Some Results on Cylindrical Cavity Resonators

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Certain hitherto unpublished theoretical results on cylindrical cavity resonators are derived. These are: an approximation formula for the total number of resonances in a circular cylinder; conditions to yield the minimum volume circular cylinder for an assigned Q; limitation of the frequency range of a tunable circular cylinder as set by ambiguity; resonant frequencies of the elliptic cylinder; resonant frequencies and Q of a coaxial resonator in its higher modes; and a brief discussion of fins in a circular cylinder.

The essential results are condensed in a number of new tables and graphs.

Introduction

THE subject of wave guides and the closely allied cavity resonators was of considerable interest even prior to 1942, as shown in the bibliography. It is believed that this bibliography includes virtually everything published up to the end of 1942. During the war, many applications of cavity resonators were made. Among these was the use of a tunable circular cylinder cavity in the $TE\ 01n$ mode as a radar test set; this has been treated in previous papers.[1,2] During this development, a number of new theoretical results were obtained; some of these have been published.[2] Here we give the derivation of these results together with a number of others not previously disclosed.

In the interests of brevity, an effort has been made to eliminate all material already published. For this reason, the topics are rather disconnected, and it is also assumed that the reader has an adequate background in the subject, such as may be obtained from a study of references 3 to 7 of the bibliography, or a text such as Sarbacher and Edson.[8]

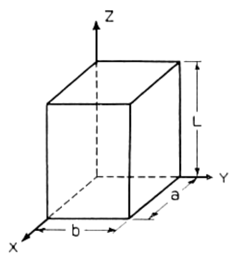
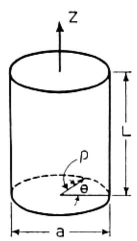
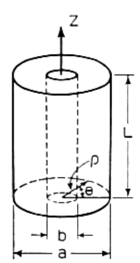
A convenient reference and starting point is afforded by Fig. 1, taken from the Wilson, Schramm, Kinzer paper.[2] This figure also explains most of the notation used herein.

Acknowledgement

In this work, as in any cooperative scientific development, assistance and advice were received from many individuals and appropriate appreciation therefor is herewith extended. In some cases, explicit credit for special contributions has been given.

Contents

TABLE I.—*Formulas for Cavity Resonators—Fields, Resonant Frequencies and Mode Shape Factors for Rectangular Prism, Circular Cylinder and Full Coaxial*

TYPE OF CAVITY & CO-ORDINATE SYSTEM	MODE	FIELD EQUATIONS *	DEFINITIONS	RESTRICTIONS ON ℓ, m, n
RECTANGULAR PRISM 	TM	$E_x = \sqrt{\frac{\mu}{\epsilon}}\,\frac{k_1 k_3}{k^2} \cos k_1x \sin k_2y \sin k_3z$ $E_y = \sqrt{\frac{\mu}{\epsilon}}\,\frac{k_2 k_3}{k^2} \sin k_1x \cos k_2y \sin k_3z$ $E_z = -\sqrt{\frac{\mu}{\epsilon}}\,\frac{k_1^2+k_2^2}{k^2} \sin k_1x \sin k_2y \cos k_3z$ $H_x = -\frac{k_2}{k} \sin k_1x \cos k_2y \cos k_3z$ $H_y = \frac{k_1}{k} \cos k_1x \sin k_2y \cos k_3z$ $H_z = 0$	$k_1 = \frac{\ell\pi}{a}$ $k_2 = \frac{m\pi}{b}$ $k_3 = \frac{n\pi}{L}$ $k^2 = k_1^2+k_2^2+k_3^2$ $\lambda = \frac{2\pi}{k}$ ℓ, m, n = INTEGRAL INDICES IDENTIFYING THE MODES. MAY ASSUME THE VALUE ZERO, SUBJECT TO RESTRICTIONS GIVEN IN ADJOINING COLUMN	$\ell > 0$ $m > 0$
	TE	$E_x = -\sqrt{\frac{\mu}{\epsilon}}\,\frac{k_2}{k} \cos k_1x \sin k_2y \sin k_3z$ $E_y = \sqrt{\frac{\mu}{\epsilon}}\,\frac{k_1}{k} \sin k_1x \cos k_2y \sin k_3z$ $E_z = 0$ $H_x = \frac{k_1 k_3}{k^2} \sin k_1x \cos k_2y \cos k_3z$ $H_y = \frac{k_2 k_3}{k^2} \cos k_1x \sin k_2y \cos k_3z$ $H_z = -\frac{k_1^2+k_2^2}{k^2} \cos k_1x \cos k_2y \sin k_3z$		$\ell + m > 0$ $n > 0$
CIRCULAR CYLINDER 	TM	$E_\rho = -\sqrt{\frac{\mu}{\epsilon}}\,\frac{k_3}{k} J_\ell'(k_1\rho) \cos \ell\theta \sin k_3z$ $E_\theta = \sqrt{\frac{\mu}{\epsilon}}\,\ell\,\frac{k_3}{k}\,\frac{J_\ell(k_1\rho)}{k_1\rho} \sin \ell\theta \sin k_3z$ $E_z = \sqrt{\frac{\mu}{\epsilon}}\,\frac{k_1}{k} J_\ell(k_1\rho) \cos \ell\theta \cos k_3z$ $H_\rho = -\ell\,\frac{J_\ell(k_1\rho)}{(k_1\rho)} \sin \ell\theta \cos k_3z$ $H_\theta = -J_\ell'(k_1\rho) \cos \ell\theta \cos k_3z$ $H_z = 0$	$k_1 = \frac{2r_{\ell m}}{a}$ $k_3 = \frac{n\pi}{L}$ $k^2 = k_1^2+k_3^2$ $\lambda = \frac{2\pi}{k}$ ℓ, m, n = DEFINED AS FOR RECTANGULAR PRISM $r_{\ell m}$ = m^{th} ZERO OF $J_\ell(x)$ FOR TM MODES $r_{\ell m}$ = m^{th} ZERO OF $J_\ell'(x)$ FOR TE MODES	$m > 0$
	TE	$E_\rho = -\sqrt{\frac{\mu}{\epsilon}}\,\ell\,\frac{J_\ell(k_1\rho)}{k_1\rho} \sin \ell\theta \sin k_3z$ $E_\theta = -\sqrt{\frac{\mu}{\epsilon}}\,J_\ell'(k_1\rho) \cos \ell\theta \sin k_3z$ $E_z = 0$ $H_\rho = \frac{k_3}{k} J_\ell'(k_1\rho) \cos \ell\theta \cos k_3z$ $H_\theta = -\ell\,\frac{k_3}{k}\,\frac{J_\ell(k_1\rho)}{k_1\rho} \sin \ell\theta \cos k_3z$ $H_z = \frac{k_1}{k} J_\ell(k_1\rho) \cos \ell\theta \sin k_3z$		$m > 0$ $n > 0$
FULL COAXIAL  $\frac{b}{a} = \eta$	TM	SAME AS FOR CIRCULAR CYLINDER, BUT SUBSTITUTE: $Z_\ell(k_1\rho)$ FOR $J_\ell(k_1\rho)$ $Z_\ell'(k_1\rho)$ FOR $J_\ell'(k_1\rho)$	SAME AS CIRCULAR CYLINDER, EXCEPT: $r_{\ell m}$ = m^{th} ZERO OF $[J_\ell(\eta x)\,Y_\ell(x) - J_\ell(x)\,Y_\ell(\eta x)]$ $A = \frac{J_\ell(r_{\ell m})}{Y_\ell(r_{\ell m})}$ FOR TM MODES	$m > 0$ SPECIAL CASE OF TM 0,0,n MODE, WITH $r_{\ell m} = 0$
	TE	WHERE $Z_\ell(k_1\rho) = J_\ell(k_1\rho) - A Y_\ell(k_1\rho)$ $Z_\ell'(k_1\rho) = J_\ell'(k_1\rho) - A Y_\ell'(k_1\rho)$	$r_{\ell m}$ = m^{th} ZERO OF $[J_\ell'(\eta x)\,Y_\ell'(x) - J_\ell'(x)\,Y_\ell'(\eta x)]$ $A = \frac{J_\ell'(r_{\ell m})}{Y_\ell'(r_{\ell m})}$ FOR TE MODES	$m > 0$ $n > 0$

CAVITY	MODE	NORMAL WAVELENGTHS	APPROXIMATION FOR TOTAL NUMBER OF MODES (TE & TM) HAVING $\lambda > \lambda_0$	FORMULAS FOR $Q\frac{\delta}{\lambda}$		DEFINITIONS
RECTANGULAR PRISM	TM	$\lambda = \frac{2}{\sqrt{\left(\frac{\ell}{a}\right)^2 + \left(\frac{m}{b}\right)^2 + \left(\frac{n}{L}\right)^2}}$	$N = 8.38\,\frac{V}{\lambda_0^3} - \frac{P}{\lambda_0}$ $V = abL$ $P = a+b+L$	$\frac{abL}{4} \cdot \frac{(p^2+q^2)(p^2+q^2+r^2)^{\frac{1}{2}}}{p^2b(a+L) + q^2a(b+L)}$	$n > 0$	$p = \frac{\ell}{a}$ $q = \frac{m}{b}$ $r = \frac{n}{L}$
				$\frac{abL}{2} \cdot \frac{(p^2+q^2)^{\frac{3}{2}}}{p^2b(a+2L) + q^2a(b+2L)}$	$n = 0$	
	TE	SAME AS TM MODES		$\frac{abL}{4} \cdot \frac{(p^2+q^2)(p^2+q^2+r^2)^{\frac{3}{2}}}{aL\left[p^2r^2 + (p^2+q^2)^2\right] + bL\left[q^2r^2 + (p^2+q^2)^2\right] + abr^2(p^2+q^2)}$	$\ell m > 0$	
				$\frac{abL}{2} \cdot \frac{(q^2+r^2)^{\frac{3}{2}}}{q^2L(b+2a) + r^2b(L+2a)}$	$\ell = 0$	
				$\frac{abL}{2} \cdot \frac{(p^2+r^2)^{\frac{3}{2}}}{p^2L(a+2b) + r^2a(L+2b)}$	$m = 0$	
CIRCULAR CYLINDER	TM	$\lambda = \frac{2}{\sqrt{\left(\frac{2r_{\ell m}}{\pi a}\right)^2 + \left(\frac{n}{L}\right)^2}}$ $(fa)^2 = \left(\frac{c r_{\ell m}}{\pi}\right)^2 + \left(\frac{cn}{2}\right)^2 \left(\frac{a}{L}\right)^2$ $c = \sqrt{\mu\epsilon}$ = VELOCITY OF ELECTROMAGNETIC WAVES IN DIELECTRIC f = FREQUENCY	$N = 4.38\,\frac{V}{\lambda_0^3} + 0.09\,\frac{S}{\lambda_0^2}$ $V = \frac{\pi a^2 L}{4}$ $S = \pi a L$	$\frac{r_{\ell m}}{2\pi}\left[1 + p^2R^2\right]^{\frac{1}{2}} \cdot \frac{1}{1+R}$	$n > 0$	$R = \frac{a}{L}$ $p = \frac{n\pi}{2r_{\ell m}}$
				$\frac{r_{\ell m}}{\pi}\,\frac{1}{2+R}$	$n = 0$	
	TE			$\frac{r_{\ell m}}{2\pi}\left[1 + p^2R^2\right]^{\frac{3}{2}} \frac{1 - \left(\frac{\ell}{r_{\ell m}}\right)^2}{1 + p^2R^3 + p^2(1-R)R^2\left(\frac{\ell}{r_{\ell m}}\right)^2}$		
FULL COAXIAL	TM	SAME FORM AS FOR CYLINDER $r_{\ell m}$ HAS DIFFERENT VALUES	$N \approx 4.4\,\frac{V}{\lambda_0^3}$ WITH SOME DOUBT AS TO VALUE OF THE COEFFICIENT	$\frac{r_{\ell m}}{2\pi}\left[1 + p^2R^2\right]^{\frac{1}{2}} \frac{(1-\eta^2H')}{(1+\eta H') + R(1-\eta^2H')}$	$n > 0$	$R = \frac{a}{L}$ $p = \frac{n\pi}{2r_{\ell m}}$ $H' = \left[\frac{Z_\ell'(\eta r_{\ell m})}{Z_\ell'(r_{\ell m})}\right]^2$ $H = \left[\frac{Z_\ell(\eta r_{\ell m})}{Z_\ell(r_{\ell m})}\right]^2$
				$\frac{r_{\ell m}}{\pi} \cdot \frac{(1-\eta^2H')}{2(1+\eta H') + R(1-\eta^2H')}$	$n = 0$	
				THESE EXPRESSIONS ARE NOT VALID FOR SMALL η WHEN $\ell = 0$		
	TE			$\frac{r_{\ell m}}{2\pi} \cdot \frac{\left[1 + p^2R^2\right]^{\frac{3}{2}} M}{(1+\eta H) + p^2R^2\frac{\ell^2}{r_{\ell m}^2}\left(1 + \frac{H}{\eta}\right) + p^2R^3M}$ WHERE $M = \left(1 - \frac{\ell^2}{r_{\ell m}^2}\right) - \eta^2 H\left(1 - \frac{\ell^2}{\eta^2 r_{\ell m}^2}\right)$		

SOURCES: HANSEN, JNL. APP. PHYS., 9, P. 654 BORGNIS, HOCHF: TECH u. ELEK: AKUS., 56, P. 47

* THE TIME FACTOR HAS BEEN OMITTED;

3. Limitation of frequency range of a tunable cavity in the *TE* 01*n* mode as set by ambiguity.
4. Resonant frequencies of an elliptic cylinder.
5. Resonant frequencies and Q of higher order modes of a coaxial resonator.
6. Fins in a circular cylinder.

Approximation Formula for Number of Resonances in a Circular Cylinder

From Fig. 1, the resonant frequencies of the cylindrical cavity are obtained from the equation:

$$(fa)^2 = \left(\frac{cr}{\pi}\right)^2 + \left(\frac{cna}{2L}\right)^2 \tag{1}$$

in which r is written in place of r_{lm}, to simplify the equations. The distribution of the resonant frequencies, starting with the lowest, can be approximated by a continuous function

$$N \approx F(f_0) \approx G(\lambda_0)$$

where N represents the total number of resonances up to a frequency f_0 or a wavelength λ_0. This is bound to be an approximation, since the true function F is discontinuous (or stepped) by virtue of the resonances being a series of discrete values. For practical purposes, if F fits the stepped curve so that the steps fluctuate above and below F, it will be a useful approximation.

Derivation of such a formula as applied to the acoustic resonances of a rectangular box has recently been a subject of investigation by Bolt[9] and Maa.[10] Only slight modifications of their method need be made to apply to the present situation.

Multiply (1) thru by $\left(\frac{\pi}{c}\right)^2$:

$$\left(\frac{\pi af}{c}\right)^2 = r^2 + \left(\frac{\pi an}{2L}\right)^2.$$

Hence, if a point $\left(r, \frac{\pi an}{2L}\right)$ is plotted on the XY plane the distance from the origin to this point will be $\frac{\pi af}{c}$ and hence a measure of the resonant frequency. If all such points are plotted, they will form a lattice representing all the possible modes of resonance. The problem, then, is to find the number of lattice points in a quadrant of a circle with radius, $R = \frac{\pi af_0}{c}$.

The values of the Bessel zero, r, are not evenly spaced along the X axis; indeed the density, or number per unit distance, increases as r increases. Let the density be $p(x)$. Then the problem becomes one of finding the weight of a quadrant of material whose density varies as $p(x)$.

Suppose the expression for M, the number of zeros r, less than some value x, is of the form

$$M = Ax^2 + Bx$$

whence, by differentiation,

$$p(x) = 2\,Ax + B.$$

The weight, W, of the quadrant of a circle of radius R is then, by integration,

$$W = \frac{2}{3}AR^3 + \frac{\pi}{4}BR^2.$$

Since there are $\frac{2L}{\pi a}$ lattice points per unit distance along the Y axis, $\frac{2LW}{\pi a}$ is apparently the total number of points in the quadrant. However, there are two small corrections to consider. First is that in this procedure a lattice point is represented by an area and for the points along the X axis half the area, i.e., a strip $\frac{\pi a}{4L}$ wide lying in the adjacent quadrant, has been omitted. Second is that the restriction $n > 0$ for TE modes eliminates half the points along the X axis. As it happens, these corrections just cancel each other. Thus we have

$$N = \frac{16\pi A}{3}\frac{V}{\lambda_0^3} + \frac{\pi B}{2}\frac{S}{\lambda_0^2}$$

in which

$$V = \frac{\pi a^2 L}{4} \qquad S = \pi a L \qquad \lambda_0 = \frac{c}{f_0}.$$

From a tabulation[11] of the first 180 values of r, the empirical values $A = 0.262$, $B = 0$ were obtained. This gives

$$N = 4.39\frac{V}{\lambda_0^3}.$$

Subsequently, from an analysis of over a thousand modes in a "square cylinder" ($a = L$), Dr. Alfredo Baños, formerly of M.I.T. Radiation Laboratory, has calculated the empirical formula

$$N = 4.38\frac{V}{\lambda_0^3} + 0.089\frac{S}{\lambda_0^2} \tag{2}$$

from which $A = 0.262$, $B = 0.057$. These values give better agreement with the 180 tabulated values of r.

There is a two-fold degeneracy in a circular cylinder for modes with $\ell > 0$, which is removed, for example, when the cylinder is made elliptical. The total number of modes, then, counting degeneracies twice, is about $2N$, which brings (2) in line with the general result that, in any cavity resonator, the total number of modes is of the order $\frac{8\pi}{3} \frac{V}{\lambda_0^3}$.

MINIMUM VOLUME OF CIRCULAR CYLINDER FOR ASSIGNED Q

In practical applications of resonant cavities, the conditions of operation may require high values of Q which can be attained only by the use of high order modes. The total number of modes, most of which are undesired, can then be reduced only by making the cavity volume as small as possible, consistent with meeting the requirement on Q.

It will be shown that, for a cylinder, operation in the *TE* 01*n* mode very probably gives the smallest volume for an assigned Q.

Statement of Problem

When the relative proportions (the shape) of a cavity and the mode of oscillation are fixed, both the Q and the volume, V, of the cavity are functions of the operating wavelength, λ. Since we are primarily interested in the relationship between Q and V, with λ fixed, some simplification can be made by eliminating λ as a parameter. This may be done by a change of variables to $Q\frac{\delta}{\lambda}$ and $\frac{V}{\lambda^3}$, respectively; to simplify the typography, these quantities will be denoted by single symbols:

$$P \equiv Q\frac{\delta}{\lambda}$$

$$W \equiv \frac{V}{\lambda^3}.$$

We are, consequently, interested in the following specific problem:

> In a circular cylindrical resonator, which is the optimum mode family and what is the corresponding shape to obtain the smallest value of W for a preassigned value of P?

A rigorous solution cannot be obtained by the methods of elementary calculus, since P is not a continuous function of the mode of oscillation. However, a possible procedure is to assume continuity, and examine the relation between P and W under this assumption. If sufficiently positive results are obtained, the conclusions may then be carried over to the discontinuous (i.e., the physical) case with reasonable assurance that, except perhaps for special values, the correct answer is obtained. We proceed on this basis.

Solution

To permit a more coherent presentation of the arguments, only their general outline follows. More mathematical details are given later.

We start with the formulas for $Q\frac{\delta}{\lambda}$ $(= P)$ as given in Fig. 1.

The first operation is to show that, under comparable conditions, i.e., λ, r, n fixed, the TE $0mn$ modes give the highest values of P. That this is plausible can be seen in a general manner from the equations as they stand. For the TE modes, if $\ell = 0$, the numerator of the fraction is largest. Also, P simplifies, and the denominator roughly reduces the expression in square brackets to the 1/2 power. Now compare this expression with those for the TM modes. That for the TM modes $(n > 0)$ is smaller because of the factor $(1 + R)$ in the denominator. Finally, that for the TM modes $(n = 0)$ is still smaller, because $1 < (1 + p^2R^2)^{1/2}$.

This leaves only the TE $0mn$ modes to be considered, and the next step is to show that $m = 1$ is the most favorable value. Since the relation between P and W is complicated, a parameter φ is introduced, with φ defined by

$$\tan \varphi = pR. \tag{3}$$

The resulting parametric equations are:

$$P = \frac{r}{2\pi} \frac{1}{\cos^3 \varphi + \frac{1}{p} \sin^3 \varphi} \tag{4}$$

$$W = \frac{pr^3}{4\pi^2} \frac{1}{\cos^2 \varphi \sin \varphi}. \tag{5}$$

For each of the discrete values of r and n (n is related to p) then, plots of P vs W can be prepared as shown in Fig. 2 for the TE $01n$ modes.

Inspection of Fig. 2 shows that the best value of Q does not correspond to a minimum of W or a maximum of P for a given value of n, but rather to a point on the "envelope" of the curves. To get the envelope, we assume p to be continuous and proceed in the standard manner. It turns out that, by solving (4) for p in terms of P, r and φ, substituting the resulting expression in W, and setting $\frac{\partial W}{\partial \varphi} = 0$ an equation is obtained which, when solved for φ, gives the values of φ which lie on the envelope.

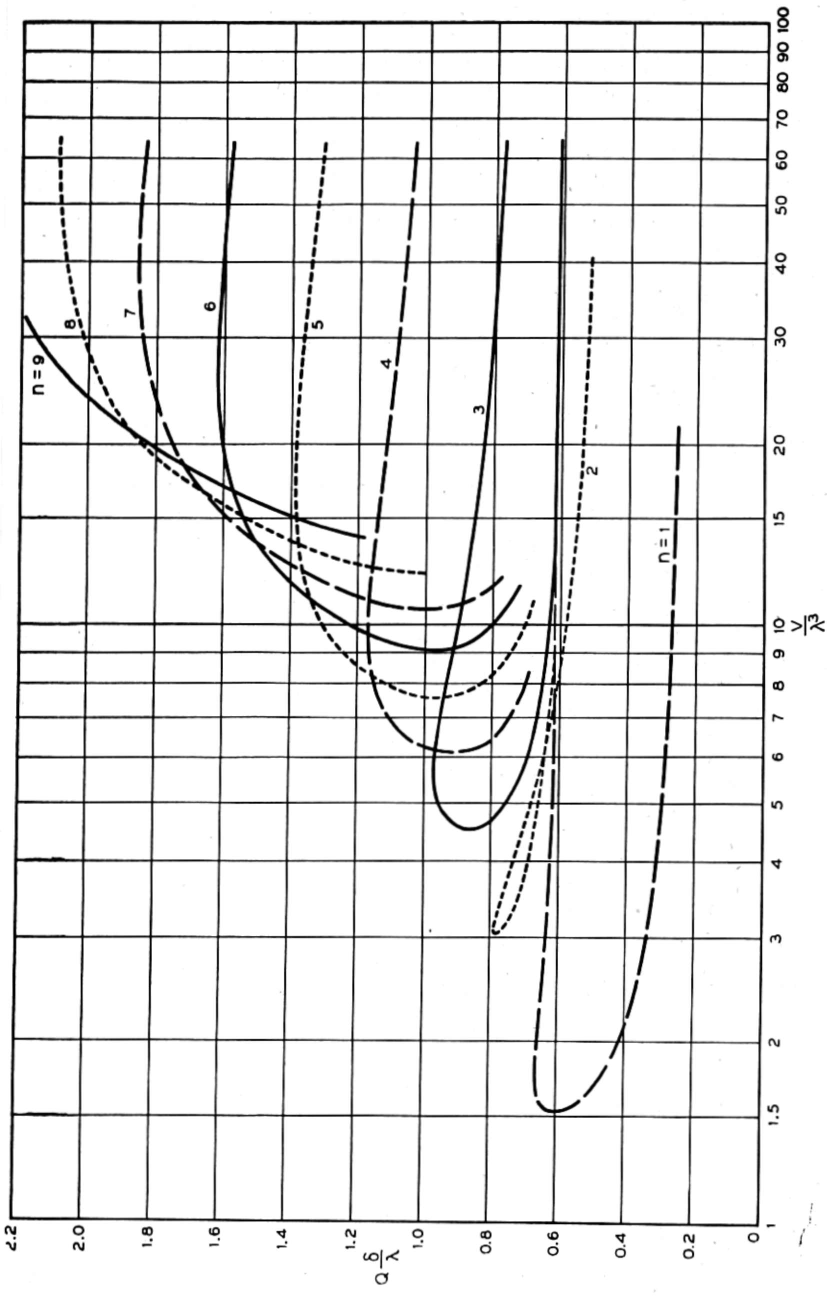


Fig. 2—Relation between $Q\,\frac{\delta}{\lambda}$ (= P) and $\frac{V}{\lambda^3}$ (= W) in cylindrical cavity resonator operating in *TE* 01*n* mode.

We next substitute this expression for φ in W and calculate $\frac{\partial W}{\partial r}$ assuming now that r is continuous, and find that W has no minimum. Practically, this means that the smallest value of r should be used, i.e., the $TE\,01n$ mode.

Finally, since from Fig. 2 it is seen that the envelope is reasonably smooth for values of $Q\frac{\delta}{\lambda} > 1$, the expression for φ derived on the assumption of continuous p is used to obtain a simple relation of great utility in practical cavity design.

Details of solution

In (3), since R must be finite for a physical cylinder, $0 < \tan\varphi < \infty$, $0 < \sin\varphi < 1$, and $0 < \cos\varphi < 1$. Hence we may always divide by $\sin\varphi$ or $\cos\varphi$. Note that φ ranges between 0° and 90°.

From Fig. 1,

$$k = \frac{2r(1 + p^2R^2)^{1/2}}{a}$$

whence

$$k \sin\varphi = \frac{2prR}{a} \tag{6}$$

$$k \cos\varphi = \frac{2r}{a}. \tag{7}$$

We define W by:

$$W \equiv \frac{V}{\lambda^3} = \frac{\pi a^3}{4R}\,\frac{k^3}{8\pi^3}. \tag{8}$$

Substituting (6) and (7) in (8),

$$W = \frac{pr^3}{4\pi^2}\,\frac{1}{\cos^2\varphi\sin\varphi}. \tag{5'}$$

Substitution of (3) into the expression for $Q\frac{\delta}{\lambda}$ $(= P)$ for the TE modes as given in Fig. 1 yields, after some manipulation

$$P = \frac{r}{2\pi}\cdot\frac{1 - (\ell/r)^2}{\cos^3\varphi + \frac{1}{p}\sin^3\varphi + \left(\cos\varphi - \frac{1}{p}\sin\varphi\right)(\ell/r)^2\sin^2\varphi}.$$

To show that any value of $\ell > 0$ reduces P below its value when $\ell = 0$, let

$$a = \cos^3 \varphi + \frac{1}{p} \sin^3 \varphi$$

$$b = \left(\cos \varphi - \frac{1}{p} \sin \varphi\right) \sin^2 \varphi$$

$$c = (\ell/r)^2.$$

It suffices to show that

$$\frac{1}{a} > \frac{1 - c}{a + bc}$$

where the question is in doubt because b may take on negative values. If the inequality is to be valid, it is necessary only that $(b + a) > 0$, that is, $\cos \varphi > 0$. Hence, for the TE modes, only $\ell = 0$ needs be considered. For this case, the expression for P simplifies to

$$P = \frac{r}{2\pi} \frac{1}{\cos^3 \varphi + \frac{1}{p} \sin^3 \varphi}. \tag{4'}$$

For the TM modes, there is similarly obtained

$$P = \frac{r}{2\pi} \frac{1}{\cos \varphi + \frac{1}{p} \sin \varphi} \qquad n > 0 \tag{9}$$

$$P = \frac{r}{2\pi} \frac{\cos \varphi}{\cos \varphi + \frac{1}{2p} \sin \varphi} \qquad n = 0. \tag{10}$$

It is easy to show, since $\cos \varphi < 1$ and $\sin \varphi < 1$, that both (9) and (10) are less than (4′).

Hence we have shown that, under comparable conditions, i.e., r and p constant, the TE $0mn$ modes have higher values of P than any others. There is one flaw in the argument, viz., r takes on discrete values and cannot be made the same for all modes. It is conceivable, therefore, that for some specific values of P, a mode other than the TE $0mn$ can be found which gives a smaller W than either of the two "adjacent" TE $0mn$ modes, one having a value of r higher, the other lower, than the supposed high-P mode. This situation requires further refinement, and hence complication, in the analysis; we pass over this point.

Having so far indicated that the TE $0mn$ modes are the best, our next objective is find the best value of m, if possible.

By use of the parametric equations (4) and (5), Fig. 2 has been plotted for $r = 3.83$ (*TE* 01n modes) and values of n from 1 to 9. This drawing shows that, for each discrete value of r, minimum W/P is given by points on the "envelope" of the family of curves.

The standard method of obtaining the envelope is to express W as a function of P with n as parameter (r is assumed fixed, for the moment), i.e., $W = F(P, n)$, and then set $\frac{\partial F}{\partial n} = 0$. However, in this case it is easier to express $W = G(P, \varphi)$ and $\varphi = H(n)$, whence

$$\frac{\partial F}{\partial n} = \frac{\partial G}{\partial \varphi} \cdot \frac{\partial \varphi}{\partial n}$$

and the envelope is obtained by setting $\frac{\partial G}{\partial \varphi} = 0$ provided $\frac{\partial \varphi}{\partial n} \neq 0$. We proceed, therefore, as follows.

Assume p is continuous, and solve (4) for p, obtaining:

$$p = \frac{\sin^3 \varphi}{\frac{r}{2\pi P} - \cos^3 \varphi}. \tag{11}$$

Now substitute (11) in (5). This gives W as a function of P and φ:

$$W = \frac{r^3}{4\pi^2}\left[\frac{\sin^2 \varphi}{\cos^2 \varphi \left(\frac{r}{2\pi P} - \cos^3 \varphi\right)}\right]. \tag{12}$$

To solve $\frac{\partial W}{\partial \varphi} = 0$, we differentiate and simplify. This yields

$$5 \cos^3 \varphi - 3 \cos^5 \varphi = \frac{r}{\pi P}. \tag{13}$$

Substituting (13) back into (11) yields

$$p = \frac{2 \sin \varphi}{3 \cos^3 \varphi} \tag{14}$$

The situation so far is that, with P and r assigned, W lies on the envelope and is a minimum when φ satisfies (13); p is then given by (14). Obviously, for (13) to hold, it is necessary that

$$\frac{r}{2\pi P} < 1.$$

To obtain the best value of r, the procedure is to differentiate W_{min} with respect to r, assuming now that r is continuous, and examine for a minimum. We can, however, first differentiate (12) by setting

$$\frac{dW}{dr} = \frac{\partial W}{\partial r} + \frac{\partial W}{\partial \varphi} \cdot \frac{\partial \varphi}{\partial r}$$

and then substitute from (13). However, when (13) is satisfied, $\frac{\partial W}{\partial \varphi} = 0.$ This process yields

$$\frac{dW}{dr} = \frac{r^2}{\pi^2} \frac{(2 - 3\cos^2\varphi)}{9\sin^2\varphi\cos^5\varphi}.$$

This shows $\frac{dW}{dr}$ to be positive, when $\cos^2\varphi < \frac{2}{3}$. Hence $\frac{dW}{dr} = 0$ corresponds to a maximum, rather than a minimum.* If $\cos^2\varphi < \frac{2}{3}$, that is, $\varphi > 35°16'$, then r should be as small as possible. The smallest r is 3.83, for the *TE* 01n modes. For $r = 3.83$, and $\varphi > 35°$, from (13) there is obtained $P > 0.75$.

The analysis thus indicates that, for values of $P = Q\frac{\delta}{\lambda}$ greater than 0.75, the *TE* 01n mode yields the smallest ratio W/P or V/Q.

An interesting and simple relation between fa and R for minimum W/P can easily be derived from the foregoing equations. Substitute (14) back into (6), thereby obtaining

$$k = \frac{4Rr}{3a\cos^3\varphi}. \tag{15}$$

Now use (7) with (15) to eliminate $\cos\varphi$, replace k by $2\pi/\lambda$, and r by 3.83, its numerical value for the *TE* 01n modes. This gives

$$\left(\frac{a}{\lambda}\right)^2 R = 2.23$$

or by substituting $\lambda = \frac{c}{f}$, $c = 3 \times 10^{10}$,

$$(fa)^2 R = 20.1 \text{ x } 10^{20}.$$

This useful relation was first discovered by W. A. Edson.

Some further discussion is of interest. It is realized that a number of points have not been taken care of in a manner entirely satisfactory mathematically, but nevertheless important practical results have been obtained. As an example, since p and r can assume only discrete values, there are

* It is for this reason that the determination of the stationary values of $W(r, p, \varphi)$, subject to the constraint $P(r, p, \varphi) =$ constant, by La Grange multipliers fails to yield the desired least value of W/P.

specific situations where some mode other than the *TE* 01n gives a smaller W/P. For example, it may be shown that for P between 0.97 and 1.14 the *TE* 021 mode yields a smaller W than the *TE* 013 or *TE* 014 modes. However, the margin is small, and for larger P, the *TE* 02n modes become progressively poorer.

Limitation on Frequency Range of Tunable Cavity as Set by Ambiguity

In the design of a tunable cylindrical resonant cavity intended for use in the *TE* 01n mode, the requirements on Q may dictate a diameter large enough to sustain *TE* 02n' or *TE* 03n' modes. Also, the range of variation of cavity length may be such that the *TE* 01$(n + 1)$ mode is supported. As the cavity is required to tune over a certain range of frequency, the maximum frequency range possible in the *TE* 01n mode without interference from the *TE* 01$(n + 1)$† or any *TE* 02 or *TE* 03 modes is of interest. The interference from the *TE* 01$(n + 1)$ limits the useful range of the *TE* 01n by the presence of extraneous responses at more than one dial setting for a given frequency or more than one frequency for a given dial setting. In applications so far made, it has been possible to eliminate extraneous responses from the *TE* 02 and *TE* 03 modes, but crossings of these modes with the main *TE* 01n mode have not been permitted. No designs have had diameters sufficiently large to support *TE* 04 modes.

The desired relations are easily obtained by simple algebraic manipulation of equation (1). For simplicity in presentation of the results, we introduce some symbols applicable to this section only:

$$A = \left[\frac{cr_{\ell,m}}{\pi}\right]^2 \qquad B = \left[\frac{c}{2}\right]^2 = 2.247 \times 10^{20}$$

$$A_0 = \text{value of } A \text{ for } TE\ 01n \text{ modes} = 13.371 \times 10^{20}$$

$$t = A/A_0$$

$$x_0 = (a/L)^2 \text{ at low frequency end of useful range of } TE\ 01n \text{ mode}$$

$$F = \text{frequency range ratio} = \frac{\text{maximum } f}{\text{minimum } f}.$$

The values of A and t depend upon the interfering mode under consideration. For the *TE* 02n modes, $A = 44.822 \times 10^{20}$, $t = 3.3522$.

The two typical cases of interest are shown on Fig. 3. For case I, am-

† It is easy to show that the extraneous response from the *TE* 01$(n - 1)$ mode is not limiting. The proof depends on the inequality $n^2 > (n + 1)(n - 1)$.

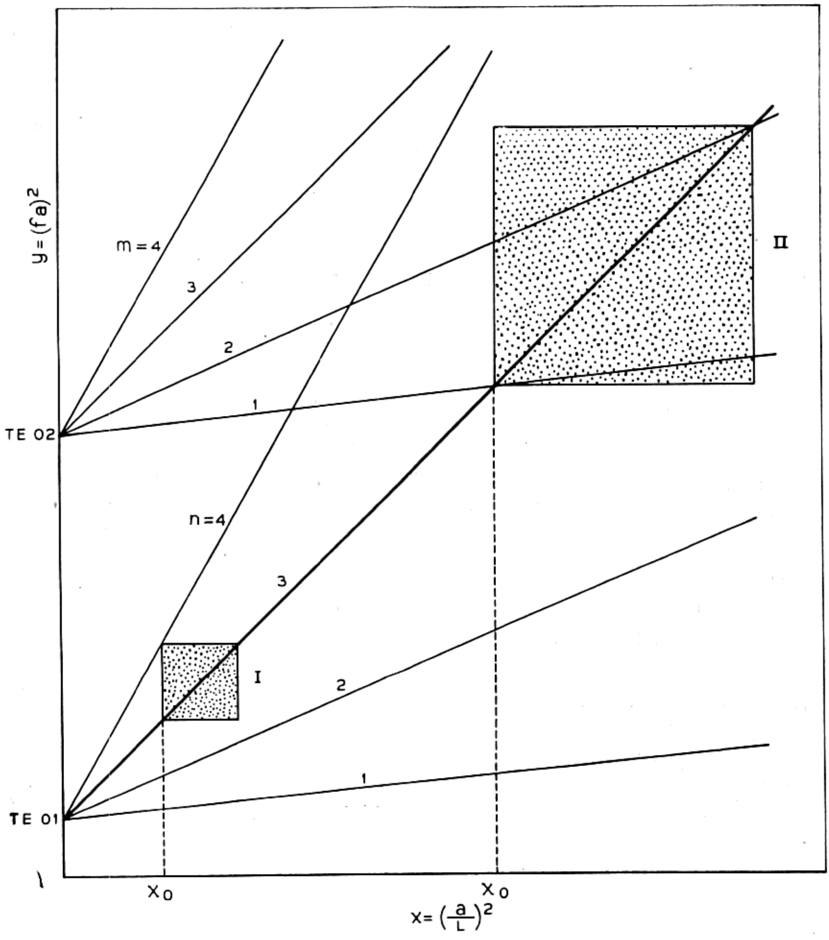


Fig. 3—Mode chart illustrating types of interference with TE 01n mode.

biguity from TE 01(n + 1) mode, it is found that

$$F^2 = \frac{A_0 + B(n+1)^2 x_0}{A_0 + Bn^2 x_0}.$$

Curves of F for this case are shown on Fig. 4.

The maximum value of F is obtained when $x_0 = \infty$ and is

$$F_{\max} = \frac{n+1}{n}.$$

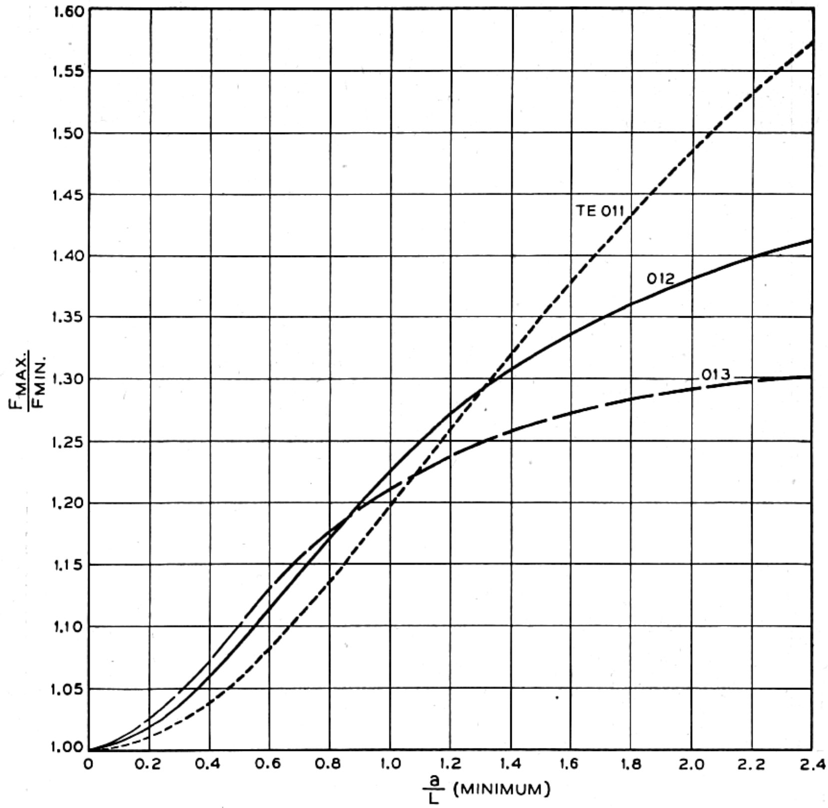


Fig. 4—Curves showing maximum value of frequency ratio without interference from $TE\ 01(n+1)$ mode (case I of Fig. 3).

TABLE I.—*Case II: Maximum Frequency Range Ratio, F, for TE 01n Mode when Limited by Mode Crossings with TE 02m and TE 02(m+1) Modes.*

m	$n = 3$		$n = 4$		$n = 12$	
	F	$(a/L)_{min}$	F	$(a/L)_{min}$	F	$(a/L)_{min}$
1	1.198	1.323	1.086	0.966	1.008	0.313
2			1.242	1.080	1.013	0.316
3					1.019	0.322
4					1.027	0.331
5					1.037	0.343
6					1.051	0.360
7					1.071	0.384
8					1.104	0.418
9					1.168	0.471
10					1.345	0.564

For case II, range limited by mode crossings, it is found that

$$x_0 = \frac{A - A_0}{B(n^2 - n'^2)}$$

$$F^2 = \frac{(n^2 - n'^2)[n^2 t - (n' + 1)^2]}{(n^2 t - n'^2)[n^2 - (n' + 1)^2]}.$$

Some values for this case are given in Table I.

The formulas above are general and may be used for any pair of mode types by using the appropriate values for A and t.

The Elliptic Cylinder

In the design of high Q circular cylinder cavity resonators operating in the TE 01n mode, it is desirable to know how much ellipticity is tolerable, so that suitable manufacturing limits may be set. The elliptical wave guide has already been studied, notably by Brillouin[12] and Chu,[13] but the results are not in suitable form or of adequate precision for the present purposes. More recently tables[14] have become available which permit the calculation of some of the properties of the elliptical cylindrical resonator.

The elliptical cavity involves Mathieu functions, which are considerably more complicated than Bessel functions.[16] The tables give the numerical coefficients of series expansions, in terms of sines, cosines, and Bessel functions, of the Mathieu functions up to the fourth order. These tables have been used for the calculation of some quantities of interest in connection with elliptical deformations of a circular cylinder in the TE 01n mode.

The Ellipse

All mathematical treatments of the ellipse (including the tables mentioned above) use the eccentricity, e, as the quantity describing the amount of departure from the circular form. The eccentricity is the ratio

$$e = \frac{\text{distance between foci}}{\text{major axis}}.$$

This is not a quantity subject to direct measurement, hence we here introduce and use throughout the ellipticity, E, defined as

$$E = \frac{\text{difference between major and minor diameters}}{\text{major diameter}}.$$

It is clear that the ellipticity is easily obtained directly.

Again, many results are given in terms of the major diameter. Since we are interested in deformations from circular, and in such deformations the perimeter remains constant, while the major diameter changes, we have expressed our results in terms of an average diameter, defined as

$$D = \frac{\text{perimeter}}{\pi}.$$

Figure 5 shows the ellipse and various relations of interest.

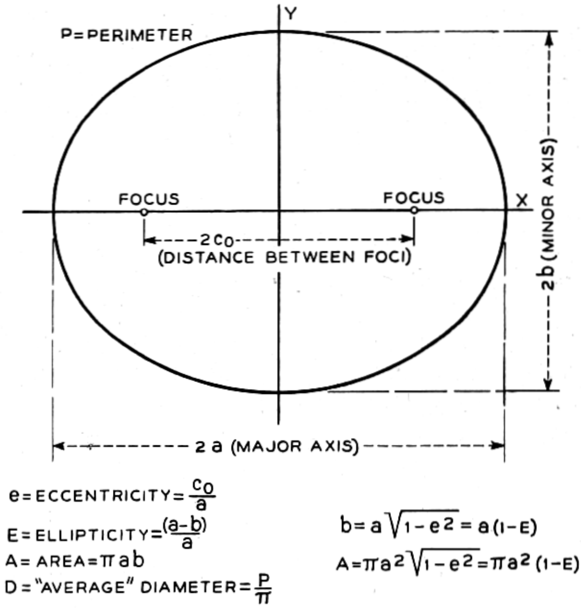


Fig. 5—The ellipse

Elliptic Coordinates and Functions

The elliptic coordinate system is shown on Fig. 6. Following Stratton,[15] we have used ξ in place of the table's z, since we wish to use z as the coordinate along the longitudinal axis. Stratton also uses $\eta = \cos\varphi$ as the angular coordinate; this is frequently convenient.

Analogous to $\cos \ell\theta$ and $\sin \ell\theta$ in the circular case, there are even and odd* angular functions, denoted by

$$^{e}S_\ell(c, \cos\varphi) \text{ and } {}^{o}S_\ell(c, \cos\varphi)$$

which reduce to $\cos \ell\theta$ and $\sin \ell\theta$ respectively when $c \to 0$. Similarly, there are even and odd* radial functions, denoted by

$$^{e}J_\ell(c, \xi) \text{ and } {}^{o}J_\ell(c, \xi)$$

* For $\ell = 0$, only even functions exist.

which both reduce to $J_\ell(k_1\rho)$ when $c \rightarrow 0$. In the above, c is a parameter related to the ellipticity.* The tables do not give values of the functions, but rather give numerical coefficients

$$D_n^\ell \text{ and } F_n^\ell$$

of expansions in series of cosine, sine and Bessel functions, which permit one to calculate the elliptic cylinder functions. The coefficients, of course,

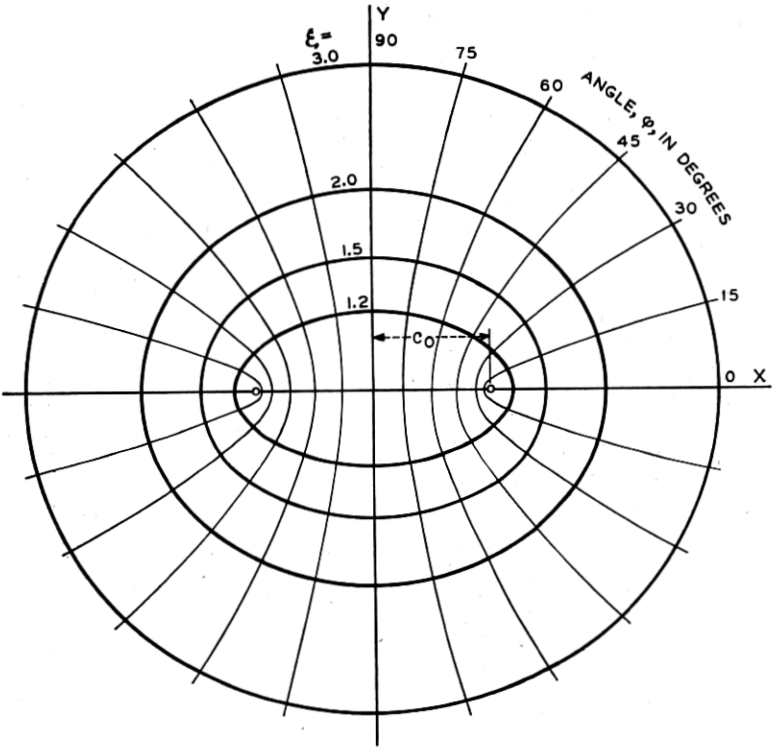


Fig. 6—Elliptic coordinate system

depend on the parameter c; the largest value of c in the tables is 4.5, which corresponds to an ellipticity of 39% in a cylinder operating in the TE $01n$ mode.** For this case, Bessel functions up to $J_{12}(x)$ and $J'_{12}(x)$ are needed for calculating the radial function. It is clear that calculations on elliptic cylinders have not been put on a simple basis.

* Not to be confused with c = velocity of electromagnetic waves; the symbol c is here carried over from the published tables.

** An ellipticity of 39% means that the difference between maximum and minimum diameters is 39% of the maximum diameter. For a given c, the ellipticity depends on the mode.

Field Equations

The equations for the fields are easily obtained from section 6.12 of Stratton's book, and are given in Table II, which is self-explanatory, except for the quantity c, which we now proceed to discuss.

Resonant Frequencies

The elliptic cylinder has the major diameter, $2a$, and the focal distance, $2c_0$. The equation of its surface is then expressed by $\xi = \frac{a}{c_0} = \alpha$. On this surface, E_η must vanish. This requires that ${}^{eo}J'_\ell(c, \alpha) = 0$ for TE modes and that ${}^{eo}J_\ell(c, \alpha) = 0$ for TM modes. The series expansions are in terms of $c\xi$ as variable. Let $c\alpha = r'_{\ell m}$ or $r_{\ell m}$ be the roots of the above equations. Then $\frac{c}{c_0} = \frac{r}{a}$ (dropping the subscripts ℓ, m). Now, in working out the solution of the differential equations, it turned out that $c = c_0 k_1$. Here k_1 is one component of the wave number, k†. Hence $k_1 = \frac{r}{a}$. Furthermore, the eccentricity is $e = \frac{c_0}{a} = \frac{c}{r}$. The indicated procedure is: 1) choose a value of c; 2) find the various values of r for which the radial function or its derivative is zero; 3) then calculate the corresponding eccentricity and resonant frequency. Notice that for a given value of c, the values of r will depend on the mode, and hence so will the eccentricity.

We now wish to express our results in terms of the ellipticity and the average diameter. To convert eccentricity to ellipticity, we use

$$E = 1 - \sqrt{1 - e^2}.$$

The perimeter of the ellipse is given by $P = 4aE(e)$ where $E(e)$ is the complete elliptic integral of the second kind.††
In terms of the average diameter we find

$$k_1 = \frac{2}{D}\left[\frac{2rE(e)}{\pi}\right]$$

or calling the quantity in brackets s, $k_1 = \frac{2s}{D}$. This is now in the same form as k_1 for a circular cylinder of diameter D. The quantity s is the reciprocal of Chu's $\frac{\lambda_c}{S}$.

† It is recalled that

$$k = \frac{2\pi}{\lambda} = \sqrt{k_1^2 + k_3^2}\,; \qquad k_1 = \frac{r}{a}; \qquad k_3 = \frac{n\pi}{L}.$$

†† This is tabulated as $E(\alpha)$ in Jahnke & Emde, p. 85, with $\alpha = \sin^{-1}e$.

We have calculated and give in Table III values of r, e, E and s for several values of c and for a few modes of special interest. For three cases, ${}^{e}TE$ 01, ${}^{e}TM$ 11 and ${}^{o}TM$ 11, we have determined an empirical formula to fit the calculated values of s. These are also given in Table III.

TABLE II. Elliptic Cylinder Fields

TE Modes

$$E_\xi = -k\sqrt{\frac{\mu}{\epsilon}}\,\frac{\sqrt{1-\eta^2}}{q}\,S'_\ell(c,\eta)J_\ell(c,\xi)\sin k_3 z\cos\omega t$$

$$E_\eta = k\sqrt{\frac{\mu}{\epsilon}}\,\frac{\sqrt{\xi^2-1}}{q}\,S_\ell(c,\eta)J'_\ell(c,\xi)\sin k_3 z\cos\omega t$$

$$H_\xi = k_3\,\frac{\sqrt{\xi^2-1}}{q}\,S_\ell(c,\eta)J'_\ell(c,\xi)\cos k_3 z\sin\omega t$$

$$H_\eta = k_3\,\frac{\sqrt{1-\eta^2}}{q}\,S'_\ell(c,\eta)J_\ell(c,\xi)\cos k_3 z\sin\omega t$$

$$H_z = k_1^2\,S_\ell(c,\eta)J_\ell(c,\xi)\sin k_3 z\sin\omega t$$

TM Modes

$$E_\xi = -k_3\,\frac{\sqrt{\xi^2-1}}{q}\,S_\ell(c,\eta)J'_\ell(c,\xi)\sin k_3 z\cos\omega t$$

$$E_\eta = -k_3\,\frac{\sqrt{1-\eta^2}}{q}\,S'_\ell(c,\eta)J_\ell(c,\xi)\sin k_3 z\cos\omega t$$

$$E_z = k_1^2\,S_\ell(c,\eta)J_\ell(c,\xi)\cos k_3 z\cos\omega t$$

$$H_\xi = -k\sqrt{\frac{\epsilon}{\mu}}\,\frac{\sqrt{1-\eta^2}}{q}\,S'_\ell(c,\eta)J_\ell(c,\xi)\cos k_3 z\sin\omega t$$

$$H_\eta = k\sqrt{\frac{\epsilon}{\mu}}\,\frac{\sqrt{\xi^2-1}}{q}\,S_\ell(c,\eta)J'_\ell(c,\xi)\cos k_3 z\sin\omega t$$

Notes:

Derivatives are with respect to ξ and η.

S_ℓ and J_ℓ carry prefixed superscripts, e or o, since they may be either even or odd.

$$q = c_0\sqrt{\xi^2-\eta^2} \qquad c = c_0 k_1$$

$$k_1 = \frac{r_{\ell,m}}{a} \qquad k_3 = \frac{n\pi}{L} \qquad k^2 = k_1^2 + k_3^2$$

$2c_0$ is distance between foci of ellipse.

a is the semi major diameter of the ellipse.

$r_{\ell,m}$ is the value of $c\xi$ that makes

$$J_\ell(c,\xi) = 0 \text{ for } TM \text{ modes}$$

$$J'_\ell(c,\xi) = 0 \text{ for } TE \text{ modes.}$$

TABLE III ROOT VALUES OF RADIAL ELLIPTIC CYLINDER FUNCTIONS

Mode	c	r	e	E	s
^{e}TE 01	0	3.8317	0	0	3.8317
	0.2	3.8343	0.05216	0.001361	3.8317
	0.4	3.8423	0.10410	0.005434	3.8318
	0.6	3.8558	0.15561	0.012181	3.8324
	0.8	3.8753	0.20643	0.021539	3.8337
	1.0	3.9015	0.25631	0.033406	3.8366
	1.2	3.9349	0.30496	0.047636	3.8417
	1.4	3.9763	0.35209	0.064033	3.8500
	1.6	4.0264	0.39738	0.082346	3.8624
	2.0	4.154	0.4814	0.12351	3.902
	3.0	4.634	0.6474	0.2378	4.101
	4.0	5.29	0.756	0.346	4.42
	4.5	5.66	0.795	0.393	4.62

$$s = 3.8317 + 4.33\,E^2 + 1.9E^3$$

Mode	c	r	e	E	s
^{e}TM 11	0	3.8317	0	0	3.8317
	0.2	3.8330	0.05218	0.001362	3.8304
	0.4	3.8370	0.10425	0.005449	3.8265
	0.6	3.8436	0.15610	0.012259	3.8201
	0.8	3.8532	0.20762	0.021791	3.8113
	1.0	3.8658	0.25868	0.034036	3.8003
	1.2	3.8818	0.30913	0.048981	3.7874
	1.4	3.9015	0.35884	0.066599	3.7727
	1.6	3.9253	0.40761	0.086844	3.7568
	4.5	5.13	0.878	0.520	3.91

$$s = 3.8317 - 0.96E + 1.1E^2$$

Mode	c	r	e	E	s
^{o}TM 11	0	3.8317	0	0	3.8317
	0.2	3.8356	0.05214	0.001361	3.8330
	0.4	3.8474	0.10397	0.005419	3.8370
	0.6	3.8670	0.15516	0.012111	3.8436
	0.8	3.8944	0.20542	0.021326	3.8530
	1.0	3.9298	0.25446	0.032918	3.8654
	1.2	3.9731	0.30203	0.046701	3.8809
	1.4	4.0243	0.34788	0.062462	3.8997

$$s = 3.8317 + 0.95E + 2.2E^2$$

Mode	c	r	e	E	s
^{e}TE 22	0	6.706	0	0	6.706
	0.4	6.712	0.0596	0.00178	6.706
	0.8	6.729	0.1189	0.00709	6.705
	1.2	6.756	0.1776	0.01590	6.702
	1.6	6.788	0.2357	0.02817	6.693
	2.0	6.826	0.2930	0.04389	6.677
^{o}TE 22	0	6.706	0	0	6.706
	0.4	6.712	0.0596	0.00178	6.706
	0.8	6.730	0.1189	0.00709	6.706
	1.2	6.762	0.1775	0.01587	6.708
	1.6	6.810	0.2350	0.02799	6.715
	2.0	6.877	0.2908	0.04323	6.729

Mode	c	r	e	E	s
${}^{e}TE$ 32	0	8.015	0	0	8.015
	0.4	8.020	0.0499	0.00124	8.015
	0.8	8.035	0.0996	0.00497	8.015
	1.2	8.059	0.1489	0.01115	8.014
	1.6	8.093	0.1977	0.01974	8.013
	2.0	8.135	0.2459	0.03070	8.010
${}^{o}TE$ 32	0	8.015	0	0	8.015
	0.4	8.020	0.0499	0.00124	8.015
	0.8	8.035	0.0996	0.00497	8.015
	1.2	8.060	0.1489	0.01115	8.015
	1.6	8.097	0.1976	0.01972	8.018
	2.0	8.146	0.2455	0.03061	8.022
${}^{e}TM$ 01	0	2.4048	0		
	0.2	2.4090	0.08302		
	0.4	2.4216	0.16518		
	0.6	2.4431	0.24559		
	0.8	2.4739	0.32337		
	1.0	2.5149	0.39762		
${}^{e}TE$ 11	0	1.8412	0		
	0.2	1.8416	0.10860		
	0.4	1.8430	0.21704		
	0.6	1.8452	0.32516		
	0.8	1.8484	0.43280		
	1.0	1.8527	0.53975		

Notes:

Superscripts e and o on mode designation signify even and odd.

c is parameter used in the Tables (Stratton, Morse, Chu, Hutner, "Elliptic Cylinder and Spheroidal Wave Functions")

r is the value of the argument which, for TM modes, makes the radial function zero and, for TE modes, makes its derivative zero.

e is the eccentricity of the ellipse;

$$e = \frac{\text{distance between foci}}{\text{major diameter}}$$

E is the ellipticity of the ellipse;

$$E = \frac{\text{difference between major and minor diam.}}{\text{major diameter}}$$

s is the root value, referred to the "average diameter"; it is related to r by:

$$s = \frac{r}{\pi}\,\frac{\text{perimeter}}{\text{major diameter}}$$

The quantity s is also related to the cutoff wavelength in an elliptical wave guide according to:

$$s = \frac{\text{perimeter of guide}}{\text{cutoff wavelength}}$$

Resonator Q

Although the calculation of the root values is straightforward and not overly laborious, the same cannot be said for the integrations involved in the determination of resonator Q. The procedure is obvious: The field equations are given; it is only necessary to integrate $H^2 d\tau$ over the volume and $H^2 d\sigma$ over the surface and get Q from

$$Q = \frac{2}{\delta} \frac{\int H^2 d\tau}{\int H^2 d\sigma} \tag{16}$$

with δ = skin depth, a known constant. Unfortunately the integrations cannot at present be expressed in closed form. A numerical solution can be obtained by a combination of integration in series and of numerical integration.

The calculations have been made for the ${}^eTE\ 01$ mode with $c = 2.0$, for which $r = 4.154$. This value of c corresponds in this case to an ellipticity of about 12%; in a 4″ cylinder this would amount to 1/2″ difference between largest and smallest diameters. Evaluation* of the integrals yields:

$$\int_V H^2\, d\tau = 12.307\, k_3^2 L + 12.294\, k_1^2 L$$

$$\int_S H^2\, d\sigma = 49.228\, k_3^2 + 0.1619\, k_1 k_3^2 L + 6.6847\, k_1^3 L$$

Substituting $k_1 = \dfrac{7.804}{D}$ and $k_3 = \dfrac{\pi n}{L}$, one obtains, finally

$$Q\delta = 0.471\, D \left(\frac{1 + 0.1622\, n^2 R^2}{1 + 0.0039\, n^2 R^2 + 0.1529\, n^2 R^3} \right).$$

For a circular cylinder,

$$Q_c \delta = 0.5\, D \left(\frac{1 + 0.1681\, n^2 R^2}{1 + 0.1681\, n^2 R^3} \right).$$

Comparison of these two formulas for $Q\delta$ shows that the losses in the end plates (n^2R^3 term) are less with respect to the side wall losses in the elliptical cylinder. The net loss in $Q\delta$, as described by the reduction in the multiplier from 0.5 to 0.471, is thus presumably ascribable to an increase in side wall losses (stored energy assumed held constant). The additional term in n^2R^2 in the denominator is responsible for the difference in the attenuation-frequency behavior of elliptical vs circular wave guide as shown by Chu, Fig. 4. Incidentally, these results agree numerically with those of Chu.

* Numerical integration was by Weddle's rule; intervals of 5° in φ and 0.1 in x were used. The calculations were made by Miss F. C. Larkey.

Corresponding expressions for the resonant wavelength are

$$\lambda = \frac{\pi D}{s\sqrt{1 + \left(\frac{\pi n D}{2sL}\right)^2}} = \frac{0.805\, D}{\sqrt{1 + 0.1622\, n^2 R^2}}$$

$$\lambda_c = \frac{0.820\, D}{\sqrt{1 + 0.1681\, n^2 R^2}}.$$

As an example, take $n = 1$, $R = 1$, then

(Circular)	$Q_c\delta = 0.500\, D$		$\lambda_c = 0.759\, D$
(Elliptical)	$Q\delta = 0.473\, D$		$\lambda = 0.747\, D$
	Ratio $= 0.946$		Ratio $= 0.984$.

Conclusions

The mathematics of the elliptic cylinder have not yet been developed to the point where the design of cavities of large ellipticity could be undertaken. On the other hand, sufficient results have been obtained to indicate that the ellipticity in a cavity intended to be circular, resulting from any reasonable manufacturing deviations, would not have a noticeable effect on the resonant frequencies or Q values, at least away from mode crossings.

Full Cylindrical Coaxial Resonator

The full coaxial resonator has been of some interest because of various suggestions for the use of a central rod for moving the tuning piston in a *TE* 01n cavity.

The cylindrical coaxial resonator, with the central conductor extending the full length of the resonator, has modes similar to the cylinder. In fact, the cylinder may be considered as a special case of the coaxial. The indices ℓ, m, n have much the same meaning and the resonant frequencies are determined by the same equation (1). However, now the value of r depends in addition (see Fig. 1) upon η, where

$$\eta = \frac{\text{diameter inner conductor}}{\text{diameter outer conductor}} = \frac{b}{a}.$$

The problem now arises of how best to represent the relations between f, a, b and L. The r's depend on η; so one possibility is to determine their values for a given η and then construct a series of mode charts, one for each value of η.

A more flexible arrangement is to plot the values of r vs η and allow the user to construct graphs suitable for the particular purpose in hand. An equivalent scheme has been used by Borgnis.[18]

It turns out that as $\eta \to 1$, $r(1 - \eta) \to m\pi$, for the *TM* modes and the

TE 0*mn* modes, and $r(1 - \eta) \rightarrow (m - 1)\pi$ for all other *TE* modes. For the former modes, r becomes very large as $\eta \rightarrow 1$, that is, as the inner conductor fills the cavity more and more, the frequency gets higher and higher. For the *TE* $\ell 1n$ modes, however, as the inner conductor grows, the frequency falls to a limiting value. This is discussed in more detail by Borgnis.[18]

Figure 7 shows $r(1 - \eta)$ vs η, for a few of the lower modes; the scale for η between 0.5 and 1.0 is collapsed since this region does not appear to be of great engineering interest. A different procedure is used for the roots of the *TE* $\ell 1n$ modes. Figure 8 is a direct plot of r vs η for a few of the lower modes. In this case, $r \rightarrow \ell$ as $\eta \rightarrow 1$.

Distribution of Normal Modes

The calculation of the distribution of the resonant modes for the coaxial case follows along the lines of that for the cylinder, as given previously. The difference lies in the distribution of the roots r, which now depend upon the parameter η. The determination of this latter distribution offers difficulties. There is some evidence, however, that the normal modes will follow, at least to a first approximation, the same law as the cylinder, viz.:

$$N = 4.4 \frac{V}{\lambda_0^3}$$

with some doubt regarding the value of the coefficient.

$Q \frac{\delta}{\lambda}$ *in Coaxial Resonator*

The integrations needed to obtain this factor are relatively straightforward, but a little complicated. The final results are given in Fig. 1.

The defining equation is (16); the components of H are given in Fig. 1. The integrations can be done with the aid of integrals given by McLachlan[17] and the following indefinite integral:

$$\int \left[\ell \frac{2Z_\ell^2(x)}{x^2} + Z_\ell'^2(x) \right] x\, dx = \frac{x^2}{2} \left[Z_\ell'^2(x) + \frac{2Z_\ell(x)Z_\ell'(x)}{x} + Z_\ell^2(x) \left(1 - \frac{\ell^2}{x^2}\right) \right]$$

which can be verified by differentiation, remembering that $y = Z_\ell(x)$ is a solution of $y'' + \frac{1}{x} y' + \left(1 - \frac{\ell^2}{x^2}\right) y = 0$.

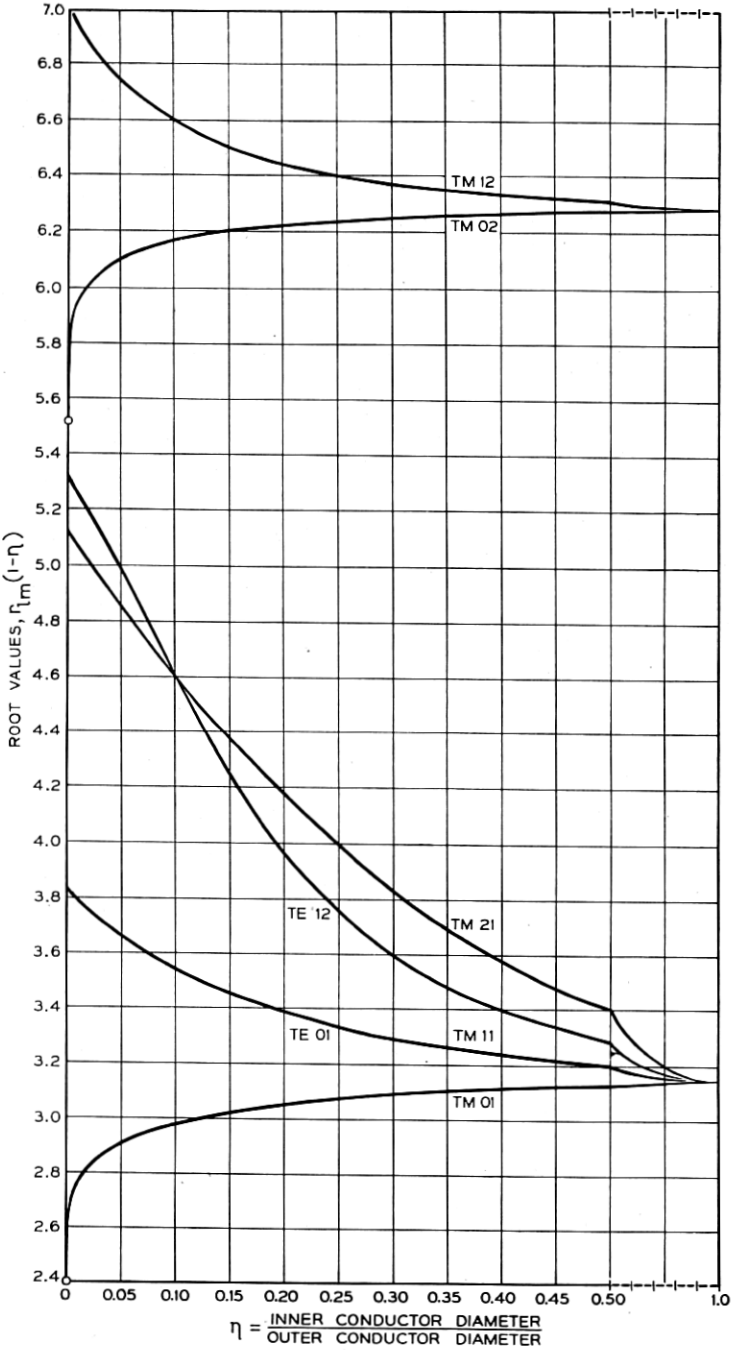


Fig. 7—Full coaxial resonator root values $r_{\ell m}\,(1 - \eta)$

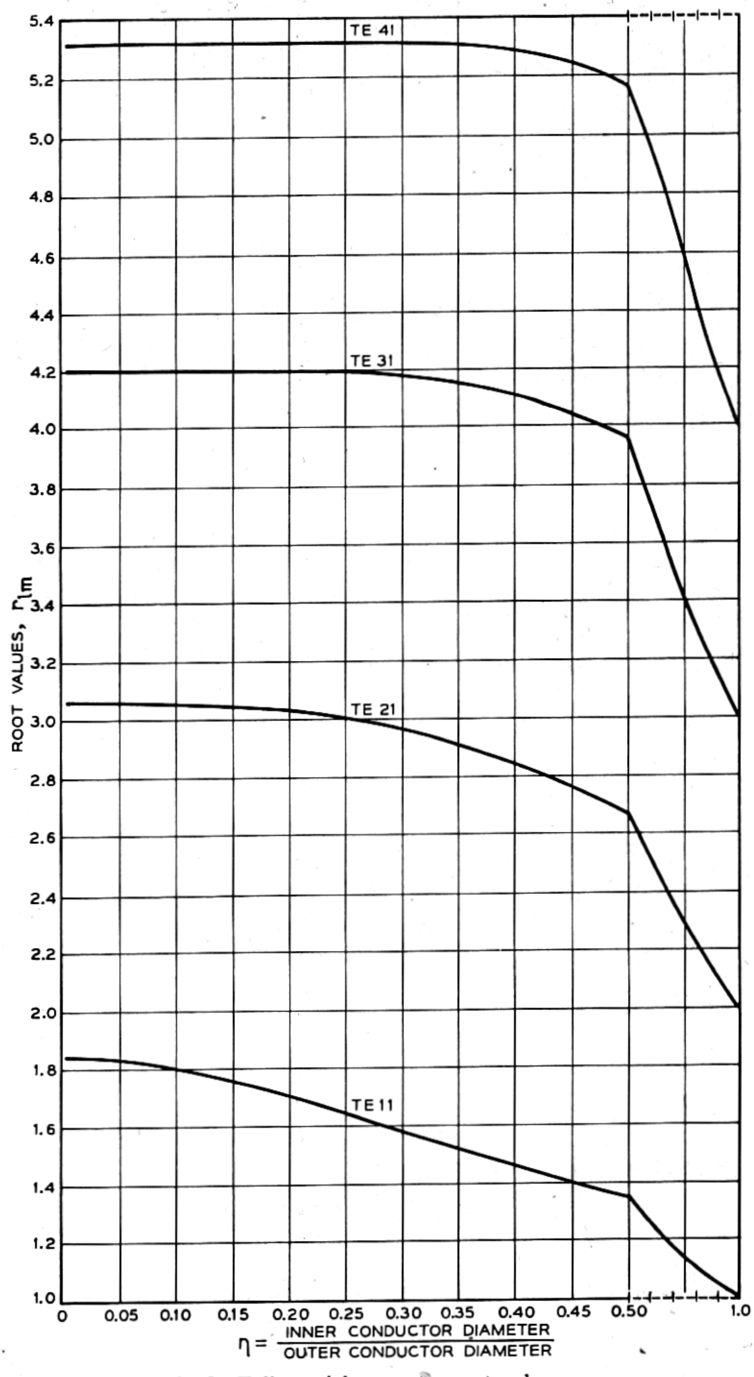


Fig. 8—Full coaxial resonator root values $r_{\ell m}$

An investigation needs to be made of the behavior of the formulas as $\eta \to 0$ before any conclusion may be drawn regarding their blending into those for the cylinder. For *TE* modes with $\ell = 0$, the term involving $\frac{H}{\eta}$ disappears, hence no question arises. Consider then $\ell > 0$, and let $x = \eta r$ for the discussion following. From expansions given in McLachlan, it is easy to show that, for small x

$$Y_\ell(x) \approx -\frac{(\ell-1)!}{\pi}\left(\frac{2}{x}\right)^\ell \qquad Y'_\ell(x) \approx \frac{\ell!}{\pi}\left(\frac{2}{x}\right)^\ell \frac{1}{x}$$

$$J_\ell(x) \approx \frac{x^\ell}{2^\ell \ell!} \qquad J'_\ell(x) \approx \frac{x^{\ell-1}}{2^\ell(\ell-1)!}.$$

Since, from Fig. 1,

$$A = \frac{J'_\ell(r)}{Y'_\ell(r)} = \frac{J'_\ell(\eta r)}{Y'_\ell(\eta r)} = \frac{J'_\ell(x)}{Y'_\ell(x)}$$

it is found, upon substitution of the approximations given above:

$$Z_\ell(x) \approx \frac{2x^\ell}{2^\ell \ell!}.$$

That is, $Z_\ell(x) \sim x^\ell$ and hence $\to 0$ as $x \to 0$. Furthermore $Z_\ell(r)$ remains finite as $\eta \to 0$. Hence $H \sim x^{2\ell}$ and $\frac{H}{\eta} \sim x^{2\ell-1}$. Therefore, for $\ell > 0$, $\frac{H}{\eta} \to 0$ as $\eta \to 0$.

Hence, the expression for $Q\frac{\delta}{\lambda}$ for the coaxial structure reduces to that for the cylinder, for any value of ℓ, in the *TE* modes.

For the *TM* modes, and for $\ell > 0$, an entirely similar argument shows that H' remains finite as $\eta \to 0$. Hence, the expression for $Q\frac{\delta}{\lambda}$ for these modes also reduces to that for the cylinder.

For the *TM* modes, and with $\ell = 0$, we have

$$Z'_0(x) = -J_1(x) + J_0(x)\frac{Y_1(x)}{Y_0(x)}.$$

For $x \to 0$, $J_1(x) \to 0$ and $J_0(x) \to 1$, hence for small x,

$$Z'_0(x) \sim \frac{Y_1(x)}{Y_0(x)}.$$

Now substitute the approximate values of the Y for small x. The result is

$$Z_0'(x) \sim \frac{1}{x \log \frac{x}{2}}.$$

Since $Z_0'(r)$ is finite, it follows that

$$\eta H' \sim \frac{1}{x\left(\log \frac{x}{2}\right)^2}$$

and it is easily shown that $\eta H' \rightarrow \infty$ as $\eta \rightarrow 0$. On the other hand, $\eta^2 H' \rightarrow 0$ as $\eta \rightarrow 0$. Hence, $Q\frac{\delta}{\lambda} \rightarrow 0$ as $\eta \rightarrow 0$. On the other hand, for $\eta = 0$, a

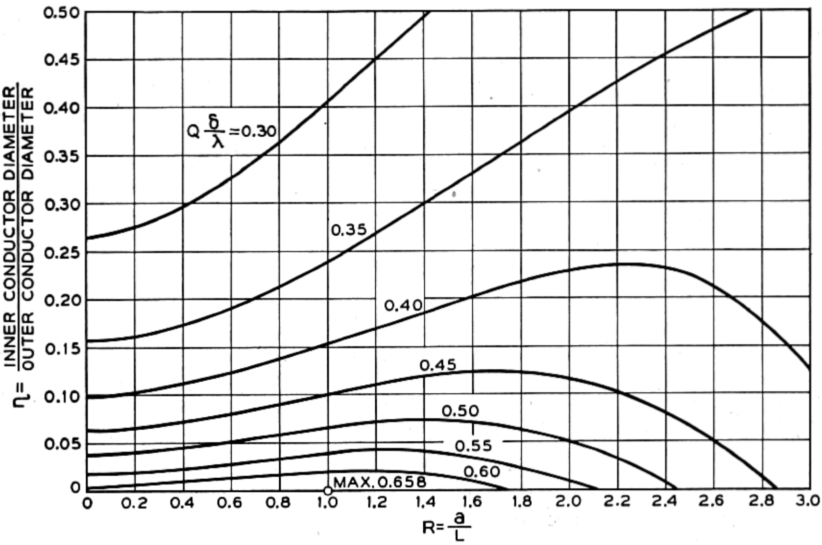


Fig. 9—Coaxial resonator. *TE* 011 mode Contour lines of $Q\frac{\delta}{\lambda}$

perfect cylinder exists whose $Q\frac{\delta}{\lambda}$ is not zero. It is concluded that the expression for $Q\frac{\delta}{\lambda}$ does not apply for small η for the *TM* modes with $\ell = 0$.

Thus it is seen that the expressions for the factor $(Q\frac{\delta}{\lambda})$ reduce to those given for the cylinder, when $\eta = 0$, except for *TM* modes with $\ell = 0$. For these latter cases, the factor approaches zero as η approaches zero, because $\eta H'$ increases without limit. This means that an assumption

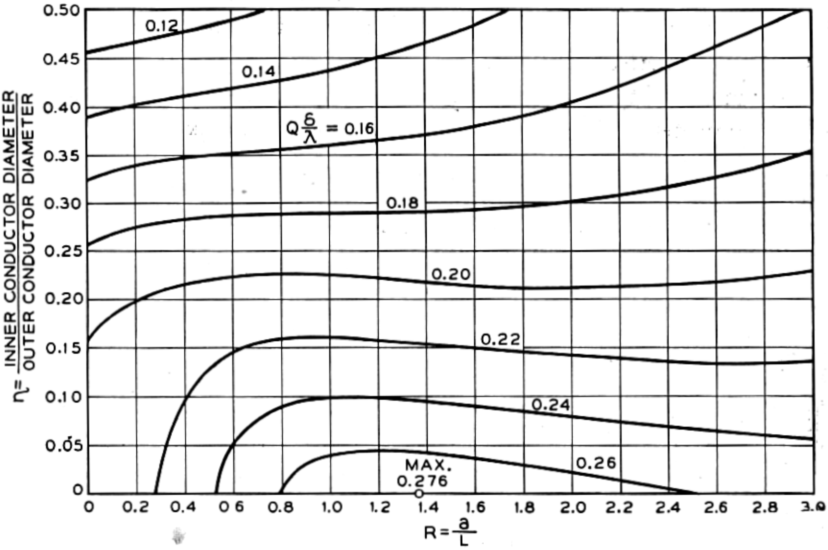


Fig. 10—Coaxial resonator. *TE* 111 mode Contour lines of $Q\frac{\delta}{\lambda}$

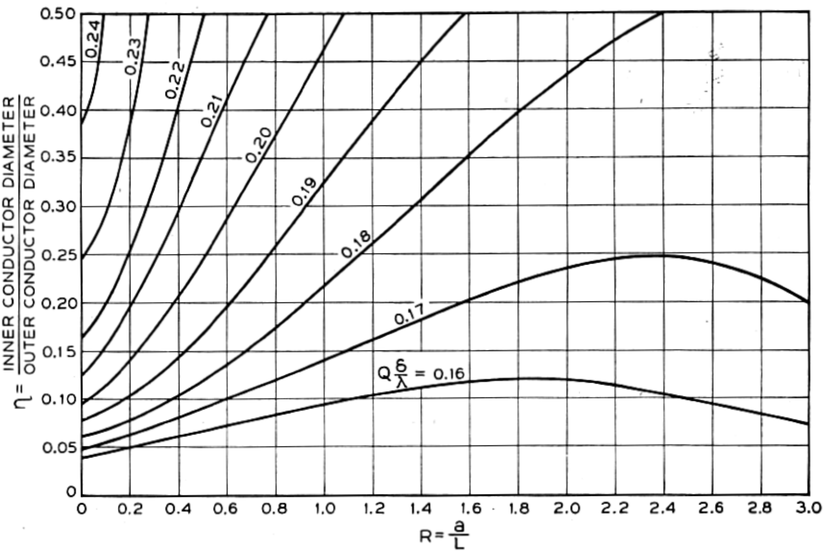


Fig. 11—Coaxial resonator. *TM* 011 mode Contour lines of $Q\frac{\delta}{\lambda}$

which was made in the derivation of the Q values is not valid for small η; that is, the fields for the dissipative case are not the same as those derived on the basis of perfectly conducting walls.

The expressions for the factor are rather complicated, as it depends on several parameters. When a given mode is chosen, the number of parameters reduces to two, η and R. Contour diagrams of $Q\frac{\delta}{\lambda}$ vs η and R are given on Figs. 9, 10, 11 and 12 for the *TE* 011, *TE* 111, *TM* 011 and *TM* 111

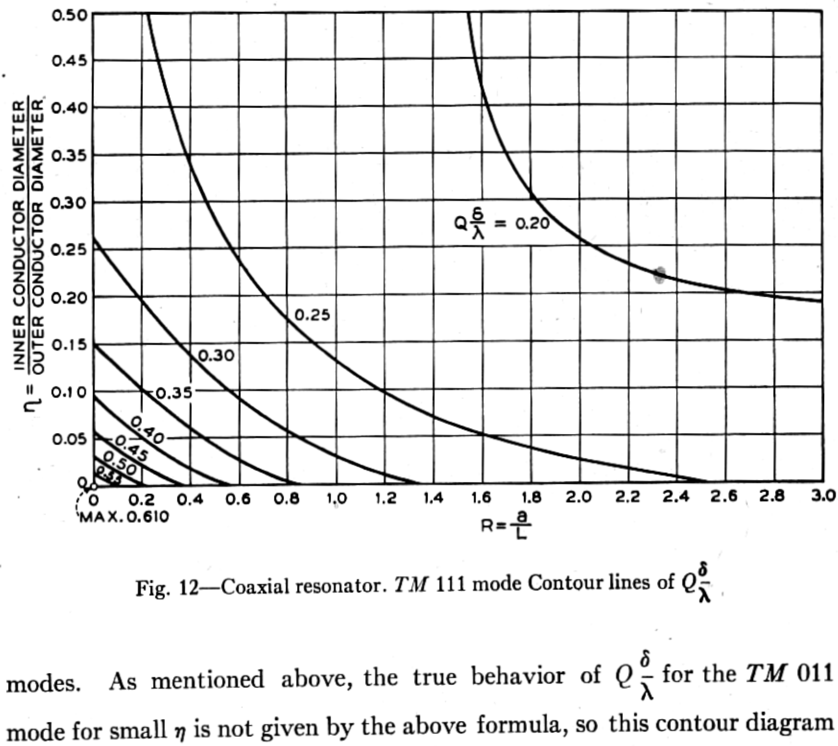


Fig. 12—Coaxial resonator. *TM* 111 mode Contour lines of $Q\frac{\delta}{\lambda}$.

modes. As mentioned above, the true behavior of $Q\frac{\delta}{\lambda}$ for the *TM* 011 mode for small η is not given by the above formula, so this contour diagram has been left incomplete.

Fins in a Cavity Resonator

The suppression of extraneous modes is always an important problem in cavity design. Among the many ideas advanced along these lines is the use of structures internal to the cavity.

It is well known that if a thin metallic fin or septum is introduced into a cavity resonator in a manner such that it is everywhere perpendicular to the *E*-lines of one of the normal modes, then the field configuration and frequency of that particular mode are undisturbed. For example, Fig. 13 shows the E-lines in a TE 11n mode in a circular cylinder. If the upper half of the cylinder wall is replaced by a new surface, shown dotted, the field and frequency in the resulting flattened cylinder will be the same as

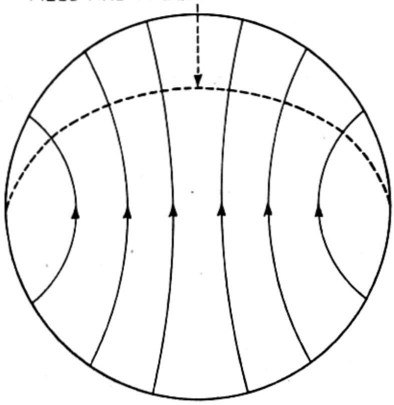


Fig. 13—E Lines in TE 11n mode

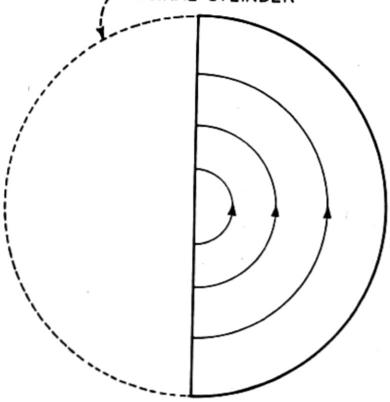


Fig. 14—"TE 01n" mode in half-cylinder

before. Indeed, they will also be the same in the crescent-shaped resonator indicated in the figure.

Except for isolated cases, all the other modes of the original cylinder will be perturbed in frequency since the old fields fail to satisfy the boundary conditions over the new surface. Furthermore, if the original cylinder was circular, its inherent double degeneracy will be lost and each of the original modes (with minor exceptions) will split into two.

Although the frequency and fields of the undisturbed mode are the same, the Q is not necessarily so. For example, Fig. 14 shows a "*TE* 01*n* mode" in a half cylinder.*

It is easy to calculate $Q\frac{\delta}{\lambda}$ for this case. The result is

$$Q\frac{\delta}{\lambda} = \frac{r}{2\pi} \cdot \frac{(1 + p^2R^2)^{3/2}}{1 + p^2R^3 + K_1 + K_2p^2R^2} \tag{17}$$

in which

$$K_1 = 1.290 \qquad K_2 = 0.653.$$

Here K_1 and K_2 are constants which account for the resistance losses in the flat side. For the full cavity, shown dotted in Fig. 14, eq. (17) holds with $K_1 = K_2 = 0$. If the circular cavity has a partition extending from the center to the rim along the full length, (17) holds with the values of K_1 and K_2 halved. If a fin projects from the rim partway into the interior, still other values of K_1 and K_2 are required. It is a simple matter to compute these for various immersions; Fig. 15 shows curves of K_1 and K_2. The following table gives an idea of the magnitudes involved:

MODE: *TE* 0,1,12 $R = 0.4$

Fin, % a	$Q\frac{\delta}{\lambda}$	Ratio
0%	2.573	1.0
10	2.536	.985
20	2.479	.965
50	2.04	.79
100	1.47	.57

The question now is asked, "Suppose a longitudinal fin were used, small enough to cause only a tolerable reduction in the Q. Would such a fin ameliorate the design difficulties due to extraneous modes?"

Some of the effects seem predictable. All modes with $\ell > 0$ will be split to some extent, into two modes of different frequencies. Consider the *TE* 12*n* mode, for example. There will be one mode, of the same frequency as the original whose orientation must be such that its *E*-lines are perpendicular to the fin. The Q of this mode would be essentially unchanged. There will be a second mode, oriented generally 90° from the first, whose *E*-lines will be badly distorted (and the frequency thereby lowered) in the vicinity

* Solutions for a cylinder of this cross-section are known and all the resonant frequencies and Q values could be computed, if they had any application.

of the fin. It would be reasonable to expect the Q of this mode to be appreciably lowered because of the concentrated field there. If two fins at 90° were present, there would be no orientation of the original TE $12n$ mode which would satisfy the boundary conditions. In this case both new modes

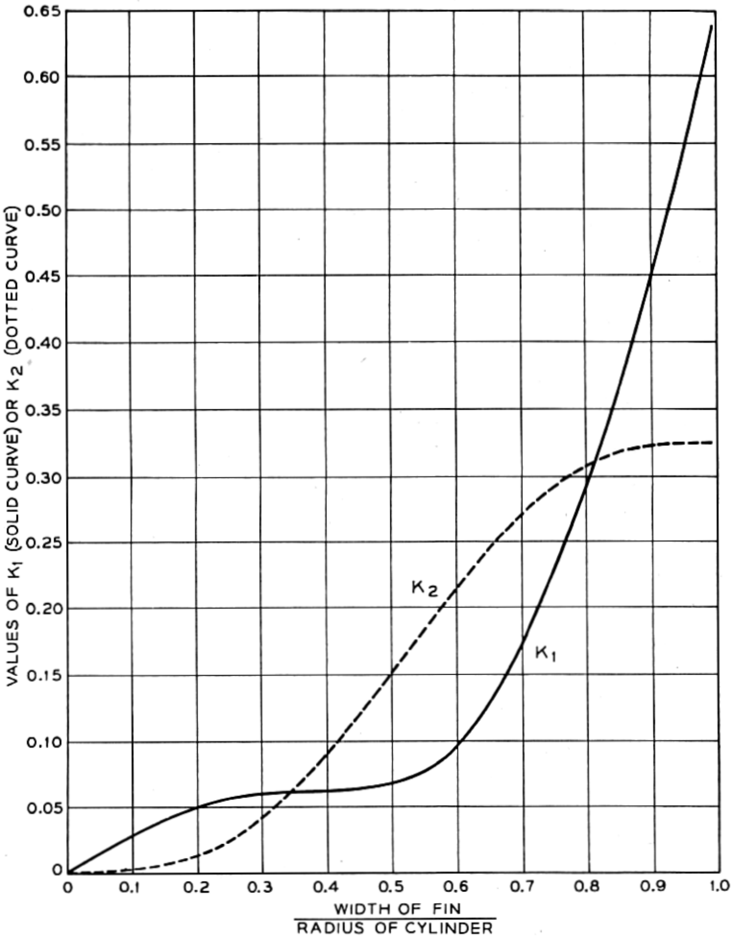


Fig. 15—Constants for calculation of Q of TE $01n$ mode in cylinder with longitudinal fin.

would be perturbed in frequency from the original value. If both fins were identical, the perturbations would be equal and a double degeneracy ensue. Similar effects would happen to the other types of modes.

The major advantage derivable from such effects would appear to be in extraneous transmissions. The fin serves to orient positively the fields in the cavity, and the input and output coupling locations can then be appropriately chosen. On the basis that internal couplings are responsible for mode crossing difficulties, one might hazard a guess that a real fin would increase such couplings.

Another application of fins might be in a wave guide feed in which it is desired to establish only a $TE\ 0m$ wave. In this case, Q is not so important and larger fins can be used. If these extended virtually to the center and x of them were present (with uniform angular spacing) all types of wave transmission having ℓ less than $x/2$, x even or ℓ less than x, x odd, would be suppressed. This use of fins is an extension of the wires that have been proposed in the past.

Conclusion

It is hoped that the foregoing, which covers some of the theoretical work done by the author during the war, will be of value to other workers in cavity resonators. There is much that needs to be done and hardly time for duplication of effort.

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