Probability Functions for the Modulus and Angle of the Normal Complex Variate

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This paper deals mainly with various 'distribution functions' and 'cumulative distribution functions' pertaining to the modulus and to the angle of the 'normal' complex variate, for the case where the mean value of this variate is zero. Also, for auxiliary uses chiefly, the distribution function pertaining to the reciprocal of the modulus is included. For all of these various probability functions the paper derives convenient general formulas, and for four of the functions it supplies comprehensive sets of curves; furthur, it gives a table of computed values of the cumulative distribution function for the modulus, serving to verify the values computed by a different method in an earlier paper by the same author.[1]

Introduction

IN THE solution of problems relating to alternating current networks and transmission systems by means of the usual complex quantity method, any deviation of any quantity from its reference value is naturally a complex quantity, in general. If, further, the deviation is of a random nature and hence is variable in a random sense, then it constitutes a 'complex random variable,' or a 'complex variate,' the word 'variate' here meaning the same as 'random variable' (or 'chance variable'—though, on the whole, 'random variable' seems preferable to 'chance variable' and is more widely used).

Although a complex variate may be regarded formally as a single analytical entity, denotable by a single letter (as Z), nevertheless it has two analytical constituents, or components: for instance, its real and imaginary constituents (X and Y); also, its modulus and amplitude ($|Z|$ and θ). Correspondingly, a complex variate can be represented geometrically by a single geometrical entity, namely a plane vector, but this, in turn, has two geometrical components, or constituents: for instance, its two rectangular components (X and Y); also, its two polar components, radius vector and vectorial angle ($R \equiv |Z|$ and θ).

This paper deals mainly with the modulus and the angle of the complex variate,[2] which are often of greater theoretical interest and practical im-

[1] "Probability Theory and Telephone Transmission Engineering," *Bell System Technical Journal*, January 1933, which will hereafter be referred to merely as the "1933 paper".

[2] Throughout the paper, I have used the term 'complex variate' for any 2-dimensional variate, because of the nature of the contemplated applications indicated in the first

portance than the real and imaginary constituents. The modulus variate and the angle variate, individually and jointly, are of considerable theoretical interest; while the modulus variate is also of very considerable practical importance, and the angle variate may conceivably become of some practical importance.

The paper is concerned chiefly with the 'distribution functions'[3] and the 'cumulative distribution functions' pertaining to the modulus (Sections 3 and 5) and to the angle (Sections 6 and 7) of the 'normal' complex variate, for the case where the mean value of this variate is zero. The distribution function for the reciprocal of the modulus is also included (Section 4).

The term 'probability function' is used in this paper generically to include 'distribution function' and 'cumulative distribution function.'

To avoid all except short digressions, some of the derivation work has been placed in appendices, of which there are four. These may be found of some intrinsic interest, besides facilitating the understanding of the paper.

1. Distribution Function and Cumulative Distribution Function in General: Definitions, Terminology, Notation, Relations, and Formulas

The present section constitutes a generic basis for the rest of the paper.

Let τ denote any complex variate, and let ρ and σ denote any pair of real quantities determining τ and determined by τ. (For instance, ρ and σ might be the real and imaginary components of τ, or they might be the modulus and angle of τ.) Geometrically, ρ and σ may be pictured as general curvilinear coordinates in a plane, as indicated by Fig. 1.1.

Let τ' denote the unknown value of a random sample consisting of a single τ-variate, and ρ' and σ' the corresponding unknown values of the constituents of τ'.

Further, let $G(\rho, \sigma)$ denote the 'areal probability density' at any point ρ,σ in the ρ,σ-plane, so that $G(\rho,\sigma)dA$ gives the probability that τ' falls in a differential area dA containing the point τ; and so that the integral of

paragraph of the Introduction, and also because the present paper is a sort of sequel to my 1933 paper, where the term 'complex variate' (or rather, 'complex chance-variable') was used throughout since there it seemed clearly to be the best term, on account of the field of applications contemplated and the specific applications given as illustrations. However, for wider usage the term 'bivariate' might be preferred because of its prevalence in the field of Mathematical Statistics; and therefore the paper should be read with this alternative in view.

[3] The term 'distribution function' is used with the same meaning in this paper as in my 1933 paper, although there the term 'probability law' was used much more frequently than 'distribution function,' but with the same meaning.

$G(\rho,\sigma)dA$ over the entire ρ,σ-plane is equal to unity, corresponding to certainty.

For the sake of subsequent needs of a formal nature, it will now be assumed that $G(\rho,\sigma) = 0$ at all points ρ,σ outside of the ρ_1, ρ_2, σ_1, σ_2 quadrilateral region in the ρ,σ-plane, Fig. 1.1, bounded by arcs of the four heavy curves, for which ρ has the values ρ_1 and ρ_2 and σ the values σ_1 and σ_2, with ρ_1 and σ_1 regarded, for convenience, as being less than ρ_2 and σ_2 respectively. Further, $G(\rho,\sigma)$ will be assumed to be continuous inside of this

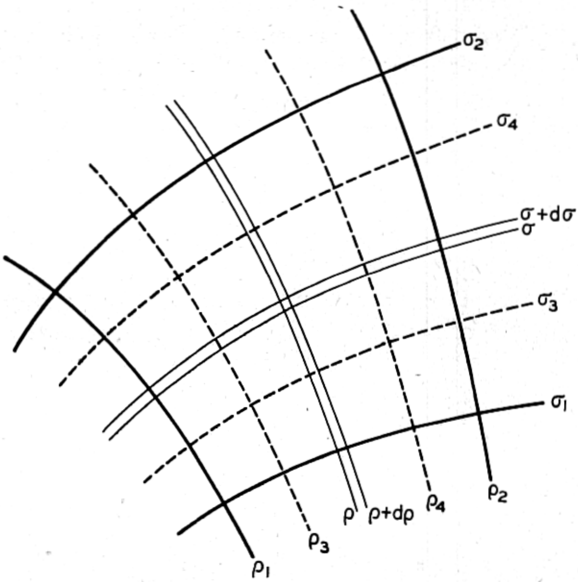

Fig. 1.1—Diagram of general curvilinear coordinates.

quadrilateral region, and to be non-infinite on its boundary. Hence, for probability purposes, it will suffice to deal with the open inequalities

$$\rho_1 < \rho < \rho_2, \qquad (1.1) \qquad\qquad \sigma_1 < \sigma < \sigma_2, \qquad (1.2)$$

which pertain to this quadrilateral region excluding its boundary; and thus it will not be necessary to deal with the closed inequalities $\rho_1 \leqq \rho \leqq \rho_2$ and $\sigma_1 \leqq \sigma \leqq \sigma_2$, which include the boundary.[4]

[4] The matters dealt with generically in this paragraph may be illustrated by the following two important particular cases, which occur further on, namely:

POLAR COORDINATES: $\rho = |\tau| = R$, $\sigma = \theta =$ angle of τ. Then $\rho_1 = R_1 = 0$, $\rho_2 = R_2 = \infty$, $\sigma_1 = \theta_1 = 0$, $\sigma_2 = \theta_2 = 2\pi$, whence (1.1) and (1.2) become $0 < R < \infty$ and $0 < \theta < 2\pi$, respectively.

RECTANGULAR COORDINATES: $\rho = \text{Re}\,\tau = x$, $\sigma = \text{Im}\,\tau = y$. Then $\rho_1 = x_1 = -\infty$, $\rho_2 = x_2 = \infty$, $\sigma_1 = y_1 = -\infty$, $\sigma_2 = y_2 = \infty$, whence (1.1) and (1.2) become $-\infty < x < \infty$ and $-\infty < y < \infty$, respectively.

A generic quadrilateral region contained within the quadrilateral region $\rho_1, \rho_2, \sigma_1, \sigma_2$ in Fig. 1.1 is the one bounded by arcs of the dashed curves $\rho_3, \rho_4, \sigma_3, \sigma_4$, where $\rho_3 < \rho_4$ and $\sigma_3 < \sigma_4$. Here, as in the preceding paragraph, it will evidently suffice to deal with open inequalities.

Referring to Fig. 1.1, the probability functions with which this paper will chiefly deal are certain particular cases of the probability functions $P(\rho, \sigma)$, $P(\rho \mid \sigma_{34})$ and $Q(\rho_{34}, \sigma_{34})$ occurring on the right sides of the following three equations respectively:

$$p(\rho < \rho' < \rho + d\rho, \sigma < \sigma' < \sigma + d\sigma) = P(\rho,\sigma)d\rho d\sigma, \tag{1.3}$$

$$p(\rho < \rho' < \rho + d\rho, \sigma_3 < \sigma' < \sigma_4) = P(\rho \mid \sigma_{34})d\rho, \tag{1.4}$$

$$p(\rho_3 < \rho' < \rho_4, \sigma_3 < \sigma' < \sigma_4) = Q(\rho_{34}, \sigma_{34}). \tag{1.5}$$

These equations serve to define the above-mentioned probability functions occurring on the right sides in terms of the probabilities denoted by the left sides, each expression $p(\quad)$ on the left side denoting the probability of the pair of inequalities within the parentheses.[5] Inspection of these equations shows that: $P(\rho,\sigma)$ is the 'distribution function' for ρ and σ jointly; $P(\rho \mid \sigma_{34})$ is a 'distribution function' for ρ individually, with the understanding that σ' is restricted to the range σ_3-to-σ_4; $Q(\rho_{34}, \sigma_{34})$ is a 'cumulative distribution function' for ρ and σ jointly.

Since the left sides of (1.3), (1.4) and (1.5) are necessarily positive, the right sides must be also. Hence, as all of the probability functions occurring in the right sides are of course desired to be positive, the differentials $d\rho$ and $d\sigma$ must be taken as positive, if we are to avoid writing $|d\rho|$ and $|d\sigma|$ in place of $d\rho$ and $d\sigma$ respectively.

Returning to (1.3), it is seen that, stated in words, $P(\rho,\sigma)$ is such that $P(\rho,\sigma)d\rho d\sigma$ gives the probability that the unknown values ρ' and σ' of the constituents of the unknown value τ' of a random sample consisting of a single τ-variate lie respectively in the differential intervals $d\rho$ and $d\sigma$ containing the constituent values ρ and σ respectively. Thus, unless $d\rho d\sigma$ is the differential element of area, $P(\rho,\sigma)$ is not equal to the 'areal probability density,' $G(\rho,\sigma)$, defined in the fourth paragraph of this section. In general, if E is such that $Ed\rho d\sigma$ is the differential element of area, then $P(\rho, \sigma) = EG(\rho, \sigma)$. (An illustration is afforded incidentally by Appendix A.)

$P(\rho,\sigma)$, defined by (1.3), is the basic 'probability function,' in the sense that the others can be expressed in terms of it, by integration. Thus

[5] Thus p in $p(\quad)$ may be read 'probability that' or 'probability of.'

$P(\rho \mid \sigma_{34})$ and $P(\sigma \mid \rho_{34})$, defined respectively by (1.4) and by the correlative of (1.4), can be expressed as 'single integrals,' as follows[6]:

$$P(\rho \mid \sigma_{34}) = \int_{\sigma_3}^{\sigma_4} P(\rho,\sigma)\, d\sigma, \quad (1.6) \qquad P(\sigma \mid \rho_{34}) = \int_{\rho_3}^{\rho_4} P(\rho,\sigma)\, d\rho. \quad (1.7)$$

$Q(\rho_{34}, \sigma_{34})$, defined by (1.5), can be expressed as a 'double integral,' fundamentally; but, for purposes of analysis and of evaluation, this will be replaced by its two equivalent 'repeated integrals':

$$Q(\rho_{34}, \sigma_{34}) = \int_{\rho_3}^{\rho_4} \left[\int_{\sigma_3}^{\sigma_4} P(\rho,\sigma)\, d\sigma \right] d\rho = \int_{\sigma_3}^{\sigma_4} \left[\int_{3}^{\rho_4} P(\rho,\sigma)\, d\rho \right] d\sigma, \quad (1.8)$$

the set of integration limits being the same in both repeated integrals because these limits are constants, as indicated by Fig. 1.1. On account of (1.6) and (1.7) respectively, (1.8) can evidently be written formally as two single integrals:

$$Q(\rho_{34}, \sigma_{34}) = \int_{\rho_3}^{\rho_4} P(\rho \mid \sigma_{34})\, d\rho = \int_{\sigma_3}^{\sigma_4} P(\sigma \mid \rho_{34})\, d\sigma, \quad (1.9)$$

but implicitly these are repeated integrals unless the single integrations in (1.6) and (1.7) can be executed, in which case the integrals in (1.9) will actually be single integrals, and these will be quite unlike each other in form, being integrals with respect to ρ and σ respectively—though of course yielding a common expression in case the indicated integrations can be executed.

The particular cases of (1.4) and (1.5) with which this paper will chiefly deal are the following three:

$$p(\rho < \rho' < \rho + d\rho, \sigma_1 < \sigma' < \sigma_2) = P(\rho \mid \sigma_{12})\, d\rho \equiv P\,(\rho)\, d\rho, \quad (1.10)$$

$$p(\rho_1 < \rho' < \rho, \sigma_1 < \sigma' < \sigma_2) = Q(< \rho,\sigma_{12}) \equiv Q(\rho), \quad (1.11)$$

$$p(\rho < \rho' < \rho_2, \sigma_1 < \sigma' < \sigma_2) = Q(> \rho,\sigma_{12}) \equiv Q^*(\rho). \quad (1.12)$$

[6] The single-integral formulation in (1.6) can be written down directly by mere inspection of the left side of (1.4). Alternatively, (1.6) can be obtained by representing the left side of (1.4) by a repeated integral, as follows:

$$P(\rho \mid \sigma_{34}) d\rho = \int_{\rho}^{\rho+d\rho} \left[\int_{\sigma_3}^{\sigma_4} P(\rho,\, \sigma) d\sigma \right] d\rho = \left[\int_{\sigma_3}^{\sigma_4} P(\rho,\, \sigma) d\sigma \right] d\rho,$$

whence (1.6); the last equality in the above chain equation in this footnote evidently results from the fact that, in general, $\int_x^{x+dx} f(x)dx = f(x)dx$, since each side of this equation represents dA, the differential element of area under the graph of $f(x)$ from x to $x + dx$.

In each of these three equations the very abbreviated notation at the extreme right will be used wherever the function is being dealt with extensively, as in the various succeeding sections. Such notation will not seem unduly abbreviated nor arbitrary if the following considerations are noted: In (1.10), σ_{12} corresponds to the entire effective range of σ, so that $P(\rho \mid \sigma_{12})$ is the 'principal' distribution function for ρ. Similarly, in (1.11), $Q(< \rho, \sigma_{12})$ is the 'principal' cumultive distribution function for ρ. In (1.12), the star indicates that $Q^*(\rho)$ is the 'complementary' cumulative distribution function, since $Q(\rho) + Q^*(\rho) = Q(\rho_{12}, \sigma_{12}) = 1$, unity being taken as the measure of certainty, of course.

For occasional use in succeeding sections, the defining equations for the probability functions pertaining to four other particular cases will be set down here:

$$p(\rho < \rho' < \rho + d\rho, \sigma_1 < \sigma' < \sigma) = P(\rho \mid < \sigma)\, d\rho, \tag{1.13}$$

$$p(\rho < \rho' < \rho + d\rho, \sigma < \sigma' < \sigma_2) = P(\rho \mid > \sigma)\, d\rho, \tag{1.14}$$

$$p(\rho_1 < \rho' < \rho, \sigma_1 < \sigma' < \sigma) = Q(< \rho, < \sigma), \tag{1.15}$$

$$p(\rho < \rho' < \rho_2, \sigma_1 < \sigma' < \sigma) = Q(> \rho, < \sigma). \tag{1.16}$$

It may be noted that (1.13) and (1.14) are mutually supplementary, in the sense that their sum is (1.10). Similarly, (1.15) and (1.16) are mutually supplementary, in the sense that their sum is $Q(\rho_{12}, < \sigma) = Q(< \sigma, \rho_{12})$, which is the correlative of (1.11).

This section will be concluded with the following three simple transformation relations (1.17), (1.18) and (1.19), which will be needed further on. They pertain to the probability functions on the right sides of equations (1.3), (1.4) and (1.5) respectively. h and k denote any positive real constants, the restriction to positive values serving to simplify matters without being too restrictive for the needs of this paper.

$$P(h\rho, k\sigma) = \frac{1}{hk} P(\rho, \sigma), \tag{1.17}$$

$$P(h\rho \mid k\sigma_{34}) = \frac{1}{h} P(\rho \mid \sigma_{34}), \tag{1.18}$$

$$Q(h\rho_{34}, k\sigma_{34}) = Q(\rho_{34}, \sigma_{34}). \tag{1.19}$$

Each of the three formulas (1.17), (1.18), (1.19) can be rather easily derived in at least two ways that are very different from each other. One way depends on probability inequality relations of the sort

$$p(t<t'<t+dt) = p(gt<gt'<gt+d[gt]), \tag{1.20}$$

$$p(t_3<t'<t_4) = p(gt_3<gt'<gt_4), \tag{1.21}$$

where t stands generically for ρ and for σ, and g is any positive real constant, standing generically for h and for k; (1.20) and (1.21) are easily seen to be true by imagining every variate in the universe of the t-variates to be multiplied by g, thereby obtaining a universe of (gt)-variates. A second way of deriving each of the three formulas (1.17), (1.18), (1.19) depends on general integral relations of the sort

$$\int_a^b f(t)\,dt = \frac{1}{g}\int_{ga}^{gb} f(t)\,d(gt) = \frac{1}{g}\int_{ga}^{gb} f\left(\frac{\lambda}{g}\right)d\lambda. \tag{1.22}$$

A third way, which is distantly related to the second way, depends on the use of the Jacobian for changing the variables in any double integral; thus,

$$\frac{P(\rho,\sigma)}{P(\lambda,\mu)} = \left|\frac{d\lambda d\mu}{d\rho d\sigma}\right| = \left|\frac{\partial(\lambda,\mu)}{\partial(\rho,\sigma)}\right| = 1 \div \left|\frac{\partial(\rho,\sigma)}{\partial(\lambda,\mu)}\right|, \tag{1.23}$$

the first equality in (1.23) depending on the fact that the two sets of variables and of differentials have corresponding values and hence are so related that

$$p(\rho<\rho'<\rho+d\rho,\ \sigma<\sigma'<\sigma+d\sigma) = p(\lambda<\lambda'<\lambda+d\lambda,\ \mu<\mu'<\mu+d\mu), \tag{1.24}$$

whence

$$P(\rho,\sigma)\,|\,d\rho d\sigma\,| = P(\lambda,\mu)\,|\,d\lambda d\mu\,|.$$

2. The Normal Complex Variate and Its Chief Probability Functions

The 'normal' complex variate may be defined in various equivalent ways. Here, a given complex variate $z = x + iy$ will be defined as being 'normal' if it is possible to choose in the plane of the scatter diagram of z a pair of rectangular axes, u and v, such that the distribution function[7] $P(u,v)$ for the given complex variate with respect to these axes can be written in the form[8]

$$P(u,v) = \frac{1}{2\pi S_u S_v}\exp\left[-\frac{u^2}{2S_u^2} - \frac{v^2}{2S_v^2}\right] = P(u)P(v). \tag{2.1}$$

We shall call $w = u + iv$ the 'modified' complex variate, as it represents the value of the given complex variate $z = x + iy$ when the latter is referred to the u,v-axes; $P(u)$ and $P(v)$ are respectively the individual distribution functions for the u and v components of the modified complex variate; and

[7] Defined by equation (1.3) on setting $\rho = u$ and $\sigma = v$.

[8] This equation is (12) of my 1933 paper. It can be easily verified that the (double) integral of (2.1) taken over the entire u, v-plane is equal to unity.

S_u and S_v are distribution parameters called the 'standard deviations' of u and v respectively. If t stands for u and for v generically, then

$$P(t) = \frac{1}{\sqrt{2\pi}S_t}\exp\left[-\frac{t^2}{2S_t^2}\right], \qquad (2.2) \qquad S_t^2 = \int_{-\infty}^{\infty} t^2 P(t)\,dt. \qquad (2.3)$$

From the viewpoint of the scatter diagram, the distribution function $P(u,v)$ is, in general, equal to the 'areal probability density' at the point u,v in the plane of the scatter diagram, so that the probability of falling in a differential element of area dA containing the point u,v is equal to $P(u,v)dA$; similarly, $P(u)$ and $P(v)$ are equal to the component probability densities. In particular, the probability density is 'normal' when $P(u,v)$ is given by (2.1).

Geometrically, equation (2.1) evidently represents a surface, the normal 'probability surface,' situated above the u, v-plane; and $P(u, v)$ is the ordinate from any point u,v in the u,v-plane to the probability surface.

The u,v-axes described above will be recognized as being the 'principal central axes,' namely that pair of rectangular axs which have their origin at the 'center' of the scatter diagram of $z = x + iy$ and hence at the center of the scatter diagram of $w = u + iv$, so that $\overline{w} = 0$, and are so oriented in the scatter diagram that $\overline{uv} = 0$ (whereas $\bar{z} \neq 0$ and $\overline{xy} \neq 0$, in general).

In equation (2.1), which has been adopted above as the analytical basis for defining the 'normal' complex variate, the distribution parameters are S_u and S_v ; and they occur symmetrically there, which is evidently natural and is desirable for purposes of definition. Henceforth, however, it will be preferable to adopt as the distribution parameters the quantities S and b defined by the pair of equations[9]

$$S^2 = S_u^2 + S_v^2, \qquad (2.4) \qquad bS^2 = S_u^2 - S_v^2, \qquad (2.5)$$

whence

$$b = \frac{S_u^2 - S_v^2}{S_u^2 + S_v^2} = \frac{1 - (S_v/S_u)^2}{1 + (S_v/S_u)^2}. \qquad (2.6)$$

From (2.4), S is seen to be a sort of 'resultant standard deviation.' The last form of (2.6) shows clearly that the total possible range of b is

$$-1 \leqq b \leqq 1, \qquad \text{corresponding to} \qquad \infty \geqq S_v/S_u \geqq 0.$$

The pair of simultaneous equations (2.4) and (2.5) give

$$2S_u^2 = (1+b)S^2, \qquad (2.7) \qquad 2S_v^2 = (1-b)S^2, \qquad (2.8)$$

which will be used below in deriving (2.11).

[9] Equations (2.4) and (2.6) are respectively (14) and (13) of my 1933 paper.

With the purpose of reducing the number of parameters by 1 and of dealing with variables that are dimensionless, we shall henceforth deal with the 'reduced' modified variate $W = U + iV$ defined by the equation

$$W = w/S = u/S + iv/S = U + iV. \tag{2.9}$$

Thus we shall be directly concerned with the scatter diagram of $W = U + iV$ instead of with that of $w = u + iv$.

The distribution function $P(U,V)$ for the rectangular components U and V of any complex variate $W = U + iV$ is defined by (1.3) on setting $\rho = U$ and $\sigma = V$; thus,

$$P(U,V)dUdV = p(U<U'<U+dU,\ V<V'<V+dV). \tag{2.10}$$

When the given variate $z = x + iy$ is normal, so that the modified variate $w = u + iv$ is normal, as represented by (2.1), then, since S is a mere constant, the reduced modified variate $W = U + iV$ defined by (2.9) will evidently be normal also, though of course with a different distribution parameter. Its distribution function $P(U,V)$ is found to have the formula[10]

$$P(U,V) = \frac{1}{\pi\sqrt{1-b^2}} \exp\left[-\frac{U^2}{1+b} - \frac{V^2}{1-b}\right] = P(U)P(V), \tag{2.11}$$

where $P(U)$ and $P(V)$ are the component distribution functions:

$$P(U) = \frac{1}{\sqrt{\pi(1+b)}} \exp\left[-\frac{U^2}{1+b}\right], \tag{2.12}$$

$$P(V) = \frac{1}{\sqrt{\pi(1-b)}} \exp\left[-\frac{V^2}{1-b}\right]. \tag{2.13}$$

These three distribution functions each contain only one distribution parameter, namely b; moreover, the variables $U = u/S$ and $V = v/S$ are dimensionless.

The distribution function $P(R,\theta)$ for the polar components R and θ of any complex variate $W \equiv R(\cos\theta + i\sin\theta)$ is defined by (1.3) on setting $\rho = R$ and $\sigma = \theta$; thus

$$P(R,\theta)dRd\theta = p(R<R'<R+dR,\ \theta<\theta'<\theta+d\theta). \tag{2.14}$$

For the case where W is 'normal,' it is shown in Appendix A that

$$P(R,\theta) = \frac{R}{\pi\sqrt{1-b^2}} \exp\left[\frac{-R^2}{1-b^2}(1 - b\cos 2\theta)\right] \tag{2.15}$$

$$= \frac{\sqrt{L}}{\pi} \exp[-L(1 - b\cos 2\theta)], \tag{2.16}$$

[10] This formula can be obtained from (2.1) by means of (2.7), (2.8), (2.9) and (1.17) after specializing (1.17) by the substitutions $\rho = u$, $\sigma = v$ and $h = k = 1/S$. It is (16) of my 1933 paper, but was given there without proof.

where

$$L = R^2/(1-b^2). \tag{2.17}$$

In $P(R,\theta)$ it will evidently suffice to deal with values of θ in the first quadrant, because of symmetry of the scatter diagram.

The fact that $P(R,\theta)$ depends on b as a parameter when W is 'normal' may be indicated explicitly by employing the fuller symbol $P(R,\theta;b)$ when desired; thus the former symbol is here an abbreviation for the latter.

In $P(R,\theta) \equiv P(R, \theta; b)$ it will suffice to deal with only positive values of b, that is, with $0 \leqq b \leqq 1$ (whereas the total possible range of b is $-1 \leqq b \leqq 1$). For (2.15) shows that changing b to $-b$ has the same effect as changing 2θ to $\pi \pm 2\theta$, or θ to $\pi/2 \pm \theta$; that is, $P(R,\theta; -b) = P(R, \pi/2 \pm \theta; b)$.

Seven formulas which will find considerable use subsequently are obtainable from the integrals corresponding to equations (1.13) to (1.16), by setting $\rho = R$ and $\sigma = \theta$ or else $\rho = \theta$ and $\sigma = R$, whichever is appropriate, and thereafter substituting for $P(R,\theta)$ the expression given by (2.16), and lastly executing the indicated integrations wherever they appear possible.[11] The resulting formulas are as follows:

$$P(R \mid < \theta) = \frac{\sqrt{L}}{\pi} \exp(-L) \int_0^\theta \exp(bL \cos 2\theta)\, d\theta, \tag{2.18}$$

$$P(\theta \mid < R) = \frac{\sqrt{1-b^2}}{2\pi} \frac{1 - \exp[-L(1 - b\cos 2\theta)]}{1 - b\cos 2\theta}, \tag{2.19}$$

$$P(\theta \mid > R) = \frac{\sqrt{1-b^2}}{2\pi} \frac{\exp[-L(1 - b\cos 2\theta)]}{1 - b\cos 2\theta}. \tag{2.20}$$

$$Q(< R, < \theta) = \frac{1}{\pi} \int_0^R \left[\sqrt{L} \exp(-L) \int_0^\theta \exp(bL\cos 2\theta)\, d\theta \right] dR \tag{2.21}$$

$$= \frac{\sqrt{1-b^2}}{2\pi} \int_0^\theta \frac{1 - \exp[-L(1 - b\cos 2\theta)]}{1 - b\cos 2\theta}\, d\theta, \tag{2.22}$$

$$Q(> R, < \theta) = \frac{1}{\pi} \int_R^\infty \left[\sqrt{L} \exp(-L) \int_0^\theta \exp(bL\cos 2\theta)\, d\theta \right] dR \tag{2.23}$$

$$= \frac{\sqrt{1-b^2}}{2\pi} \exp(-L) \int_0^\theta \frac{\exp(bL\cos 2\theta)}{1 - b\cos 2\theta}\, d\theta. \tag{2.24}$$

Formulas (2.21) to (2.24) are obtainable also by substituting (2.18) to (2.20) into the appropriate particular forms of (1.9).

When a θ-range of integration is 0-to-$q(\pi/2)$, where $q = 1, 2, 3$ or 4, this

[11] Except that in (2.22) the part $1/(1 - b\cos 2\theta)$ is integrable, as found in Sec. 7, equations (7.6) and (7.7).

range can be reduced to 0-to-$\pi/2$ provided the resulting integral is multiplied by q; that is,

$$\int_0^{q(\pi/2)} F(\theta)d\theta = q\int_0^{\pi/2} F(\theta)d\theta, \tag{2.25}$$

because of symmetry of the scatter diagram.

3. The Distribution Function for the Modulus

The distribution function $P(R \mid \theta_{12}) \equiv P(R)$ for the modulus R of any complex variate $W \equiv R(\cos\theta + i\sin\theta)$ is defined by equation (1.10) on setting $\rho = R$, $\sigma = \theta$, $\sigma_1 = \theta_1 = 0$ and $\sigma_2 = \theta_2 = 2\pi$; thus

$$P(R)dR = p(R<R'<R+dR,\ 0<\theta'<2\pi). \tag{3.1}$$

An integral formula for $P(R)$ is immediately obtainable from (1.6) by setting $\rho = R$, $\sigma = \theta$, $\sigma_3 = \sigma_1 = \theta_1 = 0$ and $\sigma_4 = \sigma_2 = \theta_2 = 2\pi$; thus

$$P(R) = \int_0^{2\pi} P(R,\theta)\, d\theta. \tag{3.2}$$

The rest of this section deals with the case where $W \equiv R(\cos\theta + i\sin\theta)$ is 'normal.' Since this case depends on b as a parameter, $P(R)$ is here an abbreviation for $P(R;b)$. A formula for $P(R;b)$ can be obtained by substituting $P(R,\theta)$ from (2.15) into (3.2) and executing the indicated integration by means of the known Bessel function formula

$$\int_0^{\pi} \exp(\eta\cos\psi)\, d\psi = \pi I_0(\eta), \tag{3.3}$$

$I_0(\ )$ being the so-called 'modified Bessel function of the first kind,' of order zero.[12] The resulting formula is found to be[13]

$$P(R;b) = \frac{2R}{\sqrt{1-b^2}}\exp\left[\frac{-R^2}{1-b^2}\right] I_0\left[\frac{bR^2}{1-b^2}\right]. \tag{3.4}$$

This can also be obtained as a particular case of the more general formula (2.18) by setting $\theta = 2\pi$ in the upper limit of integration and then applying (3.3).

In $P(R;b)$ it will suffice to deal with positive values of b, that is, with $0 \leqq b \leqq 1$, as (3.4) shows that $P(R;-b) = P(R;b)$.

[12] It may be recalled that $I_0(z) = J_0(iz)$, and in general that $I_n(z) = i^{-n}J_n(iz)$.
In the list of references on Bessel functions, on the last page of this paper, the 'modified Bessel function' is treated in Ref. 2, p. 20; Ref. 3, p. 102; Ref. 4, p. 41; Ref. 1, p. 77.
Regarding formula (3.3), see Ref. 1, p. 181, Eq. (4), $\nu = 0$; Ref. 1, p. 19, Eq. (9), fourth expression, $\nu = 0$; Ref. 2, p. 46, Eq. (10), $n = 0$; Ref. 3, p. 164, Eq. 103, $n = 0$.

[13] This formula was given in its cumulative forms, $\int P(R;b)dR$, as formulas (51-A) and (53-A) of the unpublished Appendix A to my 1933 paper.

It will often be advantageous to express $P_{R;\,b}$ in terms of b and one or the other of the auxiliary variables L and T defined by the equations

$$L = \frac{R^2}{1-b^2}, \quad (3.5) \qquad\qquad T = bL = \frac{bR^2}{1-b^2}. \quad (3.6)$$

Formula (3.4) thereby becomes, respectively,

$$P(R;b) = 2\sqrt{L}\exp(-L)I_0(bL), \tag{3.7}$$

$$P(R;b) = 2\sqrt{\frac{T}{b}}\exp\left[\frac{-T}{b}\right]I_0(T). \tag{3.8}$$

Formula (3.8) will often be preferable to (3.7) because the argument of the Bessel function in (3.8) is a single quantity, T.

Because tables of $I_0(X)$ are much less easily interpolated than tables of $M_0(X)$ defined by the equation

$$M_0(X) = \exp(-X)I_0(X), \tag{3.9}$$

extensive tables of which have been published,[14] it is natural, at least for computational purposes, to write (3.4) in the form

$$P(R;\,b) = \frac{2R}{\sqrt{1-b^2}}\exp\left[\frac{-R^2}{1+b}\right]M_0\left[\frac{bR^2}{1-b^2}\right]. \tag{3.10}$$

For use in equation (3.16), it is convenient to define here a function $M_1(X)$ by the equation

$$M_1(X) = \exp(-X)I_1(X), \tag{3.11}$$

corresponding to (3.9) defining $M_0(X)$. $M_1(X)$ has the similar property that it is much more easily interpolated than is $I_1(X)$; and extensive tables of $M_1(X)$ are constituent parts of the tables in Ref. 1 and Ref. 6.

The quantity $bR^2/(1-b^2) \equiv T$, which occurs in (3.4) and (3.8) as the argument of $I_0(\ \ )$, and in (3.10) as the argument of $M_0(\ \ )$, evidently ranges from 0 to ∞ when R ranges from 0 to ∞ and also when b ranges from 0 to 1. Formula (3.10) is suitable for computational purposes for all values of the above-mentioned argument $bR^2/(1-b^2) \equiv T$ not exceeding the largest values of X in the above-cited tables in Ref. 1 and Ref. 6. For larger values of the argument, and partiularly for dealing with the limiting

[14] Ref. 1, Table II (p. 698–713), for $X = 0$ to 16 by .02. Ref. 6, Table VIII (p. 272–283), for $X = 5$ to 10 by .01, and 10 to 20 by 0.1. Each of these references conveniently includes a table of $\exp(X)$ whereby values of $I_0(X)$ can be readily and accurately evaluated if desired. Values of $I_0(X)$ so obtained would enable formulas (3.4), (3.7) and (3.8) of the present paper to be used with high accuracy without any difficult interpolations, since the table of $\exp(X)$ is easily interpolated by utilizing the identity $\exp(X_1 + X_2) = \exp(X_1)\exp(X_2)$.

case where the argument becomes infinite, formula (310)—and hence (3.4)—may be advantageously written in the form

$$P(R;\,b) = \frac{2}{\sqrt{2\pi b}} \exp\left[\frac{-R^2}{1+b}\right] N_0\left[\frac{bR^2}{1-b^2}\right], \tag{3.12}$$

where

$$N_0(X) = \sqrt{2\pi X}\,\exp(-X) I_0(X) = \sqrt{2\pi X}\,M_0(X), \tag{3.13}$$

an extensive table of which has been published.[15] The natural suitability of the function $N_0(X)$ for dealing with large values of X is evident from the structure of the asymptotic series for $N_0(X)$, for sufficiently large values of X, which runs as follows:[16]

$$N_0(X) \sim 1 + \frac{1^2}{1!8X} + \frac{1^2 3^2}{2!(8X)^2} + \frac{1^2 3^2 5^2}{3!(8X)^3} + \cdots, \tag{3.14}$$

whence it is evident that

$$N_0(\infty) = 1. \tag{3.15}$$

For use in Appendix C, it is convenient to define here a function $N_1(X)$ by the equation[17]

$$N_1(X) = \sqrt{2\pi X}\,\exp(-X) I_1(X) = \sqrt{2\pi X}\,M_1(X), \tag{3.16}$$

corresponding to (3.13) defining $N_0(X)$, with $M_1(X)$ defined by (3.11). The asymptotic series for $N_1(X)$, which will be needed in Appendix C, is[18]

$$N_1(X) \sim 1 - 3\left[\frac{1}{1!8X} + \frac{(1\cdot5)}{2!(8X)^2} + \frac{(1\cdot5)(3\cdot7)}{3!(8X)^3} + \cdots\right], \tag{3.17}$$

whence it is evident that

$$N_1(\infty) = 1. \tag{3.18}$$

When b is very nearly but not exactly equal to unity, so that

$$\frac{bR^2}{1-b^2} \approx \frac{R^2}{1-b^2} \approx \frac{R^2}{2(1-b)}, \tag{3.19}$$

it is seen from (3.4) that $P(R;b)$ is, to a very close approximation, a function

[15] Ref. 7, pp. 45–72, for $X = 10$ to 50 by 0.1, 50 to 200 by 1, 200 to 1000 by 10, and for various larger values of X.

[16] Ref. 1, p. 203, with (ν, m) defined on p. 198; Ref. 5, p. 366; Ref. 2, p. 58; Ref. 3, p. 163, Eq. 84; Ref. 4, pp. 48, 84.

[17] $N_1(X)$ is tabulated along with $N_0(X)$ in Ref. 7 already cited in connection with equation (3.13).

[18] Ref. 1, p. 203, with (ν, m) defined on p. 198; Ref. 5, p. 366; Ref. 2, p. 58; Ref. 3, p. 163, Eq. 84.

of only a single quantity, which may be any one of the three very nearly equal expressions in (3.19)—but the last of them is evidently the simplest.

Fig. 3.1 gives curves of $P(R;b)$, with the variable R ranging continuously

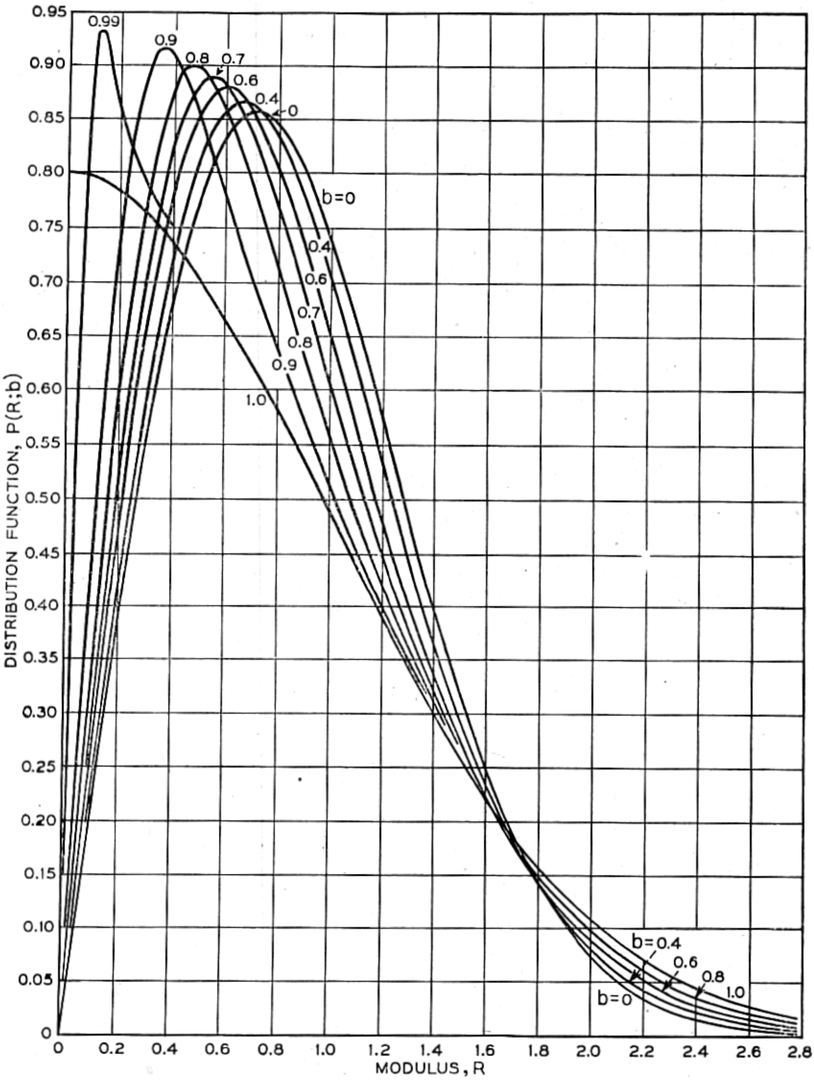

Fig. 3.1—Distribution function for the modulus ($R = 0$ to 2.8).

from 0 to 2.8 and the parameter b ranging by steps from 0 to 1 inclusive, which is the complete range of positive b. Fig. 3.2 gives an enlargement (along the R-axis) of the portion of Fig. 3.1 between $R = 0$ and $R = 0.4$,

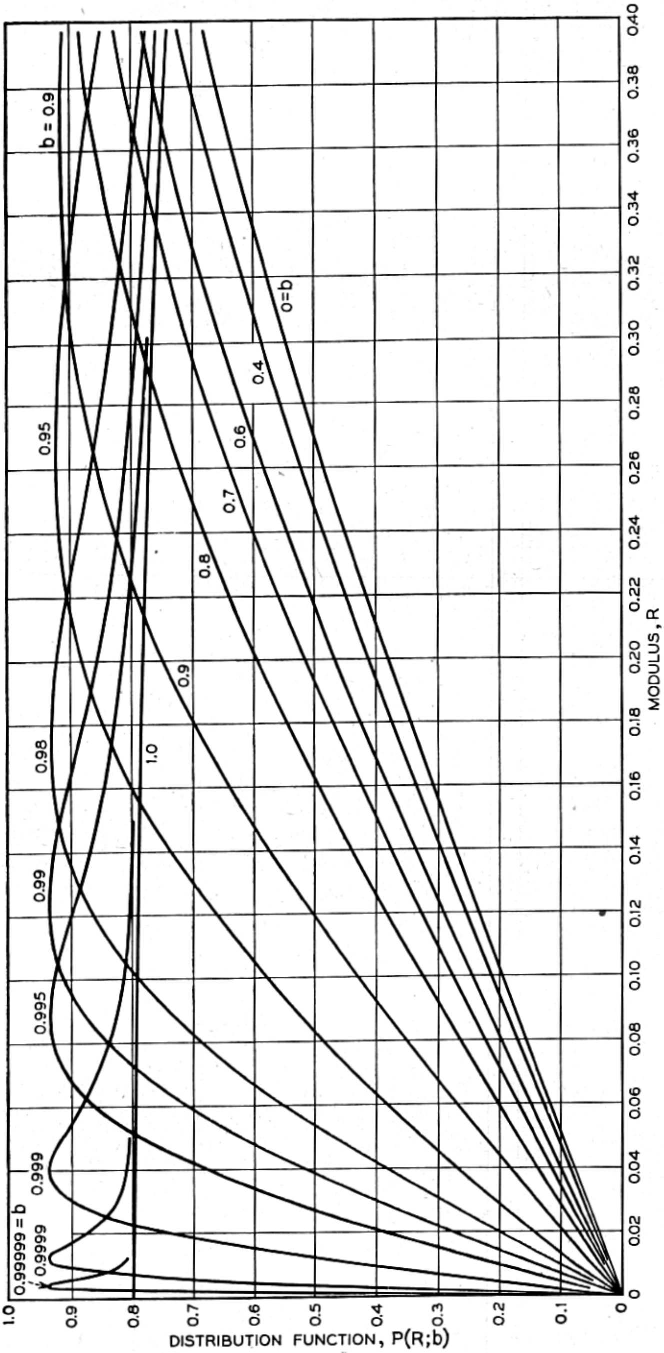

Fig. 3.2—Distribution function for the modulus (R = 0 to 0.4).

and includes therein curves for a considerable number of additional values of b between 0.9 and 1 so chosen as to show clearly how, with b increasing toward 1, the curves approach the curve for $b = 1$ as a limiting particular curve; or, conversely, how the curve for $b = 1$ constitutes a limiting particular curve—which, incidentally, will be found to be a natural and convenient reference curve. This curve, for $b = 1$, will be considered more fully a little further on, because it is a limiting particular curve and because of its resulting peculiarity at $R = 0$, the curve for $b = 1$ having at $R = 0$ a projection, or spur, situated in the $P(R;b)$ axis and extending from 0.7979 to 0.9376 therein (as shown a little further on).

The formulas and curves for $b = 0$ and $b = 1$, being of especial interest and importance, will be considered before the remaining curves of the set.

For the case $b = 0$, formula (3.4) evidently reduces immediately to

$$P(R;0) = 2R \exp(-R^2). \tag{3.20}$$

This case, $b = 0$, is that degenerate particular case in which the equiprobability curves in the scatter diagram of the complex variate, instead of being ellipses (concentric), are merely circles, as noted in my 1933 paper, near the bottom of p. 44 thereof (p. 10 of reprint).

For the case $b = 1$, the formula for the entire curve of $P(R; b) = P(R;1)$, except only the part at $R = 0$, can be obtained by merely setting $b = 1$ in[19] (3.12) as this, on account of (3.15), thereby reduces immediately to

$$P'(R; 1) = \frac{2}{\sqrt{2\pi}} \exp\left[-\frac{R^2}{2}\right], \qquad (R \neq 0), \tag{3.21}$$

$P'(R;1)$ denoting the value of $P(R;b)$ when $b = 1$ but $R \neq 0$, the restriction $R \neq 0$ being necessary because the quantity $R^2/(1-b^2)$ in (3.12)—and in (3.4)—does not have a definite value when $b = 1$ if $R = 0$. Thus, in Figs. 3.1 and 3.2, the curve of $P'(R;1)$ is that part of the curve for $b = 1$ which does not include any point in the $P(R; b)$ axis (where $R = 0$) but extends rightward from that axis toward $R = +00$. The curve of $P'(R;1)$ is the 'effective' part of the curve of $P(R;1)$, in the sense that the area under the former is equal to that under the latter, since the part of the curve of $P(R;1)$ at $R = 0$ can have no area under it.

$P(0;1)$ denoting (by convention) the value, or values, of $P(R;b)$ when $R = 0$ and $b = 1$, that is, the value, or values, of $P(R;1)$ when $R = 0$, it is seen, from consideration of the curves of $P(R;b)$ in Figs. 3.1 and 3.2 when b approaches 1 and ultimately becomes equal to 1, that the curve of $P(0;1)$ consists of all points in the vertical straight line segment extending upward in the $P(R;b)$ axis, from the origin to a height 0.9376 [= Max $P(R;1)$],[20]

[19] Use of (3.12) instead of (3.4), which is transformable into (3.12), avoids the indefinite expression $\infty .0. \infty$ which would result directly from setting $b = 1$ in (3.4).

[20] As shown near the end of Appendix B, Max $P(R;1)$ is situated at $R = 0$ and is equal to 0.9376.

together with all points in the straight line segment extending downward from the point at 0.9376 to the point at 0.7979 [$= 2/\sqrt{2\pi} = P'(R;1)$ for $R = 0+$]. The curve of $P(0; 1)$, because it has no area under it, is the 'non-effective' part of the curve of $P(R;1)$.

Starting at the origin of coordinates, where $R = 0$, the complete curve of $P(R;1)$ consists of the curve of $P(0;1)$, described in the preceding paragraph, in sequence with the curve of $P'(R;1)$, given by (3.21). Thus the complete curve of $P(R;1)$ is the locus of a tracing point moving as follows: Starting at the origin of coordinates, the tracing point first ascends in the $P(R; b)$ axis to a height 0.9376 [$=$ Max $P(R;1)$]; second, descends from 0.9376 to 0.7979 [$= 2/\sqrt{2\pi} = P'(R;1)$ for $R = 0+$]; and, third, moves rightward along the graph of $P'(R;1)$ [$b = 1$] toward $R = +\infty$. The locus of all of the points thus traversed by the tracing point is the complete curve[21] of $P(R;1)$.

In addition to being the principal part ('effective' part) of the curve of $P(R;1)$, the curve of $P'(R;1)$, whose formula is (3.21), has a further important significance. For the right side of (3.21), except for the factor 2, will be recognized as being the expression for the well-known 1-dimensional 'normal' law; the presence of the factor 2 is accounted for by the fact that the variable $R = |R|$ can have only posiive values and yet the area under the curve must be equal to unity. This case, $b = 1$, is that degenerate particular case in which the equiprobability curves, instead of being ellipses, are superposed straight line segments, so that the resulting 'probability density' is not constant but varies in accordance with the 1-dimensional 'normal' law (for real variates), as noted in my 1933 paper, at the top of p. 45 thereof (p. 11 of reprint).

All of the curves of $P(R;b)$, where $0 \leqq b \leqq 1$, pass through the origin, the curve of $P(R;1)$ [$b = 1$] being no exception, since the part $P(0;1)$ passes through the origin.

Formula (3.12), supplemented by (3.15), shows that $P(R; b) = 0$ at $R = \infty$; and this is in accord with the consideration that the total area under the curve of $P(R;b)$ must be finite (equal to unity).

Since $P(R;b) = 0$ at $R = 0$ and at $R = \infty$, every curve of $P(R;b)$ must have a maximum value situated somewhere between $R = 0$ and $R = \infty$— as confirmed by Figs. 3.1 and 3.2. These figures show that when b increases from 0 to 1 the maximum value increases throughout but the value of R where it is located decreases throughout.

The maxima of the function $P(R;b)$ and of its curves (Figs. 3.1 and 3.2) are of considerable theoretical interest and of some practical importance.

21 The presence, in the curve of $P(R; 1)$, of the vertical projection, or spur, situated in the $P(R; b)$ axis and extending from 0.7979 to 0.9376 therein, is somewhat remindful (qualitatively) of the 'Gibbs phenomenon' in the representation of discontinuous periodic functions by Fourier series.

The cases $b = 0$ and $b = 1$ will be dealt with first, and then the general case $(b = b)$.

For the case $b = 0$ it is easily found by differentiating (3.20) that $P(R; b) = P(R; 0)$ is a maximum at $R = 1/\sqrt{2} = 0.7071$ and hence that its maximum value is $\sqrt{2}\exp(-1/2) = 0.8578$, agreeing with the curve for $b = 0$ in Fig. 3.1.

For the case $b = 1$, which is a limiting particular case, the maximum value of $P(R;b) = P(R;1)$ apparently cannot be found driectly and simply, as will be realized from the preceding discussion of this case. Near the end of Appendix B, it is shown that the maximum value of $P(R;1)$ occurs at $R = 0$ (as would be expected) and is equal to 0.9376. This is the maximum value of the part $P(0;1$ of $P(R;1)$. The remaining part of $P(R;1)$, namely $P'(R;1)$, whose formula is (3.21), is seen from direct inspection of that formula to have a right-hand maximum value at $R = 0+$, whence this maximum value is $2/\sqrt{2\pi} = 0.7979$.

For the general case when b has any fixed value within its possible positive range $(0 \leqq b \leqq 1)$, it is apparently not possible to obtain an explicit expression (in closed form) either for the value of R at which $P(R;b)$ has its maximum value or for the maximum value of $P(R;b)$; and hence it is not possible to make explicit computations of these quantities for use in plotting curves of them, versus b, of which they will evidently be functions. However, as shown in Appendix B, these desired curves can be exactly computed, in an indirect manner, by temporarily taking b as the dependent variable and taking T, defined by (3.6), as an intermediate independent variable. For let R_C denote the critical value of R, that is, the value of R at which $P(R;b)$ has its maximum value; and let T_C denote the corresponding value of T, whence, by (3.6),

$$T_C = bR_C^2/(1-b^2). \tag{3.22}$$

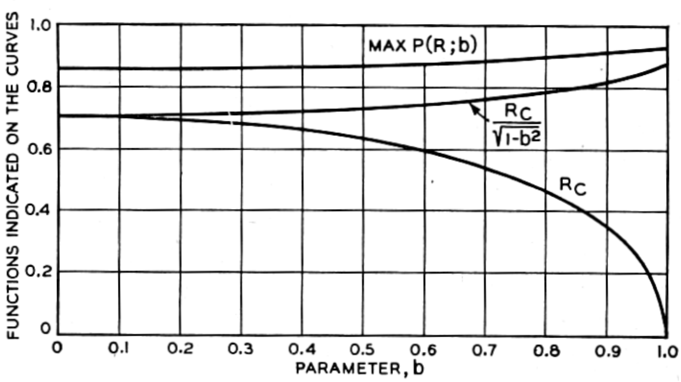

Fig. 3.3—Functions relating to the maxima of the distribution function for the modulus.

Then, computed by means of the formulas derived in Appendix B, Fig. 3.3 gives a curve of R_C and a curve of Max $P(R;b)$, each versus b. Since the curve of R_C cannot be read accurately at $b \approx 1$, there is included also a curve of $R_C/\sqrt{1 - b^2}$, from which R_C can be accurately and easily computed for any value of b; incidentally, the curve of $R_C/\sqrt{1 - b^2}$ is simultaneously a curve of $\sqrt{T_C/b}$, on account of (3.22). From Fig. 3.3 it is seen that R_C varies greatly with b but that Max $P_{R;b}$ varies only a little, as also is seen from inspection of Figs. 3.1 and 3.2 giving curves of $P(R;b)$ as function of R with b as parameter.

In Fig. 3.3, the curve of R_C shows that for $b = 1$ the maximum of $P(R;b)$ occurs at $R = 0$; and the curve of Max $P(R;b)$ shows that Max $P(R;1) \approx 0.94$, agreeing to two significant figures with the value 0.9376 found near the end of Appendix B.

4. The Distribution Function for the Reciprocal of the Modulus

At first, let R denote any real variate, and $P(R)$ its distribution function. Also let r denote the reciprocal of R, so that $r = 1/R$; and let $P(r)$ denote the distribution function for r. Then [22]

$$P(r) = R^2 P(R) = P(R)/r^2. \tag{4.1}$$

If $P(R)$ depends on any parameters, $P(r)$ will evidently depend on the same parameters.

The rest of this section deals with the case where $W \equiv R(\cos\theta + i\sin\theta)$ is 'normal.' Since this case depends on b as a parameter, $P(R)$ and $P(r)$ are here abbreviations for $P(R;b)$ and $P(r;b)$ respectively.

As $P(R;b)$ has the distribution function given by (3.4), the distribution function for r will be

$$P(r;b) = \frac{2}{(\sqrt{1 - b^2})r^3} \exp\left[\frac{-1}{(1 - b^2)r^2}\right] I_0\left[\frac{b}{(1 - b^2)r^2}\right], \tag{4.2}$$

obtained from the right side of (3.4) by changing R to $1/r$ and multiplying

[22] For if r and R denote any two real variates that are functionally related, say $F(r, R) = 0$, and if dr and dR are corresponding small increments, then evidently

$$P(r)\,|\,dr\,| = P(R)\,|\,dR\,| \quad \text{whence} \quad \frac{P(r)}{P(R)} = \left|\frac{dR}{dr}\right| = \left|\frac{\partial F/\partial r}{\partial F/\partial R}\right|.$$

In particular, if $r = 1/R$, whence $F = r - 1/R$, then (4.1) results immediately.

For a somewhat different and more detailed treatment of change of the variable in distribution functions, see Thorton C. Fry, "Probability and its Engineering Uses," 1928, pp. 153–155. (Cases of more than one variate are treated on pp. 155–174 of the same reference.)

the result by $1/r^2$, in accordance with (4.1). Evidently $P(r;-b) = P(r;b)$.

By means of (4.1), formulas (3.7) and (3.8) give, respectively,

$$P(r;b) = 2(1-b^2)L^{3/2}\exp(-L)I_0(bL), \tag{4.3}$$

$$P(r;b) = 2(1-b^2)\left[\frac{T}{b}\right]^{3/2}\exp\left[\frac{-T}{b}\right]I_0(T), \tag{4.4}$$

wherein L and T are defined by (3.5) and (3.6) respectively, but will now be written in the equivalent forms

$$L = \frac{1}{(1-b^2)r^2}, \quad (4.5) \qquad T = bL = \frac{b}{(1-b^2)r^2}, \quad (4.6)$$

which are evidently more suitable for the present section.

A few particular cases that are especially important will be dealt with in the following brief paragraph, ending with equation (4.8).

For the two extreme values of r, namely 0 and ∞, $P(r;b)$ is zero for all values of b in the b-range $(0 \leqq b \leqq 1)$.

When $b = 0$,

$$P(r;b) = P(r;0) = \frac{2}{r^3}\exp\left[\frac{-1}{r^2}\right]. \tag{4.7}$$

When $b = 1$,

$$P(r;b) = P(r;1) = \frac{2}{\sqrt{2\pi}}\frac{1}{r^2}\exp\left[\frac{-1}{2r^2}\right]. \tag{4.8}$$

Fig. 4.1 gives curves of $P(r;b)$, with the variable r ranging continuously from 0 to 1.4 and the parameter b ranging by steps from 0 to 1; however, in the r-range where r is less than about 0.6, alternate curves had to be omitted to avoid undue crowding. Fig. 4.2 gives an enlargement of the section betwen $r = 0.2$ and $r = 0.5$, and includes therein the curves that had to be omitted from Fig. 4.1.

In Fig. 4.1 it will be noted that with the scale there used for $P(r;b)$ the values of $P(r;b)$ are too small to be even detectable for values of r less than about 0.25. Even in the enlargement supplied by Fig. 4.2, the values of $P(r;b)$ are not detectable for r less than about 0.2.

The curves of $P(r;b)$ in Figs. 4.1 and 4.2 would have had to be computed from the lengthy formula (4.2)—or its equivalents—except for the fact that curves of $P(R;b)$ had already been computed in the preceding section of the paper. The last circumstance enabled the $P(r;b)$ curves to be obtained from the $P(R;b)$ curves by means of the very simple relation (4.1).

It will be observed that each curve of $P(r;b)$ [Fig. 4.1] has a maximum ordinate, whose value and location depend on b. When b increases from 0 to 1, the maximum ordinate decreases throughout but the value of r where it is located remains nearly constant, at about 0.82, until b becomes about

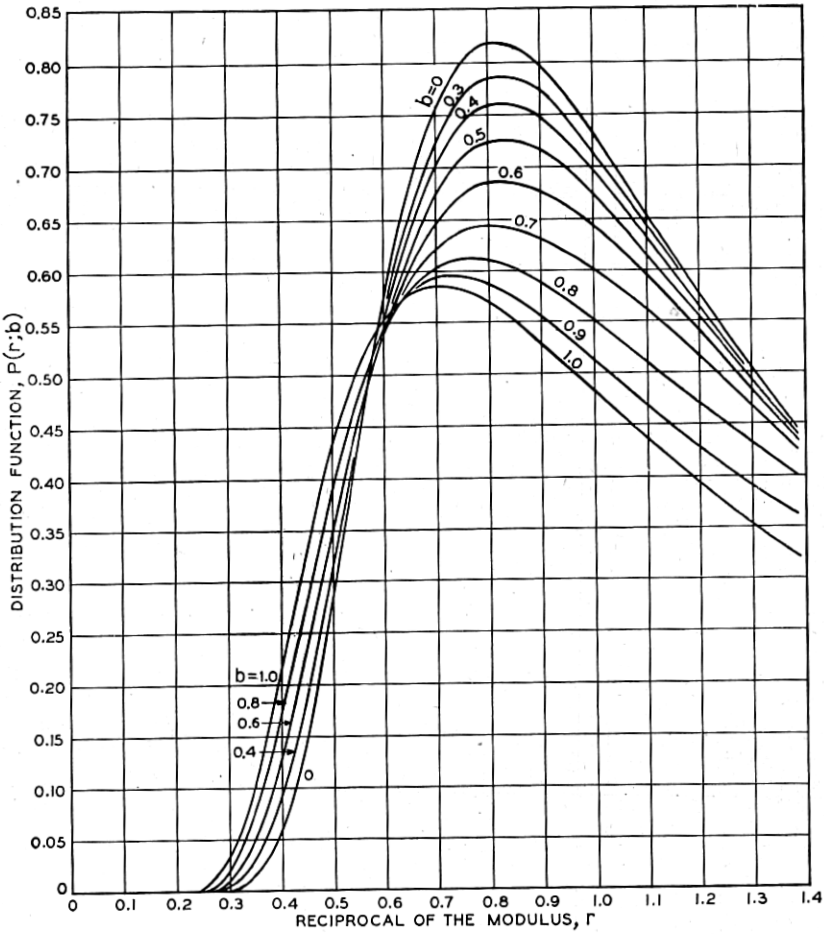

Fig. 4.1—Distribution function for the reciprocal of the modulus (r = 0 to 1.4).

0.7, after which the location of the maximum value moves rather rapidly to about 0.71 for $b = 1$.

For the cases $b = 0$ and $b = 1$, it is easily found, by differentiating (4.7) and (4.8), that the maximum ordinates are located at $r = \sqrt{2/3} = 0.8165$ and at $r = 1/\sqrt{2} = 0.7071$ respectively; and hence, by (4.7) and (4.8), that the values of these maximum ordinates are $(3\sqrt{3/2}\exp(-3/2) =$

0.8198 and $(4/\sqrt{2\pi}) \exp(-1) = 0.5871$ respectively. These results for the cases $b = 0$ and $b = 1$ agree with the corresponding curves in Fig. 4.1.

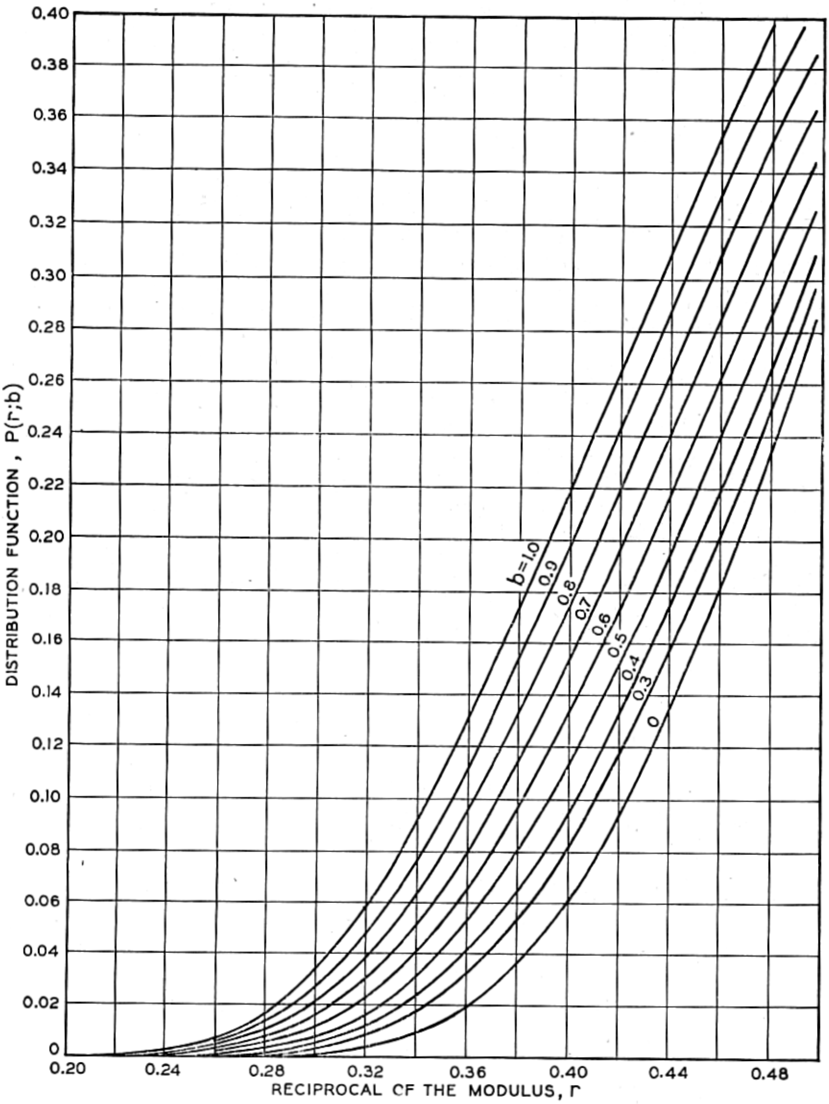

Fig. 4.2—Distribution function for the reciprocal of the modulus ($r = 0.2$ to 0.5).

For the general case where b has any fixed value in the b-range ($0 \leqq b \leqq 1$), it is apparently not possible to obtain an explicit expression (in closed form) either for the value of r at which $P(r;b)$ has its maximum value or for the maximum value of $P(r;b)$. However, as shown in Appendix C, curves of these quantities versus b can be computed, in an indirect manner, by temporarily taking b as the dependent variable and taking T, defined by (4.6), as an intermediate independent variable. For let r_c denote the critical value of r, that is, the value of r at which $P(r;b)$ has its maximum value; and let T_c denote the corresponding value of T, whence, by (4.6),

$$T_c = b/(1-b^2)r_c^2. \tag{4.9}$$

Then, computed by means of the formulas derived in Appendix C, Fig. 4.3 gives a curve of r_c and a curve of Max $P(r;b)$, each versus b. From these curves it is seen that r_c and Max $P(r;b)$ do not vary greatly with b, as also is seen from inspection of Fig. 4.1 giving curves of $P(r;b)$ as function of r with b as parameter.

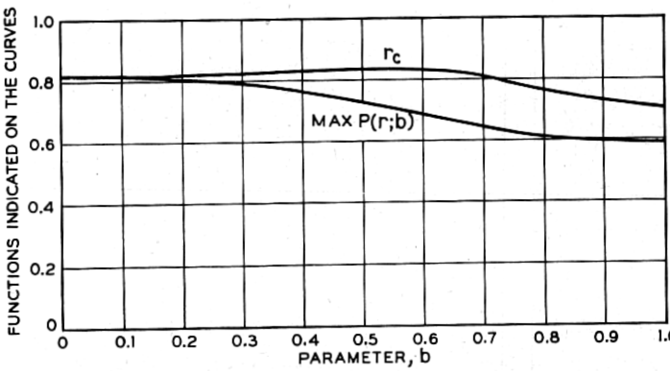

Fig. 4.3—Functions relating to the maxima of the distribution function for the reciprocal of the modulus.

5. The Cumulative Distribution Function for the Modulus

The cumulative distribution function $Q(<R,\theta_{12}) \equiv Q(R)$ for the modulus R of any complex variate $W \equiv R(\cos\theta + i\sin\theta)$ is defined by equation (1.11) on setting $\rho = R$, $\sigma = \theta$, $\rho_1 = R_1 = 0$, $\sigma_1 = \theta_1 = 0$ and $\sigma_2 = \theta_2 = 2\pi$; thus

$$Q(R) = p(0<R'<R, 0<\theta'<2\pi). \tag{5.1}$$

Similarly, from (1.12), the complementary cumulative distribution function $Q(>R,\theta_{12}) \equiv Q^*(R)$ is defined by the equation

$$Q^*(R) = p(R<R'<\infty, 0<\theta'<2\pi). \tag{5.2}$$

$Q^*(R)$ is usually more convenient than $Q(R)$ for use in engineering applications, because it is usually more convenient to deal with the relatively small probability of exceeding a preassigned rather large value of R than to deal with the corresponding rather large probability (nearly equal to unity) of being less than the preassigned value of R.

A 'double integral' for $Q(R)$, in the form of two 'repeated integrals,' can be written down directly by inspection of the $p(\ \ )$ expression in (5.1) or by specialization of (1.8); thus

$$Q(R) = \int_0^R \left[\int_0^{2\pi} P(R,\theta)\, d\theta \right] dR = \int_0^{2\pi} \left[\int_0^R P(R,\theta)\, dR \right] d\theta. \qquad (5.3)$$

Evidently these can be written formally as two 'single integrals,'

$$Q(R) = \int_0^R P(R)\, dR = \int_0^{2\pi} P(\theta \mid < R)\, d\theta, \qquad (5.4)$$

by means of the distribution functions $P(R) = P(R \mid \theta_{12})$ and $P(\theta \mid <R)$ given by the formulas

$$P(R) = \int_0^{2\pi} P(R,\theta)\, d\theta, \quad (5.5) \qquad P(\theta \mid < R) = \int_0^R P(R,\theta)\, dR. \quad (5.6)$$

(5.5) is the same as (3.2). (5.6) is a special case of (1.6), and the left side of (5.6) is a special case of $P(\rho \mid <\sigma)$ defined by (1.13).

Similarly, from (5.2), we arrive at the following formulas corresponding to (5.3), (5.4), (5.5), and (5.6) respectively:

$$Q^*(R) = \int_R^\infty \left[\int_0^{2\pi} P(R,\theta)\, d\theta \right] dR = \int_0^{2\pi} \left[\int_R^\infty P(R,\theta)\, dR \right] d\theta, \qquad (5.7)$$

$$Q^*(R) = \int_R^\infty P(R)\, dR = \int_0^{2\pi} P(\theta \mid > R)\, d\theta, \qquad (5.8)$$

$$P(R) = \int_0^{2\pi} P(R,\theta)\, d\theta, \quad (5.9) \qquad P(\theta \mid > R) = \int_R^\infty P(R,\theta)\, dR. \quad (5.10)$$

The rest of this section deals with the case where $W \equiv R(\cos\theta + i \sin\theta)$ is 'normal.'[23] Since this case depends on b as a parameter, $Q(R)$ and $Q^*(R)$ are here abbreviations for $Q(R;b)$ and $Q^*(R;b)$ respectively.

A natural and convenient way for deriving formulas for $Q(R)$ is afforded by the general formula (5.4) together with the auxiliary general formulas (5.5) and (5.6), beginning with the two latter.

For the 'normal' case, $P(R,\theta)$ is given by (2.15). When this is substituted into (5.5) and (5.6), it is found that each of the indicated integra-

[23] For the 'normal' case, the cumulative distribution function was treated in a very different manner in my 1933 paper and its unpublished Appendix A. That paper included applications to two important practical problems, and its unpublished Appendix C treated a third such problem. (The unpublished appendices, A, B and C, are mentioned in footnote 3 of the 1933 paper.)

tions can be executed, giving the two previously obtained formulas (3.4) and (2.19) for $P(R) \equiv P(R;b)$ and $P(\theta \mid <R)$ respectively. When these are substituted into (5.4), there result two types of single-integral formulas for $Q(R)$: A primary type, involving an indicated integration as to R; and a secondary type, involving an indicated integration as to θ. Formulas of these two types for $Q(R)$ will now be derived.

An integral formula of the primary type for $Q(R) \equiv Q(R;b)$ can be obtained by substituting $P(R) \equiv P(R;b)$ from (3.4) into the first integral in (5.4), giving

$$Q(R) = 2\int_0^R \frac{\lambda}{\sqrt{1-b^2}} \exp\left[\frac{-\lambda^2}{1-b^2}\right] I_0\left[\frac{b\lambda^2}{1-b^2}\right] d\lambda. \qquad (5.11)$$

This can also be obtained as a particular case of the more general formula (2.21) by setting $\theta = 2\pi$ in the upper limit of integration and then applying (3.3).

In (5.11), λ is used instead of R as the integration variable in order to avoid any possible confusion with R as an integration limit. Thus the integrand is a function of λ with b as a parameter. Evidently $Q(R;b) = Q(R;-b)$. Formula (5.11) is evidently suitable for evaluation of $Q(R)$ by numerical integration.[24]

By suitably changing the variable in (5.11), we arrive at the following various additional formulas, which, though equivalent to (5.11), are very different as regards the integrand and the limits of integration. As previously, L denotes $R^2/(1-b^2)$.

$$Q(R) = \frac{1}{\sqrt{1-b^2}} \int_0^{R^2} \exp\left[\frac{-\lambda}{1-b^2}\right] I_0\left[\frac{b\lambda}{1-b^2}\right] d\lambda, \qquad (5.12)$$

$$Q(R) = \sqrt{1-b^2} \int_0^L \exp(-\lambda)\, I_0(b\lambda)\, d\lambda, \qquad (5.13)$$

$$Q(R) = L\sqrt{1-b^2} \int_0^1 \exp(-L\lambda)\, I_0(bL\lambda)\, d\lambda, \qquad (5.14)$$

$$Q(R) = \sqrt{1-b^2} \int_{\exp(-L)}^1 I_0(b \log \lambda)\, d\lambda. \qquad (5.15)$$

These four additional formulas are of some theoretical interest, but apparently they are less suitable than (5.11) for numerical integration with respect to R. A formula differing slightly from (5.11) could evidently be obtained by taking $\lambda/\sqrt{1-b^2}$ as a new variable, and hence $R/\sqrt{1-b^2}$ as the upper limit of integration.

Corresponding formulas for $Q^*(R) \equiv Q^*(R;b)$ can of course be obtained from the preceding formulas (5.11) to (5.15) inclusive for $Q(R) \equiv Q(R;b)$

[24] In this connection, Appendix D may be of interest.

by merely changing the integration limits correspondingly—for instance, in (5.11), from 0, R to R, ∞; in (5.13), from 0, L to L, ∞; and so on. However, the first four formulas for $Q^*(R)$ so obtained would suffer the disadvantage of each having an infinite limit of integration, rendering those formulas unsatisfactory for numerical integration purposes. This difficulty can be avoided by making the substitution $R = 1/r$ in each of those formulas for $Q^*(R)$. The resulting formulas are the following five, corresponding to (5.11) to (5.15) respectively:[24]

$$Q^*(R) = \frac{2}{\sqrt{1-b^2}} \int_0^r \frac{1}{\lambda^3} \exp\left[\frac{-1/\lambda^2}{1-b^2}\right] I_0\left[\frac{b/\lambda^2}{1-b^2}\right] d\lambda, \tag{5.16}$$

$$Q^*(R) = \frac{1}{\sqrt{1-b^2}} \int_0^{r^2} \frac{1}{\lambda^2} \exp\left[\frac{-1/\lambda}{1-b^2}\right] I_0\left[\frac{b/\lambda}{1-b^2}\right] d\lambda, \tag{5.17}$$

$$Q^*(R) = \sqrt{1-b^2} \int_0^{1/L} \frac{1}{\lambda^2} \exp\left[-\frac{1}{\lambda}\right] I_0\left[\frac{b}{\lambda}\right] d\lambda, \tag{5.18}$$

$$Q^*(R) = L\sqrt{1-b^2} \int_0^{1} \frac{1}{\lambda^2} \exp\left[-\frac{L}{\lambda}\right] I_0\left[\frac{bL}{\lambda}\right] d\lambda, \tag{5.19}$$

$$Q^*(R) = \sqrt{1-b^2} \int_0^{\exp(-L)} I_0(b \log \lambda)\, d\lambda. \tag{5.20}$$

As a check on (5.16), it is obtainable from (4.2) by integrating the latter as to r.

For purposes of evaluation by numerical integration, formulas (5.11) to (5.15) inclusive may evidently differ greatly as regards the amount of labor involved and the numerical precision practically attainable. In each of these formulas except (5.14) the integrand contains only one parameter, b, while the integration range involves either R or $L \equiv R^2/(1-b^2)$. In (5.14) the integrand contains two independent parameters, b and L, while the integration range is a mere constant, 0-to-1. Similar statements apply to formulas (5.16) to (5.20) inclusive.

A partial check on any formula for $Q(R)$ can be applied by setting $R = \infty$, since $Q(\infty)$ should be equal to unity (representing certainty). If, for instance, this procedure is applied to formula (5.13), the right side is found to reduce to unity by aid of the known relation[25]

$$\int_0^\infty \exp(-A\lambda)\, J_0(B\lambda)\, d\lambda = \frac{1}{\sqrt{A^2 + B^2}} \tag{5.21}$$

together with $I_0(B\lambda) = J_0(iB\lambda)$.

An integral formula of the secondary type for $Q^*(R) \equiv Q^*(R;b)$ can be obtained by substituting (2.20) into the last integral in (5.8), utilizing (2.25),

[25] Ref. 1, p. 384, Eq. (1); Ref. 2, p. 65, Eq. (2); Ref. 4, p. 58, Eq. (4.5).

changing the variable of integration by the substitution $\theta = \phi/2$, and rearranging; thus it is found that[26]

$$Q^*(R) = \frac{\sqrt{1 - b^2}}{\pi \exp L} \int_0^\pi \frac{\exp(bL \cos \phi)}{1 - b \cos \phi} \, d\phi. \tag{5.22}$$

This formula can also be obtained as a particular case of the more general formula (2.24) by setting $\theta = 2\pi$ in the upper limit of integration, utilizing (2.25), and changing the variable of integration by the substitution $\theta = \phi/2$.

Two partial checks on any general formula for $Q(R) \equiv Q(R;b)$ or for $Q^*(R) \equiv Q^*(R;b)$ can be applied by setting $b = 0$ and $b = 1$, and comparing the resulting particular formulas with those obtained by integrating the formulas for $P(R;0)$ and $P'(R;1)$ obtained in Section 3, namely formulas (3.20) and (3.21) there. It is thus found that

$$Q^*(R; 0) = \exp(-R^2) = \int_R^\infty P(R; 0) \, dR, \tag{5.23}$$

$$Q(R; 1) = 2\left\{\frac{1}{\sqrt{2\pi}} \int_0^R \exp\left[-\frac{R^2}{2}\right] dR\right\} = \int_0^R P'(R; 1) \, dR. \tag{5.24}$$

It will be recalled that the quantity between braces in (5.24) is extensively tabulated, and that it is sometimes called the 'normal probability integral.'

Several of the above general formulas for $Q(R) \equiv p(R'<R)$ and for $Q^*(R) \equiv p(R'>R)$ are closely connected with my 1933 paper.[27] Indeed, formulas (5.11), (5.14), (5.16), (5.19) and (5.22) above are the same as (53-A), (56-A), (52-A), (55-A) and (22-A), respectively, of the unpublished Appendix A to the 1933 paper; and (5.12), (5.13), (5.15), (5.17), (5.18) and (5.20) above were derived in the same connection, although they were not included in the Appendix A.

Formula (5.22) was employed in the unpublished Appendix A of the 1933 paper, being (22-A) there, as a basis for deriving two very different kinds of series type formulas for computing the values of $p(R'>R) \equiv Q^*(R)$ underlying the values of $p_{b,0}(R'>R)$ constituting Table I (facing Fig. 8) in that paper.[28]

[26] This formula, (5.22), was derived by me in a somewhat different manner in the unpublished Appendix A to my 1933 paper. Later I found that an equivalent formula, easily transformable into (5.22), had been given by Bravais as formula (51) in his classical paper "Analyse mathématique sur les probabilités des erreurs de situation d'un point," published in Mémoires de l'Académie Royale des Sciences de l'Institut de France, 2nd series, vol. IX, 1846, pp. 255–332. (This is available in the Public Library of New York City, for instance.)

[27] There the abbreviated symbols $p(R' < R)$ and $p(R' > R)$ were used with the same meanings as the complete symbols on the right sides of equations (5.1) and (5.2), respectively, of the present paper.

[28] Each of the two kinds of series type formulas comprised a finite portion of a convergent series plus an exact remainder term consisting of a definite integral. In the

In the present paper, formulas (5.11) and (5.16) have been used for numerical evaluation of $Q(R) \equiv p(R'<R)$ and of $Q^*(R) \equiv p(R'>R)$ by numerical integration (employing 'Simpson's one-third rule'), aided by some of the considerations set forth in Appendix D. However, only a moderate number of values of these quantities have been thus evaluated—merely enough to afford a fairly comprehensive check on Table I of my 1933 paper, by means of a sample consisting of 60 values (about 26%) distributed in a somewhat representative manner over that table. These new values of $Q^*(R) \equiv p(R'>R) = 1 - Q(R)$ are presented in Table 5.1 (at the end of this section) in such a way as to facilitate comparison with the old values, namely those in the 1933 paper. Thus, for any fixed value of R in Table 5.1, there are two horizontal rows of computed values of $Q^*(R)$, the first row (top row) coming from the 1933 paper, and the second row coming from the present paper. The third row of each set of four rows gives the deviations of the second row from the first row; and the fourth row expresses these deviations as percentages of the values in the first row.

In the first row of any set of four rows, any value represents $Q^*(R) \equiv p_b(R'>R)$ obtained, in accordance with Eq. (22) of my 1933 paper, by adding $\exp(-R^2)$ to $p_{b,0}(R'>R)$ given in Table I there. In the second row of a set, any value represents $Q^*(R) = 1 - Q(R)$ as computed by formula (5.11) or (5.16) of the present paper: more specifically, the values for $R = 0.2$, 0.4, 0.6 and 0.8 were computed by (5.11); and the values for $R = 1.6$ and $R = 2$ by (5.16), taking $r = 1/1.6 = 0.625$ and $r = 1/2 = 0.5$ respectively.[29]

In the 1933 paper, the values of $p_b(R'>R) \equiv Q^*(R;b)$ for $b = 0$ and for $b = 1$ were omitted as being unnecessary there because their values could be easily obtained from the simple exact formulas to which the general formulas there reduced, for $b = 0$ and $b = 1$. Those reduced formulas were the same as (5.23) and (5.24) of the present paper, except that (5.24) gives $Q(R;1)$ instead of giving $Q^*(R;1) = 1 - Q(R;1)$. The values obtained from these two formulas, exact to the number of significant figures here retained, are given in Table 5.1 at the intersections of the first row of each set of four rows with the columns $b = 0$ and $b = 1$. Therefore in these two columns the deviations (in the third row of each set of four rows) are deviations from exact values; the values in the second row of each set are, as

use of such a formula for numerical computations, the expansion producing the convergent series was carried far enough to insure that the remainder definite integral would be relatively small, though usually not negligible; and then this remainder definite integral was evaluated sufficiently accurately by numerical integration.

[29] In the work of numerical integration, 'Simpson's one-third rule' was employed for $R = 0.2$, 0.4, 0.6, 0.8 and 2. For $R = 1.6$, so that $r = 1/1.6 = 0.625$, 'Simpson's one-third rule' was employed up to $r = 0.620$, and the 'trapezoidal rule' from $r = 0.620$ to $r = 0.625$.

already stated, those obtained by the methods of the present paper, employing numerical integration.

From detailed inspection of Table 5.1 it will presumably be considered that the agreement between the two sets of values of $Q^*(R;b) \equiv p_b(R'>R)$ is to be regarded as satisfactory, at least from the practical viewpoint, the largest deviation being less than one per cent (for $R = 0.8$, $b = 0.9$).

TABLE 5.1
VALUES OF $Q^*(R) \equiv p(R' > R)$

b....... R	0	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95	1.00
0.2	.9608	.9590	.9574	.9550	.9516	.9463	.9372	.9168	.8930	.84148
"	.9623	.9605	.9590	.9567	.9528	.9473	.9387	.9206	.8925	.84124
"	.0015	.0015	.0016	.0017	.0012	.0010	.0015	.0038	−.0005	−.00024
"	.16	.16	.17	.18	.13	.11	.16	.41	−.06	−.03
0.4	.8521	.8462	.8410	.8335	.8228	.8071	.7830	.7420	.7127	.68916
"	.8537	.8477	.8427	.8351	.8240	.8081	.7841	.7459	.7125	.68897
"	.0016	.0015	.0017	.0016	.0012	.0010	.0011	.0039	−.0002	−.00019
"	.19	.18	.20	.19	.15	.12	.14	.53	−.03	−.03
0.6	.6977	.6880	.6799	.6686	.6531	.6324	.6055	.5721	.5578	.54851
"	.6992	.6892	.6814	.6698	.6540	.6334	.6065	.5764	.5572	.54831
"	.0015	.0012	.0015	.0012	.0009	.0010	.0010	.0043	−.0006	−.00020
"	.22	.17	.22	.18	.14	.16	.17	.75	−.11	−.04
0.8	.5273	.5167	.5081	.4969	.4826	.4656	.4477	.4316	.4261	.42371
"	.5290	.5183	.5099	.4982	.4840	.4672	.4488	.4357	.4266	.42355
"	.0017	.0016	.0018	.0013	.0014	.0016	.0011	.0041	.0005	−.00016
"	.32	.31	.35	.26	.29	.34	.25	.95	.12	−.04
1.6	.07730	.07986	.08207	.08522	.0891	.0938	.0990	.1042	.1070	.10960
"	.07727	.07988	.08210	.08536	.0892	.0938	.0989	.1042	.1069	.10958
"	−.00003	.00002	.00003	.00014	.0001	.0000	−.0001	.0000	−.0001	−.00002
"	−.04	.03	.04	.16	.11	.00	−.10	.00	−.09	−.02
2.0	.01832	.02153	.02394	.02681	.0301	.0337	.0375	.0414	.0435	.04550
"	.01823	.02145	.02383	.02685	.0302	.0338	.0376	.0415	.0436	.04552
"	−.00009	−.00008	−.00011	.00004	.0001	.0001	.0001	.0001	.0001	.00002
"	−.49	−.37	−.46	.15	.33	.30	.27	.24	.23	.04

6. THE DISTRIBUTION FUNCTION FOR THE ANGLE

The distribution function $P(\theta \mid R_{12}) \equiv P(\theta)$ for the angle θ of any complex variate $W \equiv R(\cos\theta + i\sin\theta)$ is defined by equation (1.10) on setting $\rho = \theta$, $\sigma = R$, $\sigma_1 = R_1 = 0$ and $\sigma_2 = R_2 = \infty$; thus

$$P(\theta)d\theta = p(\theta<\theta'<\theta+d\theta, 0<R'<\infty). \tag{6.1}$$

An integral formula for $P(\theta)$ is immediately obtainable from (1.6) by setting $\rho = \theta$, $\sigma = R$, $\sigma_3 = \sigma_1 = R_1 = 0$ and $\sigma_4 = \sigma_2 = R_2 = \infty$; thus

$$P(\theta) = \int_0^\infty P(R, \theta)\, dR. \tag{6.2}$$

The rest of this section deals with the case where $W \equiv R(\cos\theta + i\sin\theta)$ is 'normal.' Since this case depends on b as a parameter, $P(\theta)$ is here an abbreviation for $P(\theta;b)$.

A formula for $P(\theta;b) \equiv P(\theta)$ can be obtained by substituting $P(R,\theta)$ from (2.15) into (6.2) and executing the indicated integration, which can be easily accomplished. The resulting formula is found to be

$$P(\theta;\, b) = \frac{\sqrt{1 - b^2}}{2\pi(1 - b\cos 2\theta)}. \tag{6.3}$$

This formula can also be obtained as a particular case of either of the more general formulas (2.19) and (2.20) by setting $R = \infty$ in (2.19) or $R = 0$ in (2.20); also by adding (2.19) to (2.20) and then utilizing (1.10).

In $P(\theta) \equiv P(\theta;b)$ it will evidently suffice to deal with values of θ in the first quadrant, because of symmetry of the scatter diagram.

In $P(\theta;b)$ it will suffice to deal with only positive values of b, as (6.3) shows that changing b to $-b$ has the same effect as changing 2θ to $\pi\pm 2\theta$, or θ to $\pi/2\pm\theta$; that is, $P(\theta;-b) = P(\pi/2\pm\theta;b)$.

Fig. 6.1 gives curves of $P(\theta;b)$, computed from (6.3), as function of θ with b as parameter, for the ranges[30] $0 \leqq \theta \leqq 90°$ and $0 \leqq b \leqq 1$.

The curves in Fig. 6.1 indicate that $P(\theta;b)$ is a maximum at $\theta = 0°$ and a minimum at $\theta = 90°$. These indications are verified by formula (6.3), as this formula shows that:

$$\text{Max } P(\theta;b) = P(0°;b) = \frac{1}{2\pi}\sqrt{\frac{1+b}{1-b}}, \tag{6.4}$$

$$\text{Min } P(\theta;b) = P(90°;b) = \frac{1}{2\pi}\sqrt{\frac{1-b}{1+b}}. \tag{6.5}$$

Thence

$$\text{Min } P(\theta;b)/\text{Max } P(\theta;b) = (1-b)/(1+b), \tag{6.6}$$

$$P(\theta;b)/\text{Max } P(\theta;b) = P(\theta;b)/P(0°;b) = (1-b)/(1-b\cos 2\theta). \tag{6.7}$$

The curves in Fig. 6.1 indicate also that $P(\theta;b)$ is independent of θ in the case $b = 0$. This is verified by formula (6.3), as this formula shows that

$$P(\theta;0) = 1/2\pi. \tag{6.8}$$

Thence (6.3) can be written

$$P(\theta;b)/P(\theta;0) = (\sqrt{1 - b^2})/(1-b\cos 2\theta). \tag{6.9}$$

[30] Beginning here, θ will usually be expressed in degrees instead of radians, for practical convenience.

By setting $\cos 2\theta = 0$ in (6.3), so that $\theta = 45°$, it is found that

$$(\sqrt{1 - b^2})/2\pi = P(45°;b), \tag{6.10}$$

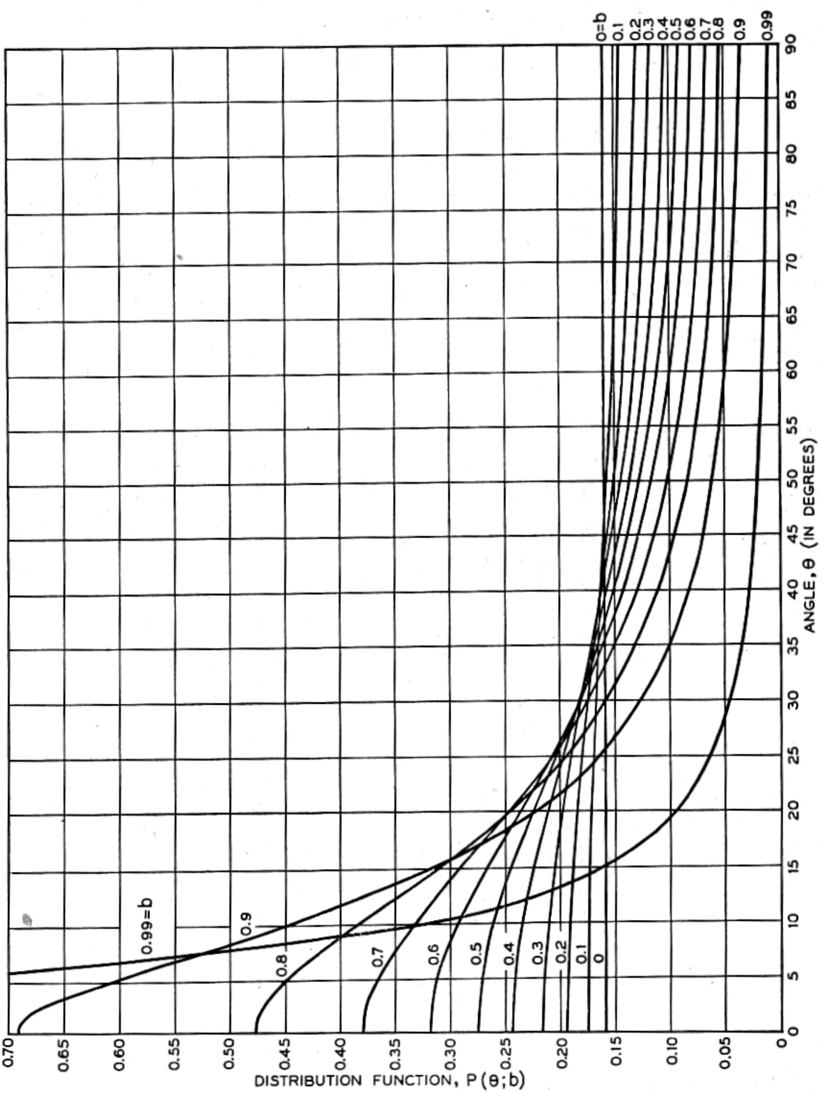

Fig. 6.1—Distribution function for the angle.

whence (6.3) can be written

$$P(\theta;b)/P(45°;b) = 1/(1-b\cos 2\theta). \tag{6.11}$$

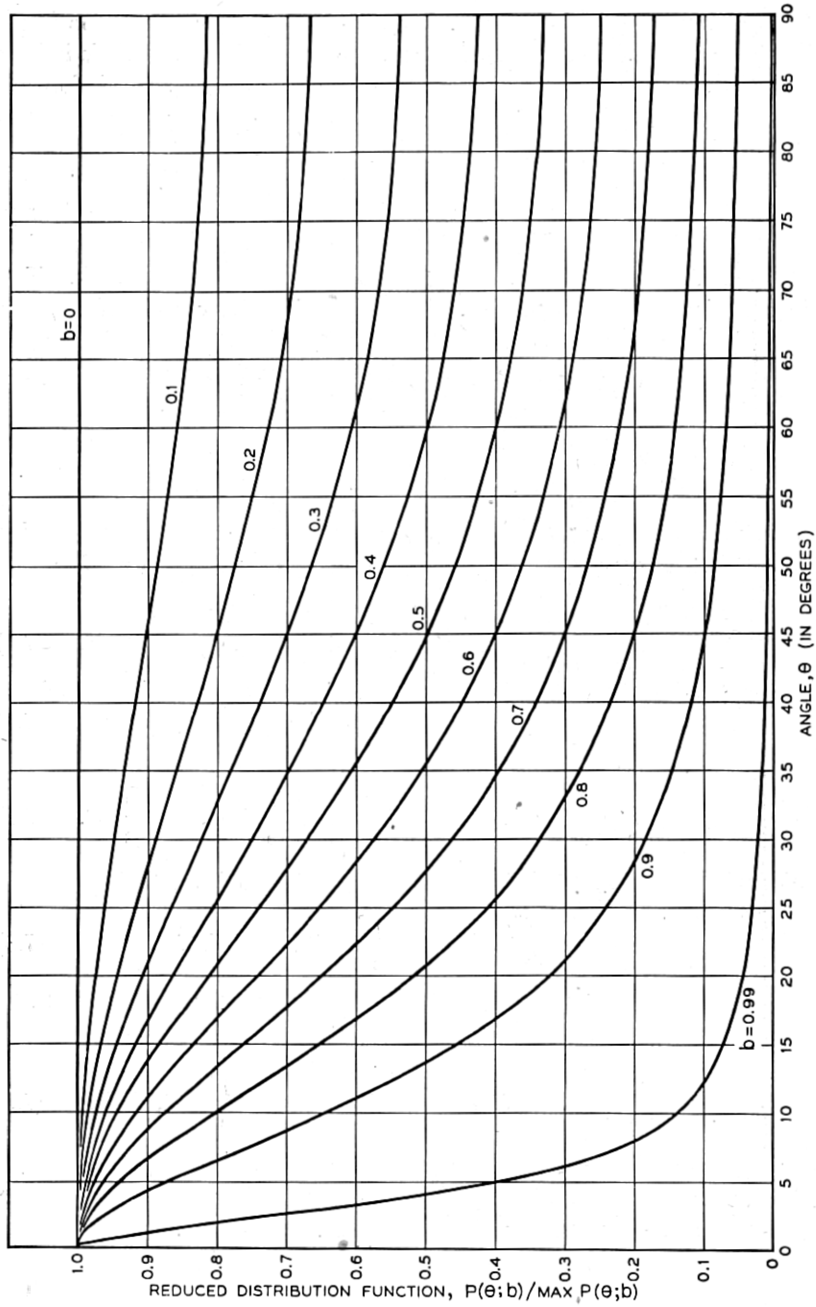

Fig. 6.2—Reduced distribution function for the angle.

In the case $b = 1$, the curves in Fig. 6.1 suggest, by limiting considerations, that $P(\theta;1)$ is zero for all θ except $\theta = 0°$, and that $P(\theta;1)$ is infinite for $\theta = 0°$. These conclusions are verified by formula (6.3), as this formula shows that:

$$P(\theta;1) = 0 \text{ for } 0°<\theta<180°;\; P(\theta;1) = \infty \text{ for } \theta = 0°,\ 180°.$$

The curves in Fig. 6.1, though having the advantage of directly representing $P(\theta;b)$ as function of θ with b as parameter, are somewhat troublesome to use because of their numerous crossings of each other. This difficulty is not present in Fig. 6.2, which gives curves of $P(\theta;b)/\text{Max } P(\theta;b)$, obtained by dividing the ordinates $P(\theta;b)$ of the curves in Fig. 6.1 by the respective maximum ordinates of those curves, as given by (6.4), so that the equation of the curves in Fig. 6.2 is formula (6.7).

7. The Cumulative Distribution Function for the Angle

The cumulative distribution function $Q(<\theta,R_{12}) \equiv Q(\theta)$ for the angle θ of any complex variate $W \equiv R(\cos\theta + i\sin\theta)$ is defined by equation (1.11) on setting $\rho = \theta$, $\sigma = R$, $\rho_1 = \theta_1 = 0$, $\sigma_1 = R_1 = 0$ and $\sigma_2 = R_2 = \infty$; thus

$$Q(\theta) = p(0<\theta'<\theta, 0<R'<\infty). \tag{7.1}$$

A 'double integral' for $Q(\theta)$, in the form of two 'repeated integrals,' can be written down directly by inspection of the $p(\ \ )$ expression in (7.1) or by specialization of (1.8); thus

$$Q(\theta) = \int_0^\theta \left[\int_0^\infty P(R,\theta)\,dR\right] d\theta = \int_0^\infty \left[\int_0^\theta P(R,\theta)\,d\theta\right] dR. \tag{7.2}$$

Evidently these can be written formally as two 'single integrals,'

$$Q(\theta) = \int_0^\theta P(\theta)\,d\theta = \int_0^\infty P(R \mid <\theta)\,dR, \tag{7.3}$$

by means of the distribution functions $P(\theta) \equiv P(\theta \mid R_{12})$ and $P(R \mid <\theta)$ given by the formulas

$$P(\theta) = \int_0^\infty P(R,\theta)\,dR, \quad (7.4) \qquad P(R \mid <\theta) = \int_0^\theta P(R,\theta)\,d\theta. \quad (7.5)$$

(7.4) is the same as (6.2). (7.5) is a special case of (1.6), and the left side of (7.5) is a special case of $P(\rho \mid <\sigma)$ defined by (1.13).

The rest of this section deals with the case where $W \equiv R(\cos\theta + i\sin\theta)$ is 'normal.' Since this case depends on b as a parameter, $Q(\theta)$ is here an abbreviation for $Q(\theta;b)$.

A natural and convenient way for deriving formulas for $Q(\theta)$ is afforded by the general formula (7.3) together with the auxiliary general formulas (7.4) and (7.5), beginning with the two latter.

It will be convenient to dispose of (7.5) before dealing with (7.4), as (7.5) turns out to be the less useful. For when $P(R,\theta)$ given by (2.16) is substituted into (7.5), the indicated integration cannot be executed in general, as (7.5) becomes (2.18), wherin the indicated integration can be executed only for certain special values of the integration limit θ—by means of the special Bessel function formula (3.3).

When $P(R,\theta)$ given by (2.15), which is equivalent to (2.16) used above, is substituted into (7.4), it is found that the indicated integration can be executed, giving the previously obtained formula (6.3) for $P(\theta) \equiv P(\theta;b)$.

A θ-integral formula for $Q(\theta) \equiv Q(\theta;b)$ can be obtained by substituting $P(\theta) \equiv P(\theta;b)$ from (6.3) into the first integral in (7.3), giving

$$Q(\theta;\, b) = \frac{\sqrt{1-b^2}}{2\pi} \int_0^\theta \frac{d\theta}{1 - b\cos 2\theta} = \frac{\sqrt{1-b^2}}{4\pi} \int_0^{2\theta} \frac{d\phi}{1 - b\cos\phi}. \tag{7.6}$$

This formula can also be obtained as a particular case of the more general formulas (2.22) and (2.24) by setting $R = \infty$ in (2.22) or $R = 0$ in (2.24); also by adding (2.22) to (2.24) and then utilizing (1.11).

The integral in (7.6) is of well-known form, and the indicated integration can be executed, yielding the following two equivalent formulas for $Q(\theta;b)$:

$$\begin{aligned} Q(\theta;\, b) &= \frac{1}{2\pi}\left|\tan^{-1}\left[\sqrt{\frac{1+b}{1-b}}\tan\theta\right]\right| \\ &= \frac{1}{4\pi}\left|\cos^{-1}\left[\frac{\cos 2\theta - b}{1 - b\cos 2\theta}\right]\right|. \end{aligned} \tag{7.7}$$

In $Q(\theta;b)$ it will evidently suffice to deal with values of θ in the first quadrant, because of symmetry of the scatter diagram, and the resulting fact that $Q(n\ 90°) = n/4$, where $n = 1, 2, 3$ or 4.

In $Q(\theta;b)$ it will suffice to deal with positive values of b, as (7.7) shows that[31]

$$Q(\theta;\, -b) = \left|\frac{1}{4} - Q\left(\frac{\pi}{2} \pm \theta;\, b\right)\right|.$$

Fig. 7.1 gives curves of $Q(\theta;b) \equiv Q(\theta)$ computed from (7.7), as function of θ with b as parameter, for the ranges $0 \leqq \theta \leqq 90°$ and $0 \leqq b \leqq 1$.

Consideration of the scatter diagram of W or of its equiprobability curves, which are concentric similar ellipses, affords several partial checks on the curves in Fig. 7.1 and on formula (7.7) from which they were plotted.

[31] This relation can also be derived geometrically from the fact that the scatter diagram for $-b$ is obtainable by merely rotating that for b through 90°, as shown by (2.6), or (2.7) and (2.8), or (2.11).

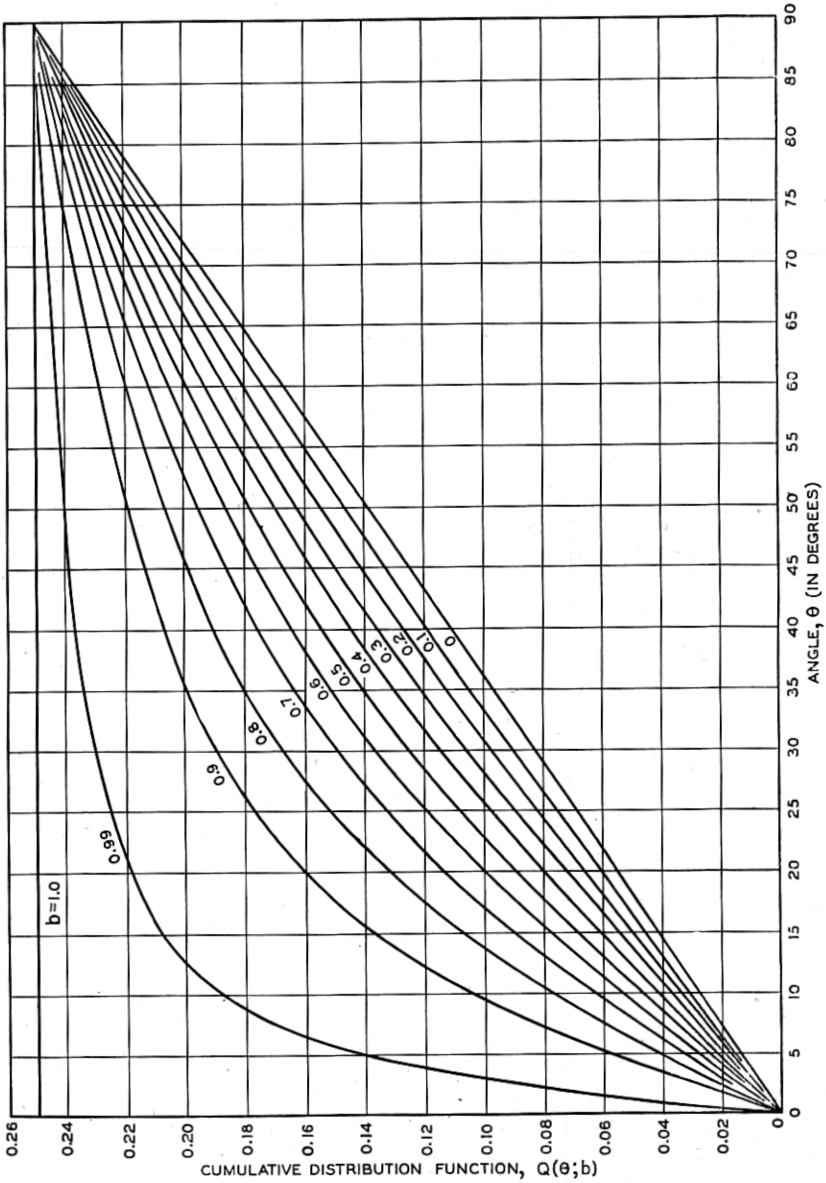

Fig. 7.1—Cumulative distribution function for the angle.

Thus, the fact that the curve for $b = 0$ is a straight line, whose equation is

$$Q(\theta;0) = \theta/2\pi = \theta^\circ/360^\circ, \qquad (b = 0),$$

corresponds to the fact that for $b = 0$ the equiprobability curves are circles.

The fact that the curve for $b = 1$ is the straight line $Q(\theta;1) = 1/4 = 0.25$ corresponds to the fact that for $b = 1$ the scatter diagram has degenerated to be merely a straight line coinciding with the real axis, so that no point outside of this line makes any contribution to $Q(\theta;1)$.

The fact that, at $\theta = 90°$, $Q(\theta;b) = Q(90°;b)$ has for all b the value $1/4 = 0.25$ corresponds to the fact that the area of a quadrant of the scatter diagram is one-fourth the area of the entire scatter diagram. Hence $Q(360°;b) = 4Q(90°;b) = 1$, which is evidently correct.

Acknowledgment

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APPENDIX A

Derivation of Formula (2.15) for $P(R,\theta)$

(2.15) will here be derived from (2.11) by utilizing the fact that the 'areal probability density', G, at any fixed point in the scatter diagram must be independent of the system of coordinates; for $G\,dA$ gives the probability of falling in any differential element of area dA, and this probability must evidently be independent of the shape of dA (assuming that all linear dimensions of dA are differential, of course). Thus, indicating the element of area by an underline, we have, in rectangular coordinates,

$$G\underline{dUdV} = P(U,V)dUdV, \qquad \text{(A1)} \qquad \text{whence} \quad G = P(U,V). \qquad \text{(A2)}$$

In polar coordinates,

$$G\underline{Rd\theta dR} = P(R,\theta)dRd\theta, \qquad \text{(A3)} \qquad \text{whence} \quad G = P(R,\theta)/R. \qquad \text{(A4)}$$

Comparing these two expressions for G shows that[32]

$$P(R,\theta) = RP(U,V). \qquad \text{(A5)}$$

Thus, a formula for $P(R,\theta)$ can be obtained from (2.11) by merely multiplying both sides of that formula by R. However, in the resulting formula it will remain to express U and V in terms of R and θ, by means of the relations

$$U = R\cos\theta, \qquad \text{(A6)} \qquad V = R\sin\theta. \qquad \text{(A7)}$$

The final result, after a simple reduction, is (2.15), which is thus proved.

APPENDIX B

Formulas of the Curves in Fig. 3.3

As in equation (3.22), R_C will here denote the critical value of R, that is, the value of R at which $P(R) \equiv P(R;b)$ has its maximum value; and T_C

[32] Formula (A5) can be easily verified by the entirely different method which utilizes (1.23).

will denote the corresponding value of T, whence T_C is given in terms of R_C and b by (3.22).

A formula for $dP(R)/dR$ could of course be obtained directly from (3.4) but it will be found preferable to obtain it indirectly from the less cumbersome formula (3.8) containing the auxiliary variable T defined by (3.6). Evidently, since b does not depend on R,

$$\frac{dP(R)}{dR} = \frac{dP(R)}{dT}\frac{dT}{dR} = \frac{2bR}{1-b^2}\frac{dP(R)}{dT}. \tag{B1}$$

Thus, since the factor $2bR/(1-b^2)$ cannot vanish for any value of R (except $R = 0$), the only critical value of R must be that corresponding to the value of T at which $dP(R)/dT$ vanishes, namely T_C, since T_C has been defined to be the value of T corresponding to R_C. (Incidentally, equation (B1) shows that T_C is equal to the value of T at which $P(R)$ is an extremum when $P(R)$ is regarded as a function of T.) From (3.22),

$$\frac{R_C^2}{1-b^2} = \frac{T_C}{b}. \tag{B2}$$

Evidently T_C and R_C must ultimately be functions of only b. The next paragraph deals with T_C, which evidently has to be known before R_C can be evaluated.

From (3.8) it is found that, since $dI_0(T)/dT = I_1(T)$,

$$\frac{dP(R)}{dT} = P(R)\left[\frac{1}{2T} + \frac{I_1(T)}{I_0(T)} - \frac{1}{b}\right]. \tag{B3}$$

Hence, since $P(R)$ does not vanish for any value of R (except $R = 0$ and $R = \infty$), T_C will be a root of the conditional equation obtained by equating to zero the expression in brackets in (B3). This conditional equation is transcendental in T_C and apparently has no closed form of explicit solution for T_C; and its solution by successive approximation, or otherwise, would likely be rather slow and laborious. However, the bracket expression in (B3) shows that b can be immediately expressed explicitly in terms of T_C by the equation

$$b = \frac{2T_C}{1 + 2T_C I_1(T_C)/I_0(T_C)}. \tag{B4}$$

For some purposes, the following two equations, each equivalent to (B4), will be found more convenient:

$$\frac{T_C}{b} = \frac{1}{2} + T_C\frac{I_1(T_C)}{I_0(T_C)}, \tag{B5}$$

$$\frac{T_C}{b} = \frac{1/2}{1 - bI_1(T_C)/I_0(T_C)}. \tag{B6}$$

On account of (B2), the right sides of (B5) and (B6) are equal not only to T_C/b but also to $R_C^2/(1-b^2)$.

Since the utilization of formulas (B4), (B5) and (B6) for computing the curves in Fig. 3.3 will involve taking T_C as the independent variable and assigning to it a set of chosen numerical values, the natural first step is to find approximately the range of T_C corresponding to the b-range, $0 \leqq b \leqq 1$, in order to be able to choose only useful values of T_C. This step will be taken in the next paragraph.

Equation (B6) shows that $T_C/b = 1/2$ when $b = 0$, and hence that $T_C = 0$ when $b = 0$; and this last is verified by (B4). The other end-value of the T_C-range, namely the value of T_C for $b = 1$, cannot be found explicitly and exactly. However, rough values of limits between which it must lie can be found fairly easily as follows: To begin with, each of the equations (B5) and (B6) shows that $T_C \geqq b/2$, for all values of b in $0 \leqq b \leqq 1$; in particular, $T_C > 1/2$ when $b = 1$. An upper limit for T_C for any value of b can be found from (B5) by utilizing the power series expressions for $I_1(T_C)$ and $I_0(T_C)$, whereby it is found that

$$\frac{I_1(T_C)}{I_0(T_C)} = H\frac{T_C}{2}, \qquad \text{(B7)} \qquad \text{where} \qquad H \approx 1 - \frac{T_C^2}{8} < 1. \qquad \text{(B8)}$$

On substituting (B7) into (B5) and then solving for T_C in terms of b and H, it is found that

$$T_C = b/(1 + \sqrt{1 - Hb^2}). \qquad \text{(B9)}$$

On account of (B8), (B9) shows that

$$T_C < b/(1 + \sqrt{1 - b^2}), \qquad \text{(B10)}$$

whence, in particular, $T_C < 1$ when $b = 1$. By successive approximation or otherwise, it can now be rather quickly found that, when $b = 1$, $T_C = 0.79$ (to two significant figures).[33]

From the preceding paragraph, it is seen that, when b ranges from 0 to 1, T_C ranges from 0 to about 0.79; T_C/b ranges from 0.5 to about 0.79; and, on account of (B2), R_C ranges from $\sqrt{0.5} = 0.707$ down to 0.

The curves in Fig. 3.3 are constructed with the aid of the formulas and methods of this appendix as follows: First, a set of values of T_C is chosen, ranging from 0 to 0.79 and slightly larger. Second, for each such chosen T_C the right side of (B5) is computed, thereby evaluating T_C/b and also $R_C^2/(1-b^2)$, these two quantities being equal by (B2). Third, the corresponding value of b is found by dividing T_C by T_C/b; less easily, it could

[33] Because of the special importance of $b = 1$ in other connections, T_C for $b = 1$ was later evaluated to four significant figures and found to be $T_C = 0.7900$; thence, by substituting this value of T into (3.8), along with $b = 1$, it was found that Max. $P(R;1) = 0.9376$, which occurs at $R = R_C = 0$, by (B2).

be found by substituting T_C into (B4). Fourth, from T_C/b the value of $\sqrt{T_C/b}$ is found, and thereby the value of $R_C/\sqrt{1-b^2}$ and thence R_C. Finally, Max. $P(R;b)$ is computed by inserting the critical values into any of the various (equivalent) formulas for $P(R;b)$, namely (3.4), (3.7), (3.8), (3.10) or (3.12).

APPENDIX C

FOMULAS OF THE CURVES IN FIG. 4.3

The first six equations of this appendix are given without derivation and almost without any comments because they correspond exactly and simply to the first six equations, respectively, of Appendix B. Beginning with the second paragraph of the present appendix, the close correspondence ceases.

$$\frac{dP(r)}{dr} = \frac{dP(r)}{dT}\frac{dT}{dr} = \frac{-2b}{(1-b^2)r^3}\frac{dP(r)}{dT}. \tag{C1}$$

$$\frac{1}{(1-b^2)r_c^2} = \frac{T_c}{b}. \tag{C2}$$

$$\frac{dP(r)}{dT} = P(r)\left[\frac{3}{2T} + \frac{I_1(T)}{I_0(T)} - \frac{1}{b}\right]. \tag{C3}$$

$$b = \frac{2T_c}{3 + 2T_c\, I_1(T_c)/I_0(T_c)}. \tag{C4}$$

$$\frac{T_c}{b} = \frac{3}{2} + T_c\,\frac{I_1(T_c)}{I_0(T_c)}. \tag{C5}$$

$$\frac{T_c}{b} = \frac{3/2}{1 - bI_1(T_c)/I_0(T_c)}. \tag{C6}$$

The bracketed expression in (C3) is seen to be obtainable from that in (B3) by merely changing T to $T/3$ wherever T does not occur as the argument of a function; hence the three equations following (C3) are obtainable from the three equations following (B3) by correspondingly changing T_C to $T_c/3$. (In this appendix, as in Section 4, small c is purposely used as a subscript to indicate a 'critical' value, whereas in Section 3 and in Appendix B, capital C is used for that purpose.)

For use below, it will here be noted that

$$I_1(T_c)/I_0(T_c) = N_1(T_c)/N_0(T_c), \tag{C7}$$

as will be seen by dividing (3.16) by (3.13). On account of (3.17) and (3.14), (C7) shows that for large values of T_c the right side of (C7) is equal to 1 as a first approximation, and to $1 - 1/2T_c$ as a second approximation; thus, for large T_c,

$$I_1(T_c)/I_0(T_c) \approx 1 - 1/2T_c \approx 1. \tag{C8}$$

The first step toward computing the curves in Fig. 4.3 is to find approximately the T_c-range corresponding to the b-range, $0 \leqq b \leqq 1$. This is done in the course of the next four paragraphs.

When $b = 0$, equation (C6) shows that $T_c/b = 3/2$ and hence that $T_c = 0$; or, what is equivalent, $b/T_c = 2/3$ and hence $1/T_c = \infty$ (since $b = 0$).

When $b = 1$, $T_c = \infty$, as can be easily verified from equation (C4), (C5) or (C6) by utilizing (C8).

Thus, from the two preceding paragraphs, it is seen that, when b ranges from 0 to 1, b/T_c ranges from 2/3 to 0; T_c/b from 3/2 to ∞; and T_c from 0 to ∞.

Since $T_c = \infty$ when $b = 1$, the choosing of a set of finite values of T_c will necessitate an approximate formula for computing T_c for values of b nearly equal to 1, which means for very large values of T. Such a formula is easily obtainable from (C5) by utilizing the approximation $1 - 1/2T_c$ in (C8), whereby it is found that, for large T_c,

$$T_c \approx b/(1-b), \quad \text{(C9)} \qquad b/T_c \approx 1-b. \quad \text{(C10)}$$

As examples, these approximate formulas give: When $b = 0.99$, $T_c \approx 99$, $b/T_c \approx 0.01$; when $b = 0.9$, $T_c \approx 9$, $b/T_c \approx 0.1$. It will be found that even in the second example the results are pretty good approximations.

The curves in Fig. 4.3 are constructed with the aid of the formulas and methods of this appendix as follows: First, a set of values of T_c is chosen, ranging from 0 to about 100 (the latter figure corresponding approximately to $b = 0.99$). Second, for each such chosen T_c the right side of (C5) is computed, thereby evaluating T_c/b and also $1/(1-b^2)r_c^2$, these two quantities being equal by (C2). Third, the corresponding value of b is found by dividing T_c by T_c/b; less easily, it could be found by substituting T_c into (C4). Fourth, from T_c/b the value of $\sqrt{T_c/b}$ is found, and thereby the value of $1/r_c\sqrt{1 - b^2}$ and thence r_c. Finally, Max $P(r;b)$ is computed by inserting the critical values into any of the (equivalent) formulas for $P(r;b)$, namely (4.2), (4.3) or (4.4).

APPENDIX D

Some Simple General Considerations Regarding the Evaluation of Cumulative Distribution Functions by Numerical Integration

This appendix gives some simple general considerations and relations that may sometimes facilitate and render more accurate the evaluation of cumulative distribution functions by numerical integration.

Some of these considerations and relations have found application in Section 5 in the evaluation of the cumulative distribution function for the modulus $R \equiv |W|$. For this reason, the variate in the present section will be denoted by R, though without thereby restricting R to denote the modulus; rather, R will here denote any positive real variate, though it should preferably be a 'reduced' variate, so as to be dimensionless, as in equation (2.9). The restriction of R to positive values is imposed because it is strongly conducive to simplicity and brevity of treatment, without constituting an ultimate limitation. The reciprocal of R will be denoted by r, as previously.[34]

We may wish to evaluate numerically the cumulative distribution function $p(R'<R) \equiv Q(R)$ or $p(R'>R) \equiv Q^*(R)$ or both. Since these are not independent, their sum being equal to unity, the evaluation of either one determines the other, theoretically. However, when the evaluated one is nearly equal to unity, the remaining one may perhaps not be evaluable with sufficient accuracy (percentagewise) by subtracting the evaluated one from unity. Then it would presumably be advantageous to introduce for auxiliary purposes the variable $r = 1/R$, since evidently

$$p(R'>R) = p(1/R'<1/R) = p(r'<r), \tag{D1}$$

$$p(R'<R) = p(r'>r) = 1 - p(r'<r). \tag{D2}$$

Thus, if $p(R'>R)$, in (D1), is small compared to unity, it is presumably evaluable with higher accuracy percentagewise by dealing with $p(r'<r)$ than with $1 - p(R'<R)$. Incidentally, after $p(r'<r)$ has been evaluated, it might be used in (D2) to arrive at a still more accurate value of $p(R'<R)$ than had originally been obtained directly by numerical integration.

Assuming that we have a plot (or a table) of the distribution function $P(R)$, we can evidently evaluate

$$P(R'<R^0) = \int_0^{R^0} P(R)dR \tag{D3}$$

directly by numerical integration, provided the plot is sufficiently extensive to include R^0; if not, we can, by (D2), resort to

$$p(R'<R^0) = 1 - p(r'<r^0) = 1 - \int_0^{r^0} P(r)dr, \tag{D4}$$

assuming that a sufficiently extensive plot (or table) of $P(r)$ is available and applying numerical integration to it.

Even if the plot of $P(R)$ used in (D3) is sufficiently extensive to include

[34] The restriction of R, and hence of r, to positive values is seen to be absent from equations (D1), (D2), (D5) and (D6) but present in (D3), (D4), (D7) and (D8).

R^0, so that (D3) could be evaluated, it might be that (D4) would result in greater accuracy; this would presumably be the case when $p(R'<R^0)$ is nearly equal to unity.

Evidently an evaluation of

$$p(R'>R^0) = \int_{R^0}^{\infty} P(R)dR \tag{D5}$$

directly by numerical integration would be less satisfactory than the evaluation of $p(R'<R^0)$ in the preceding paragraph. For, due to the presence of the infinite limit in the integral in (D5), the plot of $P(R)$ would have to be carried to a large enough value of R so that the integral from there to ∞ would be known to be negligible. This difficulty can be avoided by starting with the relation

$$p(R'>R^0) = 1 - p(R'<R^0) \tag{D6}$$

and substituting therein the value of $p(R'<R^0)$ given by (D3) or (D4), resulting respectively in the following two formulas:

$$p(R'>R^0) = 1 - \int_0^{R^0} P(R)dR, \tag{D7}$$

$$p(R'>R^0) = p(r'<r^0) = \int_0^{r^0} P(r)dr, \tag{D8}$$

the integrals in which are evidently suitable for evaluation by numerical integration, none of the integration limits being infinite. If $p(R'>R^0)$ is small compared to unity, (D8) would presumably be more accurate (percentagewise) than (D7). If the plot of $P(R)$ is not sufficiently extensive to include R^0, (D7) evidently could not be used; but, instead, (D8) could be used if the plot of $P(r)$ were sufficiently extensive to include r^0.

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