Potential Coefficients for Ground Return Circuits

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This paper is concerned with the effect of the finite conductivity and dielectric constant of the earth on the potential coefficient for a 1-wire ground return circuit. It has been customary to say that the potential coefficient V/Q is

$$p_{12} = c^2 2 \log \rho''/\rho', \text{ elm units per cm.}$$

It is generally realized of course that this is just a good approximation to the true p_{12}. To see that it is just an approximation one has only to imagine the earth turning into air, in which case the distance ρ'' will eventually cease to have significance. The object of this paper is to derive the complete expression for p_{12}.

It turns out that

$$p_{12} = c^2[2 \log \rho''/\rho' + 4(M + iN)] \tag{7}$$

where

$$M + iN = \int_0^\infty \frac{e^{-(h+z)\zeta\sqrt{\alpha}\,t} \cos y\,\zeta\sqrt{\alpha}t}{\sqrt{t^2 + ie^{i2\eta}} + (\epsilon - i2c\lambda\sigma)t}\,dt \tag{8}$$

$\alpha = 4\pi\sigma\omega$, as in Carson's work on Z_{12} and in mine on Z_{12} at high frequencies[1,2]

$$\zeta e^{i\eta} = \sqrt{1 + i(\epsilon - 1)/2c\lambda\sigma} = s$$

ϵ = dielectric constant in electrostatic units
σ = conductivity in electromagnetic units
= 10^{-13} to 10^{-14} in ordinary soil
λ = wavelength in centimeters
c = velocity of light, in cm per sec.

$$M + iN \text{ vanishes as } f \to 0, f \to \infty, \epsilon \to \infty \text{ or } \sigma \to \infty.$$

Ordinarily $4(M + iN)$ will not be an important correction to $2 \log \rho''/\rho'$; but if the frequency is high and h or z is small it can be a worthwhile correction. For example, if a .02535 inch wire be thrown out on the ground to be a 2 mc antenna and we assume that $\sigma = 10^{-13}$, $\epsilon = 15$ and $h = 3$ cm. then, with a for wire radius,

$$\begin{aligned} p_{11} &= c^2[2 \log 2h/a + 4(M + iN)] \\ &= c^2[10.455 + .152 + i\,.319]. \end{aligned}$$

$1/p_{11}$ is the capacity to ground. If there were two parallel wires the scalar potential at the first wire would be $V_1 = p_{11}Q_1 + P_{12}Q_2$.

Derivation of the Formula

WE BEGIN with the wave-function for an exponentially propagated current in a straight wire parallel to a flat earth. The wave-function for a horizontal current-element dipole has been formulated as an infinite

[1] John R. Carson: "Wave Propagation in Overhead Wires with Ground Return," *Bell Sys. Tech. Jour.* 5, pp. 539–554, 1926.
[2] W. Howard Wise: "Propagation of High-Frequency Currents in Ground Return Circuits," *Proc. I. R. E.* 22, pp. 522–527, 1934.

integral by H. von Hoerschelmann[3]. The wave-function for the current in the wire is obtained by integrating the wave-functions of the current-element dipoles along the wire from minus infinity to plus infinity. It is

$$\Pi = a \times e^{-\gamma x} \int_{-\infty}^{\infty} I_0 e^{-\gamma x} \left(\frac{e^{-ikR_1}}{R_1} - \frac{e^{-ikR_2}}{R_2} + \int_0^{\infty} \frac{2J_0(\nu\rho)}{l+m} e^{-wl} \nu \cdot d\nu \right) dx + b \times 0 \qquad (1)$$

$$- c \times 2e^{-\gamma x} \int_{-\infty}^{\infty} I_0 e^{-\gamma x} \frac{\partial}{\partial x} \int_0^{\infty} \frac{(1-\tau^2)J_0(\nu\rho)\nu}{(l+m)(l+\tau^2 m)} e^{-wl} d\nu \cdot dx.$$

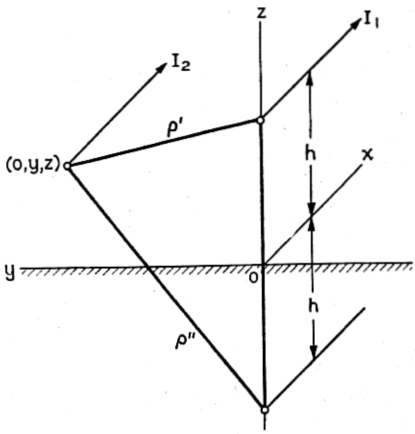

Fig. 1

The time factor is $e^{i\omega t}$. a, b and c are unit vectors pointing in the x, y and z directions.

$$R_1 = (x^2 + y^2 + (h - z)^2)^{1/2}$$
$$R_2 = (x^2 + y^2 + (h + z)^2)^{1/2}$$
$$\rho = (x^2 + y^2)^{1/2}$$
$$k = 2\pi/\lambda$$
$$k_2^2 = \epsilon\mu\omega^2 - i4\pi\sigma\mu\omega \text{ in electromagnetic units}$$

By supposing ϵ to be measured in electrostatic units we can write

$$k_2^2 = k^2(\epsilon - i2c\lambda\sigma)\mu.$$

[3] H. Von Hoerschelmann, Jahrb. der draht. Teleg. 5, pp. 14–188, 1912.

It is assumed that μ is everywhere unity in electromagnetic units.

$$l = (\nu^2 - k^2)^{1/2}, \qquad m = (\nu^2 - k_2^2)^{1/2}$$
$$w = h + z, \qquad \tau^2 = k^2/k_2^2$$

$\gamma = \alpha + i\beta$ is the desired propagation constant

The electric field parallel to the wire is

$$E_x = -i\omega\left[\Pi_x + k^{-2}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\Pi_x + \frac{\partial}{\partial y}\Pi_y + \frac{\partial}{\partial z}\Pi_z\right)\right]$$
$$= -I_0\, e^{-\gamma x}\, Z_{12} - \frac{\partial V}{\partial x} \tag{2}$$

It has previously been shown that[2]

$$Z_{12} = i\omega[2 \log \rho''/\rho' + 4(Q - iP)] \tag{3}$$

where

$$Q - iP = \frac{1}{is^2}\int_0^\infty (\sqrt{\nu^2 + is^2} - \nu)e^{-w'\nu} \cos y'\nu \cdot d\nu,$$
$$w' = w\sqrt{\alpha} \text{ and } y' = y\sqrt{\alpha}.$$

To get the potential coefficient for a ground return circuit it is necessary to compute the scalar potential.

$$V = i\omega k^{-2}\left(\frac{\partial}{\partial x}\Pi_x + \frac{\partial}{\partial y}\Pi y + \frac{\partial}{\partial z}\Pi_z\right)$$
$$= Q\, p_{12} \tag{4}$$

As in previous work the propagation constant γ is assigned the value ik as a first approximation. This is an ideal value for γ but the following considerations make it an imperative choice: (1) to assume that the current is propagated down the line with a velocity less than that of light makes the integrals very hard to evaluate, (2) to assume that the attenuation is not zero on an infinite line amounts to assuming an infinite source of energy and makes the integrals diverge.

It should not be inferred that the resulting formulas are necessarily poor if the physical system does not closely approximate the ideal one in which γ is ik. ik is employed as a convenient first approximation in evaluating the correction terms in Z_{12} and p_{12}. Eventually, if there were but one wire, one would compute $\gamma = \sqrt{(z + Z_{11})(G + i\omega/p_{11})}$, wherein Z_{11} and p_{11} have been evaluated with ik for γ, and this would be a second approximation to γ. Past experience with the second approximation so obtained has justified the expectation that it would be a satisfactory final result. Since the integrals diverge if the attenuation is not zero the use of an infinite line formula presupposes reasonably efficient transmission.

Since $\Pi \propto e^{-\gamma x}$ we have

$$i\omega k^{-2}\frac{\partial}{\partial x}\Pi_x = \frac{\omega}{k}\Pi_x = \frac{I}{ik}Z_{12} = \frac{cI}{i\omega}Z_{12}\,.$$

Since $-\dfrac{\partial I}{\partial x} = \dfrac{\partial Q}{\partial t}$ or $ikI = i\omega Q$ or $I = cQ$ this is

$$i\omega k^{-2}\frac{\partial}{\partial x}\Pi_x = Qc^2\,[2\log \rho''/\rho' + 4(Q - iP)]. \tag{5}$$

We have next to consider

$$\frac{i\omega}{k^2}\frac{\partial}{\partial z}\Pi_z = \frac{2\omega(1-\tau^2)}{ik^2}I\frac{\partial}{\partial z}\int_{-\infty}^{\infty}e^{-\gamma x}\frac{\partial}{\partial x}\int_0^{\infty}\frac{J_0(\nu\rho)e^{-wl}\,\nu\cdot d\nu}{(l+m)(l+\tau^2 m)}\,dx$$

$$= Qc^2\,\frac{2(1-\tau^2)}{ik}\frac{\partial}{\partial z}\int_0^{\infty}\frac{e^{-wl}\,\nu\cdot d\nu}{(l+m)(l+\tau^2 m)}\int_{-\infty}^{\infty}e^{-\gamma x}\frac{\partial}{\partial x}J_0(\nu\rho)\,dx.$$

The infinite integral is

$$\int_0^{\infty}\frac{e^{-wl}\,\nu\cdot d\nu}{(l+m)(l+\tau^2 m)}\left[e^{-\gamma x}J_0(\nu\rho)\Big|_{-\infty}^{\infty} + \gamma\int_{-\infty}^{\infty}e^{-\gamma x}J_0(\nu\rho)\,dx\right].$$

Since $J_0(\nu\sqrt{x^2+y^2})$ and $\cos kx$ are even functions of x and $\sin kx$ is an odd function of x

$$\int_{-\infty}^{\infty}J_0(\nu\sqrt{x^2+y^2})e^{-ikx}\,dx = 2\int_0^{\infty}J_0(\nu\sqrt{x^2+y^2})\cos kx\cdot dx$$

$$= 0 \quad \text{if} \quad \nu < k$$

$$= 2\,\frac{\cos y\sqrt{\nu^2-k^2}}{\sqrt{\nu^2-k^2}} \quad \text{if} \quad k \le \nu$$

and so our integral is

$$2ik\int_k^{\infty}\frac{e^{-wl}\,\nu}{(l+m)(l+\tau^2 m)}\cdot\frac{\cos yl}{l}\,d\nu$$

or, since $l^2 = \nu^2 - k^2$,

$$2ik\int_0^{\infty}\frac{e^{-wl}\cos yl\cdot dl}{(l+\sqrt{l^2+i^2(k_2^2-k^2)})(l+\tau^2\sqrt{l^2+i^2(k_2^2-k^2)})}$$

or, if we put $l = \nu\sqrt{\alpha}$

and $i(k_2^2 - k^2) = 4\pi\sigma\omega\left(1 + i\frac{\epsilon - 1}{2c\lambda\sigma}\right) = \alpha s^2$,

$$\frac{2ik}{\sqrt{\alpha}}\int_0^\infty \frac{e^{-w'\nu}\cos y'\nu\cdot d\nu}{(\nu + \sqrt{\nu^2 + is^2})(\nu + \tau^2\sqrt{\nu^2 + is^2})}$$

where, as in $Q - iP$, $w' = w\sqrt{\alpha}$ and $y' = y\sqrt{\alpha}$.

Noting next that $\frac{\partial}{\partial z} e^{-w'\nu} = -\sqrt{\alpha}\nu e^{-w'\nu}$

we have

$$\frac{i\omega}{k^2}\frac{\partial}{\partial z}\Pi_z = -Qc^2 4\int_0^\infty \frac{(1 - \tau^2)e^{-w'\nu}\cos y'\nu\cdot\nu\cdot d\nu}{(\nu + \sqrt{\nu^2 + is^2})(\nu + \tau^2\sqrt{\nu^2 + is^2})}$$

$$= -Qc^2\frac{4}{is^2}\int_0^\infty \frac{\sqrt{\nu^2 + is^2} - \nu}{\nu + \tau^2\sqrt{\nu^2 + is^2}}e^{-w'\nu}\cos y'\nu\cdot(1 - \tau^2)\nu\cdot d\nu.$$

Since $(1 - \tau^2)\nu = \nu + \tau^2\sqrt{\nu^2 + is^2} - \tau^2(\sqrt{\nu^2 + is^2} + \nu)$
this is

$$-Qc^2\frac{4}{is^2}\int_0^\infty\left[\sqrt{\nu^2 + is^2} - \nu - \frac{\tau^2 is^2}{\nu + \tau^2\sqrt{\nu^2 + is^2}}\right]e^{-w'\nu}\cos y'\nu\cdot d\nu$$

$$= Qc^2\left[-4(Q - iP) + 4\int_0^\infty \frac{e^{-w'\nu}\cos y'\nu\cdot d\nu}{\sqrt{\nu^2 + is^2} + \nu/\tau^2}\right]. \quad (6)$$

On adding (6) to (5) we have

$$Qp_{12} = Qc^2[2\log\rho''/\rho' + 4(M + iN)] \quad (7)$$

where

$$M + iN = \int_0^\infty \frac{e^{-w'\nu}\cos y'\nu}{\sqrt{\nu^2 + is^2} + \nu/\tau^2}d\nu$$

$$= \int_0^\infty \frac{e^{-(h+z)\zeta\sqrt{\alpha}t}\cos y\sqrt{\alpha}\zeta t}{\sqrt{t^2 + ie^{i2\eta}} + t/\tau^2}dt. \quad (8)$$

$M + iN$ vanishes as $f \to 0, f \to \infty, \epsilon \to \infty$ or $\sigma \to \infty$.

When $k_2^2 - k^2$ is minute the leading terms in the approximation (9)

for M + iN are $\frac{\pi}{8} - \frac{C}{2} - \frac{1}{2}\log(\text{p}''\sqrt{(k_2^2 - k^2)/2}) - \text{i}\frac{\pi}{4}$.

An Approximation for $M + iN$

It is possible to get series expansions for $M + iN$ but those which have been obtained do not facilitate computation. A fairly good approximation to $M + iN$ is arrived at as follows.

Let $ie^{i2\eta} = u^2, \qquad u = e^{i(\eta+\pi/4)}$

$\epsilon - i2c\lambda\sigma = a = 1/\tau^2$

$e^{-(h+z)\zeta\sqrt{\alpha}t} \cos y\zeta\sqrt{\alpha}t = \frac{1}{2}(e^{-g't} + e^{-g''t})$

$g' = (h + z - iy)\zeta\sqrt{\alpha}$

$g'' = (h + z + iy)\zeta\sqrt{\alpha}$

$g = (h + z)\zeta\sqrt{\alpha}$

Since $(t^2 + u^2)^{1/2} = (t^2 + 2tu + u^2 - 2tu)^{1/2}$

$= t + u - tu/(t + u) + \dots$

we put

$$1/(\sqrt{t^2 + u^2} + at) = 1/(t + u - tu/(t + u) + at)$$
$$= (t + u)/[(a + 1)t^2 + (a + 1)tu + u^2]$$
$$= (t + r_1 + r_2)/(a + 1)(t + r_1)(t + r_2)$$

where $r_1 = u(1 - \sqrt{1 - 4/(a + 1)})/2$.

and $\quad r_2 = u(1 + \sqrt{1 - 4/(a + 1)})/2$.

Then

$$M + iN \approx \frac{1}{(a + 1)(r_2 - r_1)} \int_0^\infty \left(\frac{r_2}{t + r_1} - \frac{r_1}{t + r_2}\right) \frac{e^{-g't} + e^{-g''t}}{2} dt$$

$$\doteq \frac{1}{2(a + 1)(r_2 - r_1)} [-r_2 e^{g'r_1} \operatorname{li}(e^{-g'r_1}) + r_1 e^{-g'r_2} \operatorname{li}(e^{-g'r_2})$$
$$- r_2 e^{g''r_1} \operatorname{li}(e^{-g''r_1}) + r_1 e^{g''r_2} \operatorname{li}(e^{-g''r_2})]. \qquad (9)$$

When $y = 0$ this reduces to

$$M + iN \approx \frac{1}{(a + 1)(r_2 - r_1)} [-r_2 e^{gr_1} \operatorname{li}(e^{-gr_1}) + r_1 e^{gr_2} \operatorname{li}(e^{-gr_2})]. \qquad (10)$$

$$\operatorname{li}(e^{-z}) = -\int_z^\infty \frac{e^{-t}}{t} dt = C + \log z + \sum_{\iota=1}^{\infty} \frac{(-z)^\iota}{\iota!\,\iota},$$

where $C = .577215665$ and, if $z = re^{i\theta}, \quad -\pi \leq \theta \leq \pi$.

$$\operatorname{li}(e^{-z}) \sim \frac{e^{-z}}{-z} \sum_{\iota=0}^{n} \frac{\iota!}{(-z)^\iota} + R_n .$$

The accompanying charts give M and N for $y = 0$ and $\epsilon = 15$. With $z = h$ they give M and N for p_{11}. The computed points are indicated by solid dots on the chart for M; they were obtained by numerical integration.

The approximation (10) was checked against the values obtained by numerical integration at a number of points. The discrepancy in each case amounted to less than one per cent for both M and N. This approximation is a much easier way to evaluate the integral than is numerical integration but it is a tedious computation with many chances for error. Conse-

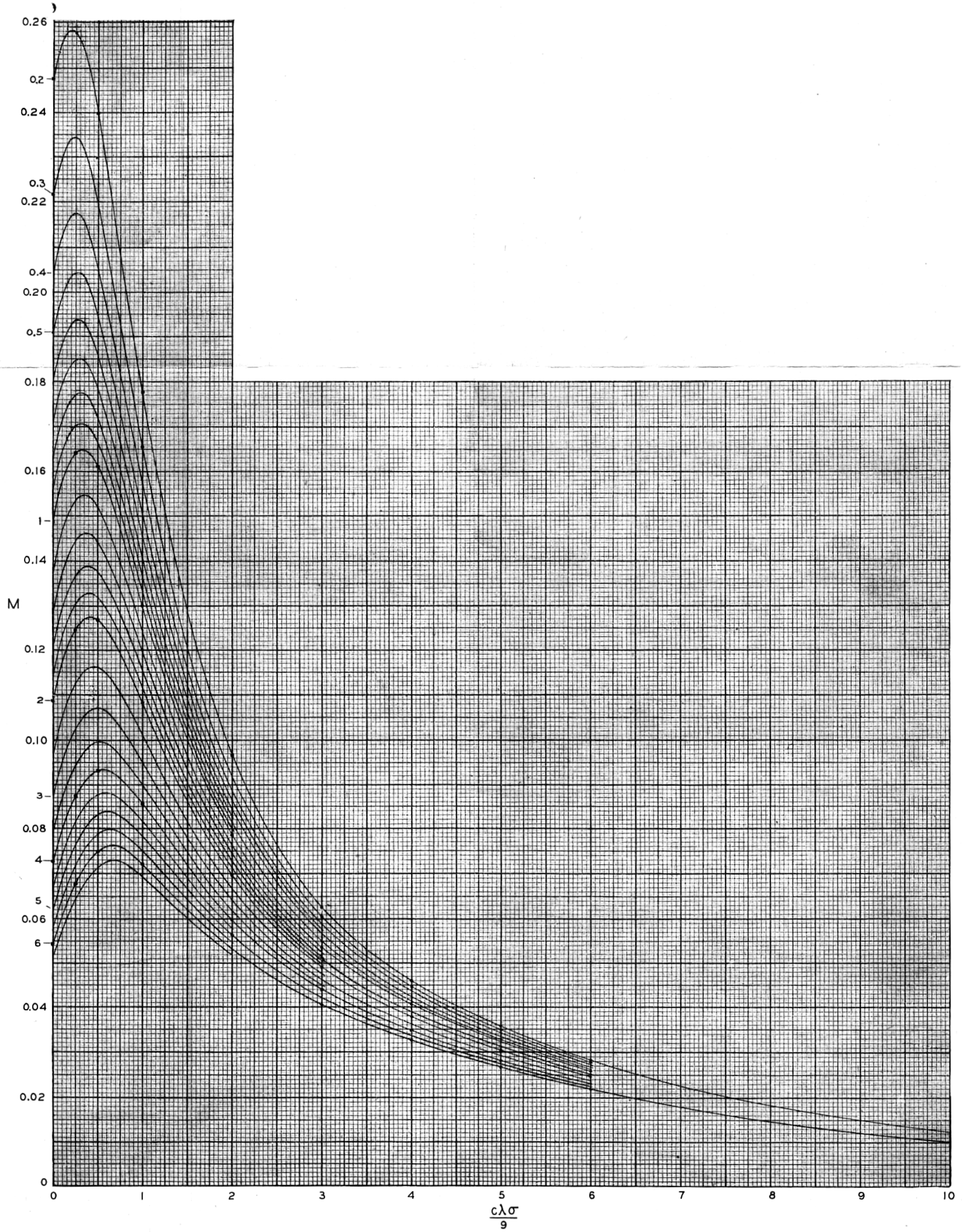

Fig. 2—M, computed with $\epsilon = 15$ and $y = 0$. The number associated with a curve is the value of $(h + z)\zeta\sqrt{a}$.

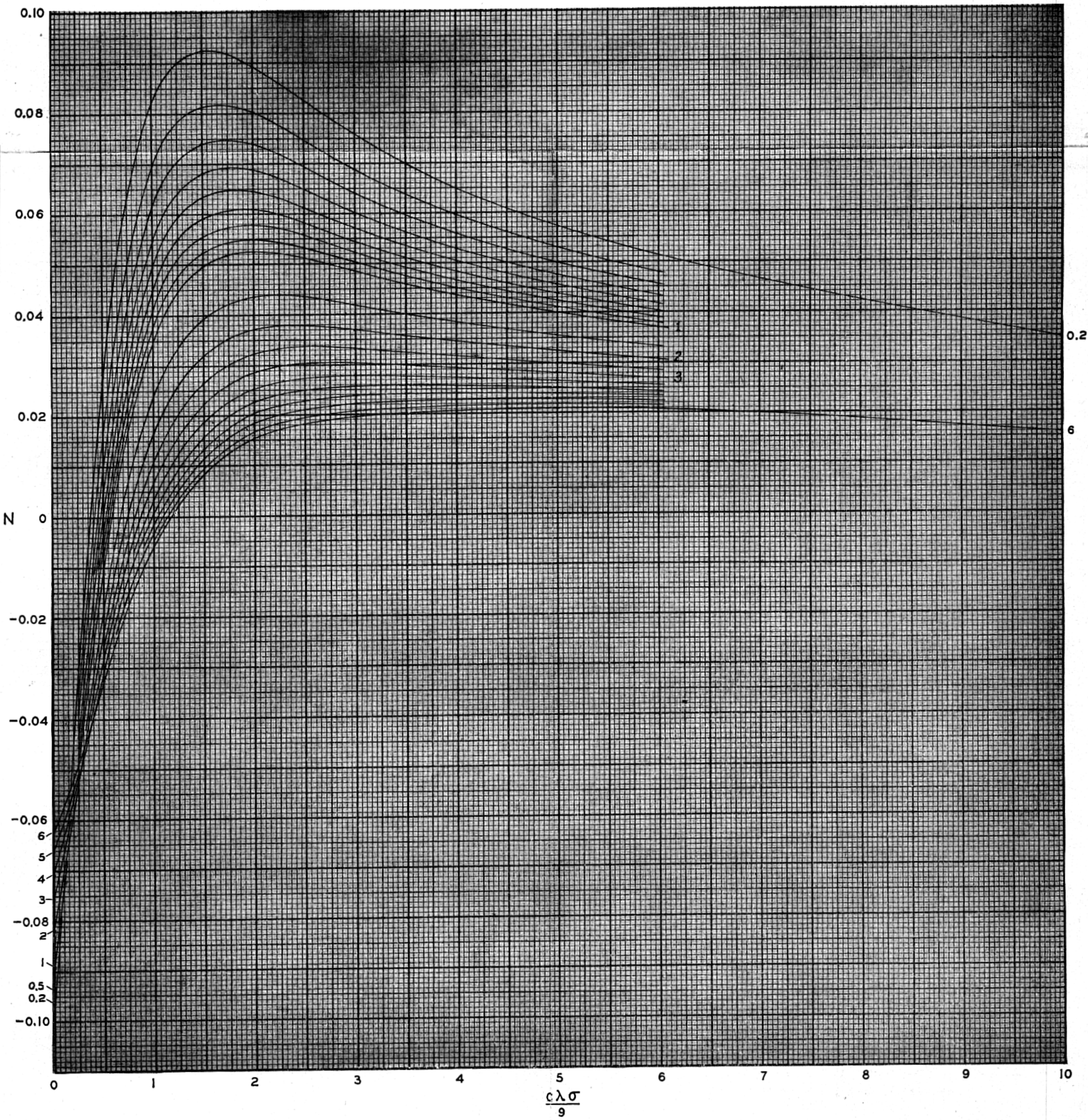

Fig. 3—N, computed with $\epsilon = 15$ and $y = 0$. The number associated with a curve is the value of $(h + z)\zeta\sqrt{a}$.

quently it is important to observe that a coarser kind of approximation may often be good enough. Thus, taking y to be zero,

$$M + iN \approx \int_0^\infty \frac{e^{-gt}}{u + at}\, dt = -\frac{1}{a} e^{gu/a} \operatorname{li}(e^{-gu/a}).$$

If g is very small one might use

$$M + iN \approx \int_0^1 \frac{dt}{u + at} + \int_1^\infty \frac{e^{-gt}}{(1 + a)t}\, dt$$

$$= \frac{1}{a} \log\left(1 + \frac{a}{u}\right) - \frac{1}{1 + a} \operatorname{li}(e^{-g}).$$

Ordinarily precision is not required in $M + iN$ because $4(M + iN)$ is a small term in p_{12}.