The Approximate Solution of Linear Differential Equations

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Linear differential equations with variable coefficients occur in many fields of applied mathematics: in the theories of acoustics, elastic waves, electromagnetic waves in stratified media, nonuniform transmission lines, wave guides, antennas, wave mechanics. The "Wave Perturbation" method described in greater detail elsewhere[1] is particularly useful in those ranges of the independent variable in which the "WKB Approximation" is not sufficiently accurate. The present paper endeavors to illustrate the remarkable accuracy of this method, particularly when compared with Picard's method.

I. Introduction

IN A recent paper[1] the approximate solution of linear differential equations by a wave perturbation method was described. When the method was applied to equations whose exact solutions were known we were greatly impressed by the rapidity of convergence of the successive approximations. Hence the purpose of this note is to present some illustrations in the hope that others may be interested and may find the proposed method an improvement on those now in use.

In essence the wave perturbation method dates back to Liouville[2], but in his mémoires he was interested in a problem of heat conduction involving a non-homogeneous differential equation with homogeneous boundary conditions, whereas we consider a homogeneous equation

$$y'' = F(x)y \tag{1}$$

with non-homogeneous initial conditions

$$y(a) = 1,\ y'(a) = 0 \tag{2a}$$

or

$$y(a) = 0 \qquad y'(a) = 1, \tag{2b}$$

the solution being desired in an interval $a \leqq x \leqq b$. Since the solution for any assigned initial or boundary conditions can be expressed as a linear combination of the solutions satisfying (2a) and (2b) we have not imposed any real limitation.

II. Theory

Comparison of the wave perturbation method with Picard's method (which is essentially a linear perturbation method) is particularly instructive. It will be recalled that in Picard's formulation the differential equation (1) is replaced by an integral equation

$$y(x) = y(a) + (x - a)y'(a) + \int_a^x F(u)y(u)(x - u)\,du \tag{3}$$

where $y(a)$ and $y'(a)$ are assigned initial values*. Writing

$$L_0(x) = y(a) + (x - a)y'(a), \tag{4}$$

$$L_n(x) = \int_a^x F(u)L_{n-1}(u)(x - u)\,du, \qquad n = 1, 2, 3, \cdots,$$

the series

$$y(x) = L_0(x) + L_1(x) + L_2(x) + \cdots \tag{5}$$

is shown to converge to a solution of the original equation. In practical applications, unfortunately, it is usually found that the successive approximations converge rather slowly unless the interval (a, b) is small.

In the wave perturbation method we first rewrite equation (1) in the form

$$y'' = -\beta^2 y + [\beta^2 + F(x)]y = -\beta^2 y + f(x)y, \tag{6}$$

and instead of the integral equation (3) we use

$$\begin{aligned} y(x) = {} & y(a)\cos\beta(x - a) + \frac{1}{\beta}y'(a)\sin\beta(x - a) \\ & + \frac{1}{\beta}\int_a^x f(u)y(u)\sin\beta(x - u)\,du. \end{aligned} \tag{7}$$

The parameter β is arbitrary and might be defined in various ways. We have found it convenient to use the definition

$$\beta^2 = -\frac{1}{b - a}\int_a^b F(x)\,dx, \tag{8}$$

so that if $F(x)$ is negative β is real and our first approximation

$$W_0(x) = y(a)\cos\beta(x - a) + \frac{1}{\beta}y'(a)\sin\beta(x - a) \tag{9}$$

is sinusoidal. If F is positive β is imaginary and we start with an exponential approximation. If F changes sign in (a, b) the best procedure is to

* This is not quite the usual form of the integral equation but it is substantially equivalent.

subdivide the interval and obtain separate approximations, though this is not necessary if F is predominantly of one sign throughout (a, b). To (9) we now add the sequence

$$W_n(x) = \frac{1}{\beta} \int_a^x f(u) W_{n-1}(u) \sin \beta(x - u) \, du, \tag{10}$$

and the series

$$y(x) = W_0(x) + W_1(x) + W_2(x) + \cdots \tag{11}$$

is the desired solution.

The flexibility of the wave perturbation method as compared with Picard's linear method lies essentially in the introduction of the variable parameter β. Since we make β depend on the length of the interval (a, b) in which a solution is desired the approximations may be extended over much longer intervals than is feasible in Picard's method. If $F(x)$ is a slowly varying function throughout (a, b), so that $f(x)$ is small, it will be found that the first approximation $W_0(x)$ is good, and the second, $W_0 + W_1$ is generally adequate.

Another choice for β is

$$\beta = \frac{1}{b - a} \int_a^b \sqrt{-F(x)} \, dx. \tag{12}$$

However, the integration in (8) will often be simpler than in (12).

Picard's method is a special case of the wave perturbation method, with $\beta = 0$. In fact, if $F(x)$ changes sign in (a, b), then in some cases β as defined by (8) will reduce to zero.

If $F(x)$ is a rapidly varying function, or if the solution is desired over an infinite interval, it is usually advantageous to transform equation (1) by first introducing a new independent variable

$$\theta = \int^x \sqrt{-F(x)} \, dx, \tag{13}$$

and then removing the first order term in the new equation by an appropriate transformation of the dependent variable.

III. Examples

For our illustrations we have used mainly the simple equation

$$y'' = -xy \tag{14}$$

whose exact solution can be expressed in terms of Bessel functions of order $\pm 1/3$. Since the Bessel functions are oscillatory in nature it might be suggested that comparison with Picard's method is weighted in our favor. This does not seem to be the case, as will be illustrated in example 4 where the exact solution is a monotonically increasing function. It has also been suggested that Picard's solution might be improved by starting from a better initial approximation, say W_0, rather than from the linear approximation L_0, but we have not found any marked improvement in the succeeding approximations (see examples 1 and 2). The various points of interest will be brought out in our examples, with the accompanying figures, which we shall now briefly describe. In each figure the heavy curve is the accurate solution while the approximations are indicated by self-explanatory letters.

Example 1, Fig. 1

$$y'' = -xy, \; 0 \leqq x \leqq 2$$

$$y(0) = 1, \; y'(0) = 0$$

Exact solution: $$y(x) = \Gamma(\tfrac{2}{3})3^{-1/3}x^{1/2}J_{-\frac{1}{3}}(\tfrac{2}{3}x^{3/2})$$

(a) Wave perturbation

$$W_0 = \cos x$$

$$W_1 = -\tfrac{1}{4}x\cos x + \tfrac{1}{4}(1 + 2x - x^2)\text{ si}$$

(b) Linear perturbation

$$L_0 = 1$$

$$L_1 = -\frac{x^3}{6}$$

(c) Linear perturbation using initial sinusoidal approximation

$$\bar{L}_0 = \cos x = W_0$$

$$\bar{L}_1 = x + x\cos x - 2\sin x$$

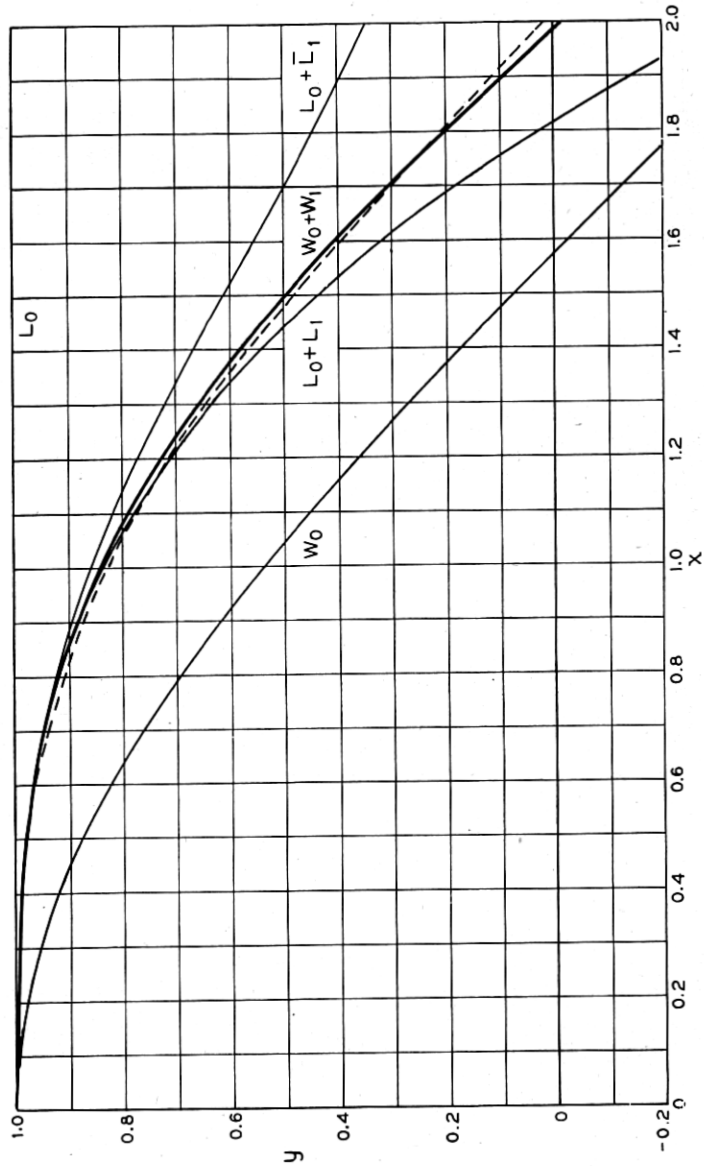
Example 2, Figs. 2, 3 and 4

$$y'' = -xy, \; 2 \leqq x \leqq 6$$

$$y(2) = 0, \; y'(2) = 1$$

Exact solution:

$$y(x) = -.84423x^{1/2}J_{-1/3}(\tfrac{2}{3}x^{3/2}) - .019291x^{1/2}J_{1/3}(\tfrac{2}{3}x^{3/2})$$



L0
L0+L̄1
W0+W1
L0+L1
W0
y
1.0
0.8
0.6
0.4
0.2
0
-0.2
X
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0

Fig. 1

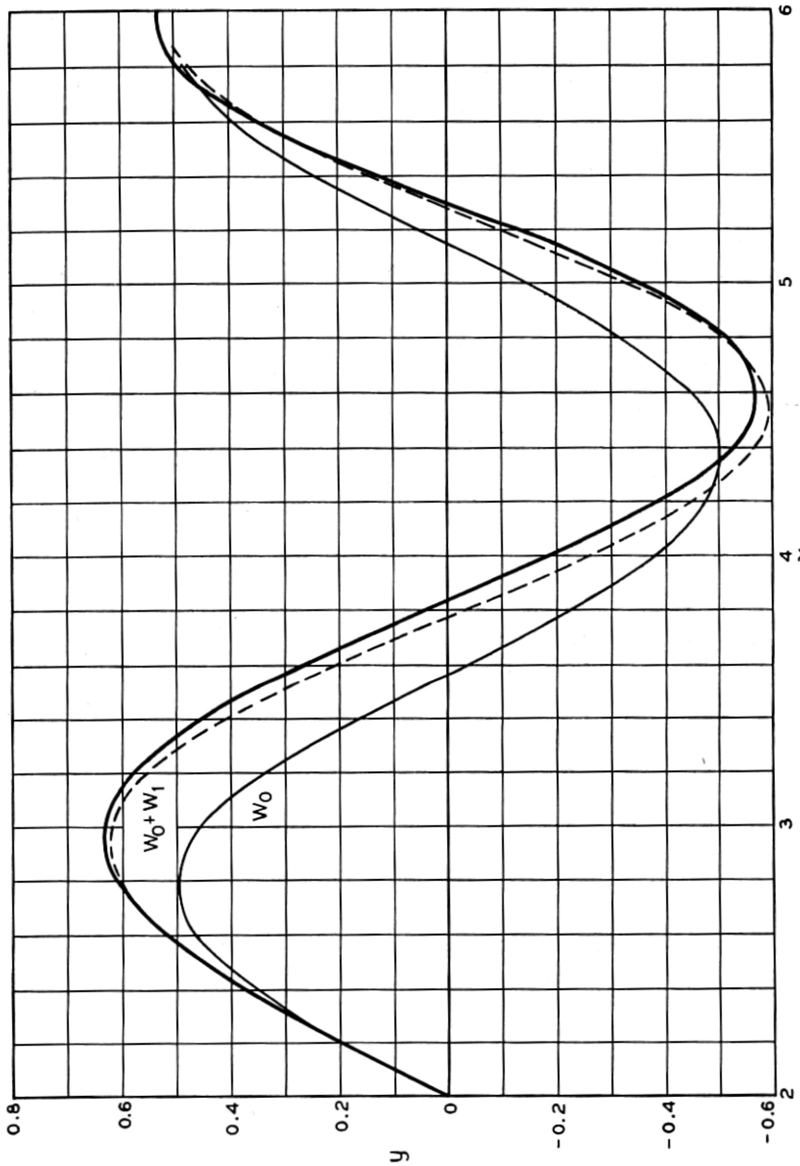


Fig. 2

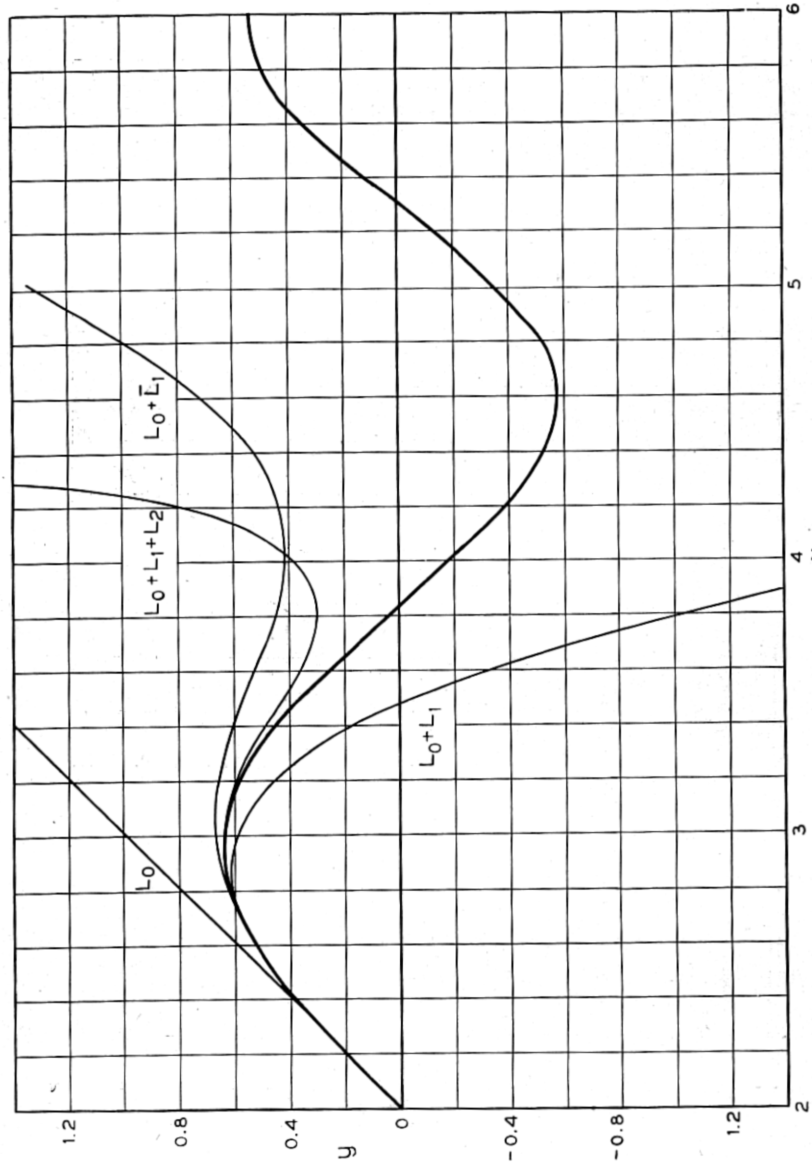


Fig. 3

(a) Wave perturbation, Fig. 2

$$W_0 = \tfrac{1}{2}\sin 2(x-2)$$

$$W_1 = \frac{6-x}{32}\sin 2(x-2) + \frac{12-8x+x^2}{16}\cos 2(x-2).$$

Figure 2 exhibits rapid pulling of the successive approximate waves toward the exact even though the interval has been chosen deliberately unfavorable to the straight wave perturbation method [see example (c) and Fig. 4 for the improved treatment].

(b) Linear perturbation, Fig. 3

$$L_0 = x - 2$$

$$L_1 = \tfrac{1}{12}(16 - 16x + 4x^3 - x^4)$$

$$L_2 = -\frac{16}{63} + \frac{16x}{45} - \frac{2x^3}{9} + \frac{x^4}{9} - \frac{x^6}{90} + \frac{x^7}{504}.$$

Using W_0 instead of L_0

$$\bar{L}_1 = \tfrac{7}{8} - \frac{x}{2} + \frac{x}{8}\sin 2(x-2) + \tfrac{1}{8}\cos 2(x-2).$$

(c) Preliminary transformation of variables, Fig. 4
Introduce $\theta = \frac{2}{3}x^{3/2}$, $y = \theta^{-1/6}v$
and the modified equation is

$$v'' = -\left(1 + \frac{5}{36\theta^2}\right)v \qquad \frac{4\sqrt{2}}{3} \leqq \theta \leqq 4\sqrt{6}.$$

Then, using for simplicity $\beta = 1$

$$v_0 = 2^{-1/2}\theta_0^{1/6}\sin(\theta - \theta_0),\ \theta_0 = 4\sqrt{2}/3$$

or

$$W_0 = (2x)^{-1/4}\sin \tfrac{2}{3}(x^{3/2} - 2^{3/2})$$

It will be seen that W_0 is a very good approximation throughout the range (2, 6). Adding W_1 obtained from

$$v_1 = \frac{5\theta_0^{1/6}}{36\sqrt{2}}\left[\cos(\theta+\theta_0)(Si\,2\theta - Si\,2\theta_0) - \sin(\theta+\theta_0)(Ci\,2\theta - Ci\,2\theta_0)\right]$$

the accurate curve y is reproduced.

In Fig. 2 the third approximation could not be distinguished from the accurate curve though numerically the values are not identical. For purposes of comparison the table of numerical values (Table A) may be found interesting.

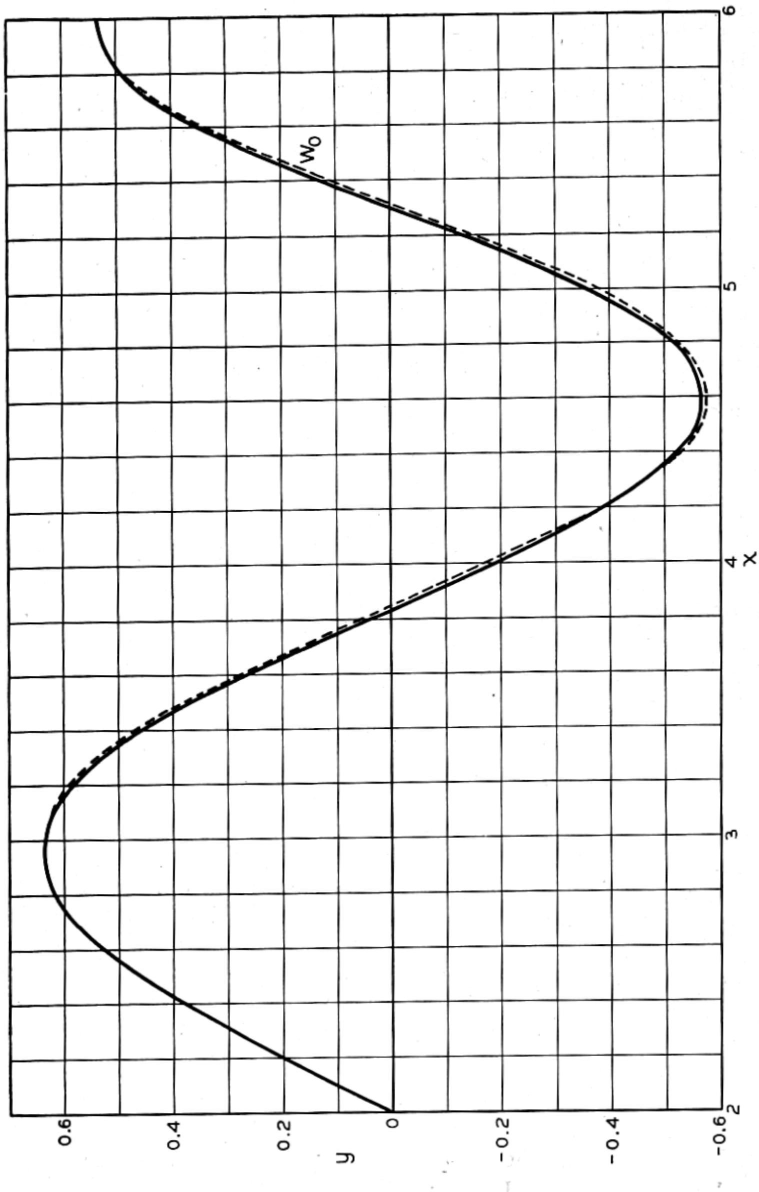


Fig. 4

TABLE A

x	y	W_0	$\sum_0^1 W$	$\sum_0^2 W$	$\sum_0^3 W$
2.0	0.	0.	0.	0.	0.
2.2	.19721	.19471	.19720	.19721	.19721
2.4	.37694	.35868	.37668	.37694	.37694
2.6	.52056	.46602	.51885	.52053	.52056
2.8	.61035	.49979	.60442	.61020	.61034
3.0	.63236	.45465	.61792	.63180	.63232
3.2	.57922	.33773	.55169	.57781	.57918
3.4	.45287	.16749	.40907	.44995	.45726
3.6	.26584	−.02919	.20603	.26085	.26561
3.8	.04126	−.22126	−.02974	.03408	.04087
4.0	−.18921	−.37840	−.26229	−.19800	−.18974
4.2	−.38951	−.47580	−.45326	−.39852	−.39011
4.4	−.52506	−.49808	−.56889	−.53240	−.52557
4.6	−.56943	−.44173	−.58697	−.57328	−.56964
4.8	−.51062	−.31563	−.50217	−.51000	−.51044
5.0	−.35548	−.13971	−.32847	−.35076	−.35494
5.2	−.13068	.05827	−.09772	−.12364	−.13006
5.4	−.12052	.24706	.14547	.12725	.12080
5.6	.34582	.39683	.35200	.34988	.34536
5.8	.49485	.48396	.47807	.49525	.49347
6.0	.53114	.49467	.49467	.52903	.52903

Example 3, Fig. 5

$$y'' = -y + \frac{2}{x^2}y, \qquad 1 \leqq x \leqq \infty$$

$$y(1) = 1,\ y'(1) = 0$$

Exact solution: $y(x) = \sin(x-1) + \frac{1}{x}\cos(x-1)$

(a) Wave perturbation, with the initial conditions satisfied exactly

$$W_0 = \cos(x-1)$$

$$W_1 = 2\sin(x-1) - 2\cos(x+1)(Ci\,2x - Ci\,2) - 2\sin(x+1)(Si\,2x - Si\,2)$$

(b) Wave perturbation, matching the exact solution at infinity

$$\tilde{W}_0 = \sin(x-1)$$

$$\tilde{W}_1 = 2\sin(x+1)\,Ci\,2x - 2\cos(x+1)(Si\,2x - \pi/2)$$

This is an example of a solution in an infinite interval, where the perturbation term is not small throughout. It is interesting to note that the second form gives good agreement with the accurate solution in most of the range of integration.

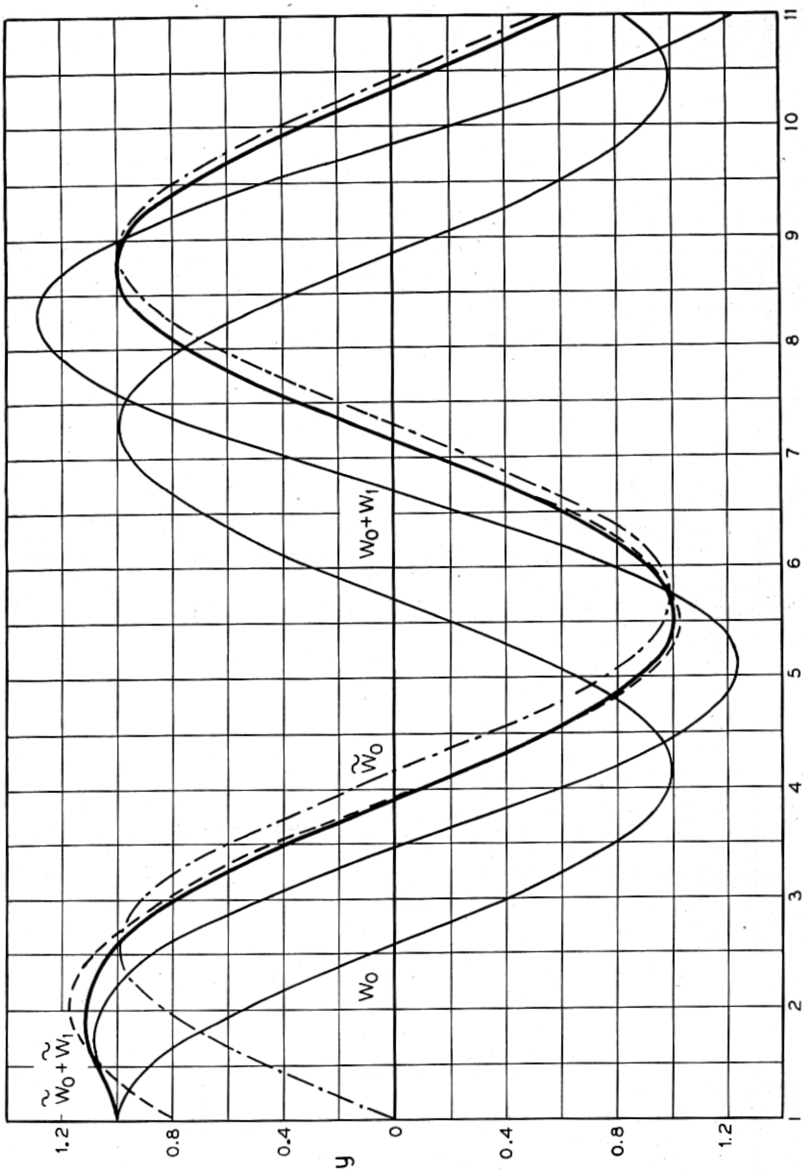

y
x
1.2
0.8
0.4
0
0.4
0.8
1.2
1
2
3
4
5
6
7
8
9
10
11
W_0
$W_0 + W_1$
$\tilde{W}_0$
$\tilde{W}_0 + \tilde{W}_1$

Fig. 5

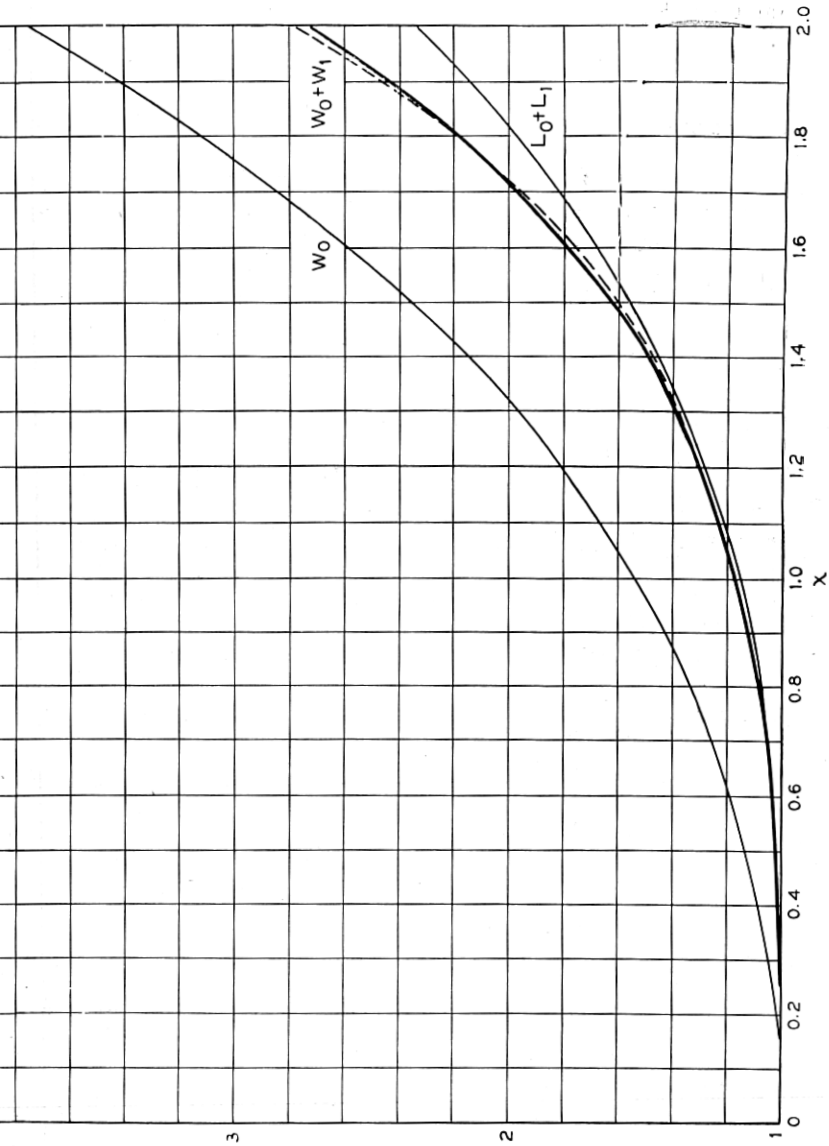


Fig. 6

Example 4, Fig. 6

$$y'' = + xy, \qquad 0 \leqq x \leqq 2$$
$$y(0) = 1, y'(0) = 0$$

Exact solution:

$$y(x) = \Gamma(\tfrac{2}{3})3^{-1/3} x^{1/2} I_{-1/3}(\tfrac{2}{3}x^{3/2}).$$

(a) Wave perturbation

$$W_0 = \cosh x$$
$$W_1 = -\frac{x}{4} \cosh x + \tfrac{1}{4}(x^2 - 2x + 1) \sinh x$$

(b) Linear perturbation

$$L_0 = 1$$
$$L_1 = \frac{x^3}{6}.$$

This is an example in which the exact solution is non-oscillatory yet even in the short interval (0, 2) $W_0 + W_1$ is a better approximation than $L_0 + L_1$.

Example 5, Table I

$$y'' + \frac{1}{x} y' + y = 0, \quad 0 < x \leqq \infty$$

Solution required to match the accurate solution

$$y(x) = J_0(x) - i\, N_0(x)$$

at infinity:

$$\tilde{W}_0 = \frac{1 + i}{\sqrt{\pi x}} e^{-ix}$$
$$\tilde{W}_1 = \frac{1 + i}{4\sqrt{\pi x}} e^{ix} \left[Ci2x - i\left(Si2x - \frac{\pi}{2}\right)\right].$$

TABLE I

x	$J_0 - iN_0$	$\tilde{W}_0$	$\tilde{W}_0 + \tilde{W}_1$	$\tilde{W}_0 + \tilde{W}_1 + \tilde{W}_2$
10	$-.2459 - i\ .0557$	$-.2468 - i\ .0526$	$-.2460 - i\ .0557$	
9	$-.0903 - i\ .2499$	$-.0938 - i\ .2489$	$-.0903 - i\ .2500$	
8	$.1717 - i\ .2235$	$.1683 - i\ .2264$	$.1717 - i\ .2235$	
7	$.3001 + i\ .0259$	$.3009 + i\ .0207$	$.3001 + i\ .0260$	
6	$.1506 + i\ .2882$	$.1568 + i\ .2855$	$.1507 + i\ .2883$	
5	$-.1776 + i\ .3085$	$-.1704 + i\ .3135$	$-.1777 + i\ .3086$	
4	$-.3975 + i\ .0169$	$-.3979 + i\ .0291$	$-.3973 + i\ .0169$	
3	$-.2601 - i\ .3769$	$-.2765 - i\ .3684$	$-.2601 - i\ .3772$	
2	$.2239 - i\ .5104$	$.1967 - i\ .5288$	$.2246 - i\ .5109$	
1	$.7652 - i\ .0883$	$.7796 - i\ .1699$	$.7683 - i\ .0860$	$.7651 - i\ .0882$
0.8	$.8463 + i\ .0868$	$.8920 - i\ .0130$	$.8499 + i\ .0916$	$.8461 + i\ .0868$
0.6	$.9120 + i\ .3086$	$1.0124 + i\ .1899$	$.9152 + i\ .3183$	$.9116 + i\ .3084$

For values of x less than 1 $\tilde{W}_2$ was evaluated numerically.

Example 6, Table II

$$y'' + \frac{1}{x} y' = \left(\frac{1}{x^2} - 1\right) y, \qquad 1 \leqq x \leqq 3.$$

$$y(1) = 1, \qquad y'(1) = 0.$$

The solution of this equation using Picard's method and the integraph has been described by Thornton C. Fry.[3] We compare his results with those obtained by the wave perturbation method. The equation is first reduced to normal form by the substitution $y = x^{-1/2} u$, so that

$$u'' = \left(-1 + \frac{3}{4x^2}\right) u$$

and we have $\beta = \frac{1}{2}\sqrt{3}$. Then

$$x^{1/2} W_0 = \cos \beta (x - 1) + \frac{1}{2\beta} \sin \beta (x - 1)$$

$$x^{1/2} W_1 = \frac{1}{4\beta} \int_1^x \left(\frac{3}{u^2} - 1\right) \left[\cos \beta (u - 1) + \frac{1}{2\beta} \sin \beta (u - 1)\right] \sin \beta (x - u)\, du.$$

While W_1 may be evaluated in terms of Ci and Si functions the values tabulated below were obtained by numerical integration. The values of the accurate solution

$$y = 1.4034\, J_1(x) - 0.3251\, N_1(x),$$

and of the third and eighth Picard approximations, are copied from Fry's paper.

TABLE II

x	y	y_3	y_8	W_0	$W_0 + W_1$
1.0	1.000	1.000	1.000	1.000	1.000
1.2	.998	.998	.998	.990	.998
1.4	.985	.984	.986	.961	.985
1.6	.956	.951	.955	.913	.956
1.8	.908	.894	.910	.848	.908
2.0	.842	.809	.844	.769	.842
2.2	.759	.694	.760	.677	.758
2.4	.659	.548	.661	.575	.659
2.6	.547	.370	.549	.466	.547
2.8	.425	.156	.427	.352	.427
3.0	.297	−.096	.300	.236	.300

References

1. S. A. Schelkunoff, Solution of linear and slightly nonlinear equations, *Quart. App. Math.*, vol. 3, p. 348, Jan., 1946.
2. J. Liouville, Mémoires sur le développement des fonctions ou parties de fonctions en séries dont les divers termes sont assujétis à satisfaire à une même équation différentielle du second ordre, contenant un paramètre variable, *Jl. de Math: Pures et Appl.*, v. *1*, p. 253–265, 1836, v. *2*, p. 16–35 and p. 418–436, 1837. A brief discussion of the method will be found in "Numerical studies in differential equations" by H. Levy and E. A. Baggott, London, 1934; but these authors apply it to the numerical solution of non-linear equations and do not seem to have appreciated that its real potentialities lie in the field of linear equations, where they use only the better known methods.
3. Thornton C. Fry, The use of the integraph in the practical solution of differential equations by Picard's method of successive approximations, *Proc. Int. Math. Congress of Toronto*, pp. 405–428, 1924.