We do this by dividing the interval of integration into two parts, $(-\infty, 0)$ and $(0, t)$; thus

$$\begin{aligned} a(t) + ib(t) &= hke^{-(i\omega_0+\gamma)t}\left(\int_{-\infty}^{0} e^{(i\omega_0+\gamma)\tau}\,d\tau + e^{i\Theta}\int_0^t e^{(i\omega_0+\gamma)\tau}\,d\tau\right) \\ &\quad + h\bar{k}e^{-(i\omega_0+\bar{\gamma})t}\left(\int_{-\infty}^{0} e^{(i\omega_0+\bar{\gamma})\tau}\,d\tau + e^{i\Theta}\int_0^t e^{(i\omega_0+\bar{\gamma})\tau}\,d\tau\right) \\ &= hk\,\frac{(1 - e^{i\Theta})e^{-(i\omega_0+\gamma)t} + e^{i\Theta}}{i\omega_0 + \gamma} + h\bar{k}\,\frac{(1 - e^{i\Theta})e^{-(i\omega_0+\bar{\gamma})t} + e^{i\Theta}}{i\omega_0 + \bar{\gamma}}. \end{aligned} \tag{30}$$

Let the radian frequency interval by which the applied frequency ω_0 is set off from the resonant frequency ω_c be

$$\Delta = \omega_0 - \omega_c \tag{31}$$

as indicated on the curves of Fig. 2. When α/ω_c is small compared to unity, as it is in practical circuits, β is very nearly equal to ω_c. (See the formulas following equations (26).) Then

$$i\omega_0 + \gamma = i\omega_0 + \alpha + i\omega_c = \alpha - i\Delta + 2i\omega_0$$

and

$$i\omega_0 + \bar{\gamma} = i\omega_0 + \alpha - i\omega_c = \alpha + i\Delta. \tag{32}$$

From this it is evident that the first term of (30) contains the demodulation sum product of frequency on the order of $2\omega_0$. This frequency will be suppressed by the diode load circuit and consequently the second term of (30) is an adequate representation of the envelope function. Therefore we write

$$a(t) + ib(t) = h\bar{k}\,\frac{(1 - e^{i\Theta})e^{-(i\omega_0+\bar{\gamma})t} + e^{i\Theta}}{i\omega_0 + \bar{\gamma}} \tag{33}$$

and with the above approximations this is very nearly equal to

$$a(t) + ib(t) = \frac{h}{2C}\,\frac{(1 - e^{i\Theta})e^{-(\alpha+i\Delta)t} + e^{i\Theta}}{\alpha + i\Delta}. \tag{34}$$

One deduction that can be made immediately from this formula is that the frequency of the oscillation in the output of the rectifier caused by the phase jump at the input is Δ, the radian frequency interval by which the applied carrier frequency differs from the resonant frequency. The oscillation is heavily damped, however, because α, while being very small compared to ω_c, is comparable in magnitude with Δ in circuits commonly used.

The angle of the complex envelope function (34) represents merely a phase shift of the carrier frequency ω_0. We are interested only in the magnitude of the function, viz.:

$$c(t) = [a^2(t) + b^2(t)]^{1/2}. \tag{35}$$

After some algebraic work, the desired formula comes out of (34) in the following form:

$$c(t) = \frac{h}{2C}\,\frac{\left[1 - 2m(t)\sin\left(\Delta t + \frac{\Theta}{2}\right) + m^2(t)\right]^{1/2}}{(\alpha^2 + \Delta^2)^{1/2}}, \quad t > 0$$

where

$$m(t) = 2e^{-\alpha t}\sin\frac{\Theta}{2} \tag{36}$$

The discussion so far has dealt with a single impedance (or network) and has been concerned with obtaining formulas for the voltage across the impedance, and the envelope thereof, when a frequency-modulated current is sent through it. It is necessary now to refer to the construction of the balanced frequency detector, which is the particular object of our study. Figure 1 shows two impedances having the variation with frequency sketched in Fig. 2. The carrier current is driven through the two impedances in series and linear rectifiers are connected across each in such polarity that their low-frequency output voltages are opposed. We assume that the output voltage of each rectifier is the envelope of the voltage existing across its associated impedance. Therefore, to find the total output of the balanced frequency detector, we have to find the difference between the envelopes of these voltages.

It is necessary to specify the two impedances more precisely. It appears that the best operation is obtained if the frequency of the carrier is midway between the resonant frequencies of the two impedances. That is

$$\Delta = \omega_0 - \omega_1 = \omega_2 - \omega_0$$

where ω_1, ω_2 are the resonant frequencies of Z_1, Z_2, (previously written as ω_c, for any impedance). Furthermore it appears that the two impedances should have identical values of C and very nearly the same damping constants. The design of the detector circuit is accordingly specified by

$$\begin{aligned} C_1 &= C_2 = C \\ R_1 &= R_2 = R \\ L_1 &= 1/\omega_1^2 C \\ L_2 &= 1/\omega_2^2 C \end{aligned}$$

and then

$$\begin{aligned} \alpha_1 &= \alpha_2 = \alpha = 1/2RC \\ k_1 &= k_2 = k = 1/2C \qquad (a/\omega_0 \ll 1) \\ \sqrt{\frac{L_1}{L_2}} &= \frac{\omega_2}{\omega_1} = \frac{\omega_0 + \Delta}{\omega_0 - \Delta}. \end{aligned} \tag{37}$$

All the quantities are assumed to be substantially constant over the significant frequency range.

With the circuit constants so proportioned, it can be seen from (36) that the envelope of the voltage across Z_1 differs from that across Z_2 only in the sign of Δ. Therefore, the output voltage of the balanced frequency detector, when the instantaneous frequency variation is an impulse of moment Θ at $t = 0$, is

$$\begin{aligned} V_0(t) &= e_2(t) - e_1(t) \\ &= \frac{h}{2C}(\alpha^2 + \Delta^2)^{-1/2}\left(\left[1 + 2m \sin\left(\Delta t - \frac{\Theta}{2}\right) + m^2\right]^{1/2}\right. \\ &\qquad \left. - \left[1 - 2m \sin\left(\Delta t + \frac{\Theta}{2}\right) + m^2\right]^{1/2}\right), \qquad t > 0 \\ &= 0, \qquad t < 0, \quad \text{where } m = 2e^{-\alpha t}\sin\frac{\Theta}{2}. \end{aligned} \tag{38}$$

On Fig. 5 is given a plot of this function for a value $\alpha/\Delta = 1$ of the relative damping. Calculations for other values of α/Δ show that the output oscillates only weakly for $\alpha/\Delta = \frac{1}{2}$ and is nearly dead-beat for $\alpha/\Delta = 2$

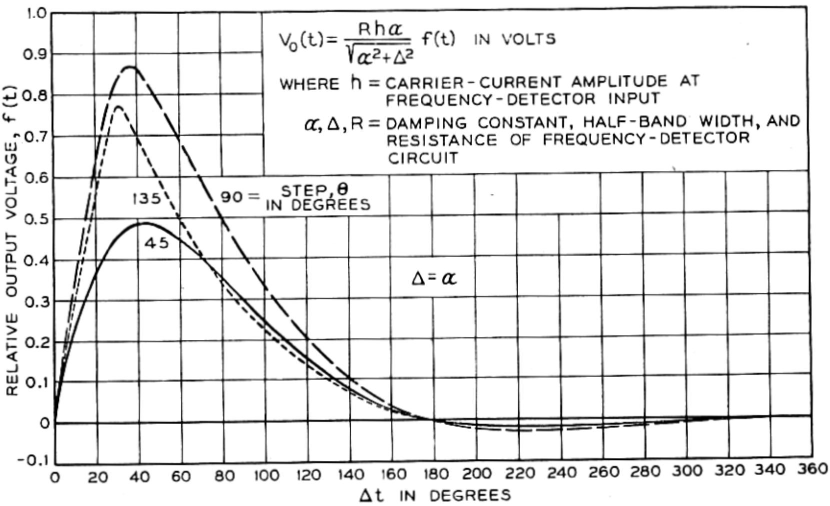

Fig. 5—Transient response of a balanced frequency detector for a step Θ in the carrier phase.

Example 2: Signal Reception

If the carrier is frequency-modulated by a signal

$$s(t) = S \cos qt \tag{39}$$

and the frequency modulation factor of the transmitter is μ, the phase of the carrier wave is made to vary in accordance with the relation

$$\theta(t) = \mu \int_{-\infty}^{t} s(t)\, dt = \frac{\mu S}{q} \sin qt = x \sin qt \tag{40}$$

μS is then the radian frequency deviation of the transmitter and $\mu S/q$, which is the ratio of this frequency deviation to the frequency of the signal, is commonly referred to as the "frequency deviation ratio." This factor, which is denoted by x in the above equation, enters as a fundamental parameter in all FM theory.

To find the envelope function of the voltage wave produced across the impedance (26), when the frequency-modulated carrier is received at the frequency detector, we put (40) in (28). To effect the integration, the expansion

$$e^{ix \sin qt} = \sum_{n=-\infty}^{\infty} J_n(x) e^{inqt} \tag{41}$$

is used, $J_n(x)$ being the Bessel coefficient of the first kind, of the nth order and with the argument x. For brevity, $J_n(x)$ will be written J_n. Also, the first term of (28) is to be omitted, because, as was shown in the preceding example, it represents frequency sum terms which are filtered out by the diode output circuit. Then we have

$$a(t) + ib(t) = h\bar{k}e^{-(i\omega_0+\bar{\gamma})t} \int_{-\infty}^{t} e^{(i\omega+\bar{\gamma})\tau} \sum_{n=-\infty}^{\infty} J_n e^{inq\tau}\, d\tau. \tag{42}$$

When the integration is carried out, the result is

$$a(t) + ib(t) = h\bar{k} \sum_{n=-\infty}^{\infty} \frac{J_n e^{inqt}}{i\omega_0 + \bar{\gamma} + inq} = h\bar{k} \sum_{n=-\infty}^{\infty} \frac{J_n e^{inqt}}{\alpha + i\Delta + inq}. \tag{43}$$

To obtain the magnitude of $a + ib$, which is the envelope required, we multiply the above Fourier series for $a + ib$ by that for $a - ib$, obtaining a double summation, which can be written as follows:

$$\begin{aligned} c^2(t) = a^2(t) + b^2(t) &= c_0 + \sum_{n=1}^{\infty} (c_n e^{inqt} + \bar{c}_n e^{-inqt}) \\ &= a_0 + 2 \sum_{n=1}^{\infty} a_n \cos nqt \end{aligned} \tag{44}$$

where a_n is the real part of the complex coefficient c_n and $a_0 = c_0$.

The coefficients are given by

$$\begin{aligned} c_n &= \frac{h^2}{4C^2} \sum_{m=-\infty}^{\infty} \frac{J_m J_{m+n}}{(\alpha - i\Delta - imq)[\alpha + i\Delta + i(m+n)q]} \\ \bar{c}_n &= \frac{h^2}{4C^2} \sum_{m=-\infty}^{\infty} \frac{J_m J_{m+n}}{(\alpha + i\Delta + imq)[\alpha - i\Delta - i(m+n)q]}. \end{aligned} \tag{45}$$

Obtaining this result involves use of the relation

$$J_{-n}(x) = (-)^n J_n(x). \tag{46}$$

From (45)

$$a_n = \frac{h^2}{4C^2} \sum_{m=-\infty}^{\infty} \frac{J_m J_{m+n}[\alpha^2 + (\Delta + mq)(\Delta + mq + nq)]}{[\alpha^2 + (\Delta + mq)^2][\alpha^2 + (\Delta + mq + nq)^2]}. \tag{47}$$

Finally, we have to obtain, as before, the difference between the envelopes of the voltages across the two impedances of the balanced frequency detector; that is, we have to determine

$$V_0(t) = c_2(t) - c_1(t) \tag{48}$$

where $c_1(t)$ is given by (44) as it stands and $c_2(t)$ is obtained from the same expression merely by reversing the sign of Δ. The complete solution for the output voltage of the frequency detector, when a phase variation, $\theta'(t) = xq \cos qt$, is impressed on the carrier at the input, then is

$$V_0(t) = \left[a_0 + 2 \sum_{n=1}^{\infty} a_{n2} \cos nqt\right]^{1/2} - \left[a_0 + 2 \sum_{n=1}^{\infty} a_{n1} \cos nqt\right]^{1/2} \quad (49)$$

where a_{n1} is given by (47) and a_{n2} is given by the same formula with the sign of Δ reversed.

An approximation that is permissible when the frequency swing xq does not approach the available frequency range Δ is

$$V_0(t) = \sum_{n=1}^{\infty} A_n \cos nqt \quad (50)$$

where

$$A_n = \frac{a_{n2} - a_{n1}}{\sqrt{a_0}}, \qquad n \text{ odd}; = 0, n \text{ even}. \quad (51)$$

The table following gives the results of a computation of the first four coefficients from formulas (45) and (51) for the case of a frequency deviation ratio, $x = 5$, for $q/2\pi = 3000$ cycles per second, for $\Delta/2\pi = 30{,}000$ cycles per second and for $\alpha/\Delta = \frac{1}{2}$, 1 and 2.

Coefficient	$\alpha/\Delta = .5$	1.0	2.0
A_0	0	0	0
A_1	.848	.478	.191
A_2	0	0	0
A_3	.0312	−.00438	−.00281

Note: To obtain volts, multiply all values by $\frac{Rh\alpha}{(\alpha^2 + \Delta^2)^{1/2}}$

The coefficients for even values of n vanish, which confirms what can be inferred from physical considerations, namely, that the balanced construction of the frequency detector eliminates the d.c. component and all even harmonics. From the ratio of A_3 to A_1 we obtain the following ratios, expressed in db's, of the third harmonic distortion to the fundamental signal for the three circuit designs:

α/Δ	$20 \log 10 \lvert A_3/A_1 \rvert$
.5	−28.7 db
1.0	−40.8
2.0	−36.6

The results for the sinusoidal signal, when considered in conjunction with those for the impulse modulation, also permit certain conclusions regarding signal-to-noise ratios for impulsive interference in FM reception.

Ratio of Noise to Signal

It may be helpful, in conclusion, to attempt a theoretical estimate of the ratio of noise to signal in the audio output under the condition of severe impulsive interference.

The ratio of the peak value of the pulse to the signal amplitude at the frequency detector output is given by the ratio of the peak values of $f(t)$, as plotted in figure 5, to the values of A_1 in the table above. To obtain a result of practical significance, however, the effect of the audio circuit should be taken into account. In the absence of specific information on the structure of this circuit, we assume that the peak value of a pulse at its output is equal to the area, or moment, of the pulse at its input times twice the audio cutoff frequency. This is true for an ideal "square cutoff" filter and not seriously in error for actual circuits. The area of the largest pulse at the frequency detector output is approximately $2\Delta/(\alpha^2 + \Delta^2)$ and the value of A_1, the signal fundamental amplitude, can be approximated by

$$A_1 = \frac{2\Delta\, x\, q}{\alpha^2 + \Delta^2} \tag{52}$$

(For the example above, this approximation gives A_1 = .8, .5 and .2 as compared to the exact values, .848, .478 and .191.) In this way we arrive at the following estimate of the peak ratio of noise to signal in the audio output:

$$\frac{\text{Max. value of largest pulse}}{\text{Signal amplitude}} = \frac{\omega_a}{\pi\, x\, q} \tag{53}$$

where ω_a is the cutoff-frequency of the audio circuit, q the signal frequency and x the frequency deviation ratio. Then xq is the "frequency swing" of the transmitter, i. e., the maximum departure of the instantaneous frequency from its mean value. It is to be noted that this formula is free from the detector circuit parameters, α, Δ, R, and indicates that, to a first approximation, at least, the maximum ratio of noise to signal depends only upon the audio circuit cutoff frequency and the FM swing. Furthermore, this establishes a ceiling for the interference that will not be exceeded no matter how large the impulses may be.

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