Spectra of Quantized Signals

By W. R. BENNETT

1. Discussion of Problem and Results Presented

SIGNALS which are quantized both in time of occurrence and in magnitude are in fact quite old in the communications art. Printing telegraph is an outstanding example. Here, time is divided into equal divisions, and the number of magnitudes to be distinguished in any one interval is usually no more than two, corresponding to the closed or open positions of a sending switch. It is only in recent years, however, that the development of high speed electronic devices has progressed sufficiently to enable quantizing techniques to be applied to rapidly changing signals such as produced by speech, music, or television. Quantizing of time, or time division, has found application as a means of multiplexing telephone channels.[1] The method consists of connecting the different channels to the line in sequence by fast moving switches synchronized at the transmitting and receiving ends. In this way a transmission medium capable of handling a much wider band of frequencies than required for one telephone channel can be used simultaneously by a group of channels without mutual interference. The plan is the same as that used in multiplex telegraphy. The difference is that ordinary rotating machinery suffices at the relatively low speeds employed by the latter, while the high speeds needed for time division multiplex telephony can be realized only by practically inertialess electron streams. Also the widths of frequency band required for multiplex telephony are enormously greater than needed for the telegraph, and in fact have become technically feasible only with the development of wide-band radio and cable transmission systems. As far as any one channel is concerned the result is the same as in telegraphy, namely that signals are received at discrete or quantized times. In the limiting case when many channels are sent the speech voltage from one channel is practically constant during the brief switch closure and, in effect, we can send only one magnitude for each contact or quantum of time. The more familiar word "sampling" will be used here interchangeably with the rather formidable term "quantizing of time".

Quantizing the magnitude of speech signals is a fairly recent innovation. Here we do not permit a selection from a continuous range of magnitudes but only certain discrete ones. This means that the original speech signal is to be replaced by a wave constructed of quantized values selected on a minimum error basis from the discrete set available. Clearly if we assign the quantum values with sufficiently close spacing we may make the quantized wave indistinguishable by the ear from the original. The purpose of quantization of magnitudes is to suppress the effects of interference in the transmission medium. By the use of precise receiving instruments we can restore the received quanta without any effect from superposed interference provided the interference does not exceed half the difference between adjacent steps.

By combining quantization of magnitude and time, we make it possible to code the speech signals, since transmission now consists of sending one of a discrete set of magnitudes for each distinct time interval.[2,3,4,5,6,7] The maximum advantage over interference is obtained by expressing each discrete signal magnitude in binary notation in which the only symbols used are 0 and 1. The number which is written as 4 in decimal notation is then represented by 100, 8 by 1000, 16 by 10,000; etc. In general, if we have N digit positions in the binary system, we can construct 2^N different numbers. If we need no more than 2^N different discrete magnitudes for speech transmission, complete information can be sent by a sequence of N on-or-off pulses during each sampling interval. Actually a total of $2^N!$ different coding plans (sets of one-to-one correspondences between signal magnitudes and on-or-off sequences) is possible. The straightforward binary number system is taken as a representative example convenient for either theoretical discussion or practical instrumentation. We assume that absence of a pulse represents the symbol 0 and presence of a pulse represents the symbol 1. The receiver then need only distinguish between two conditions: no transmitted signal and full strength transmitted signal. By spacing the repeaters at intervals such that interference does not reach half the full strength signal at the receiver, we can transmit the signal an indefinitely great distance without any increment in distortion over that originally introduced by the quantizing itself. The latter can be made negligible by using a sufficient number of steps.

To determine the number of quantized steps required to transmit specific signals, we require a knowledge of the relation between distortion and step size. This problem is the subject of the present paper.* We divide the problem into two parts: (1) quantizing the magnitude only and (2) combined quantizing of magnitude and time. The first part can be treated by a simple model: the "staircase transducer", which is a device having the instantaneous ouput vs. input curve shown by Fig. 1. Signals impressed on the stair-

* Other features of the quantizing and coding theory are discussed in forthcoming papers by Messrs. C. E. Shannon, J. R. Pierce, and B. M. Oliver.

case transducer are sorted into voltage slices (the treads of the staircase), and all signals within plus or minus half a step of the midvalue of a slice are replaced in the output by the midvalue. The corresponding output when the input is a smoothly varying function of time is illustrated in Fig. 2. The output remains constant while the input signal remains within the boundaries of a tread and changes abruptly by one full step when the signal crosses the boundary. It is not within the scope of the present paper to discuss the internal mechanism of a staircase transducer, which may have many different physical embodiments. We are concerned rather with the distortion produced by such a device when operating perfectly.

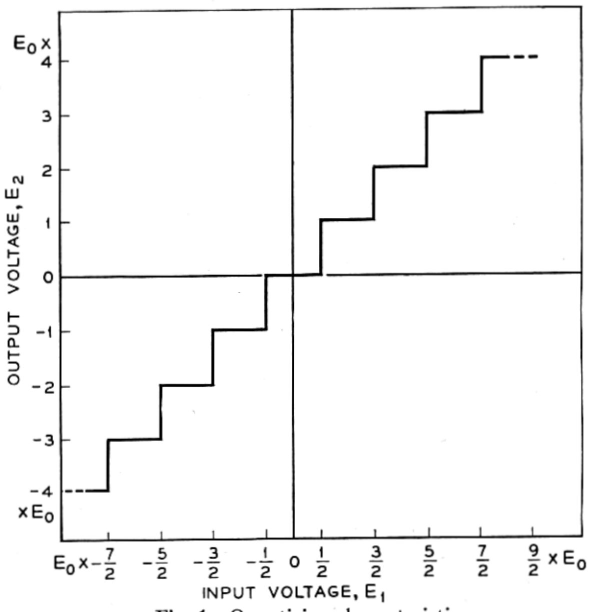

Fig. 1—Quantizing characteristic.

The distortion or error consists of the difference between the input and output signals. The maximum instantaneous value of distortion is half of one step, and the total range of variation is from minus half a step to plus half a step. The error as a function of input signal voltage is plotted in Fig. 3 and a typical variation with time is indicated in Fig. 2. If there is a large number of small steps, the error signal resembles a series of straight lines with varying slopes, but nearly always extending over the vertical interval between minus and plus half a step. The exceptional cases occur when the signal goes through a maximum or minimum within a step. The limiting condition of closely spaced steps enables us to derive quite simply an approximate value for the mean square error, which will later be shown to be sufficiently accurate in most cases of practical importance. This approximation consists of calculating the mean square value of a straight line going from minus half a step to plus half a step with arbitrary slope. If

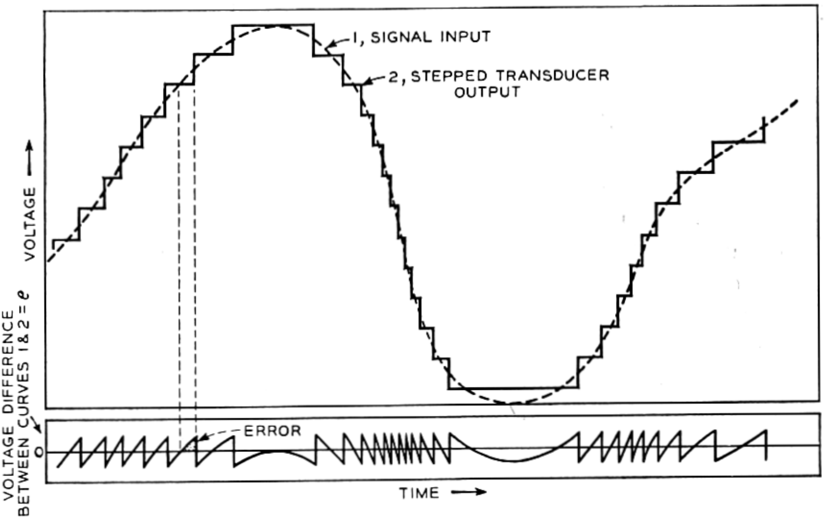

Fig. 2—A quantized signal wave and the corresponding error wave.

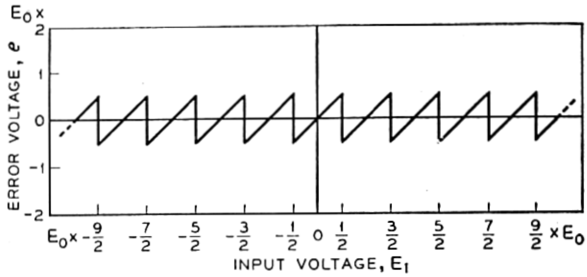

Fig. 3—Characteristic of the errors in quantizing.

E_0 is the voltage corresponding to one step, and s is the slope, the equation of the typical line is:

$$\epsilon = st, \qquad -\frac{E_0}{2s} < t < \frac{E_0}{2s} \tag{1.0}$$

where ϵ is the error voltage and t is the time referred to the midpoint as origin. Then the mean square error is

$$\overline{\epsilon^2} = \frac{s}{E_0} \int_{-E_0/2s}^{E_0/2s} \epsilon^2 \, dt = \frac{E_0^2}{12}, \tag{1.1}$$

or one twelfth the square of the step size.

Not all the distortion falls within the signal band. The distortion may be considered to result from a modulation process consisting of the application of the component frequencies of the original signal to the non-linear staircase characteristic. High order modulation products may have frequencies quite remote from those in the original signal and these can be excluded by a filter passing only the signal band. It becomes of importance, therefore, to calculate the spectrum of the error wave. This we shall do in the next section for a generalized signal using the method of correlation, which is based on the fact that the power spectrum of a wave is the Fourier cosine transform of the correlation function. The result is then applied to a particular kind of signal, namely one having energy uniformly distributed throughout a definite frequency band and with the phases of the components randomly distributed. This is a particularly convenient type of signal because it in effect averages over a large number of possible discrete frequency components within the band. Single or double-frequency signal waves are awkward for analytical purposes because of the ragged nature of the spectra produced. The amplitudes of particular harmonics or cross-products of discrete frequency components are found to oscillate violently with magnitude of input. The use of a large number of input components smooths out the irregularities.

The type of spectra obtained is shown in Fig. 4. Anticipating binary coding, we have shown results in terms of the number of binary digits used. The number of different magnitudes available are 16, 32, 64, 128, and 256 for $N = 4, 5, 6, 7$ and 8 digits, respectively. Here a word of explanation is needed with respect to the placing of the scale of quantized voltages. A signal with a continuous distribution of components along the frequency scale is theoretically capable of assuming indefinitely great values of instantaneous voltage at infrequent instants of time. An actual quantizer (staircase transducer) has a finite overload value which must not be exceeded and hence can have only a finite number of steps. This difficulty is resolved here by the experimentally observed fact that thermal noise, which has the type of spectrum we have assumed for our signal, has never been observed to exceed appreciably a voltage four times its root-mean-square value. Hence we have placed the root-mean-square value of the input signal at one-fourth the overload input to the staircase. This fixes the relation between step size and the total number of steps. In the actual calculation the number of steps is taken as infinite; the effect of the assumed additional steps beyond 2^N is negligible because of the rarity of excursion into this range.

The curves of Fig. 4 are drawn for the case in which the signal band starts at zero frequency. The original signal band width is represented by one unit on the horizontal scale. The relatively wide spread of the distortion spectrum is clearly shown. As the number of digits (or steps) is increased

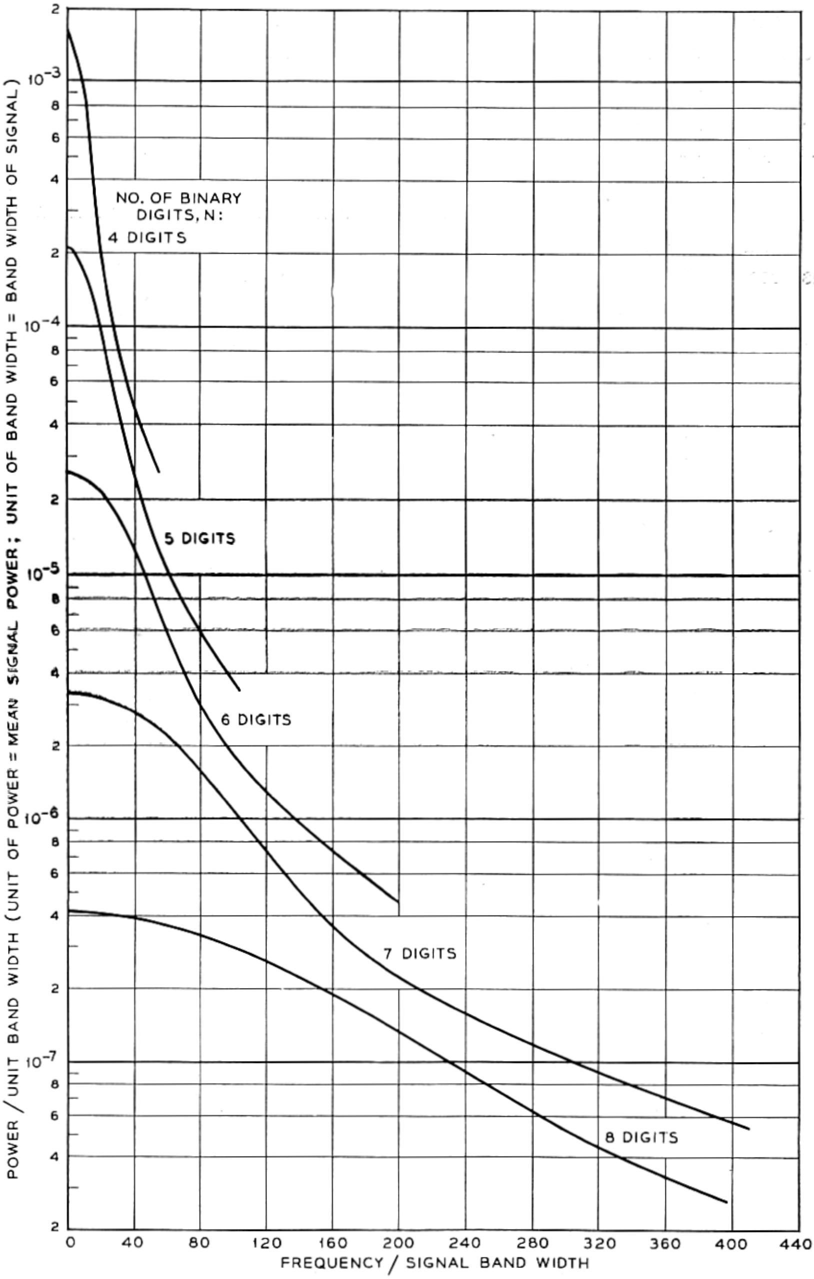

Fig. 4—Spectrum of distortion from quantizing the magnitudes of a random noise wave. Full load on the quantizer is reached by peaks 12 db above the r.m.s. value of input.

the spectrum becomes flatter over a wider range, but with a smaller maximum density. The area under each curve represents the total mean power in the corresponding error wave and is found to agree quite accurately with the approximate result of Eq. (1.1). The distortion power falling in the signal band is represented by the area included under the curve from zero to unit abscissa.

Quantizing the magnitude only is not a technically attractive method of transmission because of the wide frequency band required to preserve the discrete values of the quanta. Thus in a 128-step system, a full load sinusoidal signal passes through 64 different steps each quarter cycle and hence would require transmitting 256 successively different magnitudes during each period of the signal frequency. We therefore consider the second problem—that of sampling the quantized magnitudes.

The theory of periodic sampling of signals is a limiting case of commutator modulation theory as previously shown by the author.[1] We may think of a periodically closed switch in series with the line and source as producing a multiplication of the signal by a switching function. The switching function has a finite value during the time of switch closure and is zero at other times. It may be expanded in a Fourier series containing a term of zero frequency, the repetition frequency of switch closure, and all harmonics of the latter. Multiplication of the signal by the Fourier series representing the constant component of the switching function gives a term proportional to the signal itself. Multiplication of the signal by the fundamental component of the switching function gives upper and lower sidebands on the repetition frequency. Likewise multiplication by the harmonics gives sidebands on each harmonic. The signal is separable from the sidebands on a frequency basis if the signal band does not overlap the lower sideband on the repetition frequency. This leads to the condition for no distortion in time division: the highest signal frequency must be less than one-half the repetition frequency.

To apply the above theory to instantaneous sampling we let the duration of switch closure in one period approach zero. We then approach the condition of one signal value in each period, so that the repetition frequency now becomes the sampling frequency. Clearly the sampling frequency must slightly exceed twice the highest signal frequency. We also note that as the contact time tends toward zero, the switching function approaches a periodically repeated impulse. The important terms of the Fourier series representing the switching function accordingly become a set of harmonics of equal amplitude with a constant component equal to half the amplitude of the typical harmonic. On multiplication of this series by the signal, we get a set of sidebands of equal amplitude including the one corresponding to the original signal itself, the sideband on zero frequency.

These results may be applied to the staircase transducer. The output may be resolved into the input signal plus the error. The sampling frequency is assumed to exceed its minimum required value of twice the top signal frequency. The component of the output that is equal to the original signal can therefore be separated at the receiver by a filter passing the original signal band. A similar statement cannot be made for the error component, for it has been found to extend over a vastly greater range than the original signal. To calculate the total distortion received in the signal band, we can multiply the distortion spectrum by the switching function and sum up all sideband contributions to the original signal band. Each har-

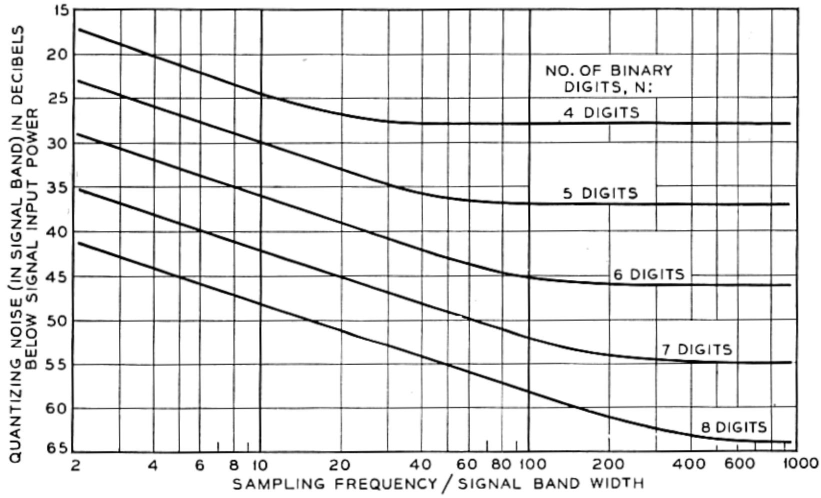

Fig. 5—Total distortion in signal band from quantizing and sampling a random noise wave. Full load on the quantizer is 12 db above the r.m.s. value of input.

monic of the switching function makes such contributions by beating with a band of the error spectrum above and below the frequency of the harmonic. These contributions add as power when the sampling frequency is independent of the individual frequencies contained in the signal. The total error power accepted by the signal band filter decreases as the sampling frequency is increased because each harmonic of the sampling frequency is thereby pushed upward into a less dense portion of the error spectrum. In the limit as the sampling frequency is made indefinitely large, we return to the non-sampled case, that of the staircase transducer only.

Figure 5 shows the calculated curves of distortion in the signal band plotted as a function of ratio of sampling frequency to signal band width. The curves have downward slopes approaching asymptotes corresponding to the area from zero to unity under the corresponding curves of Fig. 4.

The initial points at the minimum sampling rate are determined on the other hand by the total area under the curves of Fig. 4, since the accepted side-bands on the harmonics in this case exactly fill out the entire error spectrum. These initial points are therefore given quite accurately by Eq. (1.1), which, as pointed out before, is a good approximation for the total areas. We can also give a direct demonstration of the applicability of Eq. (1.1) to the initial points of the curves of Fig. 5 by means of the following theorem:

Theorem I. The mean square value of the response of an ideal low-pass filter to a train of unit impulses multiplied by instantaneous samples occurring at double the cutoff frequency is equal to the mean square value of the samples provided no harmonic of the sampling frequency is equal to twice the frequency of one component or equal to the sum or difference of two component frequencies of the sampled signal. Proof of the theorem is given in Appendix I. To apply it here we resolve the input into two components: the true signal and the error. The former is reproduced with fidelity in the output because it contains only frequencies below half the sampling rate. The error component in the output represents the response of the low-pass filter to the error samples. Except for very special types of signals, the error samples are uniformly distributed throughout the range from minus half a step to plus half a step. Calculation of the mean square value of such a distribution gives Eq. (1.1).

We have tacitly assumed above that the sampled values applied to the filter in the output of the system are infinitesimally narrow pulses of height proportional to the samples. In actual systems it is found advantageous to hold the sampled values constant in the individual receiving channels until the next sample is received. This means that the input to the channel filter is a succession of rectangular pulses of heights proportional to the samples. The resulting magnitude of recovered signal is much larger than would be obtained if very short pulses of the same heights were used; stretching the pulses in time produces in effect an amplification. The amplification is obtained, however, at the expense of a variation of channel transmission with signal frequency. Infinitesimally short pulses have a flat frequency spectrum, while pulses of finite duration do not. The frequency characteristic introduced by lengthening the pulses is easily calculated by determining the steady state admittance function of a network which converts impulses to the actual pulses used. The general formula for this admittance when a unit impulse input is converted into an output pulse $g(t)$ is easily shown to be:

$$Y(i\omega) = f_s \int_{-\infty}^{\infty} g(t)e^{-i\omega t}\, dt \tag{1.2}$$

where f_s is the repetition frequency and ω is the angular signal frequency.

We shall call this Theorem II and give the proof in Appendix II. This relation is similar to that found in television and telephotography for the "aperture effect", or variation of transmission with frequency caused by the finite size of the scanning aperture. The pulse shape $g(t)$ is analogous to a variation in aperture height $g(x)$, where x is distance along the line of scanning. Hence it has become customary to use the term "aperture effect" in the theory of restoring signals from samples. The aperture effect associated with rectangular pulses lasting from one sample to the next amounts to an amplitude reduction of $\pi/2$ or 3.9 db at the top signal frequency (one half the sampling rate) compared to a signal of zero frequency. There is also a constant delay introduced equal to half the sampling period. The latter does not cause any distortion and the amplitude effect can be corrected by properly designed equalizing networks.

The fact that many pulse spectra can be simply expressed in terms of a flat spectrum associated with sharp pulses and an aperture effect caused by the particular shape of pulse used does not appear to have been recognized in the recent literature, although applications were made by Nyquist in a fundamental paper[8] of 1928. Premature introduction of a specific finite pulse not only complicates the work, but also restricts the generality of the results.

Distortion caused by quantizing errors produces much the same sort of effects as an independent source of noise. The reason for this is that the spectrum of the distortion in the receiving filter output is practically independent of that of the signal over a wide range of signal magnitudes. Even when the signal is weak so that only a few quantizing steps are operated, there is usually enough residual noise on actual systems to determine the quantizing noise and mask the relation between it and the signal. Eq. (1.1) yields a simple rule enabling one to estimate the magnitude of the quantizing noise with respect to a full load sine wave test tone. Let the full load test tone have peak voltage E; its mean square value is then $E^2/2$. The total range of the quantizer must be $2E$ because the test signal swings between $-E$ and $+E$. The ratio $2E/E_0 = r$ is a convenient one to use in specifying the quantizing; it is the ratio of the total voltage range to the range occupied by one step. The ratio of mean square signal to mean square quantizing noise voltage is

$$\frac{E^2/2}{E_0^2/12} = \frac{6E^2}{4E^2/r^2} = \frac{3r^2}{2} \tag{1.3}$$

Actual systems fail to reproduce the full band $f_s/2$ because of the finite frequency range needed for transition from pass-band to cutoff. If we introduce a factor κ to represent the ratio of equivalent rectangular noise band to $f_s/2$, the actual received noise power is multiplied by κ. Then the signal-to-noise ratio in db for a full load test tone is

$$D = 10 \log_{10} \frac{3r^2}{2\kappa} \text{ db} \tag{1.4}$$

In practical applications the value of κ is about 3/4 which gives the convenient rule:

$$D = 20 \log_{10} r + 3 \text{ db} \tag{1.5}$$

In other words, we add 3 db to the ratio expressed in db of peak-to-peak quantizing range to the range occupied by one step. For various numbers of binary digits the values of D are:

TABLE I

Number of Digits	D
3	21
4	27
5	33
6	39
7	45
8	51

From Table I we can make a quick estimate of the number of digits required for a particular signal transmission system provided that we have some idea of the required signal-to-noise ratio for a full load test tone. The latter ratio may be expressed in terms of the full load test tone which the system is required to handle and the maximum permissible unweighted noise power at the same level point. Since quantizing noise is uniformly distributed throughout the signal band, its interfering effect on speech or other program material is probably similar to that of thermal noise with the same mean power. Requirements given in terms of noise meter readings must be corrected by the proper weighting factor before applying the table. If the signal transmitted is itself a multiplex signal with channels allotted on a frequency division basis, the noise power falling in each channel is the same fraction of the total noise power as the band width occupied by the signal is of the total band width of the system.

We have thus far considered only the case in which the quantized steps are equal. In actual systems designed for transmission of speech it is found advantageous to taper the steps in such a way that finer divisions are available for weak signals. For a given number of total steps this means that coarser quantization applies near the peaks of large signals, but the larger absolute errors are tolerable here because they are small relative to the bigger signal values. Tapered quantizing is equivalent to inserting complementary nonlinear transducers in the signal branch before and after the quantizer. In the usual case, the transducer ahead of the quantizer is of the "compressing" type in which the loss increases as the signal increases. If the full load signal just covers all the linear quantizing steps, a weak signal gets a bigger share of the steps than it would if the transducer were linear. The transducer after the quantizer must be of the "expanding" type which gives decreased loss to the large signals to make the overall combination linear.

On the basis of the theory so far discussed, we can say that the error spectrum out of the linear quantizer is virtually the same whether or not the signal input is compressed. The operation of the expandor then magnifies the errors produced when the signal is large. When weak signals are applied, the mean square error is given by Eq. (1.1), as before, but when the signal is increased an increment in noise occurs. The mean square value of noise voltage under load may be computed from the probability density of the signal values and the output-vs-input characteristic of the expandor, or its inverse, the compressor. A first order approximation, valid when the steps are not too far apart, replaces (1.1) by:

$$\overline{\epsilon^2} = \frac{E_0^2}{12} \int_{Q_2}^{Q_1} \frac{p_1(E_1)\, dE_1}{[F'(E_1)]^2} \tag{1.6}$$

where Q_1 and Q_2 are the minimum and maximum values of the input signal voltage E_1, $p_1(E_1)$ is the probability density function of the input voltage, and $F'(E_1)$ is the slope of $F(E_1)$, the compression characteristic.

Some experimental results obtained with a laboratory model of a quantizer are given in Figs. 6–9. Figs. 6–7 show measurements on the third harmonic associated with 6-digit quantizing. As mentioned before, the amplitude of any one harmonic oscillates with load. The calculated curves shown were obtained by straightforward Fourier analysis. In the measurements it was convenient to spot only the successive nulls and peaks.

In Fig. 6 the bias was set to correspond to the stair-case curve of Fig. 1, while in Fig. 7 the origin is moved to the point $(E_0/2, E_0/2)$, i.e., to the middle of a riser instead of a tread. The peaks of ratio of harmonic to fundamental decrease steadily as the amplitude of the signal is increased to full load, which is just opposite to the usual behavior of a communication system. It is difficult to extrapolate experience with other systems to specify quality in terms of this type of harmonic distortion.

Figure 8 shows measurements of the total distortion power falling in the signal band when the signal is itself a flat band of thermal noise. The technique of making such measurements has been described in earlier articles.[9,10] Measurements are shown for quantizing with both equal and tapered steps. The particular taper used is indicated by the expandor characteristic of Fig. 9. The compression curve is found by interchanging horizontal and vertical scales. The measurements were made on a quantizer with 32, 64, and 128 steps, and a sampling rate of 8,000 cycles per sec-

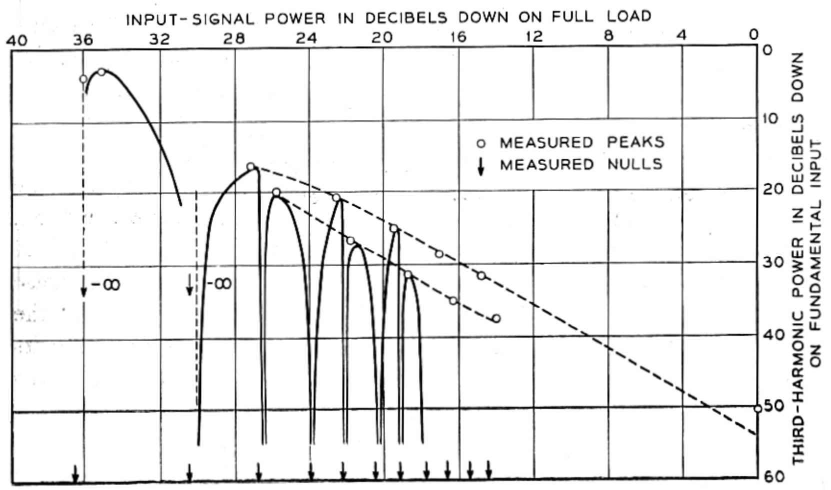

Fig. 6—Third harmonic in 64-step quantized output with bias at mid tread. The smooth curves represent computed values.

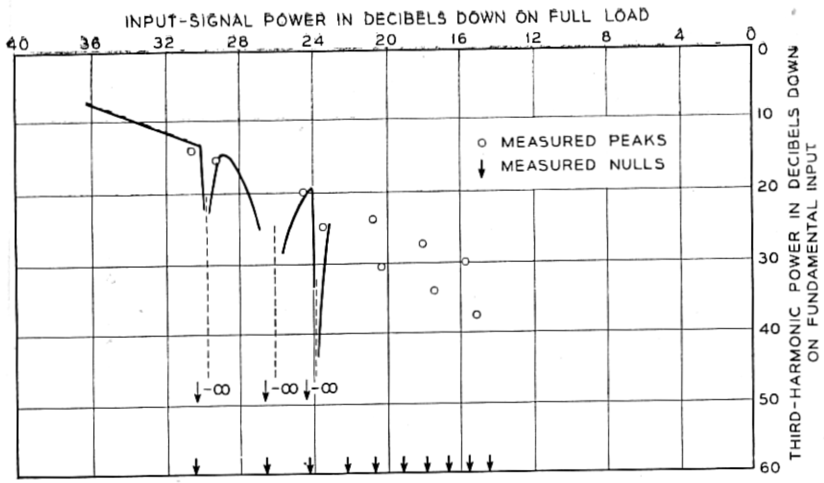

Fig. 7—Third harmonic in 64-step quantized output with bias at mid-riser. The smooth curves represent computed values.

ond. The applied signal was confined to a range below 4,000 cycles per second. With equal steps the distortion power is practically independent of load as shown by the db-for-db straight lines. With tapered steps, the distortion is less for weak signals, and only slightly greater for large signals.

The vertical line designated "full load random noise input" represents the value of noise signal power at which peaks begin to exceed the quantizing

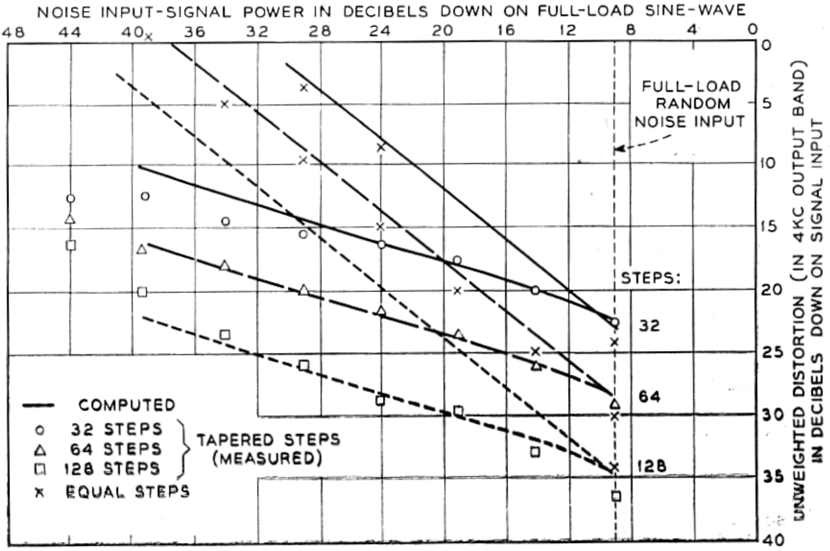

Fig. 8—Total distortion in signal band from quantizing with equal and tapered steps.

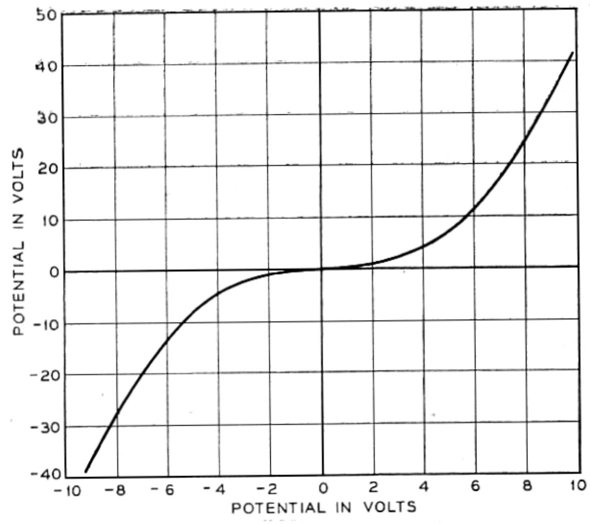

Fig. 9—Expanding characteristic applied to noise in tapered steps of Fig. (8).

range. This occurs when the rms value of input is 9 db below the rms value of the sine wave which fully loads the quantizer.

Flatness of the distortion spectrum with frequency within the signal band is demonstrated by Fig. 10. Two kinds of input were used here—a flat band of thermal noise and a set of 16 sine waves with frequencies distributed throughout the band. Results in the two cases were practically the same. The theoretical levels of distortion power for the band widths of the measuring filters (95 cps) are shown by the horizontal lines.

In the experimental results given here use has been made of laboratory studies by Messrs. A. E. Johanson, W. A. Klute, and L. A. Meacham.

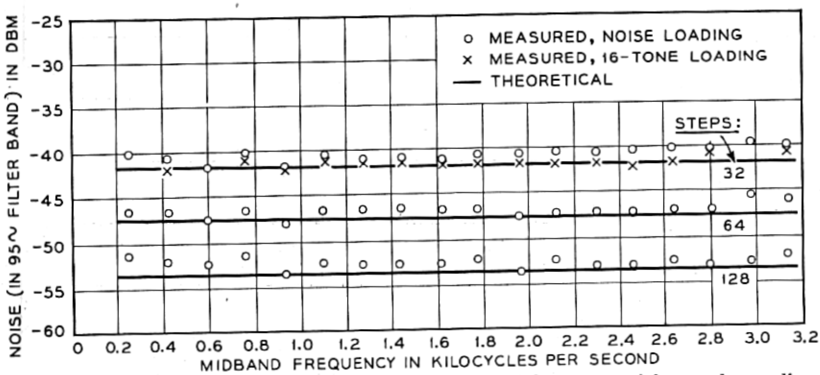

Fig. 10—Spectral density of distortion in signal band from quantizing and sampling. The quantizing steps were equal and the quantizer was fully loaded by a random noise or 16-tone input signal with mean power = −2.5 dbm.

2. Theoretical Analysis

The correlation theorem discovered by N. Wiener[11] may be stated as follows: Let ψ_τ represent the average value of the product $I(t)I(t + \tau)$, where $I(t)$ is the value of a variable such as current or voltage at time t, and $I(t + \tau)$ is the value at a time τ seconds later. Mathematically:

$$\psi_\tau = \overline{I(t)I(t + \tau)} = \lim_{T\to\infty} \frac{1}{T} \int_0^T I(t)I(t + \tau)\, dt \tag{2.0}$$

From analogy with statistical theory, ψ_τ is called the correlation of $I(t)$ with itself, or the autocorrelation function of the signal. Since we shall not deal here with the correlation of two signals, we shall shorten our terms and call ψ_τ simply the correlation of $I(t)$. Let $w_f\, df$ represent the mean power in the output of an ideal bandpass filter of width df centered at f. We assume that the ideal filter is designed to work between resistances of one ohm each and that the input signal $I(t)$ is delivered to the filter from a source with internal resistance of one ohm. (The use of unit resistances does not restrict the generality of the results, since equivalent transmission performance of any linear electrical circuit is obtained by multiplying all impedances by a constant factor. All voltages are multiplied and all currents divided by the same factor. By assuming unit values of resistance we are able to use

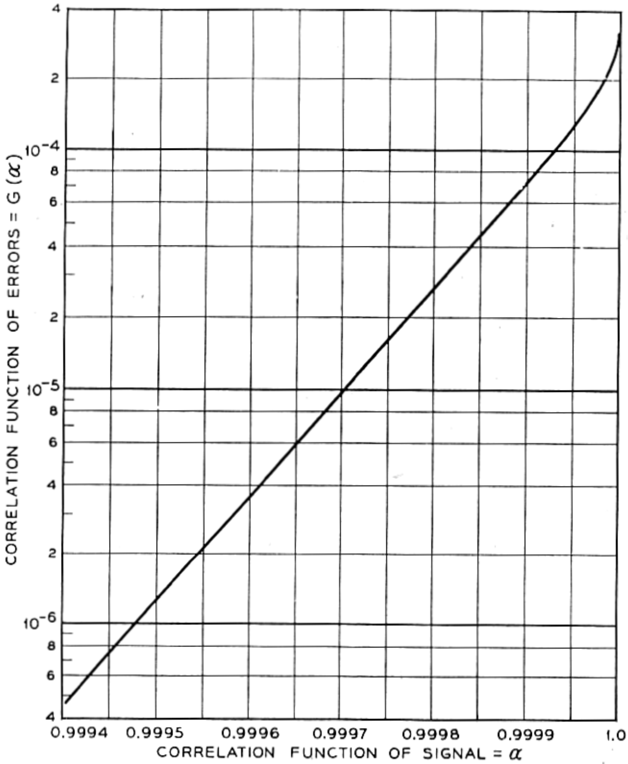

Fig. 11—Correlation function of 7-digit quantizing errors.

squared values of voltages and currents to represent power.) The theorem states that w_f and ψ are related by the equation:

$$w_f = 4 \int_0^\infty \psi_\tau \cos 2\pi f \tau \, d\tau \tag{2.1}$$

Proof may be found in the references cited. When the signal contains periodic components, the integral in (2.1) becomes divergent in the ordinary or Riemann sense, but this difficulty may be overcome by either applying the theory of divergent integrals or replacing Riemann by Stieltjes integration. We shall not require these modifications here because we shall base our analysis on signals with a continuous spectrum. We note that ψ_0 is the mean square value of the signal itself. We also point out that the inversion formula for the Fourier integral enables us to express ψ_τ in terms of w_f, thus:

$$\psi_\tau = \int_0^\infty w_f \cos 2\pi\tau f \, df \tag{2.2}$$

It also may be shown that the ratio ψ_τ/ψ_0 cannot have values outside the interval from -1 to $+1$.

The correlation theorem furnishes a powerful analytical tool for the solution of modulation problems because the calculation of the average ψ_τ is often a straightforward process, while direct calculation of w_f may be a very devious one. Once ψ_τ has been obtained, Eq. (2.1) brings the highly developed theory of Fourier integrals to bear on the computation of w_f.

We shall give the derivation of w_f for quantizing noise making use of the correlation function. In the analysis we shall apply a number of other needed theorems with appropriate references given for proof.

Our first problem is that of calculating the spectrum of the output of the staircase transducer, Fig. 1, when the spectrum of the input signal is given. Let w_f represent the power spectrum of the input signal and ψ_τ the autocorrelation function. The two quantities are related by (2.1) and it is sufficient to express our results in terms of either one. If the instantaneous value of the input signal is represented by E_1, and that of the output by E_2, the staircase function may be defined mathematically by:

$$E_2 = mE_0, \qquad \frac{2m-1}{2} E_0 < E_1 < \frac{2m+1}{2} E_0, \qquad m = 0, \pm 1, \pm 2, \cdots \tag{2.3}$$

The error is the difference between E_1 and E_2 and may be written as

$$\epsilon(t) = E_1 - E_2 = E_1 - mE_0, \qquad \frac{2m-1}{2} E_0 < E_1 < \frac{2m-1}{2} E_0 \tag{2.4}$$

The error characteristic is plotted in Fig. 3.

One approach depends on a knowledge of the probability density function $p(V_1, V_2)$ of the variables $V_1 = E_1$ at time t and $V_2 = E_2$ at time $t + \tau$. The definition of this function is that $p(V_1, V_2) \, dV_1 dV_2$ is the probability that V_1 and V_2 lie in a rectangle of dimensions dV_1 and dV_2 centered on the point V_1, V_2 of the V_1V_2-plane. The function $p(V_1, V_2)$ has been calculated for certain types of signals and in theory could be computed for any signal by standard methods. If it is assumed known, we may determine the correlation function of the error. Let

$$F(V_1, V_2) = \epsilon(t)\epsilon(t+\tau) = (V_1 - mE_0)(V_2 - nE_0),$$

$$\frac{2m-1}{2}E_0 < V_1 < \frac{2m+1}{2}E_0, \quad \frac{2n-1}{2}E_0 < V_2 < \frac{2n+1}{2}E_0, \tag{2.5}$$

$$m, n = 0, \pm 1, \pm 2, \cdots$$

Eq. (2.5) defines $F(V_1, V_2)$ as a definite constant value in each square of width E_0 in the V_1V_2-plane. By elementary statistical theory, the correlation function ξ_τ of the error wave is now

$$\xi_\tau = \overline{F(V_1, V_2)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(V_1, V_2)p(V_1, V_2)\, dV_1\, dV_2 \tag{2.6}$$

The correlation may therefore be calculated since F and p are known functions. The power spectrum Ω_f of the error wave is then equal to the right-hand member of (2.1) with ξ_τ substituted for ψ_τ.

We are interested in the case in which the signal voltage has a smoothly varying spectrum over a specified band. This is a property of a random noise function which has a normal distribution of instantaneous voltages. The two-dimensional probability density function of such a wave is known[12]. It is

$$p(V_1, V_2) = \frac{1}{2\pi\sqrt{\psi_0^2 - \psi_\tau^2}} \exp\left[\frac{\psi_0(V_1^2 + V_2^2) - 2\psi_\tau V_1 V_2}{2(\psi_0^2 - \psi_\tau^2)}\right]. \tag{2.7}$$

By inserting this value and that of $F(V_1, V_2)$ from (2.5) in (2.6), making the change of variable:

$$\left.\begin{aligned} V_1 - mE_0 &= E_0x/2 \\ V_2 - nE_0 &= E_0y/2 \end{aligned}\right\} \tag{2.8}$$

and adopting the notation,

$$k = E_0^2/\psi_0, \; \alpha = \psi_\tau/\psi_0, \; G(\alpha) = \xi_\tau/\psi_0, \tag{2.9}$$

we obtain the following integral determining ξ_τ,

$$G(\alpha) = \frac{k^2}{32\pi(1-\alpha^2)^{1/2}} \cdot \int_{-1}^{1}\int_{-1}^{1} xyH(x, y) \exp\frac{-k(x^2 + y^2 - 2\alpha xy)}{8(1-\alpha^2)}\, dx\, dy \tag{2.10}$$

$$H(x, y) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \cdot \exp\frac{-k[m^2 + m(x - \alpha y) + n^2 + n(y - \alpha x) - 2\alpha mn]}{2(1-\alpha^2)} \tag{2.11}$$

The power density spectrum of the errors is, from (2.1),

$$\Omega_f = 4 \int_0^\infty \xi_\tau \cos 2\pi f \tau \, d\tau$$

$$= 4\psi_0 \int_0^\infty G(\alpha) \cos 2\pi f \tau \, d\tau \tag{2.12}$$

If the signal band is flat from $f = 0$ to $f = f_0$, with no energy outside this band,

$$\alpha = \frac{1}{f_0} \int_0^{f_0} \cos 2\pi \tau f \, df = \frac{\sin 2\pi f_0 \tau}{2\pi f_0 \tau} \tag{2.13}$$

Letting $\gamma = f/f_0$,

$$\Omega_0(\gamma) = \frac{f_0 \, \Omega_f}{\psi_0} = \frac{2}{\pi} \int_0^\infty G\left(\frac{\sin z}{z}\right) \cos \gamma z \, dz, \tag{2.14}$$

To complete the calculation, we must evaluate the integral (2.10). The first step is to transform the double summation (2.11) into products of single sums by the change of indices:

$$\begin{pmatrix} m + n = m' \\ m - n = n' \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} m = \dfrac{m' + n'}{2} \\ n = \dfrac{m' - n'}{2} \end{pmatrix} \tag{2.15}$$

The rearrangement is permissible because the double series is absolutely convergent. The new indices m' and n' also run from minus to plus infinity, but must be either both even or both odd because $m' \pm n'$ is even. On dropping the primes after the substitution is completed, we find

$$\begin{aligned} H(x, y) = \sum_{m=-\infty}^{\infty} \exp \frac{-k[2m(x + y) + 4m^2]}{4(1 + \alpha)} \sum_{n=-\infty}^{\infty} \\ \cdot \exp \frac{-k[2n(x - y) + 4n^2]}{4(1 - \alpha)} + \sum_{m=-\infty}^{\infty} \\ \cdot \exp \frac{-k[(2m + 1)(x + y) + (2m + 1)^2]}{4(1 + \alpha)} \sum_{n=-\infty}^{\infty} \\ \cdot \exp \frac{-k[2n + 1)(x - y) + (2n + 1)^2]}{4(1 - \alpha)} \end{aligned} \tag{2.16}$$

A further simplification results from a change of the variables of integration to eliminate the terms in xy. This is done by setting

$$\begin{pmatrix} x = u + v \\ y = u - v \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} u = (x+y)/2 \\ v = (x-y)/2 \end{pmatrix} \tag{2.17}$$

By calculating the Jacobian of the transformation, we find $dx\,dy = 2\,du\,dv$. The region of integration in the uv-plane is a rhombus bounded by the lines $u \pm v = \pm 1$. We then have:

$$G(\alpha) = \frac{k^2}{16\pi(1-\alpha^2)^{1/2}}\left[\int_{-1}^{0}\int_{-1-u}^{1+u} dv + \int_0^1 du \int_{u-1}^{1-u} dv\right](u^2 - v^2)$$

$$\exp\left[-\frac{k}{4}\left(\frac{u^2}{1+\alpha} + \frac{v^2}{1-\alpha}\right]\sum_{m=-\infty}^{\infty} \exp\frac{-2mk(2u+2m)}{4(1+\alpha)}$$

$$\sum_{n=-\infty}^{\infty} \exp\frac{-2nk(2v+2n}{4(1-\alpha)} + \sum_{m=-\infty}^{\infty} \exp\frac{-(2m+1)k(2u+2m+1)}{4(1+\alpha)}$$

$$\sum_{n=-\infty}^{\infty} \exp\frac{-(2n+1)k(2v+2n+1)}{4(1-\alpha)} \qquad (2.18)$$

If we substitute $u = -x$ in the first double integral, $m = -m'$ in the first series, and $m = -m' - 1$ in the third series, we see that the two double integrals are equal. We therefore drop the first double integral and multiply the second by two. The inner integral may then be split into parts with limits from $v = 0$ to $v = 1 - u$ and $v = u - 1$ to $v = 0$. Substituting $v = -y$ in the second part and treating the series as before, we find that the two parts give equal contributions, so that the bracketed integral terms become

$$4\int_0^1 du \int_0^{1-u} dv$$

applied to the integrand.

The series in (2.18) may be written as Theta Functions, and the imaginary transformation of Jacobi then used as an aid in reduction. We may proceed in a more direct manner, however, by applying Poisson's Summation Formula:[13]

$$\sum_{n=-\infty}^{\infty} \varphi(2\pi n) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(\tau)e^{-im\tau}\,d\tau \qquad (2.19)$$

We thereby show that

$$\sum_{m=-\infty}^{\infty} \exp\left[-am(x+2m)\right] = \sqrt{\frac{\pi}{2a}}\,e^{ax^2/8}\left[1 + 2\sum_{m=1}^{\infty} e^{-m^2/\pi^2 2a}\cos\frac{m\pi x}{2}\right] \qquad (2.20)$$

$$\sum_{m=-\infty}^{\infty} \exp\left[-a(2m+1)(x+2m+1)\right] = \frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{ax^2/4}\left[1 + 2\sum_{m=1}^{\infty}(-)^m e^{-m^2\pi^2/4a}\cos\frac{m\pi x}{2}\right] \qquad (2.21)$$

When the series in (2.18) of type corresponding to the left-hand members of (2.20) and (2.21) are replaced by the equivalent righthand members, positive exponents containing the squared variables of integration are introduced which cancel the negative exponents already present in the integrand. The resulting integral may be written:

$$G(\alpha) = \frac{k}{4} \int_0^1 du \int_0^{1-u} (u^2 - v^2)[f_1(1 + \alpha, u) f_1(1 - \alpha, v) + f_2(1 + \alpha, u) f_2(1 - \alpha, v)] \, dv, \quad (2.22)$$

where

$$f_1(a, x) = 1 + 2 \sum_{m=1}^{\infty} \exp \frac{-m^2 \pi^2 a}{2k} \cos \frac{m\pi x}{2} \quad (2.23)$$

$$f_2(a, x) = 1 + 2 \sum_{m=1}^{\infty} (-)^m \exp \frac{-m^2 \pi^2 a}{2k} \cos \frac{m\pi x}{2} \quad (2.24)$$

The integrations may now be performed without difficulty. The complete result, which as we shall immediately show is hardly ever necessary to use in full is:

$$\begin{aligned} G(\alpha) = {} & \frac{k}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp\left(-\frac{4n^2\pi^2}{k}\right) \sinh \frac{4n^2\pi^2\alpha}{k} \\ & + \frac{k}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (m \neq n) \frac{1}{(m^2 - n^2)} \exp \frac{-4(m^2 + n^2)\pi^2}{k} \\ & \sinh \frac{4(m^2 - n^2)\pi^2\alpha}{k} - \frac{k}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (m \neq n) \frac{1}{(m - \frac{1}{2})^2 - (n - \frac{1}{2})^2} \\ & \exp \frac{-4[(m - \frac{1}{2})^2 + (n - \frac{1}{2})^2]\pi^2}{k} \sinh \frac{4[(m - \frac{1}{2})^2 - (n - \frac{1^2}{2})]\pi^2\alpha}{k} \end{aligned} \quad (2.25)$$

An alternative derivation of (2.25), subsequently suggested by Mr. S. O. Rice, is based on the fact that $\epsilon(t)$ as defined by (2.4) or Fig. 3 is a periodic function of E_1 which can be expanded in a Fourier series with period E_0. Substituting the series in (2.5) leads to an expression for $\epsilon(t) \, \epsilon \, (t + \tau)$ as the product of two Fourier series. After proof that it is permissible to write this product as a double series and to calculate the average sum as the sum of the averages of the individual terms the problem is reduced to a double series in which the typical term is proportional to the average value of exp $i(uV_1 + vV_2)$ where u and v are constants depending on the position of the term in the series. Rice has shown[12] that the average value of such a term is exp $[-(u^2 + v^2)\psi_0/2 - uv\psi_\tau]$. Summation of these terms leads again to (2.25).

From the defining equation (2.9) we note that k is a small quantity when more than a very few steps are used in the quantizer so that exponentials with exponent containing the factor $-1/k$ are very small except when the factor is multiplied by a number near zero. It will be seen that this can only happen in the first series and then only when α approaches the value unity. We recall that α lies in the range -1 to $+1$ and it is apparent from (2.25) that $G(\alpha)$ is an odd function of α. We thus need consider only positive values of α very slightly less than unity. Only the component of the sinh with positive exponent is then significant, and we write the very accurate approximation for $G(\alpha)$:

$$G(\alpha) \doteq \frac{k}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp \frac{-4n^2\pi^2(1 - \alpha)}{k} \tag{2.26}$$

A typical curve of $G(\alpha)$ vs. α for a fixed value of k is shown in Fig. 11. The rapidity with which it falls away at the left of the point $\alpha = 1$ is such that the curve can only be plotted by greatly expanding the scale of α in this region. The physical significance of the spike-shaped curve is that $G(\alpha)$ is a measure of the correlation of the errors as a function of the correlation of the applied signal. When there are many steps there is virtually no correlation between errors in successive samples except when there is complete correlation of successive signal values.

Use of the approximation (2.26) enables us to derive a convenient formula for the spectral density of the errors in a flat band input signal. Substituting (2.26) in (2.14) we obtain:

$$\Omega_0(\gamma) \doteq \frac{k}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^{\infty} \exp\left[\frac{-4n^2\pi^2}{k}\left(1 - \frac{\sin z}{z}\right)\right] \cos \gamma z \, dz \tag{2.27}$$

The integrand is negligible except when z is near zero, and in this region we may replace $(\sin z)/z$ by the first two terms of its power series expansion. We then find

$$\begin{aligned}\Omega_0(\gamma) &\doteq \frac{k}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^{\infty} \exp\left(\frac{-2n^2\pi^2 z^2}{3k}\right) \cos \gamma z \, dz \\ &= \frac{k}{2\pi^3} \sqrt{\frac{3k}{2\pi}} \sum_{n=1}^{\infty} \frac{1}{n^3} \exp\left(\frac{-3k\gamma^2}{8n^2\pi^2}\right).\end{aligned} \tag{2.28}$$

Only one set of calculations from the infinite series need be made since we may define a function of one variable

$$B(z) = \sum_{n=1}^{\infty} \frac{e^{-z/n^2}}{n^3}. \tag{2.29}$$

Then

$$\Omega_0(\gamma) \doteq \frac{k}{2\pi^3} \sqrt{\frac{3k}{2\pi}} \, B\left(\frac{3k\gamma^2}{8\pi^2}\right). \tag{2.30}$$

The curves of Fig. (4) were obtained in this way. The relation between k and the number of digits N is based on the assumption of the rms value of signal reaching one-fourth the instantaneous overload voltage of the quantizer. Since zero signal voltage is in the middle of the quantizing range $2^N E_0$, the overload signal measured from zero is $2^{N-1}E_0$. The mean square signal input is ψ_0. Therefore

$$2^{N-1} E_0 = 4\sqrt{\psi_0} \tag{2.31}$$

or from (2.9)

$$k = 1/4^{N-3} \tag{2.32}$$

We thus have obtained the spectrum of the quantizing errors without sampling. To apply our results to the sampling case we sum up all contributions from each harmonic of the sampling rate beating with the noise spectrum from quantizing only. The resulting power spectrum is given by

$$A_f = \Omega_f + \sum_{n=1}^{\infty} (\Omega_{nf_s - f} + \Omega_{nf_s + f}), \qquad 0 \le f \le f_s/2. \tag{2.33}$$

If y is the ratio of sampling frequency to signal band width and $A_0(y)$ is the ratio of quantizing power received in the signal band to the applied signal power,

$$A_0(y) = \Omega_0(1) + \sum_{n=1}^{\infty} [\Omega_0(ny + 1) + \Omega_0(ny - 1)]. \tag{2.34}$$

This is the equation used in calculating the curves of Fig. (5).

APPENDIX I

Relation Between Mean Squares of Signal and Its Samples

We have already shown that there is a unique relationship between a signal occupying the band of all frequencies less than f_c, and the sampled values of the signal taken at a rate $f_s = 2f_c$. If we are given the signal wave, we can obviously determine the samples; and if we are given the samples, we can determine the signal wave since it is the response of an ideal low-pass filter of cutoff frequency f_c to unit impulses multiplied by the samples. If we apply samples of a signal containing components of frequency greater than f_c, the output of the filter is a new signal with frequencies confined to the band from zero to f_c and yielding the same sampled values as the original wideband signal.

We now consider the problem of determining the mean square value of the samples of an arbitrary function $f(t)$. Let the samples be taken at $t = nT$, $n = 0, \pm 1, \pm 2, \cdots$, where $T = 1/2f_c = 1/f_s$.

We may write an expression for the squared samples as a limit of the product of the squared signal and a periodic switching function of infinitesimal contact time, thus

$$f^2(nt_0) = \operatorname*{Lim}_{\tau \to 0} f^2(t) S(\tau, t) \tag{I—1}$$

where:

$$S(\tau, t) = \begin{pmatrix} 1, & -\tau/2 < t < \tau/2 \\ 0, & \tau/2 < t < T - \tau/2 \end{pmatrix} \tag{I—2}$$

$$S(\tau, t + T) = S(\tau, t), n = 0, \pm 1, \pm 2, \cdots \tag{I—3}$$

By straightforward Fourier series expansion:

$$S(\tau, t) = \frac{\tau}{T} + \sum_{m=1}^{\infty} \frac{2 \sin m\pi\tau/T}{m\pi} \cos 2m\pi f_s t. \tag{I—4}$$

The mean square value of the samples is the limit of the average value of f^2S taken over the contact intervals of duration τ. The average value of f^2S taken over all time, including the blank intervals, is in the limit a fraction τ/T of the average over the contact intervals only. Therefore

$$\begin{aligned} \overline{f^2(nt_0)} &= \operatorname*{Lim}_{\tau \to 0} \overline{\frac{T}{\tau} f^2(t) S(\tau, t)} \\ &= \operatorname*{Lim}_{\tau \to 0} \overline{f^2(t) + \sum_{m=1}^{\infty} \frac{2T \sin m\pi\tau/T}{m\pi\tau} f^2(t) \cos 2m\pi f_s t} \\ &= \overline{f^2(t)} + \operatorname*{Lim}_{\tau \to 0} \sum_{m=1}^{\infty} \frac{2T \sin m\pi\tau/T}{m\pi\tau} \overline{f^2(t) \cos 2m\pi f_s t}. \end{aligned} \tag{I—5}$$

Now the long time average value of $f^2(t) \cos 2m\pi f_s t$ must vanish unless $f^2(t)$ contains a component of frequency mf_s. This could not happen except where $f(t)$ itself contains a component of frequency $mf_s/2$ or two components f_1 and f_2 such that

$$|f_1 \pm f_2| = mf_s \tag{I—6}$$

When no such relation of dependency exists:

$$\overline{f^2(nt_0)} = \overline{f^2(t)}. \tag{I—7}$$

As pointed out before if $f(t)$ contains no frequencies above f_c, the response of the ideal low-pass filter to the samples is $f(t)$, and $f(nt_0)$ represents the samples of $f(t)$. If $f(t)$ does contain frequencies exceeding f_c, the response of the filter is $\phi(t)$, where $\phi(t)$ is wholly confined to the band 0 to f_c and yields the same samples as $f(t)$, i.e.,

$$\phi(nt_0) = f(nt_0), n = 0, \pm 1, \pm 2, \cdots \tag{I—8}$$

Eq. (I—7) applied to $\phi(t)$ gives the result:

$$\overline{\phi^2(nt_0)} = \overline{\phi^2(t)}. \tag{I—9}$$

By combining (I—8) and (I—9), we obtain

$$\overline{f^2(nt_0)} = \overline{\phi^2(t)}. \tag{I—10}$$

APPENDIX II

Fundamental Theorem on Aperture Effect in Sampling

If we sample the wave $Q \cos qt$ at a rate f_s, and multiply each sample by a short rectangular pulse of unit height and duration τ centered at the sampling instants, we obtain by reference to Eq. (I–4) replacing $2\pi f_s$ by ω_s,

$$F(t) = Q \cos qt\, S(\tau, t) = \frac{\tau}{T} Q \cos qt + Q \sum_{m=1}^{\infty} \frac{\sin m\pi\tau/T}{m\pi} [\cos (m\omega_s + q)t + \cos (m\omega_s - q)t]. \tag{II—1}$$

The fact that pulse modulation is similar to the more familiar carrier modulation processes is brought out by this equation; the sampling frequency is in fact the carrier. The writer has found that the method of calculation he published in 1933,[14] in which the signal and carrier frequencies are taken as independent variables, is ideally suited for calculations of pulse-modulated spectra. Artificial and cumbersome devices such as assuming the signal and sampling frequencies to be harmonics of a common frequency are thereby avoided.

A unit impulse $\delta(t)$ has zero duration and unit area; hence we may write:

$$\delta(t) = \lim_{\tau \to 0} \frac{S(\tau, t)}{\tau}. \tag{II—2}$$

A train of samples in which each sample is multiplied by a unit impulse may therefore be written as

$$\sum_{n=-\infty}^{\infty} Q \cos qt\, \delta(t - t_n) = \lim_{\tau \to 0} \left[\frac{Q}{T} \cos qt + Q \sum_{m=1}^{\infty} \frac{\sin m\pi\tau/T}{m\pi\tau} [\cos (m\omega_s + qt) + \cos (m\omega_s - q)t] \right]. \tag{II—3}$$

Suppose we apply the train of waves (II–3) to a linear electrical network which delivers the response $g(t)$ when the input is a unit impulse $\delta(t)$. The steady state admittance of the network is given by[15]

$$Y_0(i\omega) = \int_{-\infty}^{\infty} g(t) e^{-i\omega t}\, dt \tag{II—4}$$

and the response of the network to (II–3) is therefore:

$$\begin{aligned} I(t) &= \frac{Q}{T} \mid Y_0(iq) \mid \cos\,[qt + ph\, Y_0(iq)] \\ &\quad + \frac{Q}{T} \sum_{m=1}^{\infty} (\mid Y_0(im\omega_s + iq) \mid \cos\,[(m\omega_s + q)t \qquad \text{(II—5)} \\ &\quad + ph\, Y_0(im\omega_s + iq)] + \mid Y_0(im\omega_s - iq) \mid \cos\,[(m\omega_s - q)t \\ &\quad + ph\, Y_0(im\omega_s - iq)]). \end{aligned}$$

But $I(t)$ evidently represents a train of pulses in which the pulse occurring at $t = nT$ is equal to the nth sample multiplied by $g(t - nT)$. We have thus obtained the spectrum of a set of samples in which the pulse representing a unit sample is the generalized wave form $g(t)$. Furthermore if the signal frequency q is less than $\omega_s/2$, an ideal low-pass filter with cutoff at $\omega_s/2$ responds only to the first component of (II—5).

The "aperture effect" or variation of transfer admittance with signal frequency is thus given by

$$Y(iq) = \frac{1}{T} Y_0(iq) = f_s Y_0(iq). \qquad \text{(II—6)}$$

This is Theorem II. Since the system is linear when the signal frequency does not exceed half the sampling frequency, the principle of superposition may be applied to composite signals. In the case of distortion from quantizing errors the aperture effect applies to the error component delivered by the low-pass output filter. For an imperfect low-pass filter in the output we multiply the aperture admittance function by the actual transfer admittance of the filter.

A theorem equivalent to the above has been derived by a different method in a recent paper[16] published after completion of the above work.

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