Dynamics of Package Cushioning

By RAYMOND D. MINDLIN

Introduction

MECHANICAL damage is a common occurrence in the transportation of packaged articles. The causes of failures are generally inadequate protective cushioning, lack of ruggedness of the outer packing container, or occasional abnormal weakness of the packaged article. The first of these difficulties is the subject of this paper.

One of the major influences in reducing the incidence of mechanical failures of packaged articles in recent years has been the use of the drop test. The drop test is performed simply by raising the package to a specified height and dropping it to the floor. The package and its contents are then examined for damage. This is a go-no-go test and requires a large number of samples before a reliable estimate of quality can be made. An adequate number of tests is prohibitive when the article packaged is costly. In such cases it is important, and in any case it is useful, to supplement the drop test data with measurements and calculations. It is also possible to evolve rational procedures for designing packages, as described in the present paper, so that a particular product will survive a drop test at any specified height, with a known factor of safety and with a minimum amount of space assigned for cushioning. The drop test then becomes only a check instead of playing an integral role in a cut and try design procedure.

Assuming that the outer container is adequate, the survival of a packaged article in a drop test still depends upon a large number of factors descriptive of the mechanical properties of both the cushioning medium and the packaged item. However, the more important properties can be grouped so that they may be replaced by knowledge of only the following factors:

(1) The magnitude of the maximum acceleration that the cushioning permits the packaged item to reach.
(2) The form of the acceleration-time relation.
(3) The strengths, natural frequencies of vibration and damping of the structural elements of the packaged article.

Part I of this paper is concerned primarily with methods for predicting maximum acceleration of the packaged article with emphasis on non-linear cushioning. Part II deals primarily with the prediction of the form of the acceleration-time relation. Part III deals with the effect of acceleration on the packaged article and gives methods for determining whether or not the strength of the packaged article will be exceeded. The strength determinations themselves are not dealt with here; but the information in Part III is essential in interpreting and applying the data obtained in strength measurements. In Part IV some consideration is given to the influence of distributed mass and elasticity.

It cannot be emphasized too strongly that the determination of the mechanical properties of the packaged article, not dealt with in this paper, is an essential preliminary to a rational design procedure for packaging. The whole purpose in designing package cushioning is to limit the forces which may act on the packaged item. If one does not know to what values to limit the forces, a rational design procedure cannot be applied.

It is interesting to observe that the methods described here for analyzing and designing package cushioning are directly applicable to the design of shock mounts intended to protect equipment from the effects of a sudden change in velocity. All of the principles, formulas and design curves given here may be used in the shock mount problem with the simple substitution of $V^2/2g$ for h, where h is the height of drop in the packaging problem, g is the acceleration of gravity and V is the velocity change in the shock mount problem.

This paper is essentially a report on a study undertaken at the Bell Telephone Laboratories, Inc., in the Electronic Apparatus Development Department. The results have been applied to the packaging of large vacuum tubes and all of the examples used to illustrate the analysis and design procedures in the paper are taken from vacuum tube applications.

Miss H. A. Lefkowitz, Member of the Technical Staff, Bell Telephone Laboratories, assisted in the mathematical studies. The oscillograms, used as illustrations, were prepared under the supervision of Mr. F. W. Stubner, Member of the Technical Staff, Bell Telephone Laboratories. Figure 3.8.2 was taken from a thesis submitted by Mr. C. Ulucay in partial fulfillment of the requirements for the degree of Master of Science in the Department of Civil Engineering at Columbia University. The calculations for Figs. 3.5.1 to 3.5.6 and Fig. 3.2.2 for $\beta_1 > 0$ were performed on the Westinghouse Mechanical Transients Analyzer under the supervision of Dr. G. D. McCann, Transmission Engineer, Westinghouse Electric and Manufacturing Company.

Assumptions

The procedures to be described for the analysis and design of package cushioning are based on applications of a few simple laws of mechanics to an idealized mechanical system representing the package and its contents.

Essentially, a package consists of

1. Elements of the packaged article which are susceptible to mechanical damage.
2a. The packaged article as a whole.
2b. A cushioning medium (excelsior, cardboard spring pads, metal springs, etc.)
3. An outer container (cardboard carton, wood packing case, etc.)

The four major components are illustrated schematically in Fig. 0.2.1. The system is further idealized by "lumping the parameters"; for example, the outer container is considered as a single mass, the cushioning is considered as a massless spring with friction losses. The result of this idealization is to lose some of the fine detail of the real distributed system such as wave propagation through the cushioning and higher modes of vibration in the package structure and in the packaged article. Some consideration of these details is given in Part IV.

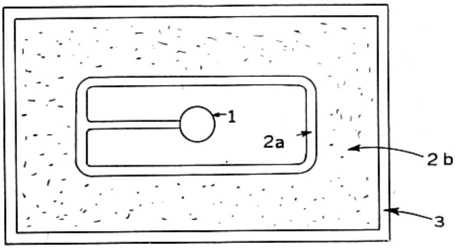


Fig. 0.2.1—Schematic representation of a package.
1. Element of packaged article
2a. Packaged article as a whole
2b. Cushioning
3. Outer container

The idealized system is illustrated in Fig. 0.2.2. The major components of the system are as follows:

1. A structural element of the packaged item is represented by a mass (m_1) supported by a linear massless spring with or without velocity damping. The mass m_1 is assumed to be small in comparison with the mass of the whole packaged item.
2a. The whole packaged item is represented by a mass m_2.
2b. The cushioning is represented by a spring which may have a linear or non-linear load-displacement characteristic and which dissipates energy through velocity damping or dry friction. Permanent deformation of the cushioning is not considered, that is, in a repetition of the drop test it is assumed that the package has the same properties as before the first test. A properly designed package will have essen-

tially this characteristic. The mass of the cushioning is assumed to be small in comparison with m_2, except in Section 4.2.

3. The outer container is represented by the mass m_3. The impact of m_3 on the floor is assumed to be inelastic and during contact the relative displacement between m_3 and the initial position of the floor is assumed to be small in comparison with the relative displacement between m_2 and m_3. In other words, no spring action is assigned to the outer container and the floor is considered rigid.

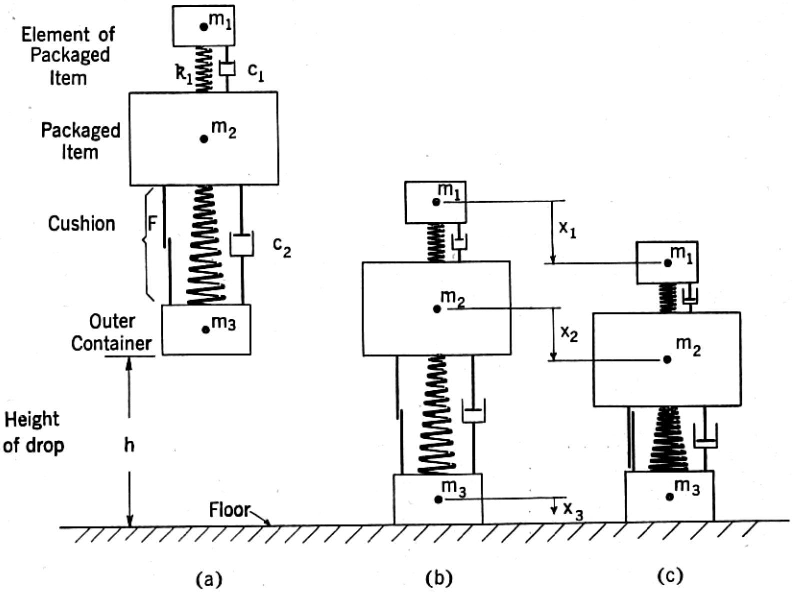


Fig. 0.2.2—Idealized mechanical system representing a package in a drop test.

PART I

MAXIMUM ACCELERATION AND DISPLACEMENT

1.1 Introduction

Most of Part I is concerned with the prediction of the maximum acceleration that the cushioning permits the packaged article (m_2) to attain. In many instances this will be all the information necessary for judging the suitability of a cushioning system. It will be all that is necessary if the shape and scale of the acceleration-time function satisfy certain criteria which are treated in detail in Parts III and IV. If these criteria are satisfied, the effect of the drop on the packaged article is found by multiplying the dead load stresses (obtained in the usual manner) by the ratio of the maximum acceleration to the acceleration of gravity. If the criteria for the use of maximum acceleration alone are not satisfied, then Parts II and III will supply a numerical factor (the Amplification Factor) by which the maximum acceleration should be multiplied, and the remainder of the procedure is the same as before.

The determination of the maximum acceleration is founded on a knowledge of the load-displacement characteristics of the cushioning. When the cushioning system is simple enough, the load-displacement relation may be found or designed by purely analytical procedures. The tension spring package, discussed in Sections 1.7 and 1.8, is an example where such a treatment is possible. In many instances, as with distributed cushioning, the load-displacement relation is more easily found by test.

A load-displacement test is made by applying successively increasing forces, with weights or in a load testing machine, to the packaged item completely assembled in its package, and measuring the corresponding displacements. The force is applied usually by means of a rod inserted in a hole cut through the outer container and the cushioning to the packaged item. It is convenient to use a low loading rate in the test, and, in doing so, the effect of resisting forces that depend on velocity is lost. These forces are often of little importance but, in certain designs, it is necessary to consider them. This is done for velocity damping in Sections 2.5, 2.6, 3.2 and 3.5.

Most of Part I is concerned with cushioning having non-linear load-displacement characteristics. Linear cushioning is rarely encountered, but it will be treated first because of its simplicity and because it will be convenient later to express the maximum acceleration in non-linear cases in terms of the maximum acceleration in a hypothetical linear case.

1.2 Derivation of Equations of Motion

To introduce the method of analysis that will be used in Part I, the simplest possible system is considered first. The m_1 system is omitted entirely, the mass of the outer container (m_3) is neglected, and the cushioning is assumed to have no damping or friction. There remain only the mass m_2 (the mass of the packaged item alone) and the supporting spring, as shown in Fig. 1.2.1. If the spring is linear its displacement is proportional to the applied load throughout the range of use (see Fig. 1.4.1). The spring rate (k_2) of a linear spring is a constant usually expressed in terms of pounds per inch. The force (P) transmitted through a linear spring is therefore given by

$$P = k_2 x_2 , \tag{1.2.1}$$

where x_2 is the displacement of m_2 measured downward from its position at first contact of the spring with the floor (see Fig. 1.2.1). For a non-linear spring P will be some other function of x_2:

$$P = P(x_2). \tag{1.2.2}$$

To write the equation of motion for the mass m_2, we consider the forces acting on it at any instant. These are (see Fig. 1.2.2(b)) the spring force P and the weight $m_2 g$, where g is the acceleration of gravity. When x_2 is positive (i.e., a downward displacement of m_2 from its position at first contact of the spring with the floor) the spring exerts an upward force P

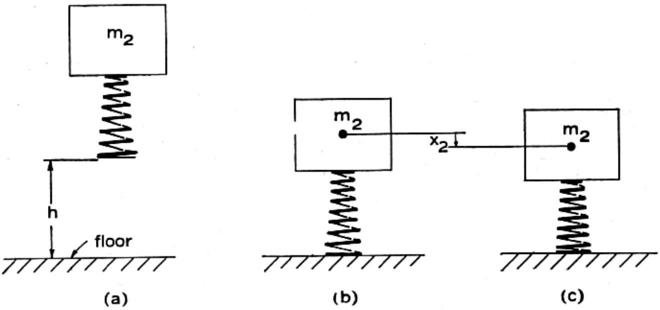


Fig. 1.2.1—Elementary system.

Fig. 1.2.2—Free body diagram for elementary system.
(a) Spring not in contact with floor.
(b) Spring in contact with floor.

on the mass, opposing the weight. The total downward force on m_2 is thus $m_2 g - P$. By the second law of motion, the product of the mass and its acceleration at any instant is equal to the applied force:

$$m_2 \ddot{x}_2 = m_2 g - P, \tag{1.2.3}$$

where the symbol $\ddot{x}_2$, representing the acceleration of m_2, stands for the second derivative of displacement with respect to time (d^2x_2/dt^2). Equation (1.2.3) is the law governing the motion of m_2 as long as the spring is in contact with the floor. When the spring is not in contact with the floor, it can exert no force on the mass so that, in writing the equation of motion that governs before or after contact, the free-body diagram of Fig. 1.2.2(a) should be used. Then

$$\ddot{x}_2 = g. \tag{1.2.4}$$

Equation (1.2.4) holds (neglecting air resistance) from the instant the package starts to fall until the instant it strikes the floor and from it we can find the package velocity at the instant of first contact. Integrating (1.2.4) with respect to time, we find

$$\dot{x}_2 = gt + A, \tag{1.2.5}$$

where $\dot{x}_2$ is the velocity (dx_2/dt) and A is a constant of integration whose value is found from the initial condition that when $t = 0$ (the instant of release) $\dot{x}_2 = 0$. Thus $A = 0$ and

$$\dot{x}_2 = gt. \tag{1.2.6}$$

Integrating again,

$$x_2 = \tfrac{1}{2} gt^2 + B. \tag{1.2.7}$$

The value of the integration constant B is found from the initial condition that $x_2 = -h$ (the height of drop) when $t = 0$. Hence $B = -h$ and

$$x_2 = \tfrac{1}{2} gt^2 - h. \tag{1.2.8}$$

At the instant of contact, $x_2 = 0$ and, from (1.2.8), the time at first contact is given by $t_0^2 = 2h/g$. Substituting this value of t in (1.2.5) we find, for the velocity at first contact,

$$[\dot{x}_2]_{x_2=0} = \sqrt{2gh}. \tag{1.2.9}$$

We now have the initial conditions for finding the values of the integration constants in the solution of equation (1.2.3), which we proceed to obtain.

First multiply both sides of (1.2.3) by dx_2/dt and write $\ddot{x}_2 = \dfrac{d}{dt}\left(\dfrac{dx_2}{dt}\right)$:

$$m_2 \frac{dx_2}{dt} \frac{d}{dt}\left(\frac{dx_2}{dt}\right) + P \frac{dx_2}{dt} = m_2 g \frac{dx_2}{dt} \tag{1.2.10}$$

or

$$\tfrac{1}{2} m_2 \frac{d}{dt}\left(\frac{dx_2}{dt}\right)^2 + P \frac{dx_2}{dt} = m_2 g \frac{dx_2}{dt}.$$

Multiplying by dt and integrating once:

$$\tfrac{1}{2} m_2 \dot{x}_2^2 + \int^{x_2} P \, dx_2 = \int^{x_2} m_2 g \, dx_2 + C, \tag{1.2.11}$$

where C is a constant of integration whose value is determined by the initial conditions that $\dot{x}_2^2 = 2gh$ and $x_2 = 0$ at the instant of contact. Hence

$$C = m_2 gh + \int^0 P\, dx_2 .$$

Substituting the above value of C in (1.2.11), we have

$$\tfrac{1}{2}m_2 \dot{x}_2^2 + \int_0^{x_2} P\, dx_2 = m_2 g(h + x_2). \tag{1.2.12}$$

It may be observed that (1.2.12) is an energy equation in which the terms have the following meanings:

$\frac{1}{2}m_2\dot{x}_2^2$ is the instantaneous kinetic energy of m_2,

$\int_0^{x_2} P\, dx_2$ is the energy stored in the spring at any instant. It is also equal to the area under the load-displacement curve up to the displacement x_2,

$m_2 g(h + x_2)$ is the potential energy of the mass at its initial height $h + x_2$ above the instantaneous position x_2.

Hence (1.2.12) expresses the law of conservation of energy.

Ordinarily h is very much larger than x_2 so that we may write, with good accuracy,

$$\tfrac{1}{2}m_2 \dot{x}_2^2 + \int_0^{x_2} P\, dx_2 = m_2 gh. \tag{1.2.13}$$

To the same approximation, equation (1.2.3) becomes

$$m_2 \ddot{x}_2 + P = 0. \tag{1.2.14}$$

Equation (1.2.14) and its first integral, equation (1.2.13), are convenient forms for calculating events at any time during contact. Their use will be illustrated in Part II. For calculating only maximum displacement and acceleration, the equations become simpler. Let

W_2 = weight of the packaged article ($=m_2 g$),

d_m = maximum displacement of the packaged article,

G_m = absolute value of maximum acceleration of the packaged article in terms of number of times gravity ($G_m = | \ddot{x}_2/g |_{\max}$),

P_m = maximum force exerted on packaged article by cushioning.

We shall limit our study to the practical regions where $P > 0$ when $x_2 > 0$. Then it may be seen from (1.2.13) that x_2 is a maximum when $\dot{x}_2$ is zero, hence

$$\int_0^{d_m} P\, dx_2 = W_2 h, \tag{1.2.15}$$

and, from (1.2.14),

$$G_m = \frac{P_m}{W_2}, \tag{1.2.16}$$

where P_m is the maximum value of P. If $P(x_2)$ is a monotonic function, P_m may be obtained from (1.2.2) by substituting d_m for x_2:

$$P_m = P(d_m). \tag{1.2.17}$$

In the unusual case where $P(x_2)$ is not monotonic, the maximum value of P in the interval $0 \leq x_2 \leq d_m$ must be chosen instead of equation (1.2.17).

The general procedure is to calculate d_m from (1.2.15), P_m from (1.2.17) and then G_m from (1.2.16). If P can be expressed analytically in terms of x_2 and if the integral in (1.2.15) can be evaluated in terms of elementary functions, simple formulas can be found for d_m and G_m. If this is not possible, then the integration can be performed graphically or numerically. Both of these procedures will be illustrated. In either case the maximum acceleration and displacement are obtained in terms of the weight of the packaged item, the height of drop and parameters descriptive of the load-displacement characteristics of the cushioning.

1.3 Linear Elasticity

For cushioning with a linear load-displacement relation, equation (1.2.1) applies. Substituting this value of P in (1.2.15), and performing the integration, we find

$$d_m = \sqrt{\frac{2hW_2}{k_2}}. \tag{1.3.1}$$

From (1.3.1) and (1.2.17),

$$P_m = \sqrt{2hW_2 k_2}, \tag{1.3.2}$$

and, from (1.3.2) and (1.2.16),

$$G_m = \sqrt{\frac{2hk_2}{W_2}}. \tag{1.3.3}$$

Notice that equation (1.3.3) holds only if there is space available for a displacement d_m and if the cushioning is linear and capable of transmitting a force P_m. Also, from (1.3.3) and (1.3.1),

$$d_m = \frac{2h}{G_m} \tag{1.3.4}$$

and

$$k_2 = \frac{W_2 G_m^2}{2h} = \frac{2hW_2}{d_m^2}. \tag{1.3.5}$$

Example: Find the properties of the linear cushioning required so that the maximum acceleration will be $50g$ in a 3 ft. drop of a 20 lb. article.

From (1.3.4),

$$\text{necessary travel, } d_m = \frac{2 \times 36}{50} = 1.44 \text{ inches.}$$

From (1.3.5),

$$\text{spring rate, } k_2 = \frac{20 \times (50)^2}{2 \times 36} = 694 \text{ lbs/in.}$$

From (1.2.16)

$$\text{Maximum force } P_m = 20 \times 50 = 1000 \text{ lbs.}$$

1.4 Cushioning with Non-Linear Elasticity

In practice it is rarely that a packaging system has linear spring characteristics. Departure from linearity may be due to

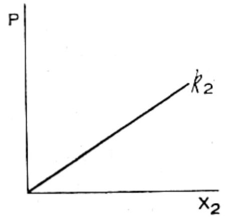


Fig. 1.4.1—Linear elasticity. Class A.

1. Non-linear geometry, such as in the tension spring package described in Section 1.7.

2. Non-linear characteristics of distributed cushioning materials such as excelsior and rubber.

3. Abrupt change of stiffness such as occurs if the packaged item can strike the wall of the container.

For the purpose of developing design formulas it is desirable to have analytical functions to represent load-displacement characteristics. It is not feasible to have only one family of functions with adjustable parameters to fit all possible shapes of load-displacement curves. Therefore, all the practical shapes have been divided into six general classes, most of which are associated with simple functions having one or two adjustable parameters. The six classes are as follows:

Class A—Linear Elasticity. This has already been treated. Its load-displacement function is

$$P = k_2 x_2. \tag{1.4.1}$$

Class B—Cubic Elasticity. This includes cushioning which does not bottom in the anticipated range of use, but the slope of the load-displacement function generally increases with increasing displacement as in the curved full line of Fig. 1.4.2. A suitable load-displacement function is

$$P = k_0 x_2 + r x_2^3. \tag{1.4.2}$$

k_0 is the initial spring rate of the cushioning, as shown by the slope of the dashed straight line in Fig. 1.4.2, and r determines the rate of increase of the spring rate. The same function can be used if the slope of the curve decreases gradually with increasing load as shown by the curved dashed line in Fig. 1.4.2. In this case the parameter r is negative.

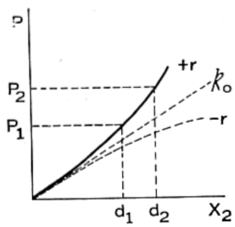


Fig. 1.4.2

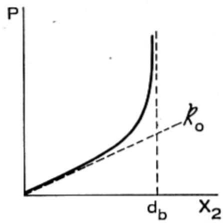


Fig. 1.4.3

Fig. 1.4.2—Cubic elasticity. Class B.
Fig. 1.4.3—Tangent elasticity. Class C.

Class C—Tangent Elasticity. Cushioning that bottoms, but not very abruptly, can be represented by the load-displacement function

$$P = \frac{2k_0 d_b}{\pi} \tan \frac{\pi x_2}{2d_b}. \tag{1.4.3}$$

Referring to Fig. 1.4.3, k_0 is the initial spring rate and d_b is the maximum available displacement. The figure shows how the stiffness of the cushioning (i.e., the slope of the curve) increases as the displacement approaches the maximum available (d_b) at hard bottoming. The shape of the curve is typical of load-displacement curves for a great variety of packages with distributed cushioning.

Figure 1.4.7 illustrates the wide variety of shapes of non-linear cushioning characteristics that can be obtained with the single function given by equation (1.4.3) simply by varying the parameter k_0; and a similar set is given by each value of d_b. Although these families of curves do not include all possible shapes, one of them can usually be found to fit a practical shape for cushioning of this class over the anticipated range of use.

Class D—Bi-linear Elasticity. This is characterized by a load-displacement curve consisting of two straight line segments. The load displacement function is (see Fig. 1.4.4)

$$\left.\begin{array}{ll} P = k_0 x_2 & 0 \leqq x_2 \leqq d_s \\ P = k_b x_2 - (k_b - k_0) d_s & x_2 \geqq d_s \end{array}\right\}. \tag{1.4.4}$$

It is useful especially in situations where very abrupt bottoming is possible.

Class E—Hyperbolic Tangent Elasticity. When the mechanism of the cushioning is such as to limit the maximum force that can be transmitted over a considerable displacement range, the load-displacement function

$$P = P_0 \tanh \frac{k_0 x_2}{P_0} \tag{1.4.5}$$

is useful. P_0 is the asymptotic value of the force and k_0 is the initial spring rate (see Fig. 1.4.5).

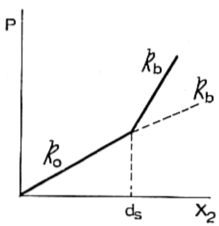


Fig. 1.4.4

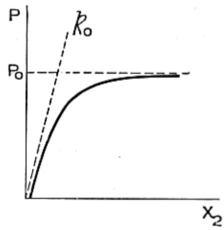


Fig. 1.4.5

Fig. 1.4.4—Bi-linear elasticity. Class D.
Fig. 1.4.5—Hyperbolic tangent elasticity. Class E.

Class F—Anomalous Elasticity. In occasional instances the load-displacement curve of the cushioning cannot be matched accurately enough by any of the five preceding functions. In such cases a numerical integration procedure can be used, as described in Section 1.15.

1.5 Cushioning with Cubic Elasticity (Class B)

Substituting (1.4.2) in (1.2.15) and performing the integration, we have:

$$\frac{k_0 d_m^2}{2} + \frac{r d_m^4}{4} = W_2 h. \tag{1.5.1}$$

Now, let

$$d_0 = \sqrt{\frac{2W_2 h}{k_0}}, \tag{1.5.2}$$

that is, d_0 is the displacement that would take place if the elasticity were linear (see equation (1.3.1)) with a constant spring rate k_0 equal to the initial spring rate of the cubic elasticity. Also let

$$B = \frac{4W_2 hr}{k_0^2}. \tag{1.5.3}$$

Then, from (1.5.1), (1.5.2) and (1.5.3)

$$\frac{d_m}{d_0} = \sqrt{\frac{2}{B}(-1 + \sqrt{1 + B})}. \tag{1.5.4}$$

Equation (1.5.4) is plotted in Fig. 1.5.1 which shows graphically how the maximum displacement d_m compares with the "equivalent linear displacement d_0" as the parameter B is varied. Note that B depends on the weight of the packaged item, the height of drop and the shape of the load displacement curve (as determined by k_0 and r).

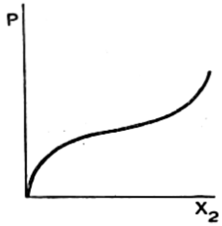


Fig. 1.4.6—Anomalous elasticity. Class F.

Similarly we can compare the maximum acceleration G_m with the maximum (G_0) that would obtain if the load displacement curve were linear with spring rate k_0. The latter acceleration is given by

$$G_0 = \sqrt{\frac{2hk_0}{W_2}} \tag{1.5.5}$$

and the former is obtained by finding P_m from (1.2.17) and then, from (1.2.16),

$$\frac{G_m}{G_0} = \sqrt{\frac{2}{B}(1 + B)(-1 + \sqrt{1 + B})}. \tag{1.5.6}$$

Equation (1.5.6) is plotted in Fig. 1.5.2.

1.6 Procedure for Finding Maximum Acceleration and Displacement for Cushioning with Cubic Elasticity

If the load-displacement curve of a cushioning system has the general appearance of Fig. 1.4.2 (where the slope increases or decreases gradually with displacement) the following procedure may be used for estimating the effectiveness of the cushioning.

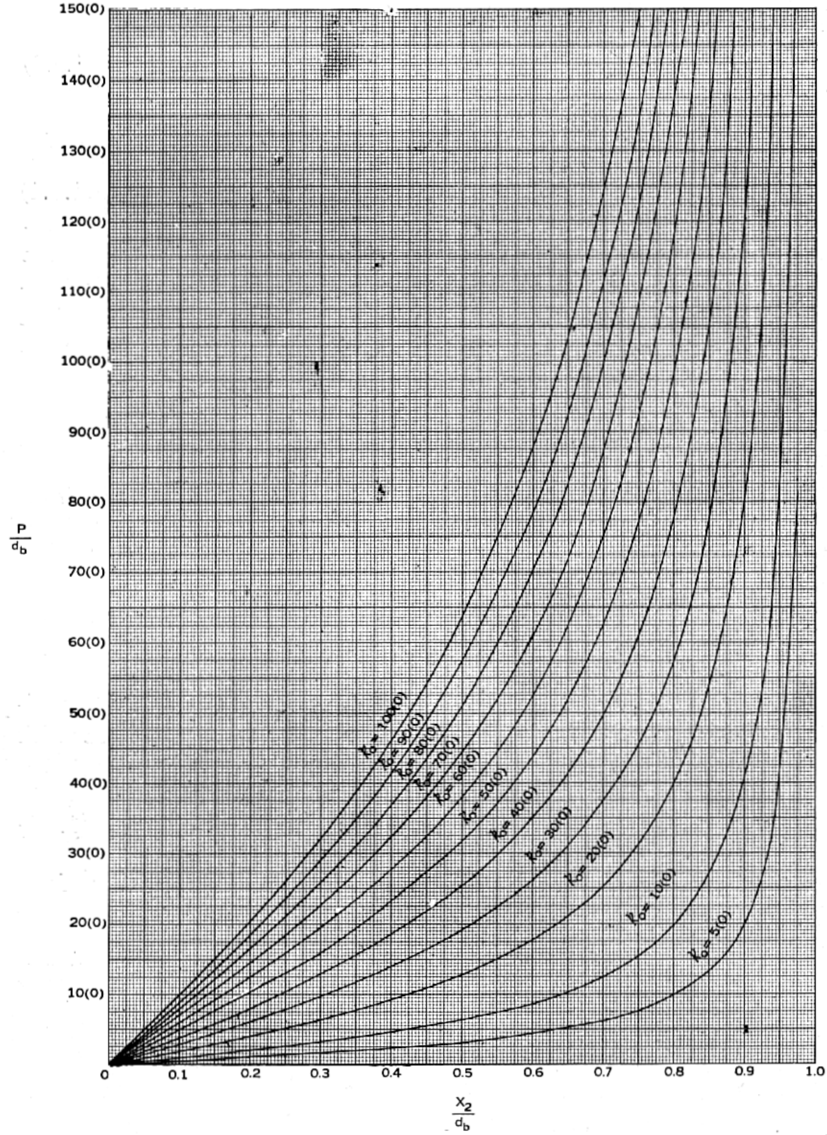


Fig. 1.4.7—Family of load displacement curves for cushioning with tangent elasticity.

a. Select the point on the load-displacement curve for which the load is equal to the weight of the packaged item multiplied by the allowable G_m. Call this load P_2 and the corresponding displacement d_2.

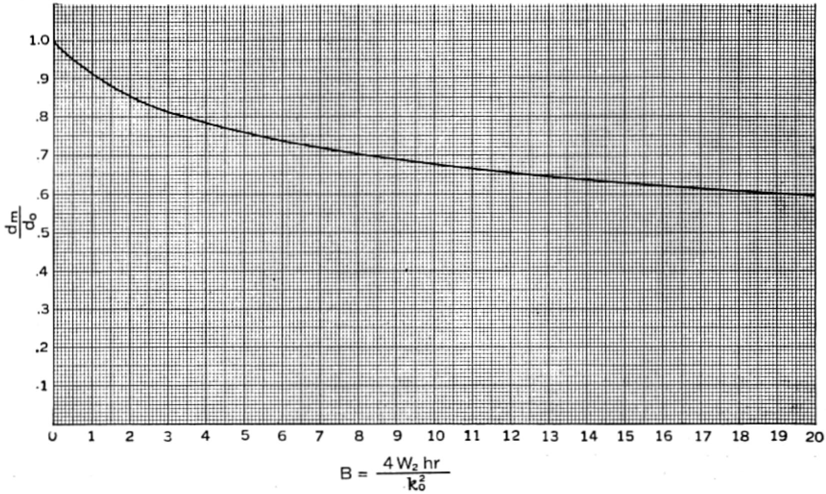


Fig. 1.5.1—Maximum displacement for cushioning with cubic elasticity. See equation (1.5.4).

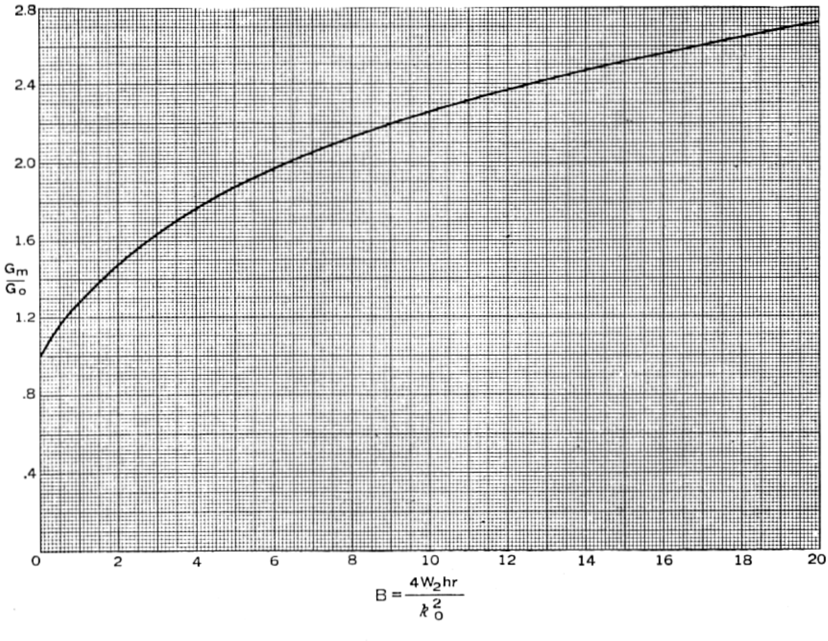


Fig. 1.5.2—Maximum acceleration for cushioning with cubic elasticity. See equation (1.5.6).

b. Select another point (d_1, P_1) about half way toward the origin from (d_2, P_2). See Fig. 1.4.2.

c. Calculate

$$k_0 = \frac{\frac{P_1}{d_1} d_2^2 - \frac{P_2}{d_2} d_1^2}{d_2^2 - d_1^2} \tag{1.6.1}$$

and

$$r = \frac{\frac{P_2}{d_2} - \frac{P_1}{d_1}}{d_2^2 - d_1^2}. \tag{1.6.2}$$

d. Using the known weight, W_2, of the packaged item, the specified height of drop h, and k_0 and r from (1.6.1) and (1.6.2), calculate B, d_0 and G_0 from (1.5.2), (1.5.3) and (1.5.5). Then calculate the maximum acceleration G_m and maximum displacement d_m from (1.5.6) and (1.5.4) or find their values from Figs. 1.5.1 and 1.5.2.

Example: A large vacuum tube, weighing 22.5 lbs, was packed in a 7″ x $7\frac{3}{4}$″ x 15″ carton which was supported on corrugated cardboard spring pads in a $10\frac{1}{2}$″ x $11\frac{1}{2}$″ x $18\frac{1}{2}$″ carton. The latter was, in turn, packed in 28 pounds of excelsior in a 25″ x 25″ x 30″ carton. The tube is rated at 50g and the package is intended for a drop of three feet.

A rod was inserted through a hole cut through the three cartons to the tube. Load was applied to the rod and the displacement of the tube was measured. The data obtained were

P (*load in lbs*)	x_2 (*displacement in inches*)
0	0
100	$\frac{5}{16}$
200	$\frac{5}{8}$
300	$\frac{7}{8}$
400	$1\frac{1}{16}$
500	$1\frac{3}{16}$
600	$1\frac{3}{8}$
700	$1\frac{1}{2}$
800	$1\frac{9}{16}$
900	$1\frac{5}{8}$
1000	$1\frac{3}{4}$
1100	$1\frac{13}{16}$
1200	$1\frac{7}{8}$
1300	$1\frac{15}{16}$
1400	2

The data are plotted in Fig. 1.6.1. The resulting curve is suitable for classification as either Class B or Class C cushioning. Considering it, for the present, as Class B, we take $P_2 = 22.5 \times 50 = 1225$, and from the curve, $d_2 = 1.9$ inches. Also, from the curve, take $d_1 = 1$ inch and $P_1 = 365$

lbs. Substituting these values in (1.6.1) and (1.6.2), we find $k_0 = 255$ and $r = 108$. Then

$$B = \frac{4W_2hr}{k_0^2} = \frac{4 \times 22.5 \times 36 \times 108}{(255)^2} = 5.4,$$

$$G_0 = \sqrt{\frac{2hk_0}{W_2}} = \sqrt{\frac{2 \times 36 \times 255}{22.5}} = 28.6.$$

Entering Fig. 1.5.2 with $B = 5.4$ we find $G_m/G_0 = 1.9$. Hence

$$G_m = 28.6 \times 1.9 = 55$$

This is close enough to the 50g rating of the tube to call the cushioning safe insofar as maximum acceleration is concerned.

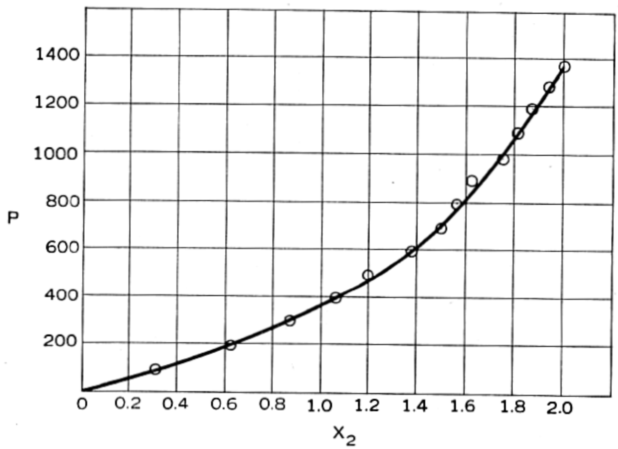


Fig. 1.6.1—Experimental load-displacement curve for a corrugated cardboard spring pad and excelsior cushion.

The maximum displacement, obtained by entering Fig. 1.5.1 with $B = 5.4$ and finding $d_m/d_0 = 0.75$. Then $d_m = 0.75 \times 2h/G_0 = 1.95$ inches. Hence, the package is much larger than necessary since approximately 8 inches of cushioning thickness is supplied to accommodate 2 inches of displacement.

1.7 The Tension-Spring Package (Class B)

The tension spring package is useful when the allowable G_m is so small and height of drop so great that a large displacement (say $d_m >$ several inches) is required. The decision as to whether or not a tension spring package is indicated may be made on the basis of a preliminary estimate of displacement based on the linear case. Suppose the height of drop is to be 60 inches and the allowable acceleration for the packaged item is

20*g*. Then, from Equation 1.3.4, the approximate displacement that will be required is

$$d_m = \frac{2 \times 60}{20} = 6 \text{ inches.} \tag{1.7.1}$$

The actual maximum displacement in a tension spring package will prove to be somewhat more than 6 inches, but the preliminary calculation shows the displacement to be large enough to warrant the use of this type of cushioning.

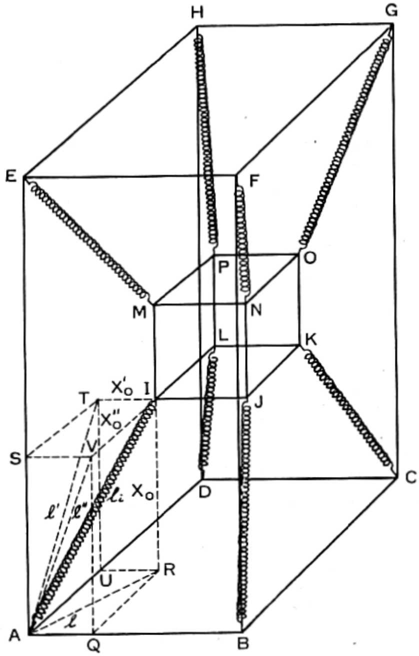


Fig. 1.7.1—Schematic diagram of a tension spring package.

A schematic diagram of a typical tension spring package is shown in Fig. 1.7.1 and a photograph of one design is given in Fig. 1.7.2. The packaged item is suspended on eight identical helical tension springs which diverge to the outer frame. The analysis and design procedures described in this and the following section apply equally well if the springs *converge* from the packaged item to the outer frame. With a slight modification, indicated in the next section, the procedure also applies if four of the springs (say, BJ, DL, EM, OG in Fig. 1.7.1) are omitted.

In all cases, however, we shall consider only systems having reflected symmetry about each of three mutually perpendicular planes through the center of gravity of the packaged article.

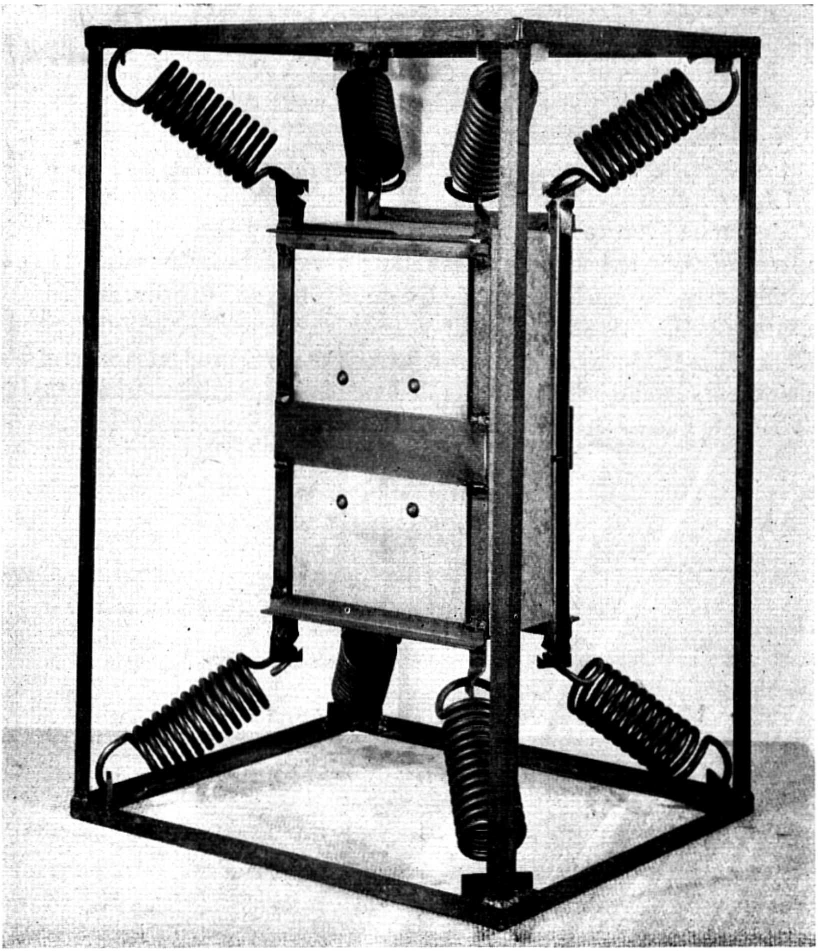


Fig. 1.7.2—A tension spring package.

The load-displacement characteristics of the spring system may be found by statical considerations. We shall examine, first, the displacement in the vertical direction in Fig. 1.7.1, using the following notations:

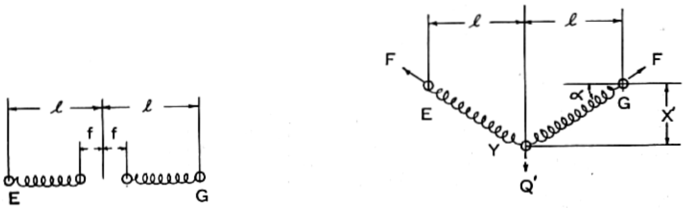
P = force applied to the suspended object,

x_2 = displacement of suspended object,

x_0 = perpendicular distance (IR, Fig. 1.7.1) from inner spring support point (I, Fig. 1.7.1) to nearest plane, perpendicular to displacement direction and containing four outer spring support points (A, B, C, D, Fig. 1.7.1);

ℓ_i = distance (IA) between spring support points when suspended article is in equilibrium position,
ℓ = projection of ℓ_i on plane $ABCD$,
f = ℓ minus length (between hooks) of unstretched spring,
k = spring rate of each spring.

Consider, first, the action of one pair of springs, say EM and GO of Fig. 1.7.1, independent of the remainder of the suspension. Since EM and GO lie in parallel vertical planes and the points M and O remain in the initial planes of their respective springs during a vertical displacement, the two springs may be considered to lie in the same plane, and to be translated horizontally in this plane so that their outer ends are separated by a distance 2ℓ. Hence Fig. 1.7.3 may be used to represent the independent action of this pair of springs and it is required to find the force Q' needed to transform Fig. 1.7.3(a) to Fig. 1.7.3(b). Initially there are two springs, each of length $\ell - f$



a b

Fig. 1.7.3—Diagram used in discussion of tension spring package.

and spring constant k, with no initial tension in them. One end of one spring is fixed at point E and one end of the other spring is fixed at a point G distant 2ℓ from E. The springs are then stretched so that the two initially free ends are located at a point Y equidistant from E and G and distant x_2' from line EG. The axis of each spring makes an angle α with EG, where

$$\sin \alpha = \frac{x_2'}{\sqrt{\ell^2 + x_2'^2}}. \tag{1.7.2}$$

In this state the axial force F in each spring is

$$F = k[\sqrt{\ell^2 + x_2'^2} - \ell + f] \tag{1.7.3}$$

and the force Q', required to equilibrate the two forces F is $2F \sin \alpha$. Considering the force Q' as a function of the displacement x_2', we write

$$Q'(x_2') = \frac{2x_2' k}{\sqrt{\ell^2 + x_2'^2}} [\sqrt{\ell^2 + x_2'^2} - \ell + f] \tag{1.7.4}$$

or

$$Q'(z') = 2k\ell\left[z' + \frac{(1-b)z'}{\sqrt{1+z'^2}}\right] \tag{1.7.5}$$

where

$$z' = \frac{x_2'}{\ell}, \qquad b = \frac{f}{\ell}.$$

Consider, next, the configuration shown in Fig. 1.7.4(a), where one end of each of four springs is fixed at a corner of a rectangle of length 2ℓ and width $2x_0$. Each spring is again of length $\ell - f$. The four free ends of the springs are drawn together at a common point X at the center of the rectangle (see Fig. 1.7.4(b)). The system is in equilibrium in this position. A force Q is then applied at X in the plane of the rectangle and normal to the side 2ℓ.

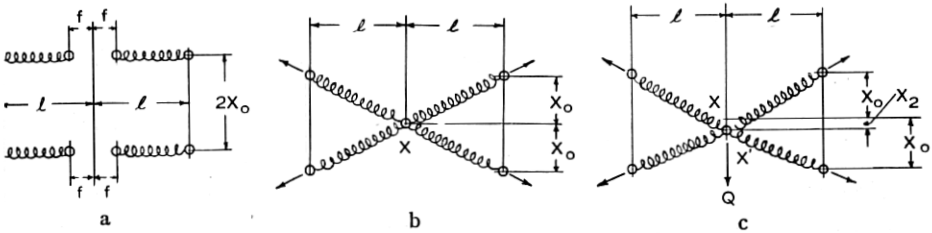


Fig. 1.7.4—Action of springs in a tension spring package.

The common point X is displaced a distance x_2 to X' (see Fig. 1.7.4(c)). Writing $z = x_2/\ell$, $a = x_0/\ell$, we observe that

$$Q(z) = Q'(z+a) + Q'(z-a), \tag{1.7.6}$$

or, from equation (1.7.5),

$$Q(z) = 2k\ell\left\{2z - (1-b)\left[\frac{z+a}{\sqrt{1+(z+a)^2}} + \frac{z-a}{\sqrt{1+(z-a)^2}}\right]\right\}. \tag{1.7.7}$$

The standard tension spring package has two sets of four springs so that the force P required to displace the common point X a distance x_2 is

$$P(z) = 2Q(z). \tag{1.7.8}$$

If x_2 is small in comparison with ℓ (i.e., z is small in comparison with unity), equation (1.7.8) may be written approximately as

$$P(z) = 4k\ell\left[2z - \frac{(1-b)z(2-z^2)}{(1+a^2)^{3/2}}\right]. \tag{1.7.9}$$

Even when x_2 becomes almost as large as ℓ, equation (1.7.9) has been found, experimentally, to be remarkably accurate.

Writing

$$K = 8k\left[1 - \frac{1-b}{(1+a^2)^{3/2}}\right] \tag{1.7.10}$$

and

$$c = \frac{(1 + a^2)^{3/2}}{1 - b} - 1, \tag{1.7.11}$$

equation (1.7.9) becomes

$$P = K\ell\left(z + \frac{z^3}{2c}\right). \tag{1.7.12}$$

It is seen, by comparison with (1.4.2) that this is Class B cushioning (cubic elasticity). K is the initial spring rate and c determines the rate of increase of stiffness with displacement. With the notation k_0, r of Section 1.5, we see that

$$k_0 = K \tag{1.7.13}$$

$$r = \frac{K}{2c\ell^2}. \tag{1.7.14}$$

Hence equations (1.5.6) and (1.5.4) may again be used to calculate maximum acceleration and displacement. B has the same meaning as before (Eq. 1.5.3).

To predict the performance, in the vertical direction (Fig. 1.7.1), of an existing tension spring package the same procedure as outlined in Section 1.6 may be used, except that it is not necessary to have a load-displacement curve for calculating k_0 and r. Instead, these parameters may be calculated directly from equations (1.7.10), (1.7.11), (1.7.13) and (1.7.14). The remainder of the procedure is the same as in Section 1.6(d).

To predict the performance perpendicular to another face, say $AEHD$ of Fig. 1.7.1, it is only necessary, in the calculation of k_0 and r, to substitute x_0' for x_0, ℓ' for ℓ (see Fig. 1.7.1) and, in place of b:

$$b' = 1 - \frac{\ell}{\ell'}(1 - b). \tag{1.7.15}$$

The initial spring rate K for any direction of acceleration may be calculated from the initial spring rates K_1, K_2, K_3 in the three directions normal to the faces of the frame by using the relation

$$\frac{1}{K^2} = \frac{s^2}{K_1^2} + \frac{t^2}{K_2^2} + \frac{u^2}{K_3^2}, \tag{1.7.16}$$

where s, t, u are the direction cosines of the acceleration direction with respect to the normals to the faces of the frame. It is seen, from (1.7.16), that the spring rate is given by the radius to the surface of an ellipsoid whose principal semi-axes are K_1, K_2, K_3.

The displacement direction does not necessarily coincide with the acceleration direction. The angle θ between them is given by

$$\cos\theta = K\left[\frac{s^2}{K_1} + \frac{t^2}{K_2} + \frac{u^2}{K_3}\right] \tag{1.7.17}$$

where K is defined by equation (1.7.16).

The spring characteristics may be made the same in all directions and the displacement direction may be made to coincide with the acceleration direction by setting

$$x_0 = x_0' = x_0'' = \frac{\ell}{\sqrt{2}}$$

and

$$\ell = \ell' = \ell''$$

(see Fig. 1.7.1). This makes $b = b' = b''$, $c = 0.828$ and $k/K = 0.274$ in the calculations of the next section.

1.8 Procedure for Designing Tension Spring Packages

The design of a tension spring package, as contrasted with the analysis of one, must proceed without initial knowledge of values for the parameters k_0 and r, since these cannot be known until the springs are designed. Therefore equations (1.5.4) and (1.5.6) cannot be used directly. For design purposes they are transformed to the following set of formulas:

$$\sqrt{c} = \sqrt{\frac{(1 + a^2)^{3/2}}{1 - b}} - 1 \tag{1.8.1}$$

$$L = \frac{h}{\sqrt{c}\,\ell G_m} = \frac{1}{8}\sqrt{(\sqrt{N} + \sqrt{N+8})^2 + \frac{16}{N}} \tag{1.8.2}$$

$$\frac{d_m}{\sqrt{c}\,\ell} = \sqrt{2(-1 + \sqrt{1 + B})} \tag{1.8.3}$$

$$M^2 = N = \frac{W_2 G_m^2}{2hK} = \frac{2}{B}(1 + B)(-1 + \sqrt{1 + B}) \tag{1.8.4}$$

$$\frac{e}{\ell} - b = -1 + \sqrt{1 + \frac{d_m + x_0}{\ell}}. \tag{1.8.5}$$

These formulas have been converted to design curves which are given in Figs. 1.8.1 to 1.8.5. The curves are for use in connection with the following routine procedure which has been found useful in designing the springs for tension spring packages. Reference should be made to Table I.

1. Enter, on Line 1, Table I, the weight (W_2) in pounds, of the suspended item. This includes the weight of the cradle or other holding arrangement and one-third the estimated weight of the springs.

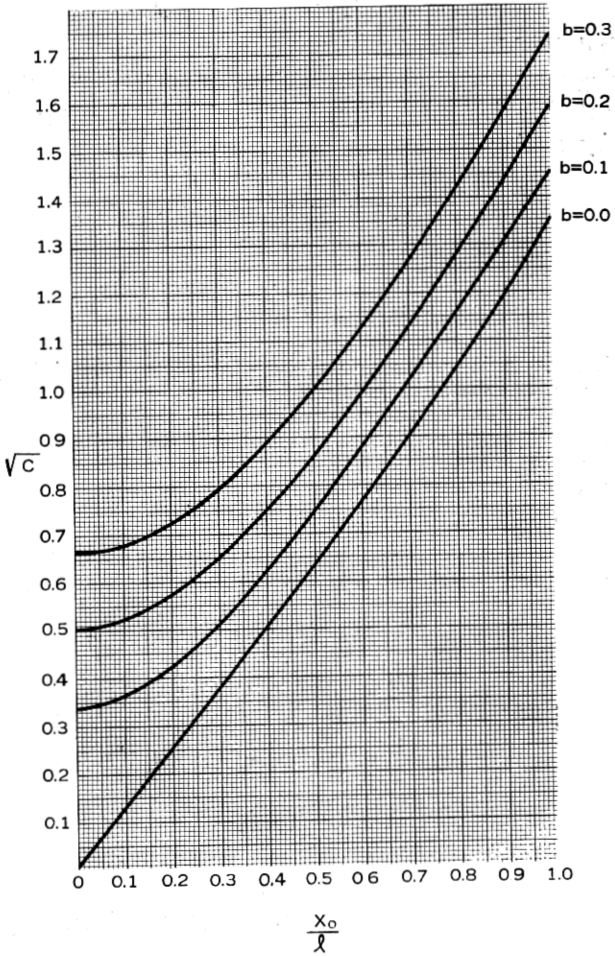


Fig. 1.8.1—Tension spring package design curve. Equation (1.8.1).

2. Enter, on Line 2, the height of drop (h) in inches.
3. Enter, on Line 3, the maximum allowable acceleration (G_m) in units of "number of times gravity." This should be determined beforehand from tests on the item to be packaged.
4. Enter, on Line 4, the dimension x_0 (inches).

5. Enter, on Line 5, the dimension ℓ (inches). For a package to have the same spring rate in all directions, $\ell = x_0\sqrt{2}$ is a necessary condition.
6. Enter, on Line 6, the value chosen for b. As b becomes greater than zero, the stiffness of the whole suspension increases for a given stiffness of individual springs. The reverse happens for b less than zero.

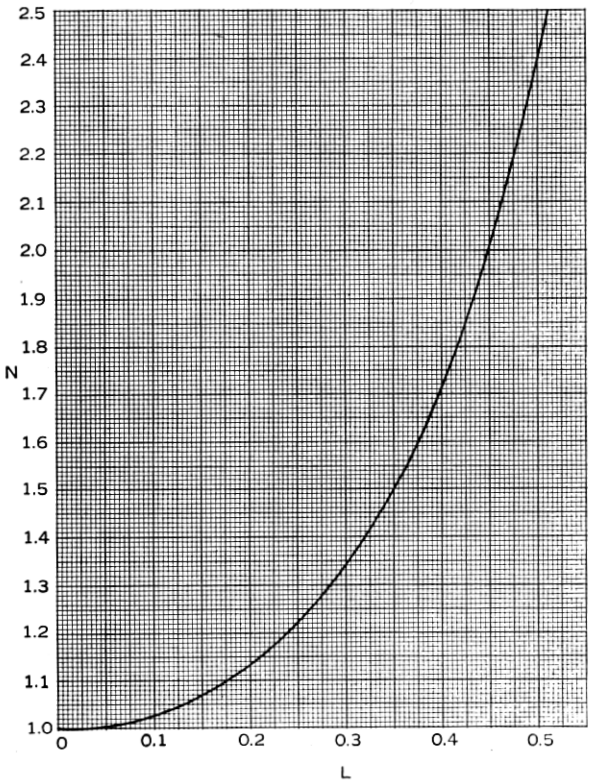


Fig. 1.8.2—Tension spring package design curve. Equation (1.8.2).

7. Calculate x_0/ℓ.
8. Enter Fig. 1.8.1 with x_0/ℓ and find $\sqrt{c}$.
9. Calculate $L = h/(\sqrt{c}\,\ell G_m)$.
10. Enter Fig. 1.8.2 with L and find N.
11. Calculate $K = (W_2 G_m^2)/(2hN)$. This is the initial spring rate of the suspension in the direction of x_0.
12. Calculate $f = 3.13\,(K/W_2)^{\frac{1}{2}}$. This is the natural frequency of vibration (cycles per second) of the suspension for small amplitudes in the x_0 direction. This should not be close to the natural frequency of

vibration of any element in the packaged item, which should be determined by test beforehand (see, also, Section 4.2). *In any case it is advisable to provide damping for the suspension.*

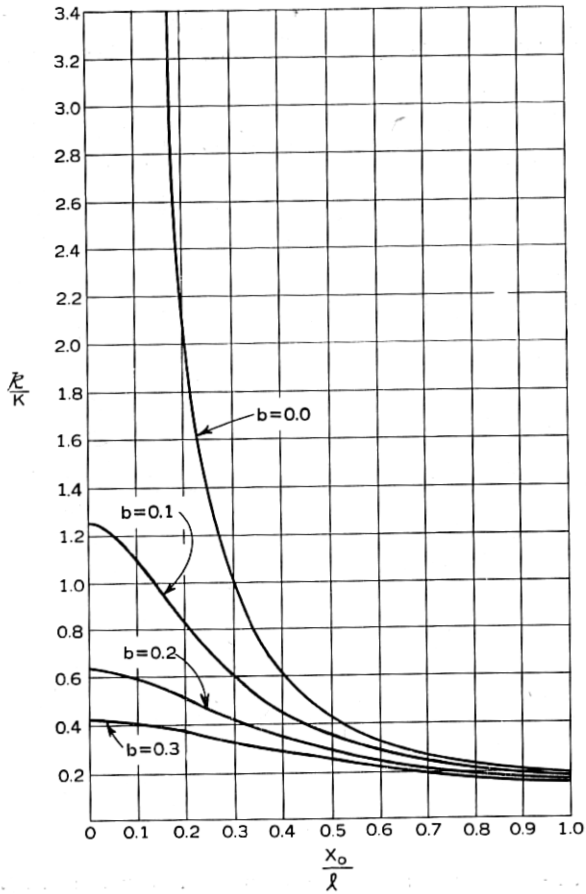


Fig. 1.8.3—Tension spring package design curve. Equation (1.7.10).

13. Enter Fig. 1.8.3 with x_0/ℓ and find k/K. If a four-spring package is desired, instead of an eight-spring package, (see Section 1.7) the value of k/K found on Fig. 1.8.3 should be multiplied by two before entering it on Line 13 in Table I. This is the only change required in the procedure.

14. Calculate $k = K\left(\frac{k}{K}\right)$. This is the spring rate of each of the springs in pounds per inch.

15. Calculate $B = (2W_2h)/(Kc\ell^2)$.
16. Enter Fig. 1.8.4(a) or (b) with B and find $d_m/(\sqrt{c}\,\ell)$.
17. Calculate $d_m/\ell = \sqrt{c} \cdot d_m/\sqrt{c}\,\ell$.
18. Calculate $d_m = \ell \cdot d_m/\ell$.
19. Calculate $(d_m/\ell) + (x_0/\ell)$.

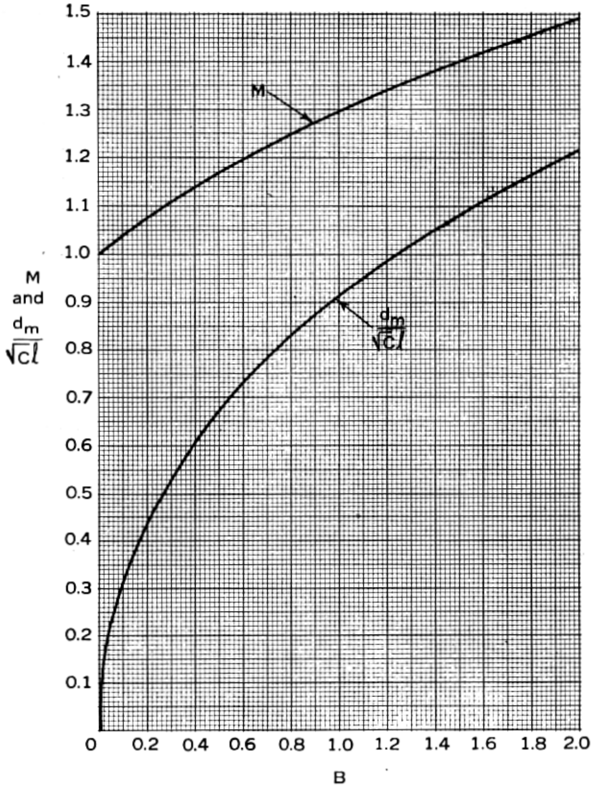


Fig. 1.8.4(a)—Tension spring package design curve. Equations (1.8.3) and (1.8.4).

20. Enter Fig. 1.8.5 with $(d_m/\ell) + (x_0/\ell)$ and find $(e/\ell) - b$. e is the stretch of each spring (in inches) when the displacement is d_m inches.
21. Calculate $F_m = k \cdot (e/\ell) \cdot \ell$. This is the maximum load (in pounds) on each spring.

22, 23, 24, 25, 26. These are the coil diameter, wire diameter, number of turns, fiber stress and length of coils. These quantities are calculated from the ordinary formulas, charts or slide rules for helical springs, using the values of k and F_m from Lines 14 and 21.

27. The length inside hooks is entered on Line 27 to group all of the spring specifications.

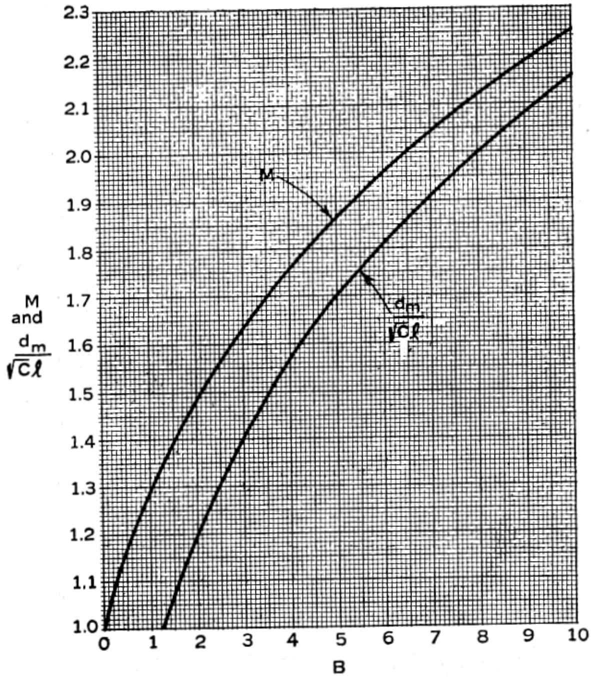


Fig. 1.8.4(b)—Tension spring package design curve. Equations (1.8.3) and (1.8.4).

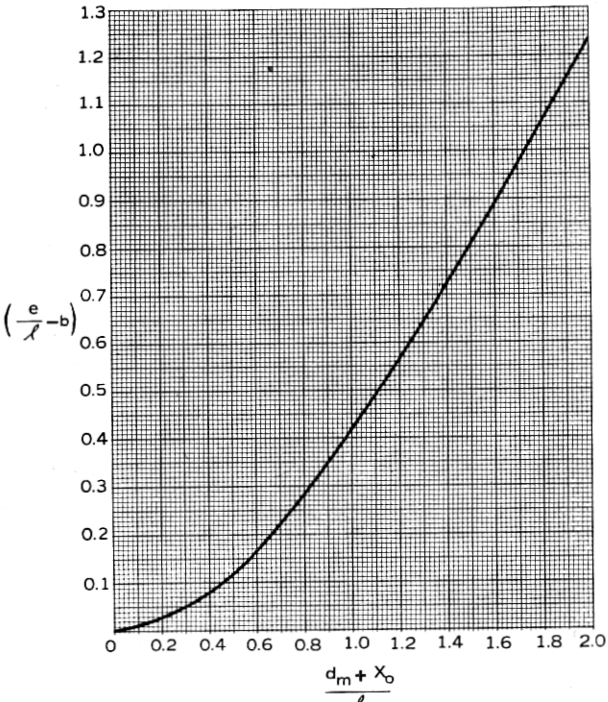


Fig. 1.8.5—Tension spring package design curve. Equation (1.8.5).

As an example of the calculations, Table I contains the entries for the design of springs for a 21 pound article (including $\frac{1}{3}$ the estimated spring weight) which is to be packaged so as not to exceed 35g in a five-foot drop.

TABLE I

Computation Form for Tension Spring Packages

1. W_2 (lbs.)	21
2. h (ins.)	60
3. G_m	35
4. x_0 (ins.)	5
5. ℓ (ins.)	7.07
6. b	0
7. Calc. x_0/ℓ	0.707
8. Find $\sqrt{c}$ from Fig. 1.8.1	0.91
9. Calc. $h/\sqrt{c}\,\ell G_m = L$	0.269
10. Find N from Fig. 1.8.2	1.265
11. Calc. $W_2G_m{}^2/2hN = K$ (lbs/in.)	169.0
12. Calc. $f = 3.13\ (K/W_2)^{\frac{1}{2}}$ (cyc./sec.)	8.9
13. Find k/K from Fig. 1.8.3	0.274
14. Calc. $k = K \cdot k/K$ (lbs/in.)	46.5
15. Calc. $B = 2W_2h/Kc\ell^2$	0.368
16. Find $d_m/\sqrt{c}\,\ell$ from Fig. 1.8.4	0.575
17. Calc. $d_m/\ell = \sqrt{c} \cdot d_m/\sqrt{c}\ell$	0.518
18. Calc. $d_m = \ell \cdot d_m/\ell$ (ins.)	3.68
19. Calc. $d_m/\ell + x_0/\ell$	1.220
20. Find e/ℓ from Fig. 1.8.5 and line 6	0.580
21. Calc. $F_m = k \cdot e/\ell \cdot \ell$ (lbs)	191.0
22. Coil diameter (ins.)	1.40
23. Wire diameter (ins.)	0.207
24. Number of turns	19
25. Fiber Stress (lbs./sq. in.) $\cdot\ 10^{-3}$	80
26. Length of Coils (ins.)	3.93
27. Length inside hooks (ins.)	7.07

1.9 Cushioning with Tangent Elasticity (Class C)

This is one of the most frequently encountered classes of cushioning since it includes a very common type of bottoming (Figs. 1.4.3 and 1.4.7). The load-displacement function (equation (1.4.3)) takes into account the fact that the cushioning can be compressed only to a definite amount d_b.

To find formulas for maximum acceleration and displacement, we proceed as follows. Substitute equation (1.4.3) in (1.2.15) and perform the integration, obtaining

$$\frac{4k_0\,d_b^2}{\pi^2}\log\cos\frac{\pi d_m}{2d_b} = -W_2h, \tag{1.9.1}$$

which may be written as

$$\tan\frac{\pi d_m}{2d_b} = \sqrt{\exp\left(\frac{\pi^2\,W_2\,h}{2k_0\,d_b^2}\right) - 1}\,. \tag{1.9.2}$$

Equation (1.9.2) can then be substituted into (1.4.3) to obtain the maximum force P_m in accordance with (1.2.17):

$$P_m = \frac{2k_0 d_b}{\pi} \sqrt{\exp\left(\frac{\pi^2 W_2 h}{2k_0 d_b^2}\right) - 1}. \tag{1.9.3}$$

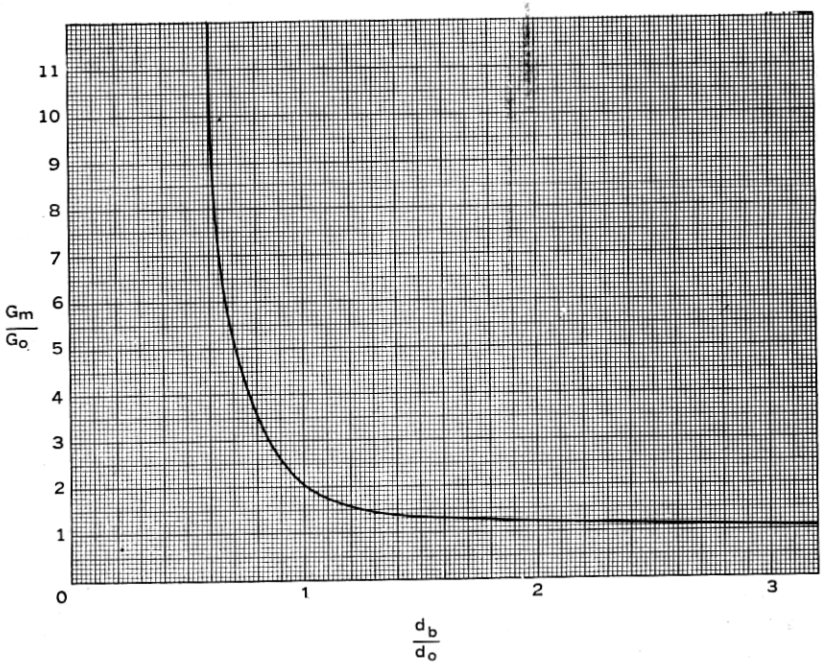


Fig. 1.9.1—Curve for finding maximum acceleration for cushioning with tangent elasticity. See equation (1.9.4).

The maximum acceleration is then obtained from (1.2.16) and may be written in the form

$$\frac{G_m}{G_0} = \frac{2d_b}{\pi d_0} \sqrt{\exp\left(\frac{\pi d_0}{2d_b}\right)^2 - 1}, \tag{1.9.4}$$

where d_0 and G_0 are defined just as in (1.5.2) and (1.5.5). G_0 is the maximum acceleration that would obtain if the cushioning did not bottom, that is, if the spring rate remained constant at its initial value k_0. d_0 is the maximum displacement that would be reached under the same linear conditions. Hence G_m/G_0 is a multiplying factor to be applied to a hypothetical linear cushioning to take into account the effect of bottoming. The multiplying factor depends only on the ratio (d_b/d_0) of the amount of space actually available to the amount of space that would be used under linear conditions.

The ratio G_m/G_0 is plotted against the ratio d_b/d_0 in Fig. 1.9.1. It may be seen that the multiplying factor increases very rapidly as the displacement ratio (d_b/d_0) falls below unity. For example, if the cushioning, with tangent elasticity, reaches hard bottoming (d_b) when only 80% of the required displacement (d_0) is attained, the acceleration is multiplied by 3.5; if only 60% of the required displacement is available, the acceleration is multiplied by 11.5.

Example: To illustrate with a numerical example, consider the case already discussed in Section 1.3, where we found that a spring rate of 694 lbs/in and a displacement of 1.44 inches were required to limit a 20-pound article

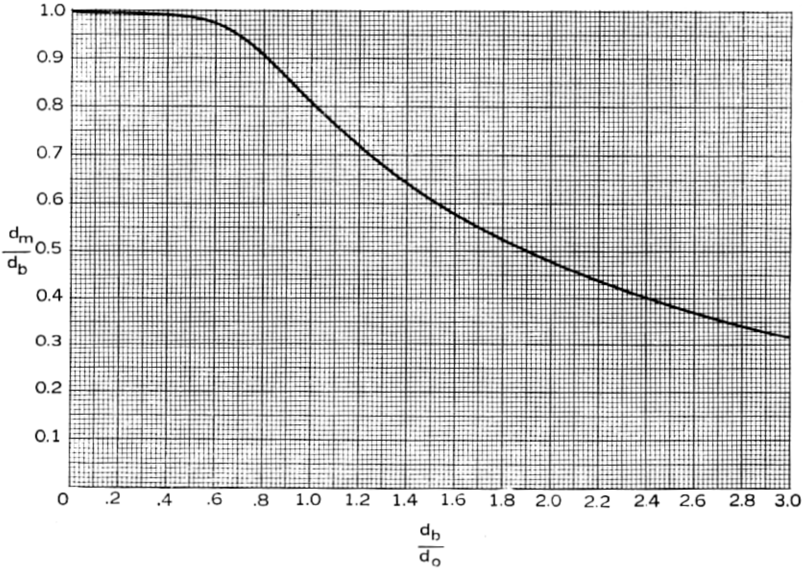


Fig. 1.9.2—Curve for finding maximum displacement for cushioning with tangent elasticity. See equation (1.9.5).

to an acceleration of $50g$ in a 3-foot drop with linear cushioning. Let us suppose that only 1.15 inches are available, instead of 1.44 inches, and that the cushioning has tangent elasticity starting with a spring rate of 694 lbs./in. Entering the curve of Fig. 1.9.1 at $d_b/d_0 = 1.15/1.44$ we find $G_m/G_0 = 3.5$. Hence the maximum acceleration will be $175g$ instead of the required $50g$. This illustrates the wide variations in acceleration that may occur as a result of minor variations of dimensions in high G packages.

It is not necessarily true that the $175g$ test is 3.5 times as severe as the $50g$ test for all elements of the supported structure, since the severity depends also on the shape and scale of the acceleration-time relation. The factor may be more or less than 3.5 but it will be very close to this value for all high-frequency elements of the structure. This subject is treated in detail in Parts II and III.

The maximum displacement d_m, in the case of tangent elasticity, may be calculated from equation (1.9.2) or, in terms of d_b/d_0, from

$$\frac{d_m}{d_b} = \frac{2}{\pi} \cos^{-1} \exp\left[-\frac{\pi^2}{8}\left(\frac{d_0}{d_b}\right)^2\right]. \quad (1.9.5)$$

The ratio d_m/d_b is plotted against d_b/d_0 in Fig. 1.9.2.

The use of Fig. 1.9.2 can be illustrated with the example already calculated, in which $d_b/d_0 = 1.15/1.44 = 0.8$. Entering the abscissa of Fig. 1.9.2 with $d_b/d_0 = 0.8$ we find $d_m/d_b = 0.915$. Hence the maximum displacement will be $0.915 \times 1.15 = 1.05$ inches.

1.10 Optimum Shape of Load-Displacement Curve for Tangent Elasticity

It is possible to choose the best shape for the load-displacement curve of the cushioning from those represented in Fig. 1.4.7. This will be, of course, not the best of all possible curves, but only the best among "tangent elasticity" curves. The best shape is defined as the one that yields the smallest maximum acceleration (G_m) for a given weight (W_2), height of drop (h) and available space (d_b). This leaves the initial spring rate (k_0) as the only remaining variable. To find its optimum value (say k_0'), set equal to zero the derivative of G_m (equation (1.9.4)) with respect to k_0, remembering that G_0 and d_0 are functions of k_0. The result is

$$\left(1 - \frac{\pi^2 W_2 h}{4d_b^2 k_0'}\right) \exp\left(\frac{\pi^2 W_2 h}{2d_b^2 k_0'}\right) - 1 = 0, \quad (1.10.1)$$

from which

$$k_0' = \frac{3.1 W_2 h}{d_b^2}. \quad (1.10.2)$$

Substituting (1.10.2) in (1.9.4) we find the minimum value (G_m') of maximum acceleration to be

$$G_m' = \frac{3.9h}{d_b}. \quad (1.10.3)$$

To illustrate the application of equations (1.10.2) and (1.10.3), consider again the case of the 20-pound article dropped from a height of three feet. We found that a linear spring, with a spring constant of 694 lbs/in, would limit the maximum acceleration to 50g if 1.44 inches of displacement were available. If only 1.15 inches of displacement are available, and the initial spring rate is kept at 694 lbs/in, we found the maximum acceleration to be

175g if the cushioning bottoms with tangent elasticity. Now, according to equation (1.10.2), the best initial spring rate for cushioning with tangent elasticity would be

$$k_0' = \frac{3.1 \times 20 \times 36}{(1.15)^2} = 1690 \text{ lbs/in.}$$

In this case, equation (1.10.3) gives, for the maximum acceleration,

$$G_m' = \frac{3.9 \times 36}{1.15} = 122\text{g}.$$

Hence, confronted with a space limitation less than that required for a 50g linear spring, it is better to use an initial spring rate higher than that for the 50g linear spring in order to strike an economical balance between displacement and bottoming. The best balance, among cushionings having tangent elasticity, is obtained by using equation (1.10.2).

If no factor of safety is considered, it would be still better not to use a bottoming type of cushion at all. From equations (1.3.5) and (1.3.3) it can be seen that a linear spring with a constant of 1090 lbs/in will give only 63g with a displacement of 1.15 inches. Such a spring, though, would bottom very sharply at a drop slightly higher than 3 ft. and would give an acceleration much greater than cushioning with tangent elasticity which bottoms more gradually. This may be important if there are high-frequency, brittle elements in the packaged article (see Part III).

1.11 Procedure for Finding Maximum Acceleration and Displacement for Cushioning with Tangent Elasticity (Class C)

To illustrate the use of the equations and curves for Class C cushioning, the same example used for Class B will be used, as it was observed that the experimental load-displacement curve in that example (Fig. 1.6.1) is a border line one which can be treated as either B or C.

By laying a straight edge along the first part of the curve (Fig. 1.6.1), the average initial spring rate is found to be 305 lbs/in. This value is taken as k_0 in the present case.

The next step is to find a value of d_b such that a graph of P/d_b vs x_2/d_b will fall slightly above the curve $k_0 = 30(0)$ in Fig. 1.4.7; d_b must be greater than 2 inches, since that displacement was obtained in the experiment. As a trial take $d_b = 2.25$ inches and test it at one point, say the experimental point $P = 300$ lbs., $x_2 = \frac{7}{8}$ in. Then $P/d_b = 133$ and $x_2/d_b = 0.39$. The point (0.39, 133) falls below the curve $k_0 = 30(0)$ in Fig. 1.4.7. Next try $d_b = 2.5$ inches. In this case, for the experimental point $P =$

300, $x_2 = \frac{7}{8}$, we find $P/d_b = 120$, $x_2/d_b = .35$. This falls slightly above the $k_0 = 30(0)$ curve as required. The whole experimental curve is then plotted to the coordinates $P/2.5$ vs $x_2/2.5$ and is found to fit as closely as necessary. Hence the parameters are adopted as $k_0 = 305$, $d_b = 2.5$.

We can now calculate the maximum acceleration that the tube will receive in, say, a three-foot drop test. First calculate, from equations (1) and (2),

$$d_0 = \sqrt{\frac{2hW_2}{k_0}} = \sqrt{\frac{2 \times 36 \times 22.5}{305}} = 2.31,$$

$$G_0 = \sqrt{\frac{2hk_0}{W_2}} = \sqrt{\frac{2 \times 36 \times 305}{22.5}} = 31.3.$$

Then $d_b/d_0 = 1.08$. Entering Fig. 4 with this value we find $G_m/G_0 = 1.82$. Hence the maximum acceleration is:

$$G_m = 31.3 \times 1.82 = 57g.$$

Finally, entering Fig. 5 with $d_b/d_0 = 1.08$ we find $d_m/d_b = 0.8$. Hence the maximum displacement is $d_m = 0.8 \times 2.5 = 2.0$ inches. This indicates that the load-displacement test was carried far enough to cover the range up to a three-foot drop.

It may be observed that the results obtained, by treating the same data as Class B or Class C cushioning, agree within a few per cent. This is because, in the example chosen, both B and C curves can be made to fit the experimental load-displacement curve.

1.12 Consequences of Abrupt Bottoming (Class D)

It is useful to examine cushioning systems that can bottom more abruptly than Class C cushioning. Abrupt bottoming is possible, for example, in a tension spring package lacking a snubbing device. An estimate of the increase in acceleration can be made by studying the case of bilinear elasticity (Fig. 1.4.4). Here we have a spring rate k_0 up to a displacement d_s, following which the cushioning has a different spring rate k_b. k_0 represents the average spring rate before bottoming and k_b can represent the much greater stiffness of the wall of the container.

If $d_0 > d_s$, that is, if

$$\sqrt{\frac{2W_2 h}{k_0}} > d_s, \tag{1.12.1}$$

the suspended article will bottom and the maximum displacement and acceleration are obtained by using both of the equations (1.4.4) in evaluating the integral in (1.2.15). Thus,

$$\int_0^{d_s} k_0 x_2 \, dx_2 + \int_{d_s}^{d_m} [k_b x_2 - (k_b - k_0) d_s] \, dx_2 = W_2 h. \tag{1.12.2}$$

The remainder of the procedure for finding G_m is the same as before. The value of d_m found from (1.12.2) is substituted for x_2 in the second of

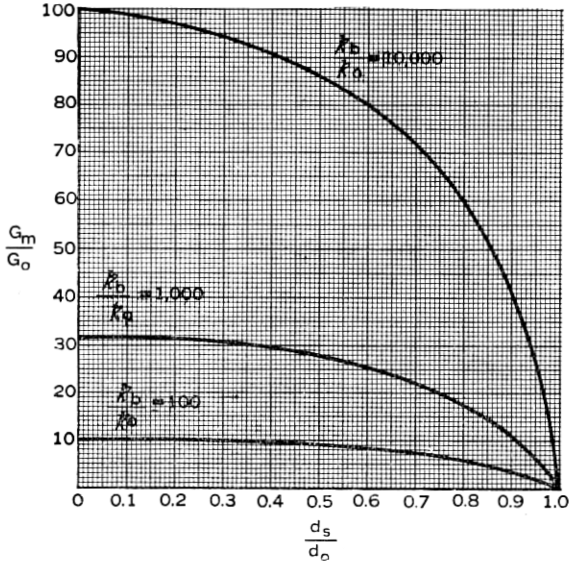


Fig. 1.12.1—Curves for finding maximum acceleration as a result of abrupt bottoming. See equation (1.12.3).

(1.4.4) and the value of P_m, thus obtained, is used, in (1.2.16), with the result:

$$G_m = G_0 \sqrt{\frac{k_b}{k_0} + \frac{d_s}{d_0^2}\left(1 - \frac{k_b}{k_0}\right)}, \tag{1.12.3}$$

where G_0 is the acceleration that would be reached if a displacement d_0 were available:

$$G_0 = \sqrt{\frac{2hk_0}{W_2}}. \tag{1.12.4}$$

The ratio G_m/G_0 is plotted against d_s/d_0 in Fig. 1.12.1 for several values of k_b/k_0. Since, in practice, k_b might be thousands of times as great as k_0, it may be seen that the increase in maximum acceleration can be very large even when d_s is only slightly less than d_0. It is apparent that a snubbing device is desirable in a tension spring suspension. This is especially true when considering high-frequency elements of the packaged article. It will be shown, in Part III, that low-frequency elements are not affected as much as might be expected from consideration of maximum acceleration alone.

1.13 Cushioning with Hyperbolic Tangent Elasticity (Class E)

In the preceding sections, there have been considered four types of elasticity (linear, cubic, tangent and bilinear) that fit the load-displacement characteristics of the more common cushioning materials and devices. There now remains the problem of finding more nearly ideal shapes of elasticity. By "more nearly ideal" is meant a shape which will result in a smaller maximum displacement for a given maximum acceleration. This is important in the packaging of very delicate articles if shipping space is limited.

It may be observed (equation (1.2.15)) that the total area under the load-displacement curve is equal to the maximum energy of the system. The maximum ordinate of the enclosed area is proportional to the maximum acceleration. Hence, if we wish to (1) limit the maximum acceleration (2) accomodate a given kinetic energy and (3) have as small a displacement as possible, the best shape for the load displacement function is $P =$ constant, where the constant is the product of the supported mass and the maximum allowable acceleration.

It is not practical to obtain this ideal shape exactly, for there will always be a finite initial spring rate and a rounding off of the load-displacement curve to the limiting maximum load. A function which represents this practical condition (and also includes the ideal case) is the hyperbolic tangent function mentioned in Section 1.4:

$$P = P_0 \tanh \frac{k_0 x_2}{P_0}. \tag{1.13.1}$$

The formulas for maximum acceleration and displacement are found in the same way as for the other classes of cushioning with the results:

$$d_m = \frac{P_0}{k_0} \cosh^{-1} \exp\left(\frac{W_2 h k_0}{P_0^2}\right) \tag{1.13.2}$$

or

$$d_m = \frac{d_0 P_0}{W_2 G_0} \cosh^{-1} \exp\left(\frac{W_2^2 G_0^2}{2P_0^2}\right) \tag{1.13.3}$$

and

$$G_m = \frac{P_0}{W_2} \tanh \frac{k_0 d_m}{P_0} \tag{1.13.4}$$

or

$$G_m = \frac{P_0}{W_2} \sqrt{1 - \exp\left(-\frac{W_2^2 G_0^2}{P_0^2}\right)} \tag{1.13.5}$$

where, as before

$$d_0 = \sqrt{\frac{2hW_2}{k_0}}, \qquad G_0 = \sqrt{\frac{2hk_0}{W_2}}.$$

Equations (1.13.3) and (1.13.5) are plotted, in Figs. 1.13.1 and 1.13.2, against the dimensionless parameter P_0/W_2G_0. The latter is the ratio of the maximum force, that the hyperbolic tangent cushioning will transmit,

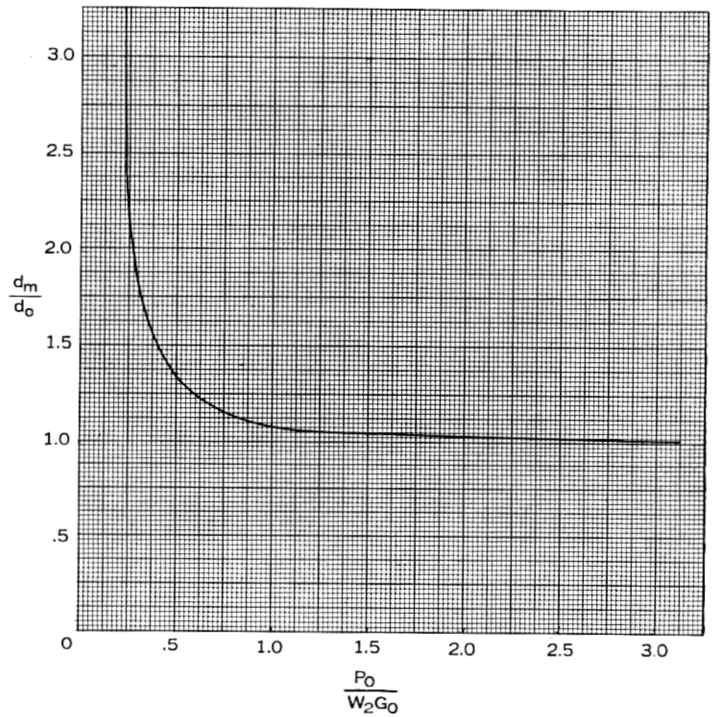


Fig. 1.13.1—Maximum displacement for cushioning with hyperbolic tangent elasticity. See equation (1.13.3).

to the force that linear cushioning would transmit under the conditions specified.

To find the value of k_0 which yields the minimum value of acceleration for a given maximum displacement, differentiate (1.13.4) with respect to k_0 and set the result equal to zero:

$$\operatorname{sech}^2 \frac{k_0 d_m}{P_0} = 0. \tag{1.13.6}$$

This is satisfied by $k_0 \to \infty$, which represents the rectangular load displacement curve and confirms the conclusion reached from energy considerations.

Taking the limit of (1.13.4) as $k_0 \to \infty$, we find the optimum acceleration to be

$$G'_m = \frac{P_0}{W_2}. \tag{1.13.7}$$

The corresponding maximum displacement is found, from (1.13.2) to be

$$d'_m = \frac{W_2 h}{P_0} = \frac{h}{G'_m}. \tag{1.13.8}$$

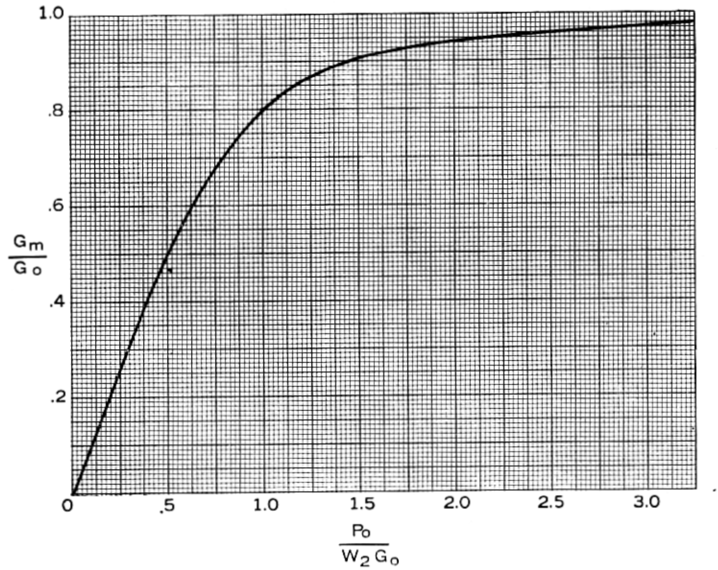


Fig. 1.13.2—Maximum acceleration for cushioning with hyperbolic tangent elasticity. See equation (1.13.5).

1.14 Minimum Space Requirements for Various Classes of Cushioning

It is interesting to compare the minimum amount of space for displacement that can be attained with the various kinds of cushioning that have been discussed.

Hyperbolic Tangent Elasticity	$d'_m = \dfrac{h}{G'_m}$
Linear Elasticity	$d_m = \dfrac{2h}{G_m}$
Tangent Elasticity	$d'_m = \dfrac{3.9h}{G'_m}$

Cubic elasticity will give a d_m somewhat more or less than $2h/G_m$ depending upon whether the parameter r is positive or negative.

It is seen that a factor of almost four can be gained, in the linear dimensions of the cushioning space required, by replacing the tangent type of cushioning with the hyperbolic tangent type.

There are several ways of obtaining a load-displacement curve with a shape similar to the hyperbolic tangent curve. One of the most interesting is suggested by the fact that the load-displacement curve of a strut has approximately this shape. Hence a bristle brush has the proper characteristics.

TABLE II

1	2	3	4	5	6	7	8
n	$\Delta(x_2)_n$	$(x_2)_n$	P_n	$\Delta A_n = \frac{1}{2}\Delta(x_2)_n \times (P_n + P_{n-1})$	$A_n = A_{n-1} + \Delta A_n$	$h_n = \frac{A_n}{W_2}$	$G_n = \frac{P_n}{W_2}$
0	—	0	0	0	0	0	0
1	0.500	0.500	120	30	30	1.6	6.5
2	0.100	0.600	150	13.5	43.5	2.4	8.1
3	0.100	0.700	205	17.0	61.3	3.3	11.1
4	0.100	0.800	290	24.8	86.1	4.7	15.7
5	0.100	0.900	410	35.0	121.1	6.6	22.2
6	0.100	1.000	585	49.8	170.9	9.2	31.6
7	0.050	1.050	730	32.9	203.8	11.1	39.5
8	0.050	1.100	950	42.0	245.8	13.3	51.4
9	0.050	1.150	1370	58.0	303.8	16.4	74.0
10	0.025	1.175	1680	38.1	341.9	18.5	91.0
11	0.025	1.200	2240	49.0	390.9	21.1	121.0
12	0.0125	1.2125	2620	30.4	421.3	22.8	141.5
13	0.0125	1.225	3200	36.4	457.7	24.7	173.0

1.15 Numerical Method for Analyzing Class F Cushioning

The numerical method to be described is one that has been adapted from a graphical one used by the Committee on Packing and Handling of Radio Valves of the British Radio Board. The method has advantages of simplicity in concept and ease of application, especially when the load-displacement curve of the cushioning does not resemble closely one of the Classes A to E described above. It has the disadvantage that it does not yield, directly, numerical factors by which the spring rate or depth of cushioning should be changed in the event that the analysis reveals inadequate or more than adequate protection.

The method is based on the fact that the area under the load-displacement curve of the cushioning represents the energy stored in, or absorbed by, the cushion. The total amount of energy that must be transferred is equal to the product of the weight (W_2) of the suspended item and the height (h) of drop. By finding the abscissa (x_2) and its ordinate (P) which include an area W_2h, the maximum displacement is immediately x_2 and the maximum acceleration is the quotient P/W_2, in accordance with equations (1.2.15) and (1.2.16).

As actually applied in the present instance, the British method was modified slightly to make the procedure a routine numerical one. The

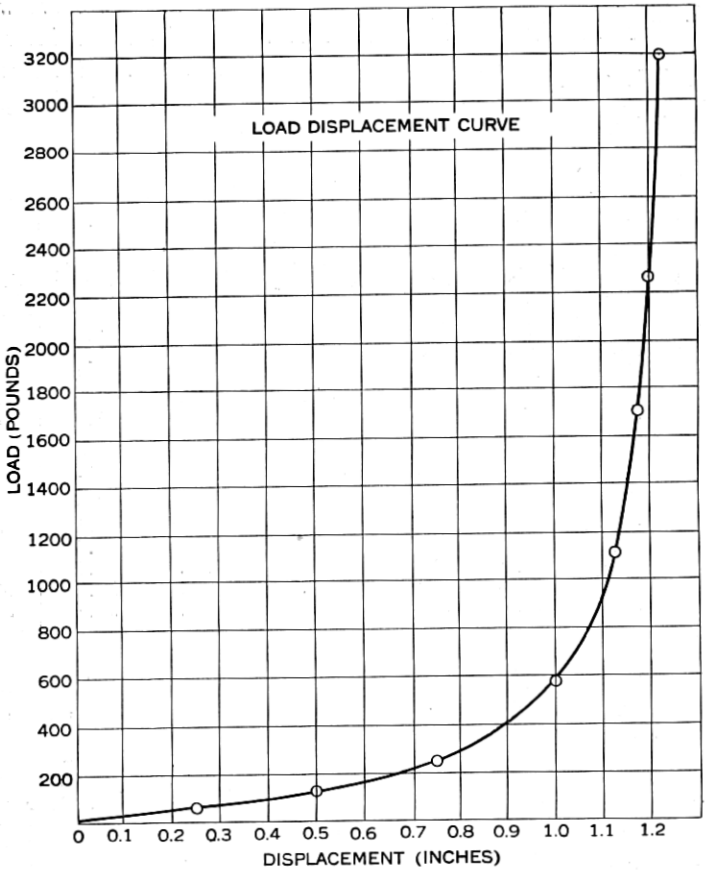


Fig. 1.15.1—Experimental load-displacement curve for Table II.

computing form is given in detail in Table II, in which the data are taken from an experimental load-displacement curve (Fig. 1.15.1) for the end spring pads of a vacuum tube package. The load-displacement data are listed in Columns 3 and 4 of Table II. The meaning of each column in the table is as follows.

Column 1. $n(= 1, 2, 3 \cdots)$ is the number that identifies the displacement (and corresponding load) up to which the area under the load-displacement curve is to be calculated.

Column 2. $\Delta(x_2)_n$ is the increment of displacement between $(x_2)_{n-1}$ and $(x_2)_n$. $\Delta(x_2)_n = (x_2)_n - (x_2)_{n-1}$, see Fig. 1.15.2. Note that, as the curve becomes steeper, $\Delta(x_2)_n$ is taken smaller for better accuracy.

Column 3. $(x_2)_n$ is the displacement associated with the n^{th} point (see Fig. 1.15.2).

Column 4. P_n is the load that produces displacement $(x_2)_n$.

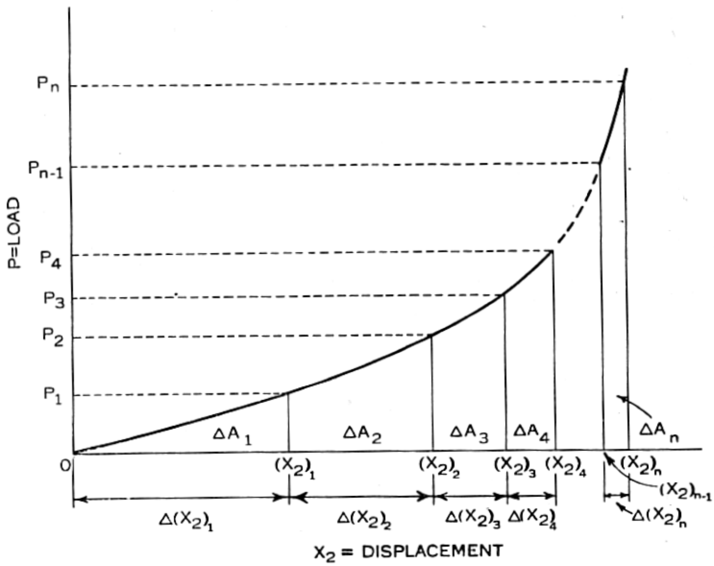


Fig. 1.15.2—Graphical illustration of numerical method of calculating area under load-displacement curve. See Table II.

Column 5. $\Delta A_n = \frac{1}{2}\Delta(x_2)_n(P_{n-1} + P_n)$ is the area of the trapezoid with altitude $\Delta(x_2)_n$ and bases P_{n-1} and P_n. It is approximately the energy absorbed by the cushioning in displacing from $(x_2)_{n-1}$ to $(x_2)_n$.

Column 6. A_n is the sum of all the trapezoidal areas from $x_2 = 0$ to $x_2 = (x_2)_n$. It is approximately the total energy the cushioning can absorb in displacing an amount $(x_2)_n$ beginning at zero displacement. Note that A_0 is always equal to zero.

Column 7. $h_n = A_n/W_2$ is the height of fall that will cause the cushioning to displace an amount $(x_2)_n$. In Table II, $W_2 = 18.5$ pounds.

Column 8. $G_n = P_n/W_2$ is the maximum acceleration experienced by the suspended mass when dropped from a height h_n.

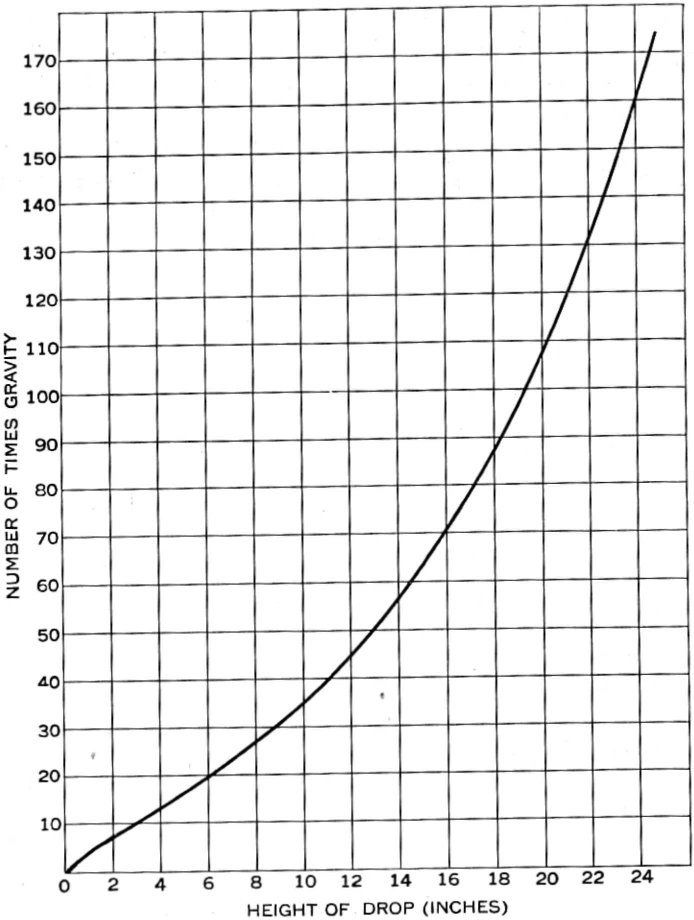


Fig. 1.15.3—Maximum acceleration vs. height of drop for an 18.5 pound article supported on cushioning with the load-displacement curve of Fig. 1.15.1. See Table II.

From the last two columns of the table a curve of height of drop vs. the corresponding acceleration may be plotted as in Fig. 1.15.3.

PART II

ACCELERATION-TIME RELATIONS

2.1 Introduction

In Part I we were concerned primarily with the *maximum* acceleration of the packaged item. In this part we shall study the details of the variation of acceleration with time in order to have this information available for our study, in Part III, of its influence on the response of elements of the packaged item.

The first case to be considered will be the simple single mass and linear spring example described in Sections 1.2 and 1.3. Following this the phenomenon of rebound of the package will be considered. The influence of velocity damping and dry friction will be studied; and, finally, the effects of non-linearity of the cushion elasticity on the acceleration-time relation will be investigated.

2.2 Acceleration-time Relation for Linear Elasticity

Returning to the elementary example studied in Sections 1.2 and 1.3, we first write the equation of motion for the mass m_2, on its linear spring of spring rate k_2 (see Fig. 1.2.1.). Equation (1.2.3) becomes

$$m_2\ddot{x}_2 + k_2x_2 = m_2g. \tag{2.2.1}$$

Using the initial conditions

$$[x_2]_{t=0} = 0, \tag{2.2.2}$$

$$[\dot{x}_2]_{t=0} = \sqrt{2gh}, \tag{2.2.3}$$

the solution of (2.2.1) is

$$x_2 = \frac{\sqrt{g^2 + 2\omega_2^2 gh}}{\omega_2^2} \sin(\omega_2 t - \alpha) + \frac{g}{\omega_2^2} \tag{2.2.4}$$

or

$$x_2 = \sqrt{\frac{W_2^2}{k_2^2} + d_m^2} \sin(\omega_2 t - \alpha) + \frac{W_2}{k_2}, \tag{2.2.5}$$

where

$$\omega_2 = \sqrt{\frac{k_2}{m_2}} = 2\pi f_2 = \frac{2\pi}{T_2} \tag{2.2.6}$$

and

$$\alpha = \tan^{-1}\frac{g}{\omega_2\sqrt{2gh}} = \tan^{-1}\frac{W_2}{k_2 d_m}. \tag{2.2.7}$$

ω_2 is the circular frequency, f_2 is the frequency and T_2 is the period of vibration of the mass m_2 on its spring; d_m has the same definition as in Section 1.3 (equation (1.3.1)).

Now, W_2/k_2 is the static displacement of the mass m_2 on its spring. This is usually very small in comparison with the maximum displacement (d_m) during impact. Hence W_2/k_2 will be neglected, and (2.2.5) becomes

$$x_2 = d_m \sin \omega_2 t. \tag{2.2.8}$$

Differentiating (2.2.8) twice with respect to t, we find, for the acceleration

$$\ddot{x}_2 = -\omega_2^2 d_m \sin \omega_2 t = -\omega_2 \sqrt{2gh} \sin \omega_2 t. \tag{2.2.9}$$

Hence the absolute magnitude of the maximum acceleration is

$$G_m = \frac{|\ddot{x}_2|_{\max}}{g} = \frac{\omega_2^2 d_m}{g} = \sqrt{\frac{2hk_2}{W_2}} \tag{2.2.10}$$

as before.

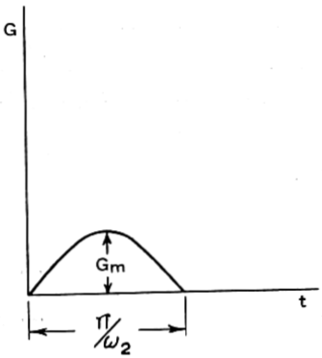


Fig. 2.2.1

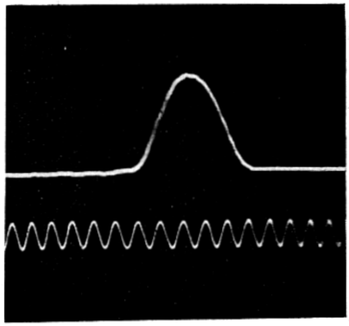


Fig. 2.2.2

Fig. 2.2.1—Half-sine-wave pulse acceleration. See equation (2.2.9).
Fig. 2.2.2—Oscillogram of a half-sine-wave pulse obtained with a piezo-crystal accelerometer.

Equation (2.2.9) shows that the acceleration varies sinusoidally with time. It rises from its initial zero value to its maximum in a time $\pi/2\omega_2$, at which time the displacement also reaches its maximum value. The acceleration returns to zero again at time π/ω_2. At this time the displacement is also zero. This is the end of the range of applicability of equation (2.2.9); for when t becomes slightly greater than π/ω_2, a tension in the spring is required. Since no mechanism, such as a large mass m_3 (Fig. 0.2.2), has been supplied, to allow a tension in the spring to develop, the system will rebound from the floor at the end of the half period π/ω_2. The acceleration is thus a half-sinusoidal pulse of duration $\pi/\omega_2 = T_2/2$ and amplitude $G_m g$ as illustrated in Fig. 2.2.1. An oscillogram of such a pulse obtained with a piezo-crystal accelerometer is shown in Fig. 2.2.2.

2.3 Package Rebound.

The presence of the mass of an outer container will affect the acceleration after the first half cycle of displacement. The outer container is represented by the mass m_3 in the general idealized system illustrated in Fig. 0.2.2 and in the simpler system (Fig. 2.3.1) that we shall consider now.

Two pairs of equations are necessary to describe the action of the system; one pair applies during the time of contact of m_3 with the floor and the second pair applies if rebound occurs.

The mass m_3 is assumed to be inelastic (see Section 0.2) so that, during the interval of its contact with the floor, the equation of motion for m_2 will be the same as before (2.2.1). In addition, there will be an equation of equilibrium for the mass m_3 :

$$R = k_2 x_2 + m_3 g \, , \tag{2.3.1}$$

where R is the upward force exerted by the floor on m_3 .

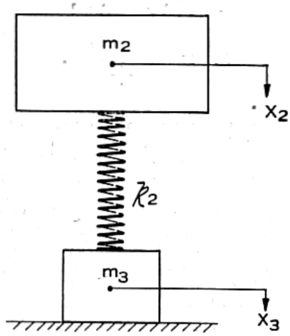


Fig. 2.3.1—Two-mass system representing packaged article, linear cushioning and outer container.

Equations (2.3.1) and (2.2.1) will hold as long as R is positive. To find out when $R > 0$, solve (2.2.1) for $k_2 x_2$ and substitute in (2.3.1):

$$R = W_2 + W_3 - m_2 \ddot{x}_2 \, . \tag{2.3.2}$$

That is, a necessary condition for rebound is that the mass of the cushioned article, multiplied by its maximum acceleration, exceeds the total weight of the package. The condition for rebound may be written

$$G_m > \frac{W_2 + W_3}{W_2} . \tag{2.3.3}$$

This is a necessary, but not a sufficient, condition for rebound because there will be energy losses as a result of damping and permanent deformation. G_m will generally have to be considerably greater than the right hand side of (2.3.3) for rebound to occur.

If rebound does not occur, equation (2.2.9) continues to apply, except for damping which will be considered in Section 2.5.

2.4 Motion After Rebound

If rebound occurs, the equations of motion for the two masses, m_2 and m_3, are

$$m_2\ddot{x}_2 + k_2(x_2 - x_3) = m_2 g, \tag{2.4.1}$$

$$m_3\ddot{x}_3 - k_2(x_2 - x_3) = m_3 g. \tag{2.4.2}$$

Multiplying (2.4.1) by m_3 and (2.4.2) by m_2 and subtracting, we find

$$m\ddot{y} + k_2 y = 0, \tag{2.4.3}$$

where

$$y = x_2 - x_3, \tag{2.4.4}$$

$$m = \frac{m_2 m_3}{m_2 + m_3}. \tag{2.4.5}$$

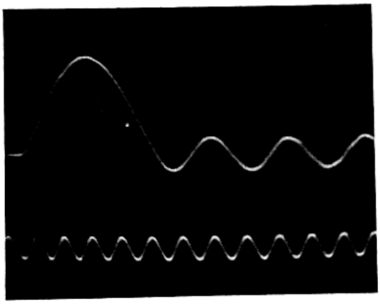


Fig. 2.4.1—Oscillogram illustrating the half-sine pulse followed by the higher frequency, lower amplitude vibration of the packaged article in a rebounding package.

Equation (2.4.3) is the equation governing the vibration of the two-mass system as a simple oscillator. The circular frequency of the vibration is

$$\omega = \sqrt{\frac{k_2}{m}} \tag{2.4.6}$$

and it may be noticed that this frequency is always greater than ω_2 (equation (2.2.6)). This fact is important in estimating the effect of vibrations on elements of the packaged item (Section 3.5).

ω is also the frequency of vibration of the packaged article during the interval of free fall. This vibration (usually of small amplitude) is initiated by the sudden release of the dead load displacement of the packaged article.

As an intermediate step in obtaining the acceleration after rebound we shall find the magnitude of the relative displacement (y) of the two masses. To do this it is necessary to solve equation (2.4.3) with the appropriate boundary conditions. Calling t_r the time at which m_3 leaves the floor, we must find y and $\dot{y}$ at $t = t_r$. Since m_3 is motionless at $t = t_r$, the relative displacement and velocity at that time are identical with x_2 and $\dot{x}_2$ respectively. The former is simply the stretch of the spring necessary to just pull the mass m_3 off the floor, i.e.,

$$[y]_{t=t_r} = [x_2]_{t=t_r} = -\frac{W_3}{k_2}. \tag{2.4.7}$$

To find the velocity at $t = t_r$, substitute (2.4.7) in (2.2.4) and also substitute t_r for t in the latter. This gives an equation for determining t_r. Then, returning to (2.2.4), differentiate it once to obtain $\dot{x}_2$ and substitute for t the value t_r just found. The result is

$$[\dot{x}_2]_{t=t_r} = [\dot{y}]_{t=t_r} = -\sqrt{2gh - \frac{gW_3(2W_2 + W_3)}{W_2 k_2}}. \tag{2.4.8}$$

The solution of (2.4.3) with initial conditions (2.4.7) and (2.4.8) is

$$y = -\frac{1}{\omega}\sqrt{2gh - \frac{W_3 g}{k_2}} \sin(\omega t - \zeta), \tag{2.4.9}$$

where
$$\zeta = \omega t_r - \tan^{-1}\frac{\omega[y]_{t=t_r}}{[\dot{y}]_{t=t_r}}.$$

We are now in a position to find the acceleration of the packaged item after rebound. Substitute y of (2.4.9) for $x_2 - x_3$ in (2.4.1) to obtain

$$\ddot{x}_2 = g + \frac{\omega_2^2}{\omega}\sqrt{2gh - \frac{W_3 g}{k_2}} \sin(\omega t - \zeta). \tag{2.4.10}$$

To obtain a simple formula for the ratio of the maximum accelerations after and before rebound, let us assume that both are much greater than gravitational acceleration. Then if

G_r = maximum number of g's after rebound, (maximum of (2.4.10))

$G_m = \sqrt{\dfrac{2hk_2}{W_2}}$ = maximum number of g's before rebound,

we find, from (2.4.10), neglecting the term g outside the radical,

$$\frac{G_r}{G_m} = \frac{W_3}{W_2 + W_3}\sqrt{1 - \frac{W_3}{2hk_2}}. \tag{2.4.11}$$

Hence, the maximum acceleration after rebound is always less than the maximum acceleration before rebound. Therefore, conditions after rebound need only be examined when the frequency after rebound (see equation (2.4.6)) is near the natural frequency of vibration of a critical element of the packaged item (see Section 3.5).

The complete acceleration history of a rebounding package with undamped linear cushioning is thus a half sine wave pulse of amplitude $G_m = \sqrt{2hk_2/W_2}$ and duration π/ω_2 followed by an oscillating acceleration of amplitude given by (2.4.11) and frequency given by (2.4.6). Such a wave shape is shown in Fig. 2.4.1.

2.5 Influence of Damping on Acceleration

The presence of damping in cushioning is always desirable to prevent the building up of large amplitudes as a result of periodic disturbances. However, damping also has an effect on the maximum acceleration that is attained in a drop test. From the latter point of view there is an optimum amount of damping and an amount that should not be exceeded if the maximum undamped acceleration is not to be exceeded.

We shall consider the case of a linear cushion with damping proportional to velocity. The system is represented in Fig. 2.5.1. With the addition

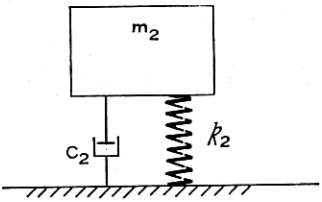


Fig. 2.5.1—Idealization of linear cushioning with velocity damping.

of the damping term the equation of motion of m_2, during contact of the package with the floor, is

$$m_2\ddot{x}_2 + c_2\dot{x}_2 + k_2x_2 = 0, \tag{2.5.1}$$

in which c_2 is the damping coefficient of the cushioning. Equation (2.5.1) is more conveniently expressed as

$$\ddot{x}_2 + 2\beta_2\omega_2\dot{x}_2 + \omega_2^2x_2 = 0, \tag{2.5.2}$$

where

$$\omega_2 = \sqrt{\frac{k_2}{m_2}}, \tag{2.5.3}$$

$$\beta_2 = \frac{c_2}{2m_2\omega_2}. \tag{2.5.4}$$

ω_2 is the undamped circular frequency of vibration of m_2 on its spring and β_2 is the fraction of critical damping. $\beta_2 = 0$ means no damping and $\beta_2 = 1$ means just enough damping so that there will be no oscillation if the packaged article is displaced and released.

The acceleration solution of (2.5.2), with the initial conditions of the drop test (see (2.2.2) and (2.2.3)) is

$$\ddot{x}_2 = -\frac{\omega_2\sqrt{2gh}}{\sqrt{1-\beta_2^2}}\, e^{-\beta_2\omega_2 t} \cos\left(\omega_2 t\sqrt{1-\beta_2^2} + \gamma\right) \tag{2.5.5}$$

where

$$\tan\gamma = \frac{2\beta_2^2 - 1}{2\beta_2\sqrt{1-\beta_2^2}}. \tag{2.5.6}$$

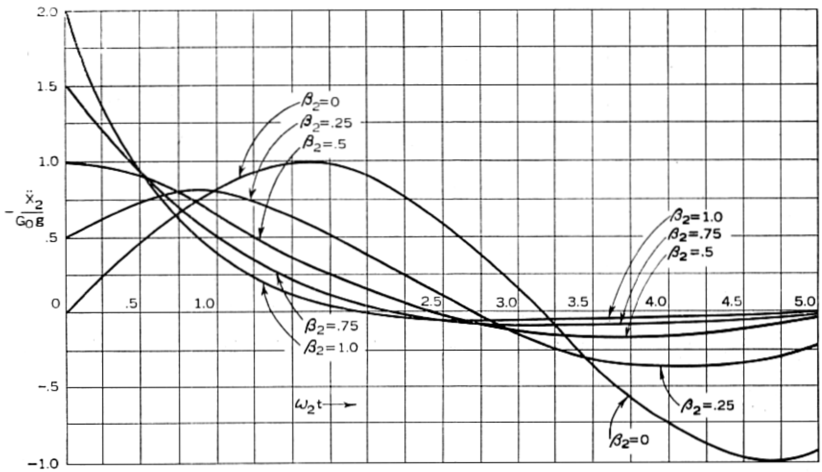


Fig. 2.5.2—Acceleration-time curves for linear cushioning with various amounts of damping (no rebound). See equation (2.5.5).

The acceleration is thus a damped sinusoid with an abruptly reached initial value whose magnitude depends upon the amount of damping. For small damping, the initial acceleration is small and then the acceleration increases, but never reaches the value that would be reached without any damping. For high damping ($\beta_2 > 0.5$) the initial value is greater than without any damping and falls off thereafter. Figure 2.5.2 shows the shapes of the acceleration time curves for several values of β_2. All of the curves are for no rebound. It may be seen, from equation (2.5.5) and Fig. 2.5.2 that the addition of damping changes the shape of the acceleration-time relation in three ways. First, a damped sinusoid replaces the pure sinusoid; second, the frequency is reduced; and, third, the initial phase is changed.

It is useful to consider in detail the effect of damping on *maximum* acceleration. Let

G_m = maximum number of g's with damping

$G_0 = \sqrt{\dfrac{2hk_2}{W_2}}$ = maximum number of g's without damping.

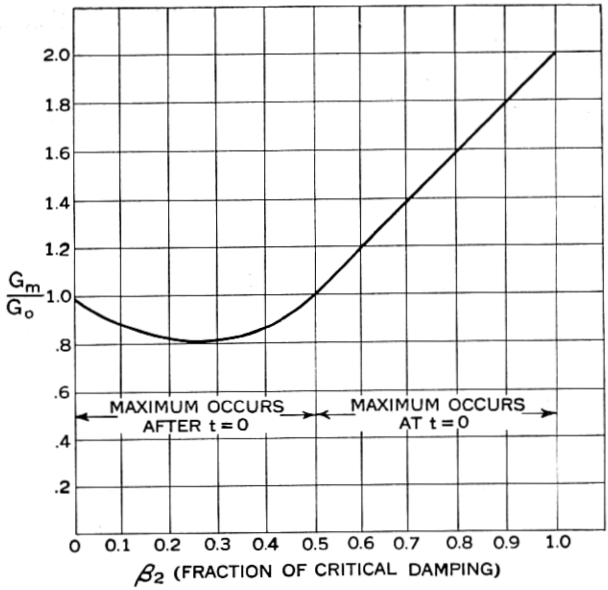


Fig. 2.5.3—Influence of velocity damping on maximum acceleration. See equation (2.5.5).

Then, from (2.5.5), at $t = 0$

$$\frac{G_m}{G_0} = 2\beta_2 \tag{2.5.7}$$

and, after $t = 0$,

$$\frac{G_m}{G_0} = e^{-\beta_2 \omega_2 t_m}, \tag{2.5.8}$$

where t_m, the time at which the maximum occurs, is given by

$$\tan \omega_2 t_m \sqrt{1 - \beta_2^2} = \frac{(1 - 4\beta_2^2)\sqrt{1 - \beta_2^2}}{\beta_2(3 - 4\beta_2^2)}. \tag{2.5.9}$$

The largest value of G_m/G_0 from (2.5.7) and (2.5.8) is plotted against β_2 in Fig. 2.5.3. It is shown there that, as the damping is increased from zero, the maximum acceleration first decreases to a minimum of 80% of G_0 and then increases to G_0 at 50% of critical damping. In this interval the maximum acceleration occurs after $t = 0$. For damping greater than $\beta_2 = 0.5$ the maximum acceleration occurs at the instant of contact and increases in direct proportion to β_2.

2.6 Influence of Damping on Rebound

In considering rebound without damping, it was found that rebound does not occur unless the product of the maximum acceleration and the suspended mass exceeds the total weight of the package. It was not necessary to distinguish between maximum acceleration on the first downstroke and first upstroke, since these are the same when there is no damping. With damping, however, the maximum acceleration on the first downstroke is

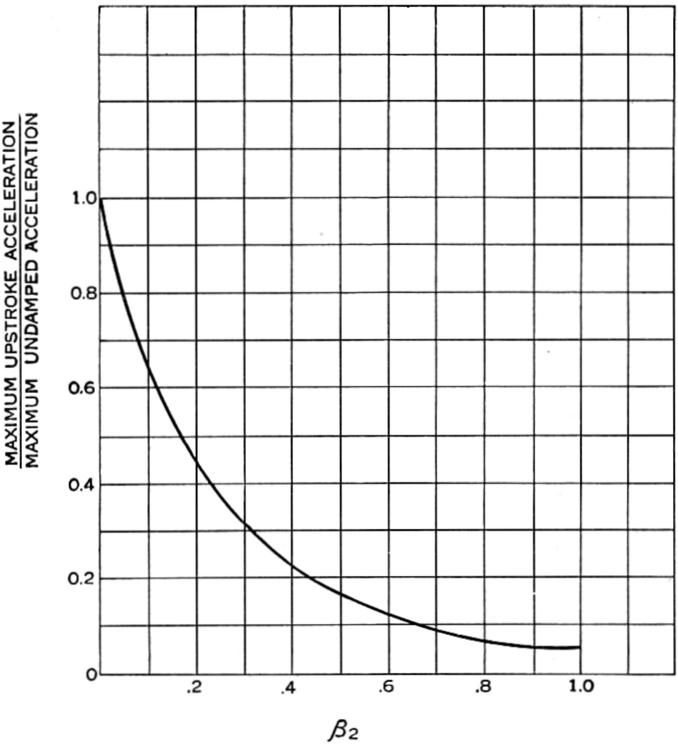


Fig. 2.6.1—Influence of velocity damping on maximum upstroke acceleration. See equation (2.5.5).

greater than that on the first upstroke (Fig. 2.5.2) and it is the latter that controls rebound. Hence damping inhibits rebound.

For example, with 50% of critical damping ($\beta_2 = 0.5$), equations (2.5.8) and (2.5.9) and Fig. 2.5.2 show that for the first downstroke $G_m/G_0 = 1$ while for the first upstroke $G_m/G_0 = 0.164$. Hence the tendency to rebound is reduced by a factor of six when damping to the extent of 50% of critical is added to an undamped package.

The ratio of the maximum acceleration on the first upstroke to the maximum undamped acceleration is plotted in Fig. 2.6.1 for various values of β_2.

2.7 Influence of Dry Friction on Acceleration and Displacement

By "dry friction" is meant friction that is independent of velocity except for sign. During contact of the package with the floor the motion of m_2 might be opposed by a constant friction force F. Such a force is developed, for example, in a package with corrugated spring pad cushioning by rubbing against the side and end pads in a top or bottom drop. A typical idealized

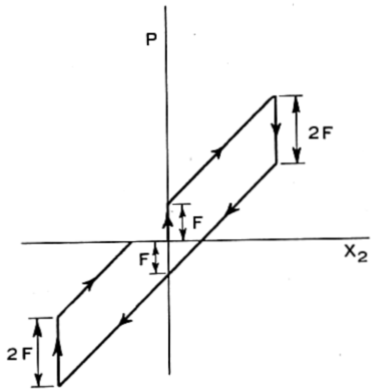


Fig. 2.7.1—Load vs. displacement for cushioning with dry friction.

load-displacement curve is shown in Fig. 2.7.1. For the first downstroke of m_2, the equation of motion of m_2 is

$$m_2\ddot{x}_2 + k_2x_2 = -F. \tag{2.7.1}$$

With initial conditions

$$[x_2]_{t=0} = 0, \qquad [\dot{x}_2]_{t=0} = \sqrt{2gh} \tag{2.7.2}$$

the solution of (2.7.1) is

$$x_2 = \sqrt{d_0^2 + \left(\frac{F}{k_2}\right)^2} \sin(\omega_2 t + \alpha) - \frac{F}{k_2} \tag{2.7.3}$$

where

$$d_0 = \sqrt{\frac{2W_2h}{k_2}},$$

$$\tan\alpha = \frac{F}{k_2 d_0} = \frac{F}{W_2 G_0},$$

$$G_0 = \sqrt{\frac{2hk_2}{W_2}}.$$

d_0 and G_0 are the maximum displacement and acceleration that would obtain if no friction were present. From (2.7.2) the maximum displacement with friction is

$$d_m = \sqrt{d_0^2 + \left(\frac{F}{k_2}\right)^2} - \frac{F}{k_2}. \tag{2.7.4}$$

Hence, the presence of friction decreases the maximum displacement since $d_m < d_0$.

From (2.7.3) the acceleration is

$$\frac{\ddot{x}_2}{g} = -\sqrt{G_0^2 + \left(\frac{F}{W_2}\right)^2} \sin(\omega_2 t + \alpha), \tag{2.7.5}$$

so that the maximum acceleration is

$$G_m = \sqrt{G_0^2 + \left(\frac{F}{W_2}\right)^2}, \tag{2.7.6}$$

which is greater than the maximum acceleration without friction.

It would appear, at first glance, that cushioning with friction always gives a greater acceleration than the corresponding cushioning without friction. However, the reverse is actually true provided we allow the same displacement in both cases. This may be done, as may be seen from (2.7.4), by decreasing the spring rate in the cushioning with friction to

$$k_F = k_2 - \frac{2F}{d_0}. \tag{2.7.7}$$

The maximum acceleration in the cushioning with friction is then, from (2.7.6),

$$G_F = G_0 - \frac{F}{W_2}, \qquad 0 \leqq \frac{F}{W_2} \leqq \frac{G_0}{2}. \tag{2.7.8}$$

That is, for the same maximum displacement, the maximum acceleration is reduced by the addition of dry friction.

2.8 Acceleration-Time Relation for Cubic Elasticity

As an example of the effect of nonlinearity of the cushioning on the shape of the acceleration-time function, the case of cubic elasticity (Class B) will be analyzed. The system to be considered is illustrated in Fig. 1.2.1, and the load-displacement relation for the cushioning is given by

$$P = k_0 x_2 + r x_2^3. \tag{2.8.1}$$

Substituting (2.8.1) in (1.2.13) and performing the indicated integration, we find

$$\dot{x}_2^2 = 2gh - \frac{k_0}{m_2} x_2^2 - \frac{r}{2m_2} x_2^4. \tag{2.8.2}$$

Remembering that $\dot{x}_2 = dx_2/dt$, we solve (2.8.2) for dt:

$$dt = \frac{dx_2}{\sqrt{2gh - \frac{k_0}{m_2} x_2^2 - \frac{r}{2m_2} x_2^4}}. \tag{2.8.3}$$

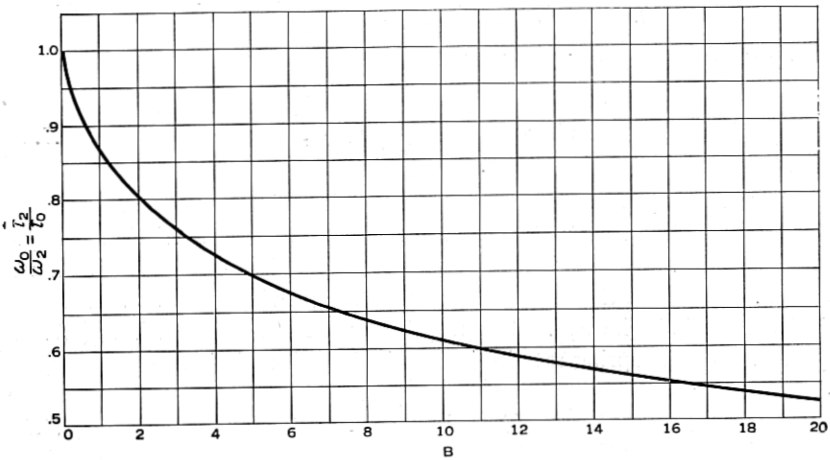


Fig. 2.8.1—Duration of acceleration pulse for cushioning with cubic elasticity. See equation (2.8.10).

Then, with the initial condition $x_2 = 0$ when $t = 0$, the integral of (2.8.3) is

$$t = \int_0^{x_2} \frac{dx_2}{\dot{x}_2} = \int_0^{x_2} \frac{dx_2}{\sqrt{2gh - \frac{k_0}{m_2} x_2^2 - \frac{r}{2m_2} x_2^4}}. \tag{2.8.4}$$

To integrate (2.8.4), let

$$Z = \frac{x_2}{\sqrt{k^2(x_2^2 - d_m)^2 + d_m^2}}, \tag{2.8.5}$$

where

$$k^2 = \tfrac{1}{2}\left\{1 - \frac{1}{\sqrt{1 + B}}\right\} \tag{2.8.6}$$

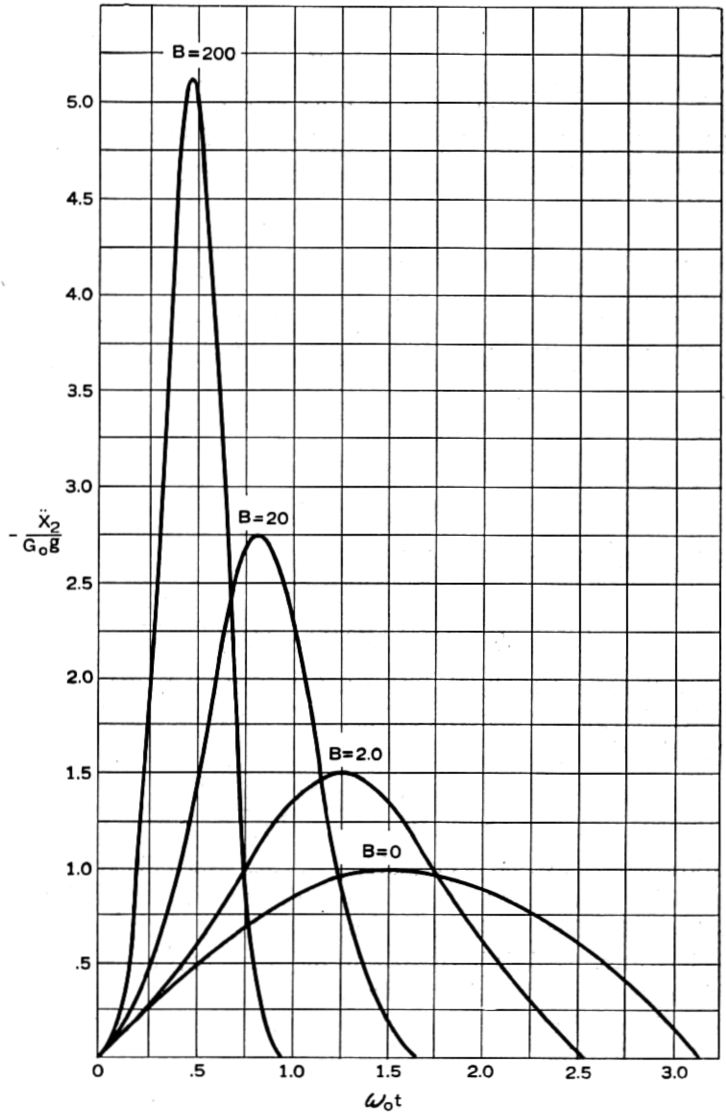


Fig. 2.8.2—Acceleration-time curves for cushioning with cubic elasticity. See equation (2.8.14).

and B and d_m are as given in Part I:

$$B = \frac{4W_2 hr}{k_0^2} \tag{1.5.3}$$

$$d_m = d_0 \sqrt{\frac{2}{B}(-1 + \sqrt{1 + B})}. \tag{1.5.4}$$

Then (2.8.4) becomes

$$t = \frac{1}{\omega_c} \int_0^Z \frac{dZ}{\sqrt{(1 - Z^2)(1 - k^2 Z^2)}}, \tag{2.8.7}$$

in which the integral is the elliptic integral of the first kind (see Hancock "Elliptic Integrals," John Wiley and Sons, New York, 1917). In (2.8.7),

$$\omega_c = \omega_0 \, (1 + B)^{1/4}, \tag{2.8.8}$$

where $\omega_0 = \sqrt{k_0/m_2}$ is the radian frequency that would obtain if the cushioning were linear with spring rate k_0. The motion for the linear case has a half period, or pulse duration $\tau_0 = \pi/\omega_0$. The half-period (τ_2) of the motion with cubic elasticity is twice the time required for x_2 to increase from 0 to d_m.

From 2.8.5

$$\begin{aligned} [Z]_{x_2=0} &= 0, \\ [Z]_{x_2=d_m} &= 1. \end{aligned} \tag{2.8.9}$$

Hence, from (2.8.7), the half-period is

$$\tau_2 = \frac{2}{\omega_c} \int_0^1 \frac{dZ}{\sqrt{(1 - Z^2)(1 - k^2 Z^2)}} = \frac{2K}{\omega_c} \tag{2.8.10}$$

where K is the complete elliptic integral of the first kind. The duration of the acceleration pulse is therefore $2K/\omega_c$. We can define a radian frequency of the acceleration by

$$\omega_2 = \frac{\pi}{\tau_2} = \frac{\pi\omega_c}{2K} = \frac{\pi\omega_0(1 + B)^{1/4}}{2K}. \tag{2.8.11}$$

The ratio ω_0/ω_2 (i.e., τ_2/τ_0) is plotted in Fig. 2.8.1 which illustrates how the pulse duration decreases as the parameter B increases. Hence, for a given cushioning with cubic elasticity, the pulse duration decreases as the height of drop increases. This is in contrast to the linear case in which the duration is independent of the height of drop.

To find the acceleration $\ddot{x}_2$, we return to (2.8.7) and write it in the form of an elliptic function:

$$\operatorname{sn} \omega_c t = Z. \tag{2.8.12}$$

Substituting the expression for Z given in (2.8.5) and solving for x_2, we find

$$x_2 = d_m cn(\omega_c t - K). \tag{2.8.13}$$

Finally, differentiating (2.8.13) twice with respect to t, we find the acceleration to be

$$\ddot{x}_2 = \omega_c^2 d_m [2k^2 sn^2(\omega_c t - K) - 1] cn(\omega_c t - K). \tag{2.8.14}$$

The ratio $-\ddot{x}_2/G_0 g$ is plotted in Fig. 2.8.2 against a radian coordinate $(\omega_0 t)$ for several values of B. It may be seen that, as B increases, the maximum acceleration increases, the duration of the pulse decreases (see Fig. 2.8.1) and the acceleration-time curve becomes bell shaped. For reference, the sinusoid for the linear case ($B = 0$) is plotted in the figure.

Figure 2.8.2 is plotted for perfect rebound. If rebound does not occur, the curves continue, mirrored in the time axis, so as to form a periodic vibration of period $2\tau_2$.

2.9 Acceleration-Time Relation for Tangent Elasticity

In this section the effect of tangent elasticity on the shape of the acceleration-time relation will be studied. The shape of the load displacement curve is given by

$$P = \frac{2k_0 d_b}{\pi} \tan \frac{\pi x_2}{2d_b}. \tag{1.4.3}$$

The system considered is again that shown in Fig. 1.2.1. Referring to the energy equation (1.2.13):

$$\frac{m_2 \dot{x}_2^2}{2} + \int_0^{x_2} P dx_2 = m_2 gh, \tag{1.2.13}$$

we substitute the above value of P and perform the indicated integration to obtain, for the velocity,

$$\dot{x}_2 = \sqrt{2gh + \frac{8k_0 d_b^2}{m_2 \pi^2} \log \cos \frac{\pi x_2}{2d_b}}. \tag{2.9.1}$$

Then, as in Section 2.8,

$$t = \int_0^{x_2} \frac{dx_2}{\dot{x}_2} = \int_0^{x_2} \frac{dx_2}{\sqrt{2gh + \frac{8k_0 d_b^2}{m_2 \pi^2} \log \cos \frac{\pi x_2}{2d_b}}} \tag{2.9.2}$$

and the half-period (τ_2) of the motion is again twice the time required for x_2 to increase from 0 to d_m. Hence

$$\tau_2 = 2 \int_0^{d_m} \frac{dx_2}{\sqrt{2gh + \frac{8k_0 d_b^2}{m_2 \pi^2} \log \cos \frac{\pi x_2}{2d_b}}} \tag{2.9.3}$$

where, from Section 1.9,

$$d_m = \frac{2d_b}{\pi} \cos^{-1} \exp\left[-\frac{\pi^2}{8} \left(\frac{d_0}{d_b} \right)^2 \right]. \tag{1.9.5}$$

The radian frequency of the acceleration is

$$\omega_2 = \frac{\pi}{\tau_2}$$

and this is to be compared with the frequency

$$\omega_0 = \frac{\pi}{\tau_0} = \sqrt{\frac{k_0}{m_2}}$$

that would obtain if the cushioning were linear with spring rate k_0. The ratio ω_0/ω_2 (i.e., τ_2/τ_0) was obtained by numerical integration of (2.9.3)

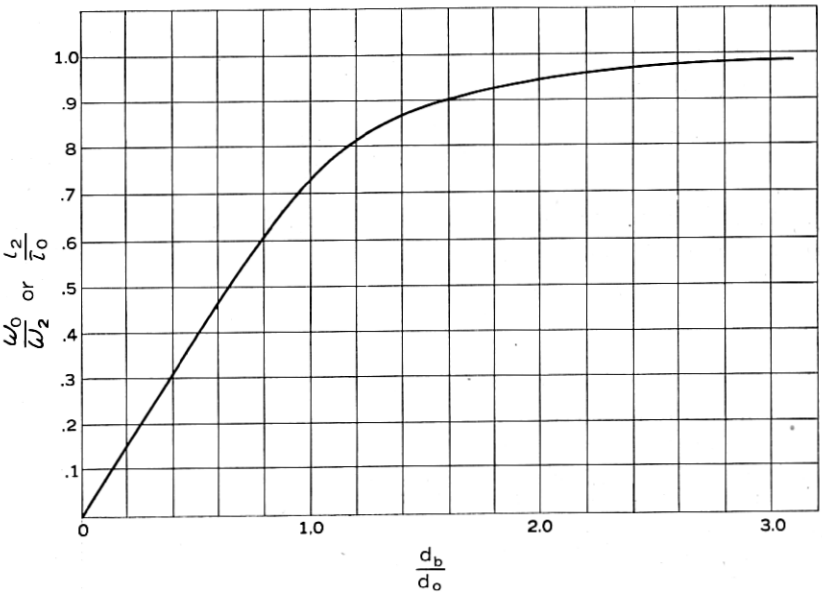


Fig. 2.9.1—Duration of acceleration pulse for cushioning with tangent elasticity. See equation (2.9.3).

and is plotted in Fig. 2.9.1 against the ratio d_b/d_0. The figure shows that for $d_b/d_0 < 1$, the pulse duration varies almost linearly with d_b/d_0. As the bottoming distance becomes larger than that required for linear cushioning with spring rate k_0, the pulse duration approaches asymptotically the duration π/ω_0 for the linear case.

As d_b/d_0 decreases, the pulse duration becomes shorter, but the maximum acceleration increases, in accordance with equation (1.9.4) and Fig. 1.9.1. The shapes of the acceleration-time curves for several values of d_b/d_0 are illustrated in Fig. 2.9.2. They are more sharply peaked than the corresponding curves for cubic elasticity (Fig. 2.8.2) as might be expected from the fact that the load-displacement curve for tangent elasticity rises more rapidly than that for cubic elasticity; that is, the bottoming is harder.

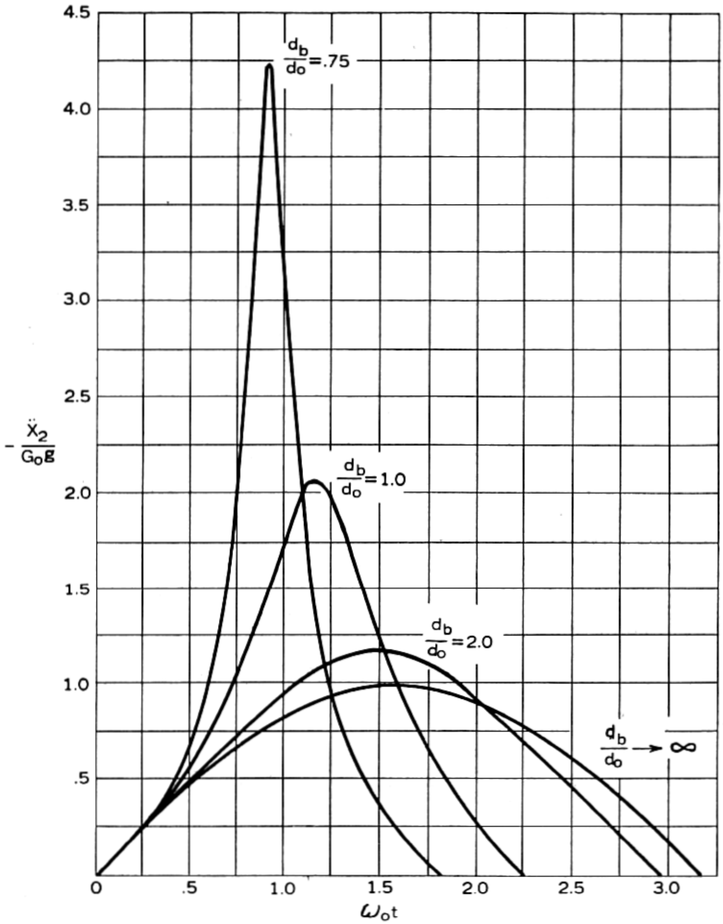


Fig. 2.9.2—Acceleration-time curves for cushioning with tangent elasticity.

The curves of Fig. 2.9.2 were obtained by numerical integration of equation (2.9.2), to obtain x_2 as a function of t, following which these values were substituted in the equation

$$m_2 \ddot{x}_2 + \frac{2k_0 d_b}{\pi} \tan \frac{\pi x_2}{2d_0} = 0$$

to obtain $\ddot{x}_2$. It may be observed that the maximum values of the curves are the values dictated by equation (1.9.4).

In performing the numerical integrations of equations (2.9.2) and (2.9.3), it is found that the integrand becomes infinite when $x_2 = d_m$ since at this point the velocity is zero. In order to avoid this difficulty, it was assumed[1] that, for a small distance in the neighborhood of d_m, the acceleration is constant with magnitude $G_m g$ as obtained from equation (1.9.4). The procedure is described in further detail in Section 2.12.

Figure 2.9.2 gives the acceleration-time curve for perfect rebound. If rebound does not occur, the acceleration is a periodic vibration, each successive half period having the shape shown, with alternating sign.

2.10 Acceleration-Time Relation for Abrupt Bottoming

By abrupt bottoming, we mean bilinear cushioning (Class D) as treated in Section 1.12. The load-displacement relation is (see equation (1.4.4) and Fig. 1.4.4)

$$\left.\begin{aligned} P &= k_0 x_2 \qquad 0 \leqq x_2 \leqq d_s \\ P &= k_b x_2 - (k_b - k_0) d_b \qquad x_2 \geqq d_s \end{aligned}\right\}. \tag{1.4.4}$$

Considering, again, the system illustrated in Fig. 1.2.1, the equation of motion of m_2, before bottoming, is

$$m_2 \ddot{x}_2 + k_0 x_2 = 0 \qquad 0 \leqq x_2 \leqq d_s \tag{2.10.1}$$

with initial conditions

$$[x_2]_{t=0} = 0, \qquad [\dot{x}_2]_{t=0} = \sqrt{2gh}. \tag{2.10.2}$$

The solution of (2.10.1) is then

$$x_2 = \frac{\sqrt{2gh}}{\omega_0} \sin \omega_0 t, \qquad 0 \leqq x_2 \leqq d_s, \tag{2.10.3}$$

where

$$\omega_0 = \sqrt{\frac{k_0}{m_2}}. \tag{2.10.4}$$

The time (t_s) at which x_2 reaches d_s is found from (2.10.3):

$$t_s = \frac{1}{\omega_0} \sin^{-1} \frac{\omega_0 d_s}{\sqrt{2gh}} = \frac{1}{\omega_0} \sin^{-1} \frac{d_s}{d_0}, \tag{2.10.5}$$

where

$$d_0 = \sqrt{\frac{2W_2 h}{k_0}};$$

[1] See Timoshenko, "Vibration problems in Engineering," D. Van Nostrand Co.. New York, Second Edition (1937) page 123.

i.e., d_0 is the displacement that would have been reached if the spring rate remained constant.

The velocity of m_2 at time t_s is

$$[\dot{x}_2]_{t=t_s} = \sqrt{2gh} \cos \omega_0 t_s = \sqrt{2gh\left(1 - \frac{d_s^2}{d_0^2}\right)}. \qquad (2.10.6)$$

If $\sqrt{2gh} > \omega_0 d_s$, the displacement will exceed d_s and the equation of motion becomes

$$m_2\ddot{x}_2 + k_b x_2 - (k_b - k_0)d_s = 0, \qquad x_2 \geqq d_s. \qquad (2.10.7)$$

The solution of (2.10.7), with initial conditions

$$\begin{aligned} [x_2]_{t=t_s} &= d_s \\ [\dot{x}_2]_{t=t_s} &= \sqrt{2gh\left(1 - \frac{d_s^2}{d_0^2}\right)}, \end{aligned} \qquad (2.10.8)$$

is

$$\begin{aligned} x_2 = \frac{k_0 d_0}{k_b} \sqrt{\frac{k_b}{k_0} + \frac{d_s^2}{d_0^2}\left(1 - \frac{k_b}{k_0}\right)} \sin (\omega_b t + \alpha - \omega_b t_s) \\ + \left(1 - \frac{k_0}{k_b}\right) d_s \qquad x_2 \geqq d_s \end{aligned} \qquad (2.10.9)$$

where

$$\left.\begin{aligned} \tan^2 \alpha &= \frac{k_0}{k_b\left(\frac{d_0^2}{d_s^2} - 1\right)} \\ \omega_b &= \sqrt{\frac{k_b}{m_2}}. \end{aligned}\right\} \qquad (2.10.10)$$

By differentiating (2.10.3) and (2.10.9) twice with respect to t, the accelerations for the two regions are found to be

$$\ddot{x}_2 = -G_0 g \sin \omega_0 t, \qquad 0 \leqq x_2 \leqq d_s, \qquad (2.10.11)$$

$$\ddot{x}_2 = -G_0 g \sqrt{\frac{k_b}{k_0} + \frac{d_s^2}{d_0^2}\left(1 - \frac{k_b}{k_0}\right)} \sin (\omega_b t + \alpha - \omega_b t_s), \qquad (2.10.12)$$

$$x_2 \geqq d_s,$$

where

$$G_0 = \sqrt{\frac{2hk_0}{W_2}}. \qquad (2.10.13)$$

Typical shapes of the acceleration pulse represented by equations (2.10.11) and (2.10.12) are shown in Fig. 2.10.1. The curves are drawn for $d_s/d_0 =$

0.5 and for several values of k_b/k_0. The peak values of the curves are the same as given by equation (1.12.3). The curve marked $k_b/k_0 = 1$ is the sinusoid of the linear case with duration

$$\tau_0 = \frac{\pi}{\omega_0}. \tag{2.10.14}$$

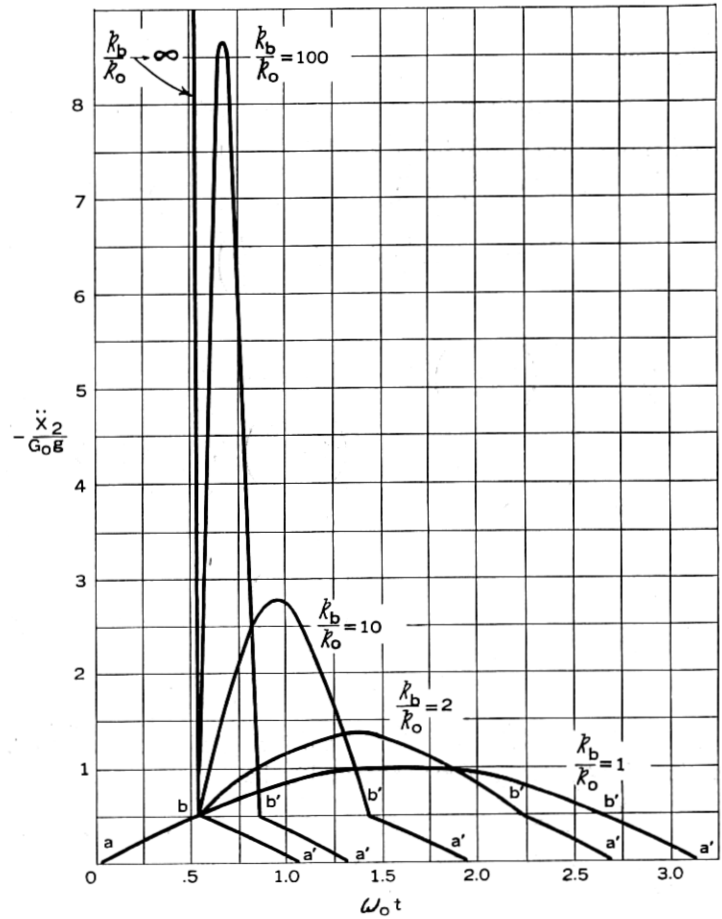


Fig. 2.10.1—Acceleration-time curves for cushioning with bi-linear elasticity. $d_s/d_0 = 0.5$. See equations (2.10.11) and (2.10.12).

As before, if the package does not rebound, the acceleration shown is mirrored in the time axis after each half cycle, to form a vibration of period $2\tau_2$.

It is useful to know the duration of the complete pulse (aa' in Fig. 2.10.1) and also the duration of bottoming (bb' in Fig. 2.10.1). Calling the former τ_2 and the latter τ_B, we have, from equations (2.10.11) and (2.10.12)

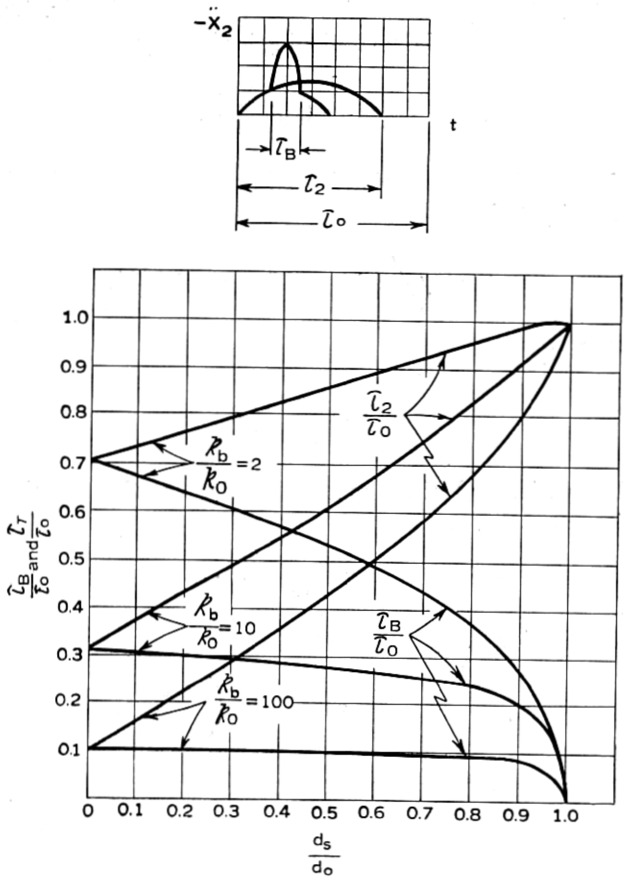


Fig. 2.10.2—Pulse durations for cushioning with bi-linear elasticity. See equations (2.10.15) and (2.10.16).

$$\frac{\tau_2}{\tau_0} = \frac{2}{\pi} \sin^{-1} \frac{d_s}{d_0} + \sqrt{\frac{k_0}{k_b}} \left[1 - \frac{2}{\pi} \tan^{-1} \sqrt{\frac{k_0}{k_b\left(\frac{d_0^2}{d_s^2} - 1\right)}} \right] \tag{2.10.15}$$

$$\frac{\tau_B}{\tau_0} = \sqrt{\frac{k_0}{k_b}} \left[1 - \frac{2}{\pi} \tan^{-1} \sqrt{\frac{k_0}{k_b\left(\frac{d_0^2}{d_s^2} - 1\right)}} \right]. \tag{2.10.16}$$

These two equations are plotted in Fig. 2.10.2 for several values of k_b/k_0.

2.11 Acceleration-Time Relation for Hyperbolic Tangent Elasticity

The relation between acceleration and time for hyperbolic tangent elasticity is found by the same procedure that was used for tangent elasticity in Section 2.9. The system considered is that shown in Fig. 1.2.1 and the load displacement curve of the cushioning is given by

$$P = P_0 \tanh \frac{k_0 x_2}{P_0}.$$

Substituting the above expression for P in the energy equation (1.2.13), we find the velocity to be

$$\dot{x}_2 = \sqrt{2gh - \frac{2P_0^2}{m_2 k_0} \log \cosh \frac{k_0 x_2}{P_0}}. \tag{2.11.1}$$

Then, as before,

$$t = \int_0^{x_2} \frac{dx_2}{\dot{x}_2} \tag{2.11.2}$$

and the half period (τ_2) of the motion is twice the time required for x_2 to increase from 0 to d_m, or

$$\tau_2 = 2 \int_0^{d_m} \frac{dx_2}{\dot{x}_2}, \tag{2.11.3}$$

where, from Section 1.13,

$$d_m = \frac{d_0 P_0}{W_2 G_0} \cosh^{-1} \exp \left(\frac{W_2^2 G_0^2}{2P_0^2} \right). \tag{1.13.3}$$

The radian frequency of the acceleration is defined as

$$\omega_2 = \frac{\pi}{\tau_2}$$

and this is to be compared with the frequency

$$\omega_0 = \frac{\pi}{\tau_0} = \sqrt{\frac{k_0}{m_2}}$$

that would obtain if the cushioning had a constant spring rate equal to the initial spring rate (k_0) of the hyperbolic tangent cushioning. The ratio ω_0/ω_2 (or τ_2/τ_0) is plotted, in Fig. 2.11.1, against the dimensionless parameter P_0/W_2G_0 (see Section 1.13). It may be observed that the pulse duration becomes very long when P_0/W_2G_0 is small, i.e., when the horizontal portion of the load displacement curve (Fig. 1.4.5) comes into play. The influence on the shape of the acceleration-time curve is illustrated in Fig. 2.11.2. The curve marked $P_0/W_2G_0 \to \infty$ is the sinusoid for the linear case. For small values of P_0/W_2G_0 the curve approaches a square wave.

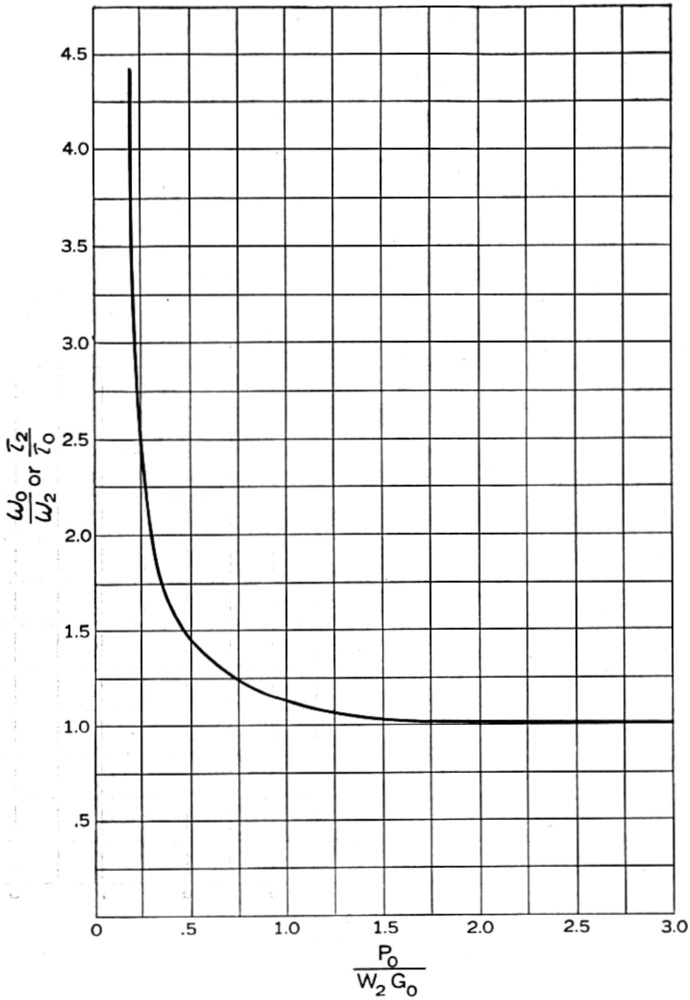


Fig. 2.11.1—Duration of acceleration pulse for cushioning with hyperbolic tangent elasticity.

2.12 Numerical Procedure for Finding Acceleration-Time Relation for Class F Cushioning

When the load-displacement curve does not resemble one of Classes A to E, the acceleration-time relation may be found by numerical integration. Combining the energy equation,

$$\frac{m_2 \dot{x}_2^2}{2} + \int_0^{x_2} P \, dx_2 = m_2 g h, \tag{1.2.13}$$

with the equation relating time and velocity,

$$t = \int_0^{x_2} \frac{dx_2}{\dot{x}_2}, \tag{2.12.1}$$

we find

$$t = \sqrt{\frac{W_2}{2g}} \int_0^{x_2} \frac{dx_2}{\sqrt{W_2 h - \int_0^{x_2} P\, dx_2}}. \tag{2.12.2}$$

As an example, consider the problem of a 15-pound article supported on cushioning with the load-displacement curve shown in Fig. 2.12.1. The package is to be dropped from a height of 3 feet. The computations are

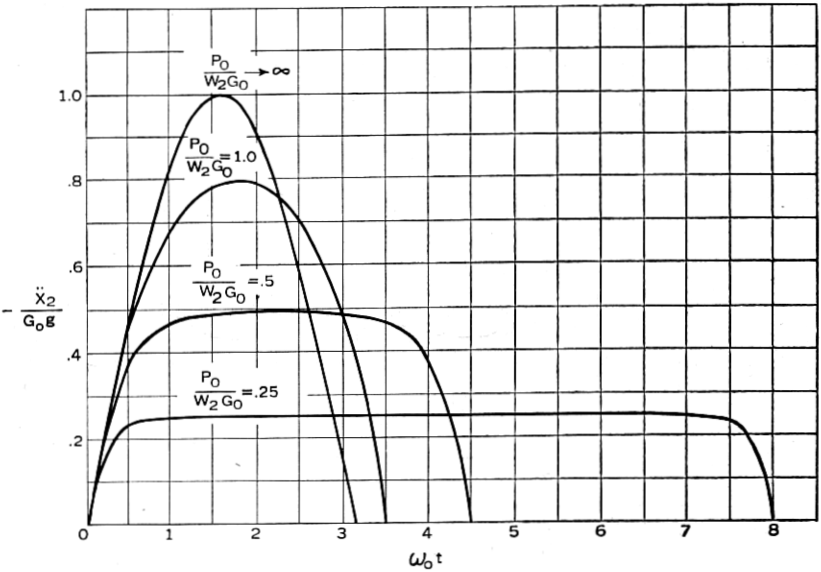


Fig. 2.11.2—Acceleration-time curves for cushioning with hyperbolic tangent elasticity.

given in detail in Tables III and IV. The headings of Columns (1) to (8) of Table III are the same as in Table II, Section 1.15. A_n is the integral under the radical of equation (2.12.2). Column (10) of Table III is the integrand of Equation (2.12.2), i.e., it is proportional to the reciprocal of the velocity expressed as a function of displacement. The function is plotted in Fig. 2.12.2 and its integration is performed in Table IV. In columns (11), (12) and (13), intervals of x_2 are chosen to suit the shape of the curve. The values for column (14) are taken from column (10). Columns (15) and (16) perform the same operations on the integrand $(W_2h - A_n)^{-\frac{1}{2}}$ that are performed in Columns (5) and (6) of Table III on the integrand P.

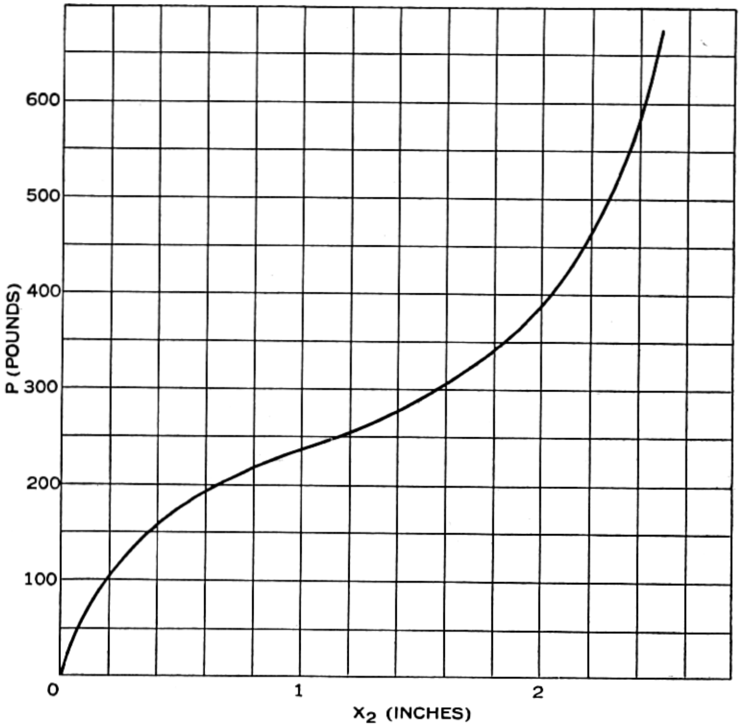


Fig. 2.12.1—A load displacement curve for Class F cushioning.

TABLE III

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
n	$\Delta(x_2)_n$	$(x_2)_n$	P_n	$\frac{\Delta(x_2)_n}{2}(P_n+P_{n-1})$	$A_n = \int_0^{(x_2)_n} P dx_2$	$h_n = \frac{A_n}{W_2}$	$G_n = \frac{P_n}{W_2}$	$W_2h - A_n$	$\frac{1}{\sqrt{W_2h - A_n}}$
0	0	0.0	0	0	0	0	0	540	0.0431
1	.20	0.2	105	10.5	10.5	0.7	7.0	529.5	0.0435
2	.20	0.4	155	26.0	36.5	2.4	10.3	503.5	0.0446
3	.20	0.6	192	34.7	71.2	4.8	12.8	468.8	0.0462
4	.20	0.8	217	40.9	112.1	7.5	14.5	427.9	0.0483
5	.20	1.0	237	45.4	157.5	10.5	15.8	382.5	0.0511
6	.20	1.2	257	49.4	206.9	13.9	17.1	333.1	0.0547
7	.20	1.4	277	52.9	259.8	17.3	18.5	280.2	0.0597
8	.20	1.6	305	58.2	318.0	21.2	20.3	222.0	0.0671
9	.20	1.8	342	64.7	382.7	25.5	22.8	157.3	0.0798
10	.20	2.0	392	73.4	456.1	30.4	26.1	83.9	0.109
11	.05	2.05	405	19.9	476.0	31.8	27.0	64.0	0.125
12	.05	2.10	422	20.7	496.7	33.2	28.1	43.3	0.152
13	.05	2.15	440	21.6	518.3	34.6	29.4	21.7	0.215
14	.01	2.16	445	4.42	522.7	34.8	29.7	17.3	0.240
15	.01	2.17	450	4.48	527.2	35.2	30.0	12.8	0.279
16	.01	2.18	455	4.52	531.7	35.5	30.3	8.3	0.347
17	.01	2.19	457	4.56	536.3	35.8	30.5	3.7	0.521
18	.01	2.20	462	4.60	540.9	36.1	30.8	0	∞

TABLE IV

(11)	(12)	(13)	(14)	(15)	(16)	(17)
n	$\Delta(x_2)_n$	$(x_2)_n$	$f_n = \frac{1}{\sqrt{W_2h - A_n}}$	$\frac{\Delta(x_2)_n}{2}(f_n + f_{n-1})$	$D_n = \int_0^{(x_2)_n} \frac{dx_2}{\sqrt{W_2h - A_n}}$	$t = D_n \times \sqrt{\frac{W_2}{2g}}$
0	0	0	0.0431	0	0	0
1	0.4	0.4	0.0446	0.0175	0.0175	0.0024
2	0.4	0.8	0.0483	0.0185	0.0360	0.0050
3	0.4	1.2	0.0547	0.0206	0.0566	0.0079
4	0.4	1.6	0.0671	0.0243	0.0809	0.0112
5	0.2	1.8	0.0798	0.0149	0.0958	0.0133
6	0.2	2.0	0.109	0.0189	0.1147	0.0160
7	0.1	2.1	0.152	0.0131	0.1278	0.0178
8	0.05	2.15	0.215	0.0092	0.1370	0.0190
9	0.03	2.18	0.347	0.0083	0.1453	0.0202
10	0.01	2.19	0.521	0.0043	0.1496	0.0208
11	0.01	2.20	∞			0.0221

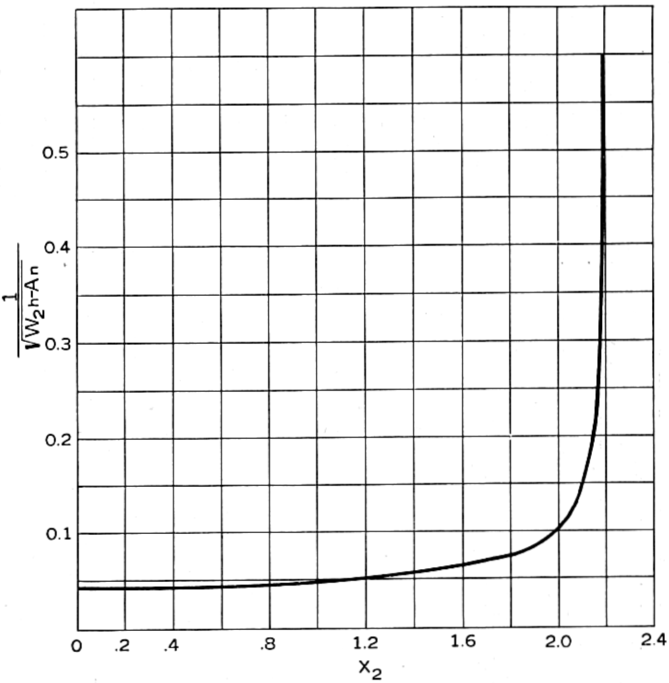


Fig. 2.12.2—Plot of Column (3) vs. Column (10) of Table III.

A difficulty arises because the integrand $(W_2h - A_n)^{-\frac{1}{2}}$ becomes infinite for the maximum displacement (see Column (14)). This is avoided by assuming that the acceleration is constant in the last interval[2] and has the

[2] Timoshenko, "Vibration Problems in Engineering," D. Van Nostrand Co., New York, Second Edition (1937) page 123.

value given in Column (8), Table III, for the maximum height of drop. Then,

$$\Delta(x_2)_n = \tfrac{1}{2}G_m g \Delta t^2 \tag{2.12.3}$$

or

$$\Delta t = \sqrt{\frac{2\Delta(x_2)_n}{G_m g}}. \tag{2.12.4}$$

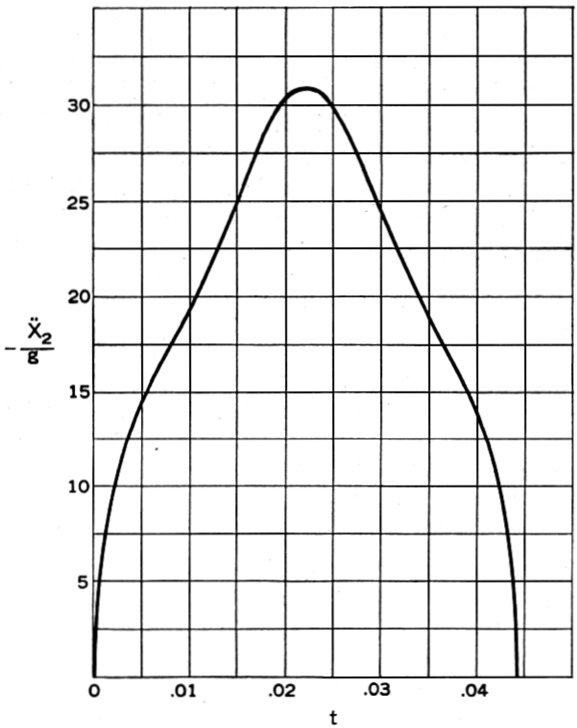


Fig. 2.12.3—Acceleration-time curve (for the cushioning shown in Fig. 2.12.1) obtained by numerical integration.

In the present instance,

$$(\Delta x_2)_n = 0.01 \text{ inches}$$

$$G_m g = 30.8 \times 386 = 11900 \text{ in/sec.}^2$$

Hence, from (2.12.4), $\Delta t = 0.0013$ sec. and the last entry in Column (17) is obtained by adding this value of Δt to the preceding entry.

The final curve of acceleration vs. time is obtained by plotting the entries of Column (17) against the entries of Column (8), Table III, for corresponding values of x_2. The result is shown in Fig. 2.12.3.

PART III

AMPLIFICATION FACTOR

3.1 Introduction

If the maximum acceleration, of the packaged article as a whole, is reached very slowly, the severity of the disturbance experienced by a structural element of the packaged article is very nearly proportional to the maximum acceleration. Roughly speaking, "very slowly" means that the time, during which the acceleration undergoes a major change in magnitude, is long in comparison with the natural period of vibration of the element under consideration. When this is so, no transient vibration is excited in the element. The displacement response of an element under very slowly varying conditions is called the "static response". Under more rapidly varying conditions the dynamic response to the same maximum acceleration may be greater or less than the static response. The ratio (A) of the maximum dynamic response to the static response is called the amplification factor. In general, for a given acceleration disturbance, very low-frequency elements have amplification factors less than unity, while the amplification factors are greater than unity for elements whose natural frequencies are near or above the disturbing frequencies. The numerical value of the amplification factor depends not only on the manner in which the disturbing acceleration varies with time, but also on the "reference acceleration", i.e., the value of acceleration for which the static response is calculated. Usually the reference acceleration chosen for calculating the static response is the maximum value (G_m) of the disturbing acceleration. However, when special circumstances are being investigated, such as the effect of damping or abrupt bottoming, the reference acceleration is taken to be G_0, which is the acceleration that would be reached if the damping or bottoming were absent. In such cases the amplification factor includes both the effect of rate of change of acceleration and the effect of the special conditions.

When the reference acceleration is G_m the amplification factor will be denoted by A_m and when the reference acceleration is G_0 the amplification factor will be denoted by A_0. The symbol G_e will be used to designate the slowly applied acceleration that would produce the same maximum displacement as the transient acceleration, i.e., $G_e = A_m G_m$ or $G_e = A_0 G_0$. The symbol G_s will be used to denote the safe value of G_e, for an element of the packaged article, as determined by a strength test or by calculation. In specifying G_s some judgement is required to take into account the effects of plastic deformation in comparing tests made on greatly different time scales. Good judgement is also necessary in deciding whether or not the assumptions listed in Section 0.2 are valid in each application. The general procedure for using amplification factors is as follows. We first find the value of the reference acceleration (in units of *number of times gravity*) from Part I. From Part II we find the properties of the acceleration-time relation which give us the information required for entering one of the curves of Part III and finding the amplification factor. Then, the product of the reference acceleration and the amplification factor ($A_m G_m$ or $A_0 G_0$) is a number (G_e) by which the weight of the structure is to be multiplied when calculating its deflection or stress by the usual static methods of elementary strength of materials. Alternatively, G_e must be found not to exceed G_s.

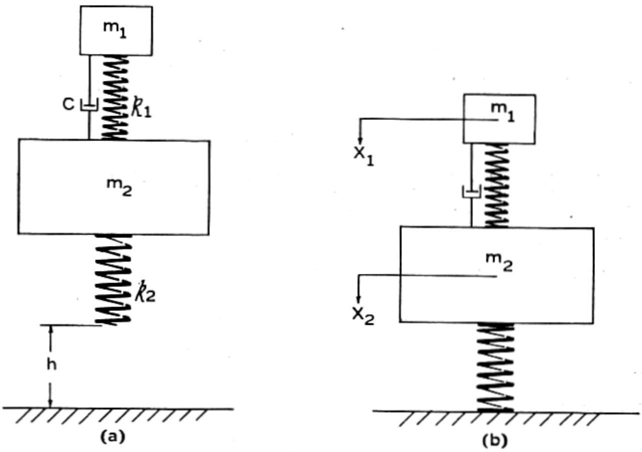


Fig. 3.2.1—Idealized system used in calculating amplification factors for linear undamped cushioning with perfect rebound. (a) initial position, (b) first contact with floor.

In the following sections the amplification factors for typical transient accelerations encountered in package drop tests are calculated. The amplification factor curves that are plotted are entirely analogous to the familiar "resonance curves" for steady sinusoidal vibration, except that in this case the disturbing forces are transients of various shapes. It will be seen from the curves that the maximum acceleration, as calculated by the methods of Part I or as measured by an accelerometer, is not always a true measure of the severity of the disturbance.

3.2 Amplification Factors for a Half-Sine-Wave Pulse Acceleration

The first case to be treated is the response of an element of the packaged item to the transient acceleration that would occur in a package with linear undamped cushioning and perfect rebound. Figure 3.2.1 illustrates the idealized system, and it may be noted that the mass m_3 is omitted, as is required for perfect rebound (Section 2.3). At first we shall consider that the mass m_1 is undamped and later we shall consider the effect of damping in this element.

The mass m_1 is taken to be small in comparison with m_2, so that the motion of the latter is the same as we found it to be in Section 2.2 where m_1 was not considered. Hence the acceleration of m_2 is a half-sine wave pulse:

$$\ddot{x}_2 = -\omega_2 \sqrt{2gh} \sin \omega_2 t, \qquad (0 \gtreqless t \gtreqless \pi/\omega_2). \tag{3.2.1}$$

The equation of motion of m_1 is

$$m_1\ddot{x}_1 + k_1(x_1 - x_2) = 0. \tag{3.2.2}$$

Let x be the relative displacement of m_1 with respect to m_2, i.e.,

$$x = x_1 - x_2 . \tag{3.2.3}$$

x is proportional to the force in the spring (k_1x) and to the acceleration of m_1 and hence is proportional to the deflection, strain and stress in the element which the system m_1, k_1 represents.

Substituting (3.2.3) in (3.2.2), we find:

$$m_1\ddot{x} + k_1x = -m_1\ddot{x}_2 . \tag{3.2.4}$$

This equation holds for the duration π/ω_2 of the pulse $\ddot{x}_2$. The initial conditions for x are

$$[x]_{t=0} = [\dot{x}]_{t=0} = 0 \tag{3.2.5}$$

so that the solution of (3.2.4) is

$$x = \frac{\omega_2\sqrt{2gh}}{\omega_2^2 - \omega_1^2}\left[\frac{\omega_2}{\omega_1}\sin \omega_1 t - \sin \omega_2 t\right], \qquad \left(0 \gtreqless t \gtreqless \frac{\pi}{\omega_2}\right). \tag{3.2.6}$$

It may be seen that x is composed of a forced displacement at the acceleration frequency ω_2, on which is superposed a free vibration at the natural frequency, ω_1, of the element. The maximum value of the relative displacement is

$$x_{\max} = \frac{\sqrt{2gh}}{\omega_1\left(\dfrac{\omega_1}{\omega_2} - 1\right)} \sin \frac{2n\pi}{\dfrac{\omega_1}{\omega_2} + 1}, \qquad \left(0 \gtreqless t \gtreqless \frac{\pi}{\omega_2}\right), \tag{3.2.7}$$

in which n is a positive integer chosen so as to make the sine term as large as possible while the argument remains less than π.

(3.2.7) gives the maximum dynamic response of the element m_1 during the interval of impact. To find the amplification factor we must compare

$x_{\max}$ with the "static response" i.e. with the value (x_{st}) that x would have if the acceleration $\ddot{x}_2$ reached the same maximum value ($\omega_2\sqrt{2gh}$) in a very long time. The resulting value may be found from (3.2.4) by omitting the transient term $m_1\ddot{x}$. Then

$$x_{st} = \omega_2 \sqrt{2gh}\,\frac{m_1}{k_1}$$

or

$$x_{st} = \frac{\omega_2}{\omega_1^2}\sqrt{2gh}. \qquad (3.2.8)$$

The amplification factor for the interval $0 \lesseqgtr t \lesseqgtr \pi/\omega_2$ is then

$$A_m = \frac{x_{max}}{x_{st}} = \frac{\dfrac{\omega_1}{\omega_2}}{\dfrac{\omega_1}{\omega_2} - 1} \sin \frac{2n\pi}{\dfrac{\omega_1}{\omega_2} + 1}, \qquad (0 \lesseqgtr t \lesseqgtr \pi/\omega_2). \qquad (3.2.9)$$

It should be observed that A_m depends only on the frequency ratio ω_1/ω_2. That is, since $\omega_1/\omega_2 = \tau_2/\tau_1$, the amplification factor depends only on the ratio of the pulse duration to the half period of vibration of the element.

Thus far we have studied only the motion in the interval $0 \lesseqgtr t \lesseqgtr \pi/\omega_2$. We must not, however, overlook the possibility of larger displacements of m_1 with respect to m_2 occurring after rebound. In fact, examination of (3.2.6) reveals that x has no maximum in the interval $0 \lesseqgtr t \lesseqgtr \pi/\omega_2$ when $\omega_1 < \omega_2$. It is very likely, then, that larger values will occur at later times.

After rebound, m_1 executes free vibrations with respect to m_2. We have to compare the magnitude of $x_{\max}$, in the interval $0 \lesseqgtr t \lesseqgtr \pi/\omega_2$, with the amplitude of the free vibration. Calling the relative displacement during free vibration x' and measuring a time coordinate t' from the instant the package leaves the floor, we have

$$m_1\ddot{x}' + k_1x' = 0, \qquad (3.2.10)$$

with initial conditions

$$\begin{aligned} [x']_{t'=0} &= [x]_{t=\pi/\omega_2}, \\ [\dot{x}']_{t'=0} &= [\dot{x}]_{t=\pi/\omega_2}. \end{aligned} \qquad (3.2.11)$$

The solution of (3.2.10) with initial conditions (3.2.11) is

$$x' = \frac{\omega_2^2\sqrt{4gh\left(1 + \cos\dfrac{\omega_1}{\omega_2}\pi\right)}}{\omega_1(\omega_2^2 - \omega_1^2)} \sin\left(\omega_1 t' + \frac{\omega_1\pi}{2\omega_2}\right), \qquad (3.2.12)$$

$$t \geqq \frac{\pi}{\omega_2}.$$

Then

$$A_m = \frac{x'_{max}}{x_{st}} = \frac{2\frac{\omega_1}{\omega_2}\cos\frac{\pi\omega_1}{2\omega_2}}{1 - \frac{\omega_1^2}{\omega_2^2}}, \qquad \left(t \geqq \frac{\pi}{\omega_2}\right). \tag{3.2.13}$$

We find, on comparing (3.2.13) with (3.2.9) that for $\omega_1 < \omega_2$ equation (3.2.13) gives the larger value of A_m, while for $\omega_1 > \omega_2$ equation (3.2.9)

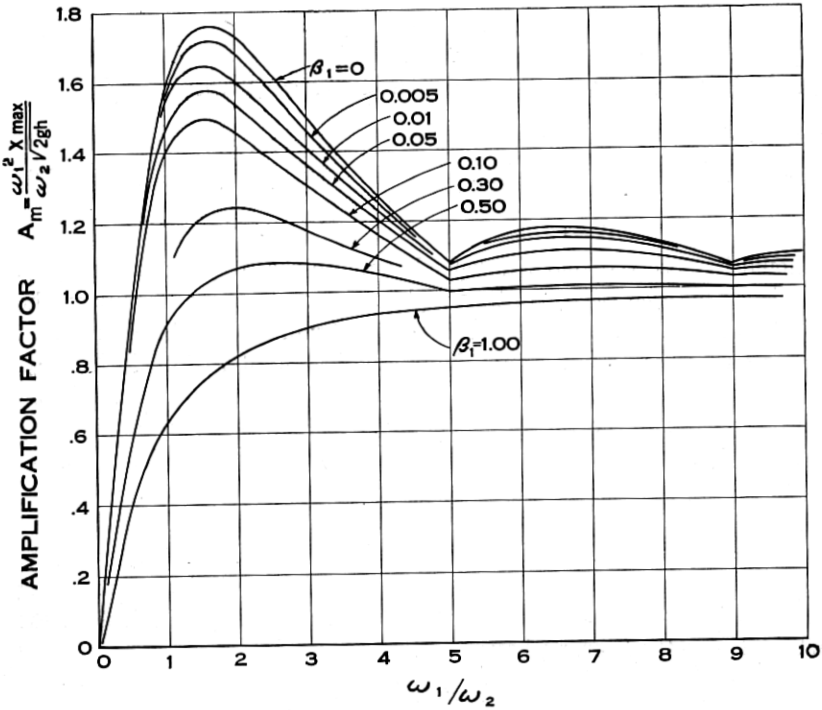


Fig. 3.2.2—Amplification factors for linear undamped cushioning with perfect rebound. See Fig. 3.2.1 and equations (3.2.9) and (3.2.13).

gives the larger value of A_m. That is, when the duration of impact is shorter than the half-period of vibration of the element, the maximum displacement (and stress) in the element occurs after the impact is over.

The curve marked $\beta_1 = 0$ in Fig. 3.2.2 is a plot of the largest value of A_m from (3.2.9) and (3.2.13) with the frequency ratio ω_1/ω_2 as abscissa. (3.2.13) was used for $\omega_1/\omega_2 \leqq 1$ and (3.2.9) for $\omega_1/\omega_2 \geqq 1$. The maximum value of A_m is 1.76 and occurs at $\omega_1/\omega_2 = 1.6$. Hence, at this frequency ratio, the deformation of the element is 1.76 times as great as would be expected from a calculation using the maximum value of acceleration alone as in Part I.

On the other hand, for frequency ratios $\omega_1/\omega_2 < 0.5$ the severity of the shock can be very much less than might be expected from the calculations of Part I. For very small values of ω_1/ω_2 the amplification factor may be seen from (3.2.13) to be equal to $2\omega_1/\omega_2$. For large values of ω_1/ω_2 (stiff elements) Fig. 3.2.2 shows that the amplification factor is very nearly unity and the methods of Part I can be used without additional calculation.

When damping of the element of the packaged article is considered, the amplification factors are less than without damping. The applicable equations of motion during and after impact are obtained by inserting velocity damping terms in (3.2.4) and (3.2.10):

$$m_1\ddot{x} + c_1\dot{x} + k_1 x = -m_1\ddot{x}_2, \qquad 0 \lesseqqgtr t \lesseqqgtr \frac{\pi}{\omega_2} \tag{3.2.14}$$

$$m_1\ddot{x}' + c_1\dot{x}' + k_1 x' = 0, \qquad t \gtreqqless \frac{\pi}{\omega_2}. \tag{3.2.15}$$

If we express the damping of the element m_1 as the fraction of critical damping

$$\beta_1 = \frac{c_1}{2\sqrt{m_1 k_1}}, \tag{3.2.16}$$

(as in Section 2.5) equations (3.2.14) and (3.2.15) become

$$\ddot{x} + 2\beta_1\omega_1\dot{x} + \omega_1^2 x = -\ddot{x}_2, \qquad 0 \lesseqqgtr t \lesseqqgtr \frac{\pi}{\omega_2}, \tag{3.2.17}$$

$$\ddot{x}' + 2\beta_1\omega_1\dot{x}' + \omega_1^2 x' = 0, \qquad t \gtreqqless \frac{\pi}{\omega_2}. \tag{3.2.18}$$

The amplification factors for equations (3.2.17) and (3.2.18), with boundary conditions (3.2.5) and (3.2.11), respectively, were obtained on the Westinghouse Mechanical Transients Analyzer[3] for $\beta_1 = 0.005, 0.01, 0.05, 0.10, 0.30, 0.50$ and 1.00. The curves are shown in Fig. 3.2.2.

3.3 Application of Half-Sine-Wave Amplification Factors

As an example of the use of the amplification factor curves of Fig. 3.2.2, let us consider the following problem:

[3] Arrangements for performing these calculations were made through the courtesy of Mr. A. C. Monteith, Manager of Industry Engineering, and Mr. C. F. Wagner, Manager of Central Station Engineering, Westinghouse Electric and Manufacturing Co. Dr. G. D. McCann, Transmission Engineer, was in immediate charge of the project. For a description of the analyzer see "A New Device for the Solution of Transient-Vibration Problems by the Method of Electrical-Mechanical Analogy" by H. E. Criner, G. D. McCann and C. E. Warren, *Journal of Applied Mechanics*, Vol. 12, No. 3 (1945) pp. A-135 to A-141.

It is required to judge the suitability of a proposed package for a large vacuum tube weighing 10 pounds. Strength tests have been made on the tube in a shock testing machine which produces a half-sine-wave acceleration pulse of 25 milliseconds duration. The weakest element of the tube is found to be the cathode structure, for which the safe maximum acceleration in the drop testing machine is 200g. The cathode structure has a natural vibration frequency of 120 cycles per second and has 1% of critical damping. The proposed package has essentially linear, undamped cushioning with a spring rate of 3300 pounds per inch and an available displacement of $\frac{3}{4}$ inch. The outer container weighs much less than the tube so that the package may be expected to rebound. Is the cushioning suitable for protecting the cathode in a drop of 5 feet?

First find the maximum G that the tube will experience in a 5 ft. drop of the package (equation 1.3.3):

$$G_m = \sqrt{\frac{2hk_2}{W_2}} = \sqrt{\frac{2 \times 60 \times 3300}{10}} = 199.$$

The accompanying maximum displacement is, from equation (1.3.4),

$$d_m = \frac{2h}{G_m} = \frac{2 \times 60}{199} = 0.6 \text{ in.}$$

The available displacement ($\frac{3}{4}$ inches) is therefore sufficient and the maximum acceleration (199g) is slightly less than the safe maximum acceleration (200g) found with the shock testing machine. However, before the cushioning is approved it is necessary to investigate the frequency effects. The duration of acceleration in both the shock machine and in the package must be considered.

The amplification factor for the element tested in the shock machine is found as follows. First find the frequency corresponding to the 25 millisecond pulse:

$$f_2 = \frac{1}{2 \times .025} = 20 \text{ c.p.s.}$$

The ratio of the element frequency to the shock machine frequency is

$$\frac{f_1}{f_2} = \frac{\omega_1}{\omega_2} = \frac{120}{20} = 6.$$

Entering Fig. 3.2.2 with $\omega_1/\omega_2 = 6$, we read, from the curve $\beta_1 = 0.01$, $A_m = 1.14$. The 200g test in the shock machine is, therefore, equivalent to a slowly applied acceleration of $G_s = 200 \times 1.14 = 228g$.

To find the corresponding quantity for the package drop, first find the cushion frequency:

$$f_2 = \frac{1}{2\pi}\sqrt{\frac{k_2}{m_2}} = \frac{1}{2\pi}\sqrt{\frac{3300 \times 386}{10}} = 57 \text{ c.p.s.}$$

The ratio of the element frequency to the package frequency is therefore

$$\frac{f_1}{f_2} = \frac{\omega_1}{\omega_2} = \frac{120}{57} = 2.1.$$

Entering Fig. 3.2.2 with $\omega_1/\omega_2 = 2.1$ we read, from the curve $\beta_1 = 0.01$, $A_m = 1.59$. The $199g$ acceleration pulse in the package drop is therefore equivalent to a slowly applied acceleration of $G_e = 199 \times 1.59 = 316g$. This is almost 40% in excess of the value ($228g$) found to be safe from the shock machine data. The cushioning is therefore judged to be inadequate. The procedure for finding the correct spring rate for the cushioning is as follows. It is known that we must have

$$G_e \lesseqgtr G_s.$$

Therefore, take

$$G_e = A_m G_m = 228.$$

Now

$$G_m = \sqrt{\frac{2hk_2}{W_2}} = \omega_2 \sqrt{\frac{2h}{g}}.$$

Therefore

$$A_m\omega_2 = 409 \text{ rad/sec.}$$

Also

$$\omega_1 = 2\pi \times 120 = 754 \text{ rad/sec.}$$

Then, with successive trial values of ω_2, we calculate ω_1/ω_2, enter Fig. 3.2.2, read the corresponding value of A_m from curve $\beta_1 = 0.01$ and test to see if the product $A_m\omega_2 = 409$. The combination which satisfies the test is found to be

$$\omega_2 = 280 \text{ rad/sec.}$$

$$\omega_1/\omega_2 = 2.69$$

$$A_m = 1.47.$$

Then

$$k_2 = \omega_2^2 m_2 = \frac{(280)^2 \times 10}{386} = 2030 \text{ lbs./in.}$$

$$G_m = \sqrt{\frac{2hk_2}{W_2}} = 155$$

$$d_m = \frac{2h}{G_m} = .77 \text{ in.}$$

Hence the spring rate of the cushioning should be reduced from 3300 lbs./in. to 2030 lbs./in. and the available space should be increased to accomodate the 0.77 inch maximum displacement before bottoming.

3.4 Special Treatment of Strong, Low Frequency Elements

The product of the amplification factor (A_m) and the maximum acceleration (G_m) must be equal to or less than the maximum allowable slowly applied acceleration (G_s):

$$G_e = A_m G_m \leqq G_s . \tag{3.4.1}$$

For frequency ratios

$$\frac{\omega_1}{\omega_2} < \frac{1}{2}, \tag{3.4.2}$$

Figure 3.2.2 shows that, approximately,

$$A_m = 2\frac{\omega_1}{\omega_2} \tag{3.4.3}$$

for $\beta_1 \lesseqgtr 0.10$. When this is so, we may combine (3.4.1) and (3.4.3) to obtain the criterion

$$2\frac{\omega_1}{\omega_2} G_m \lesseqgtr G_s . \tag{3.4.4}$$

Now,

$$G_m = \sqrt{\frac{2hk_2}{W_2}} = \omega_2 \sqrt{\frac{2h}{g}} . \tag{3.4.5}$$

Hence the criterion (3.4.4) may be written as

$$2\omega_1 \sqrt{\frac{2h}{g}} \lesseqgtr G_s \tag{3.4.6}$$

or

$$f_1 \lesseqgtr \frac{1.1\, G_s}{\sqrt{h}} \tag{3.4.7}$$

where h is in inches and f_1 is in cycles per second.

It may be observed that (3.4.7) *is independent of the properties of the cushioning.* Hence, as long as (3.4.2) is satisfied, any cushioning at all may be used for an element that satisfies (3.4.7) regardless of the magnitude of the maximum acceleration G_m. In particular, rigid mounting is suitable for such an element. The only precaution to be observed is that the maximum acceleration and duration must not be unfavorable for other elements of the packaged article.

Example: A 9-pound vacuum tube has an anode structure for which the safe maximum acceleration is $200g$ as determined in a centrifuge test. The natural vibration frequency of the anode is 35 cycles per second and the damping is 1% of critical. What cushioning around the tube is required to protect the anode from damage in a package drop of 3 feet?

Calculate

$$\frac{1.1G_s}{\sqrt{h}} = \frac{1.1 \times 200}{\sqrt{36}} = 36.7 \text{ c.p.s.}$$

This is greater than $f_1 = 35$ c.p.s and hence any cushioning is safe for the anode. The results of calculations for cushioning with spring rates of 50, 500, 5000, 5×10^5 and 5×10^7 pounds per inch are given in the following table:

k_2(lbs./in.)	G_m	$\frac{\omega_1}{\omega_2}$	A_m	A_mG_m
50	20	4.74	1.12	22
500	63	1.47	1.65	106
5×10^3	200	.474	0.9	180
5×10^5	2,000	.0474	0.09	180
5×10^7	20,000	.0047	0.009	180

In each case the product of A_mG_m is less than the allowable 200 and, as long as the combination of G_m and the amplification factors for other elements does not exceed the allowable A_mG_m for those elements, the cushioning is suitable. The precaution to observe is that higher-frequency elements shall not have amplification factors such that A_mG_m may be excessive for them.

3.5 Amplification Factors for Damped Sinusoidal Acceleration

If the outer container of the package is heavy enough (see Section 2.3) there will be no rebound and the packaged item will vibrate in the cushioning after impact. For linear cushioning with velocity damping, the acceleration produced by the vibration will be a damped sinusoid (equation (2.5.5) and Fig. 2.5.2). The system to be considered is shown in Fig. 3.5.1. To determine the effect of the damped vibration of m_2 on the mass m_1,

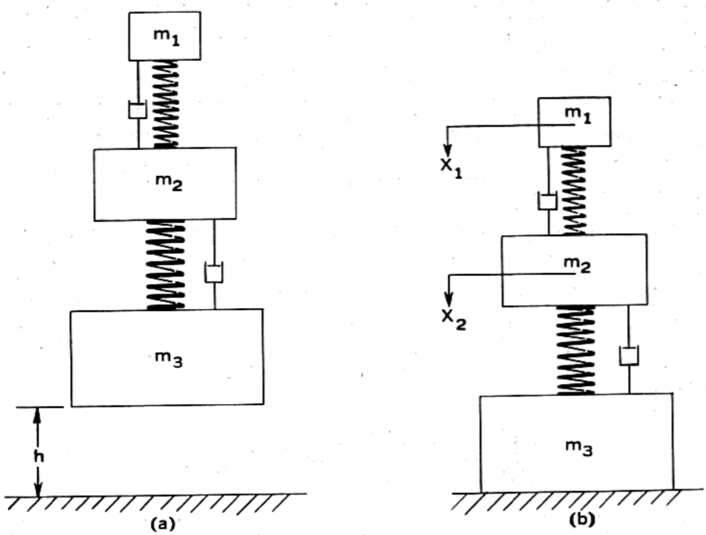


Fig. 3.5.1—Idealized system for linear damped cushioning with no rebound. (a) initial position, (b) first contact with floor.

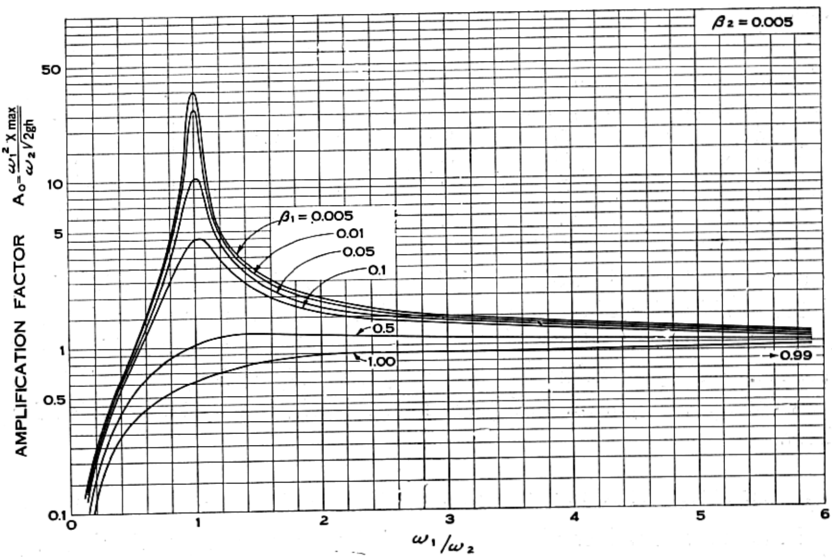


Fig. 3.5.2—Amplification factors for linear damped cushioning with no rebound. $\beta_2 = 0.005$. See equations (3.5.1) and (3.5.2).

we note that the equation of motion and initial conditions are identical with (3.2.17) except that for the acceleration $\ddot{x}_2$ we use the damped sinusoid, equation (2.5.5), instead of the half-sine pulse (3.2.1). The solution of (3.2.17), i.e. the relative displacement $(x_1 - x_2)$ of m_1 with respect to m_2, is

$$x = \frac{\omega_2^2 \sqrt{2gh}}{2\omega_2' \omega_1'} \{e^{-\beta_1\omega_1 t}[A \sin(\omega_1' t + \gamma - \delta) + B \sin(\omega_1' t - \gamma - \zeta)]$$
$$- e^{-\beta_2\omega_2 t}[A \sin(\omega_2' t + \gamma - \delta) - B \sin(\omega_2' t + \gamma + \zeta)]\} \qquad (3.5.1.)$$

where

$$\omega_1 = \sqrt{\frac{k_1}{m_1}}$$

$$\omega_2 = \sqrt{\frac{k_2}{m_2}}$$

$$\omega_1' = \omega_1\sqrt{1 - \beta_1^2}$$

$$\omega_2' = \omega_2 \sqrt{1 - \beta_2^2}$$

$$1/A = \sqrt{(\beta_2\omega_2 - \beta_1\omega_1)^2 + (\omega_1' - \omega_2')^2}$$

$$1/B = \sqrt{(\beta_2\omega_2 - \beta_1\omega_1)^2 + (\omega_1' + \omega_2')^2}$$

$$\tan \gamma = \frac{2\beta_2^2 - 1}{2\beta_2 \sqrt{1 - \beta_2^2}}$$

$$\tan \delta = \frac{\omega_1' - \omega_2'}{\beta_2 \omega_2 - \beta_1 \omega_1}$$

$$\tan \zeta = \frac{\omega_1' + \omega_2'}{\beta_2 \omega_2 - \beta_1 \omega_1}.$$

The relative displacement of m_1 with respect to m_2 is seen to consist of a forced, damped vibration (ω_2, β_2) on which is superposed the free damped oscillations (ω_1, β_1) of m_1.

The amplification factor

$$A_0 = \frac{x_{\max}}{x_{st}} = \frac{\omega_1^2 x_{\max}}{\omega_2 \sqrt{2gh}} \qquad (3.5.2)$$

is plotted in Figs. 3.5.2 to 3.5.7 for six values of β_1 and six values of β_2. These curves were obtained by direct solution of the differential equation on the Westinghouse Mechanical Transients Analyzer.[4] The amplification factor in this case includes the effect of damping; i.e., the reference acceleration is not the maximum acceleration of m_2, but is the maximum acceleration that m_2 would reach if the damping β_2 were zero. Consequently, the amplification factors for large values of ω_1/ω_2 do not approach

[4] See footnote, Section 3.2. Only enough data were obtained with the analyzer to find the general shapes of the curves, so that the fine structure is not revealed. Checks on the analyzer results were made by computing A_0 from equations (3.5.1) and (3.5.2) for $\omega_1/\omega_2 = 1$, $\beta_1 = \beta_2$; $\omega_1/\omega_2 = 0$; $\omega_1/\omega_2 \to \infty$.

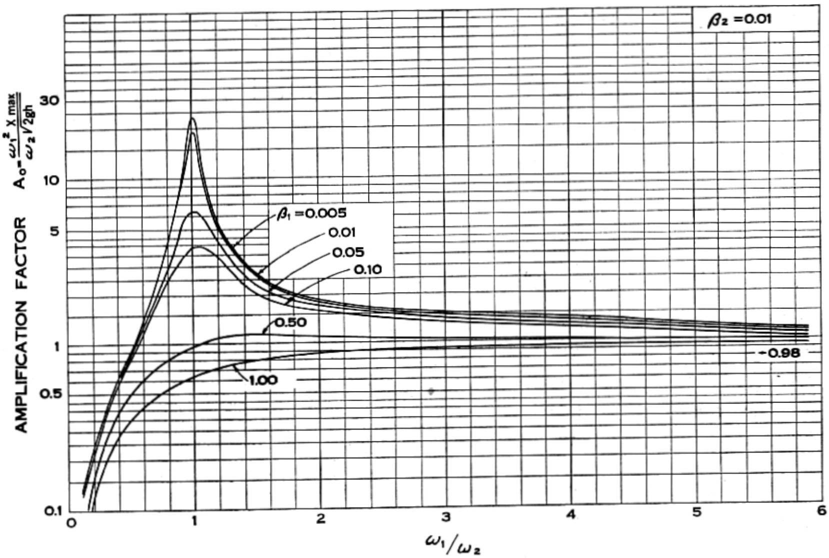


Fig. 3.5.3—Amplification factors for linear damped cushioning with no rebound. $\beta_2 = 0.01$. See equations (3.5.1) and (3.5.2).

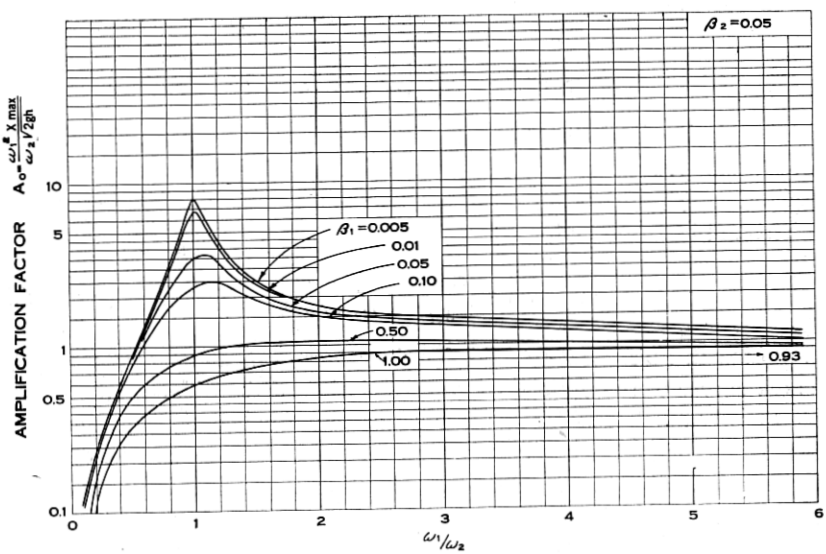


Fig. 3.5.4—Amplification factors for linear damped cushioning with no rebound. $\beta_2 = 0.05$. See equations (3.5.1) and (3.5.2).

unity. For example, the curve for $\beta_1 = 0.005, \beta_2 = 1$ (Fig. 3.5.7) approaches a value of nearly four as $\omega_1/\omega_2 \rightarrow \infty$. The factor four is composed of two

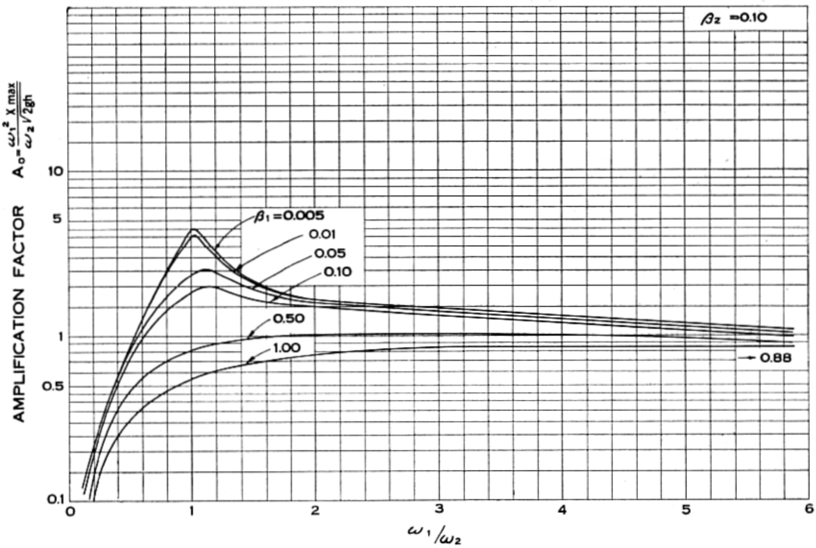


Fig. 3.5.5—Amplification factors for linear damped cushioning with no rebound. $\beta_2 = 0.1$. See equations (3.5.1) and (3.5.2).

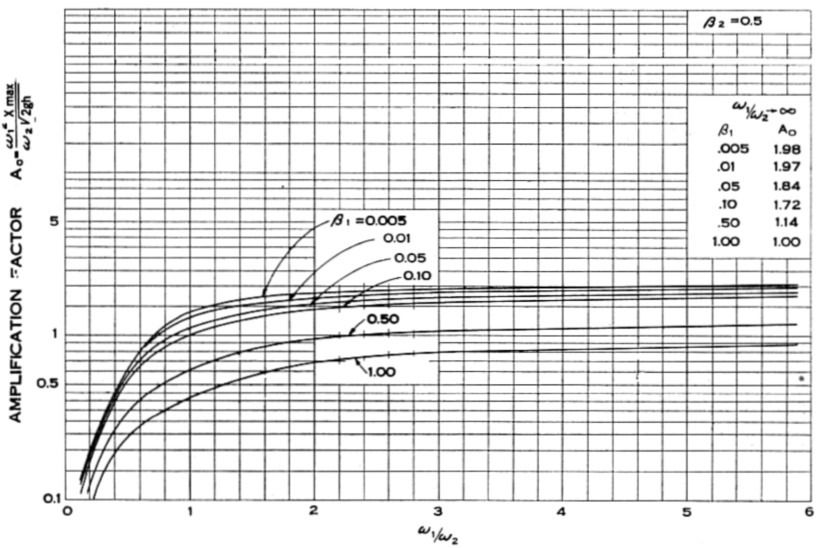


Fig. 3.5.6—Amplification factors for linear damped cushioning with no rebound. $\beta_2 = 0.5$. See equations (3.5.1) and (3.5.2).

factors of two. The first arises from the fact that the maximum value of acceleration, for $\beta_2 = 1$, is twice the value that would be reached if β_2

were equal to zero (see Fig. 2.5.3). The second factor (of nearly two) is due to the fact that the maximum acceleration is reached at time $t = 0$ when $\beta_2 = 1$ (see Fig. 2.5.2) and the response of an almost undamped system ($\beta_1 = 0.005$) to a suddenly applied and subsequently maintained acceleration is double the response to a slowly applied acceleration (see curve (a) Fig. 3.8.1). For $\beta_1 > 0$ and $\beta_2 < 1$ the amplification factor is less than four, as $\omega_1/\omega_2 \to \infty$, in accordance with the curves plotted in Fig. 3.5.8.

Example: A 1.5-pound vacuum tube is to be packed in a container whose estimated weight will be at least 50 pounds. The cathode structure of the tube has a natural frequency of 25 c.p.s. with damping 0.5% of critical

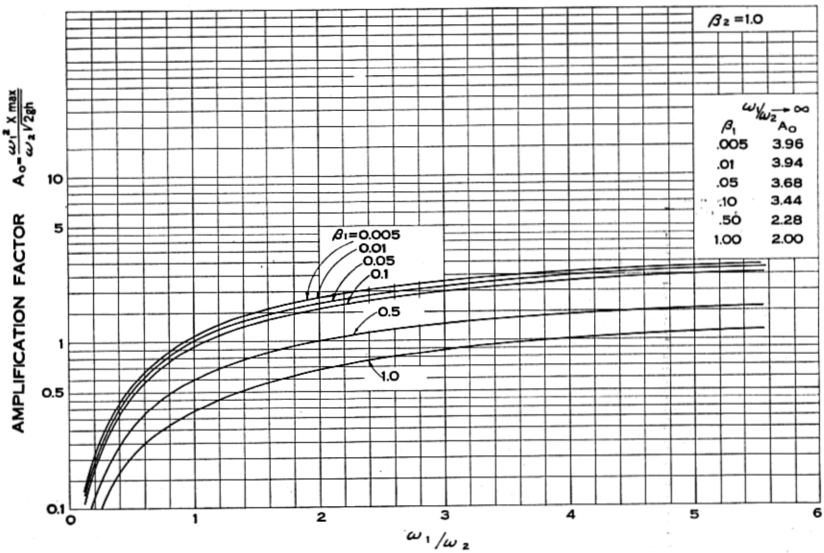


Fig. 3.5.7—Amplification factors for linear damped cushioning with no rebound. $\beta_2 = 1.0$. See equations (3.5.1) and (3.5.2).

and its safe acceleration, as determined in a centrifuge, is $90g$. What spring rate of cushioning is suitable for protecting the cathode in a drop of five feet? It is specified that the cushioning shall have damping 50% of critical.

Assuming linear cushioning, the spring rate that would be prescribed, by considering maximum acceleration alone, is

$$k_2 = \frac{W_2 G_m^2}{2h} = \frac{1.5 \times (90)^2}{2 \times 60} = 101 \text{ lbs./in.}$$

Considering damping, Fig. 2.5.3 shows that 50% of critical damping does not change G_m. To find the amplification factor we must first decide if the package will rebound. With 50% of critical damping, the maximum ac-

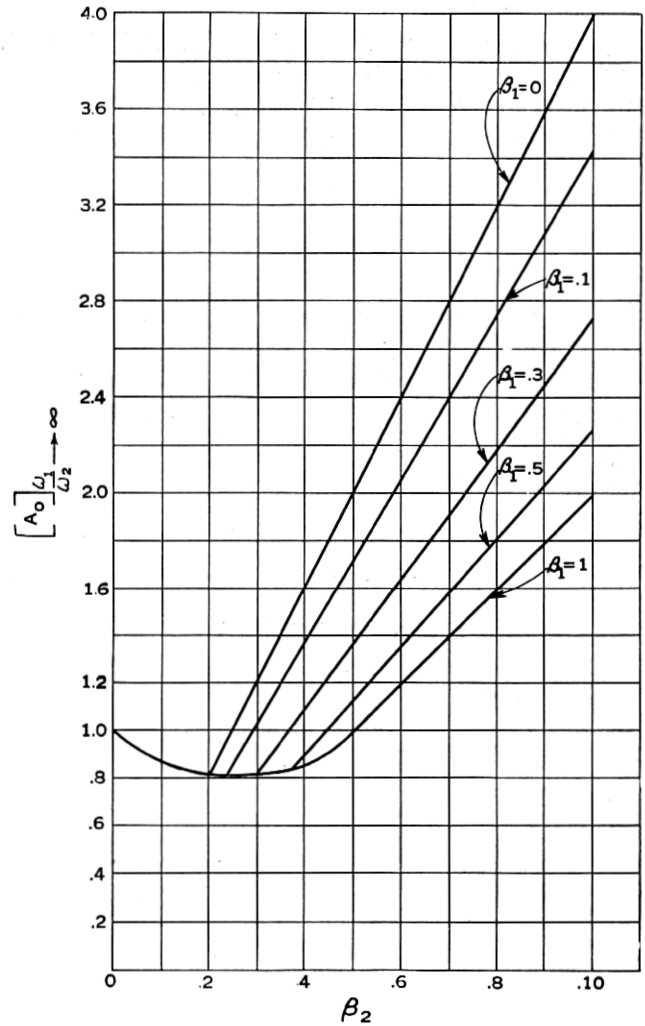


Fig. 3.5.8—Limiting values of amplification factors for linear damped cushioning with no rebound. $\omega_1/\omega_2 \rightarrow \infty$. See equations (3.5.1) and (3.5.2).

celeration on the first upstroke is 0.164 G_m (see Section 2.6 and Fig. 2.6.1). Then, $0.164 \times 90 \times 1.5 = 22$ lbs. which is less than the estimated weight of the outer container. The package will not rebound and Fig, 3.5.6 should be used for the amplification factor.

The frequency of vibration of the tube in its cushion will be

$$\omega_2 = \sqrt{\frac{k_2}{m_2}} = \sqrt{\frac{101 \times 386}{1.5}} = 159 \text{ rad./sec.}$$

Hence $\omega_1/\omega_2 = 2\pi \times 25/159 = 0.99$ and, from Fig. 3.5.6, $A_0 = 1.4$. Hence $G_e = 90 \times 1.4 = 126$, which is greater than the allowable $G_s = 90$, so that the 101 lb./in. cushion is unsatisfactory.

To obtain satisfactory cushioning, set

$$A_0 G_0 = 90,$$

that is

$$A_0 \omega_2 = \frac{90}{\sqrt{\frac{2h}{g}}} = 159 \text{ rad./sec.}$$

Noting that $\omega_1 = 2\pi \times 25 = 157$, we find from Fig. 3.5.6 that there are two values of ω_2 (90 and 600 rad/sec.) that satisfy the criterion $A_0\omega_2 = 159$ rad/sec. The first gives

$$\omega_2 = 90 \text{ rad/sec.}$$

$$A_0 = 1.8$$

$$k_2 = 31.5 \text{ lbs./in.}$$

$$G_0 = 50$$

$$G_e = 90$$

$$d_m = 2.4 \text{ in.}$$

The second gives

$$\omega_2 = 600 \text{ rad/sec.}$$

$$A_0 = 0.27$$

$$k_2 = 1400 \text{ lbs./in.}$$

$$G_0 = 335$$

$$G_e = 90$$

$$d_m = 0.36 \text{ in.}$$

The second solution requires less space for cushioning than the first but should be used only if the remainder of the tube can endure the high acceleration of $335g$. Otherwise the $50g$ package should be used.

3.6 Amplification Factors for the Pulse Acceleration of Cubic Cushioning

In a rebounding package with undamped Class B cushioning, the packaged article (m_2) will undergo a pulse acceleration of duration π/ω_2 as given by equation (2.8.11). The shape of the pulse is illustrated in Fig. 2.8.2 and its functional form is

$$\ddot{x}_2 = \frac{4K^2\,\omega_2^2\,d_m}{\pi^2}\left[2k^2\,sn^2\left(\frac{2K\omega_2 t}{\pi} - K\right) - 1\right] cn\left(\frac{2K\omega_2 t}{\pi} - K\right). \quad (2.8.14)$$

To determine the influence of the shape and duration of this pulse on the amplification factor, we proceed as before by substituting (2.8.14) in the

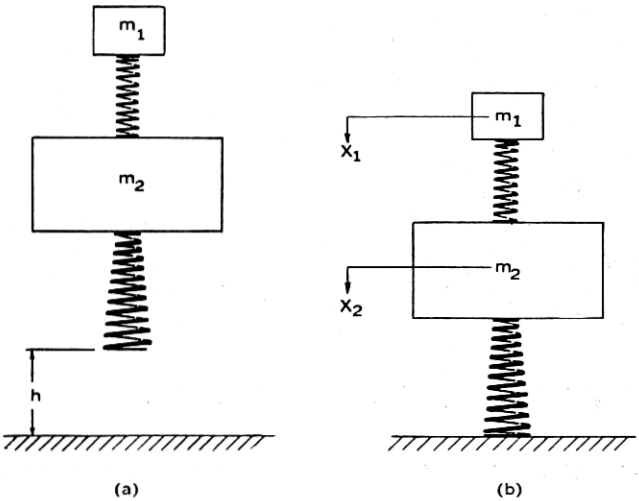


Fig. 3.6.1—Idealized system used in calculating amplification factors for non-linear, undamped cushioning with perfect rebound.

differential equation governing the relative displacement ($x = x_1 - x_2$) between m_1 and m_2 (see Fig. 3.6.1):

$$\ddot{x} + \omega_1^2 x = -\ddot{x}_2 \,. \quad (3.6.1)$$

With boundary conditions $x(0) = \dot{x}(0) = 0$, the solution of (3.6.1) may be written as

$$x = \frac{1}{\omega_1}\int_0^t \ddot{x}_2\,(\lambda)\,\sin\,\omega_1\,(\lambda - t)\,d\lambda \quad (3.6.2)$$

and the maximum value of x may be expressed by

$$x_{\max} = \frac{1}{\omega_1}\int_0^{t_m} \ddot{x}_2(\lambda)\,\sin\,\omega_1(\lambda - t_m)\,d\lambda, \quad (3.6.3)$$

where t_m is the time at which the largest value of x occurs.

The amplification factor, in this case, will be taken as the ratio of x_{max} to the relative displacement (x_{st}) resulting from a slow application of the maximum value of $\ddot{x}_2$. From (3.6.1),

$$x_{st} = \frac{|\ddot{x}_2|_{max}}{\omega_1^2} = \frac{G_m g}{\omega_1^2}, \tag{3.6.4}$$

where G_m is given by equation (1.5.6). Then

$$A_m = \frac{x_{max}}{x_{st}} = \frac{x_{max}\,\omega_1^2}{G_m g} \tag{3.6.5}$$

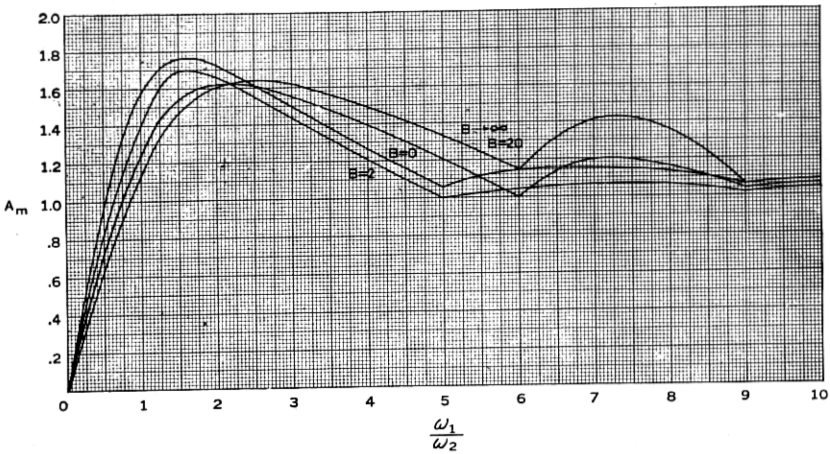


Fig. 3.6.2—Amplification factors for undamped cushioning with cubic elasticity. Perfect rebound. See equation (3.6.6).

or

$$A_m = \frac{\omega_1}{G_m g}\int_0^{t_m} \ddot{x}_2(\lambda)\sin\omega_1(\lambda - t_m)\,d\lambda. \tag{3.6.6}$$

A_m was evaluated, mostly by graphical methods, for four values of B (0, 2, 20 and ∞) and the results are plotted in Fig. 3.6.2. Observing that $B = 0$ corresponds to linear cushioning, it may be noted that cubic non-linearity in the cushioning does not change the amplification factor by more than 35% even in the most extreme case ($B \to \infty$). The severity of the shock, however, may be much greater for the cubic cushioning than for linear cushioning with a spring rate equal to the initial spring rate (k_0) of the cubic cushioning. This is because A_m is multiplied by G_m to obtain G_e and, for large values of B, G_m may be much larger than the maximum acceleration for the linear case. In other words, in comparing Class B with Class A cushioning the difference in maximum acceleration, rather than the difference in amplification factors, is usually more important.

Example: Consider the example given in Section 1.6 and let it be required to determine the effect of pulse duration on a cathode structure with a 200 c.p.s. natural frequency of vibration. In Section 1.6 we found that

$$B = 5.4 \qquad k_0 = 255$$

$$G_0 = 28.6 \qquad r = 108.$$

$$G_m = 55$$

With $B = 5.4$, enter Fig. 2.8.2 and find

$$\frac{\omega_0}{\omega_2} = 0.88.$$

Now

$$\omega_0 = \sqrt{\frac{k_0 g}{W_2}} = \sqrt{\frac{255 \times 386}{22.5}} = 66.1 \text{ rad./sec.}$$

Hence

$$\omega_2 = \frac{66.1}{0.88} = 75 \text{ rad./sec.}$$

Then, with $\omega_1/\omega_2 = 2\pi \times 200/75 = 16.7$, enter Fig. 3.6.2 and find $A_m =$ approximately 1.0. Hence G_e is about the same as G_m and the conclusions reached for this problem in Section 1.6 are not altered.

3.7 Amplification Factors for Abrupt Bottoming

The amplification factors for bilinear elasticity have not been computed in complete detail. They can be obtained approximately by using the duration curves (Fig. 2.10.2) and the amplification curves for the linear case (Figs. 3.2.2 and 3.5.2 to 3.5.7). It is useful, however, to calculate the amplification factors for extremely abrupt bottoming ($k_b \rightarrow \infty$) to obtain a general understanding of the accompanying phenomena.

The system to be considered is illustrated in Fig. 3.7.1. It is assumed that the impact between m_2 and the base (occurring at $t = t_s$, $x_2 = d_s$) has a coefficient of restitution of unity. Hence m_2 will strike the base with velocity

$$[\dot{x}_2]_{t=t_s} = \sqrt{2gh\left(1 - \frac{d_s^2}{d_0^2}\right)}$$

(see equation (2.10.8)) and leave it at a velocity of the same magnitude but opposite sign. Perfect rebound of the whole package is also assumed.

The acceleration pulse will then look like the curve marked $k_b/k_0 \to \infty$ in Fig. 2.10.1.

There will be three regions in which to consider the relative displacement x:

$$\text{Region 1} \qquad 0 < t < t_s$$

$$\text{Region 2} \qquad t_s < t < 2t_s$$

$$\text{Region 3} \qquad t > 2t_s$$

The relative displacement ($x = x_1 - x_2$) of m_1 with respect to m_2 will have

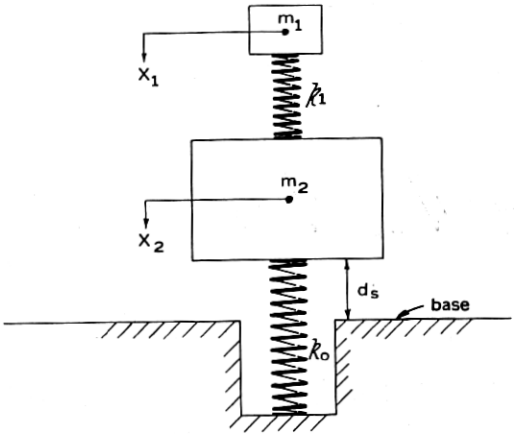


Fig. 3.7.1—Idealized system representing abrupt bottoming.

the same functional form for Region 1 as in the linear case (see equation (3.2.6)), and the amplification factor is, by analogy with (3.2.9),

$$A_0 = \frac{\dfrac{\omega_1}{\omega_0}}{\dfrac{\omega_1}{\omega_0} - 1} \sin \frac{2n\pi}{\dfrac{\omega_1}{\omega_0} + 1}, \qquad 0 < t < t_s, \tag{3.7.1}$$

where

$$\omega_0^2 = k_0/m_2 .$$

For Region 2, we use the differential equation

$$\ddot{x} + \omega_1^2 x = \omega_0 \sqrt{2gh} \sin \omega_0(t - 2t_s) \tag{3.7.2}$$

and, as initial conditions at $t = t_s$, we use the terminal conditions for Region 1 with the sign of $[\dot{x}_2]_{t=t_s}$ reversed. The amplification factor for this region is found to be (by the same method as in Section 3.2):

$$A_0 = \frac{\omega_1^2}{\omega_0 \sqrt{2gh}} \sqrt{A^2 + B^2} \sin (\omega_1 t_m + \eta - \omega_1 t_s) - \frac{1}{1 - \left(\frac{\omega_0}{\omega_1}\right)^2} \sin \left(\omega_0 t_m - \omega_0 t_s - \sin^{-1} \frac{d_s}{d_0} \right), \quad (3.7.3)$$

$$t_s < t < 2t_s$$

where

$$\frac{\omega_1^2}{\omega_0 \sqrt{2gh}} A = -\frac{\frac{\omega_0}{\omega_1}}{1 - \left(\frac{\omega_0}{\omega_1}\right)^2} \sin \left(\frac{\omega_1}{\omega_0} \sin^{-1} \frac{d_s}{d_0} \right)$$

$$\frac{\omega_1^2}{\omega_0 \sqrt{2gh}} B = \frac{\frac{\omega_0}{\omega_1}}{1 - \left(\frac{\omega_0}{\omega_1}\right)^2} \left[2 \frac{\omega_1^2}{\omega_0^2} \sqrt{1 - \frac{d_s^2}{d_0^2}} - \cos \left(\frac{\omega_1}{\omega_0} \sin^{-1} \frac{d_s}{d_0} \right) \right]$$

$$\eta = \tan^{-1} \frac{A}{B}$$

and t_m is the root of

$$\frac{\omega_1^3}{\omega_0^2 \sqrt{2gh}} \sqrt{A^2 + B^2} \cos (\omega_1 t_m + \eta - \omega_1 t_s) - \frac{1}{1 - \left(\frac{\omega_0}{\omega_1}\right)^2} \cos \left(\omega_0 t_m - \omega_0 t_s - \sin^{-1} \frac{d_s}{d_0} \right) = 0$$

that yields the largest value of A_0 in equation (3.7.3). Region 3 is governed by

$$\ddot{x} + \omega_1^2 x = 0 \quad (3.7.4)$$

and the initial conditions are the terminal conditions of Region 2. By the same method as was used in Section 3.2, we find

$$A_0 = \frac{2 \frac{\omega_1}{\omega_0}}{\left(\frac{\omega_1}{\omega_0}\right)^2 - 1} \left[\left(\frac{\omega_1}{\omega_0}\right)^2 \sqrt{1 - \frac{d_s^2}{d_0^2}} - \cos \left(\frac{\omega_1}{\omega_0} \sin^{-1} \frac{d_s}{d_0} \right) \right], \quad (3.7.5)$$

$$t > 2t_s .$$

The largest value of A_0 from equations (3.7.1), (3.7.3) and (3.7.5) is plotted against ω_1/ω_0 in Fig. 3.7.2 for several values of d_s/d_0 .

Notice that the amplification factor is A_0 rather than A_m. That is, the reference acceleration is G_0 rather than G_m. This is necessary because G_m is infinite in the present instance. Hence Fig. 3.7.2 cannot be compared directly with Figs. 3.2.2 and 3.6.2. However, it is interesting to observe that, for $\omega_1/\omega_2 < 0.5$, (low frequency elements) abrupt bottoming has no harmful effect. For high-frequency elements, the severity of bottoming is very great even when very nearly all of the required space (d_0) is available. For example, if 90% of the required space is available ($d_s/d_0 = 0.9$) and the frequency of the element is ten times the package frequency,

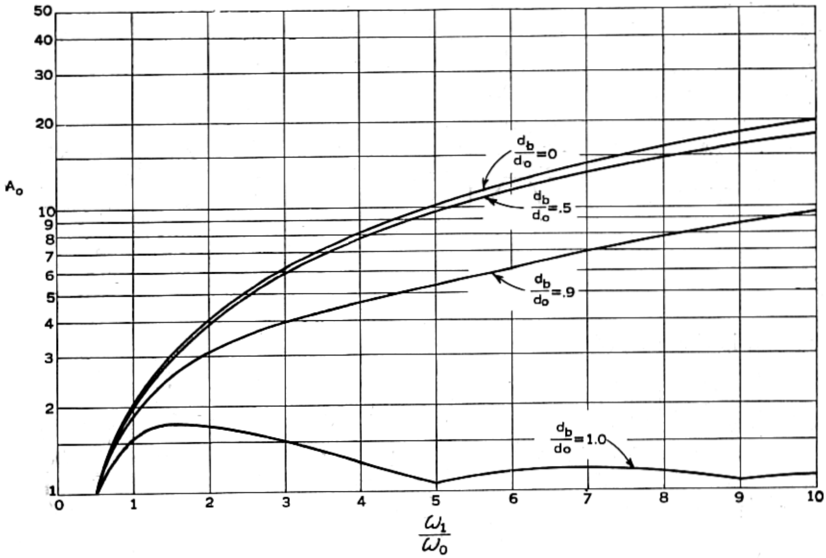


Fig. 3.7.2—Amplification factors for abrupt bottoming. See equations (3.7.1), (3.7.3) and (3.7.5).

the severity of the shock is almost ten times as great as it would be if the additional 10% of space were available.

3.8 General Influence of Shape of Acceleration-Time Curve on Amplification Factor

When amplification factor curves are not available for a special shape of acceleration-time curve, an approximate value of A_m may be obtained by interpolation between or extrapolation from the curves of the preceding sections. The shape of the acceleration-time curve and its duration (τ_2) or frequency (ω_2) should be found, first, by the methods described in Part II. The shape found should then be compared with the standard shapes shown in Part II, for which amplification factors are given in Part III.

The amplification factor found in this way will generally be within 25%

of the true value because amplification curves for pulse accelerations do not differ greatly even for very different acceleration-time curves as long as the

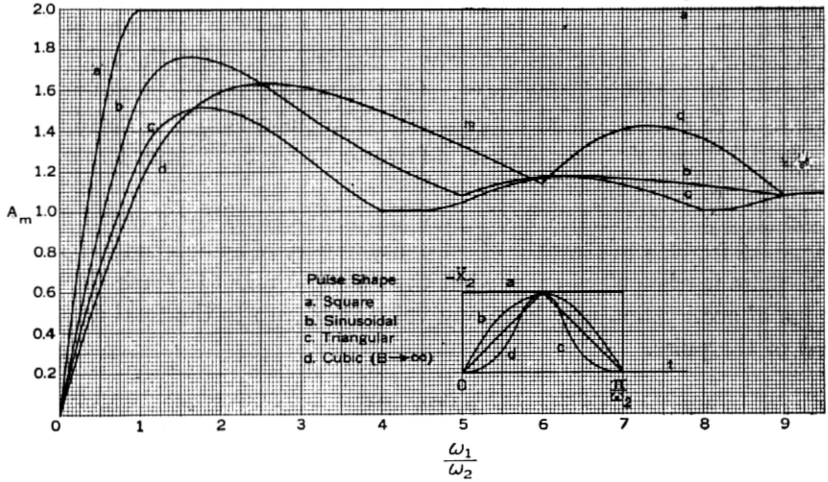


Fig. 3.8.1—Dependence of amplification factor on shape of symmetrical acceleration pulse.

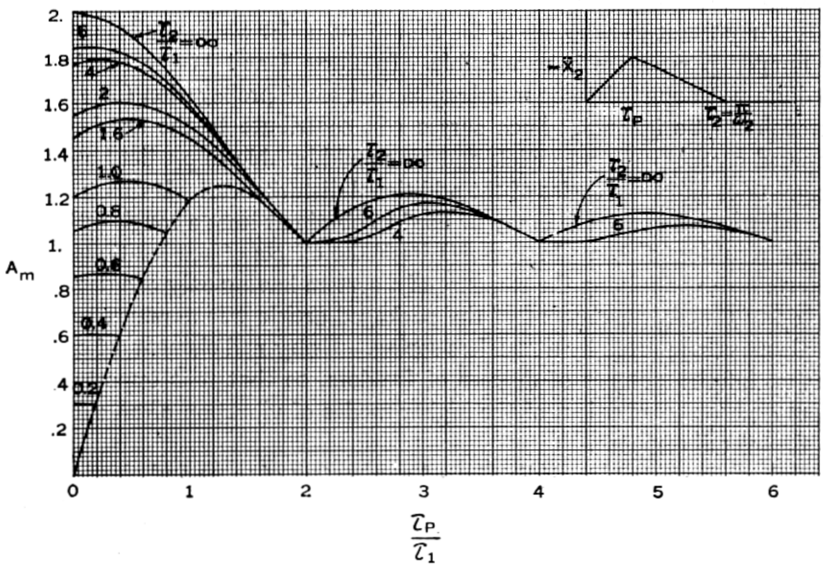


Fig. 3.8.2—Effect of asymmetry of an acceleration pulse on amplification factor.

amplitudes and frequencies are adjusted to the same scales. This is illustrated in Fig. 3.8.1 where the amplification factor curves are drawn for square wave, half-sine wave, triangular and cubic pulses.

Amplification factors for small values of ω_1/ω_2 may be calculated very accurately if it is observed that the initial slope of the amplification factor curve for a pulse acceleration is proportional to the area under the acceleration-time curve. For example, noting that the initial slope of the amplification factor curve for the half-sine wave pulse is 2, we assign the value 2 to the area under the half-sine wave. On the same scale, the area under a square wave pulse is π and under a triangular pulse is $\pi/2$. Accordingly, the initial slopes of the amplification factor curves for the latter two pulses are π and $\pi/2$ respectively.

As an additional aid in finding amplification factors for unusual cases, Fig. 3.8.2 is given to show the effect of asymmetry of an acceleration pulse. The pulse is triangular in shape but the time (τ_P) taken to reach the peak value of acceleration may have any value from zero to the total duration (τ_2) of the pulse.

PART IV

DISTRIBUTED MASS AND ELASTICITY

4.1 INTRODUCTION

It is important to be aware of the conditions under which the assumption of lumped parameters is permissible. In Parts I and II the cushioning medium was assumed to be massless, so that wave propagation (or surges) through it was ignored. Such surges will contribute to the acceleration imposed on the packaged article and we should be able to predict both the magnitudes and frequencies of the additional disturbances. If this is done, the information in Part III may be used to obtain at least a rough estimate of the resulting effects. In Part III itself the effects of accelerations were determined by studying the response of a system having only one degree of freedom; that is, an element of the packaged article was assumed to be a single mass supported by a massless spring. Every real element, of course, has an infinite number of degrees of freedom, so that it is important to discover the contribution, of the higher modes of vibration of an element, to the overall response.

Both of these problems (distributed parameters of mass and elasticity in the cushioning medium and in an element of the packaged article) are studied in this part. One example of each type is considered, and the choice of the example in each case was influenced by considerations of expediency, namely that the mathematical derivations be relatively simple and lead to solutions for which not too lengthy computations are necessary to yield results that can be applied practically. At the same time, the examples chosen are believed to give some insight into several of the physical phenomena involved. The treatment is by no means complete, but a more detailed investigation is beyond the scope of this paper.

4.2 Effect of Distributed Mass and Elasticity of Cushioning on Acceleration of Packaged Article

Referring to Fig. 4.2.1, we consider the packaged article, of mass m_2, to be supported by distributed cushioning of mass m_c and depth ℓ. The cushioning may be a pad, say of rubber, in which case ℓ is the pad thickness, or it may be a helical metal spring, in which case ℓ is the coil length. The package is dropped vertically from a height h and has attained a velocity v at the instant of contact ($t = 0$) of the outer container and the floor. The outer container is assumed to be heavy enough so that there is no rebound. A horizontal plane in the cushioning is located by a coordinate x measured from the end of the cushioning attached to the outer container. The vertical

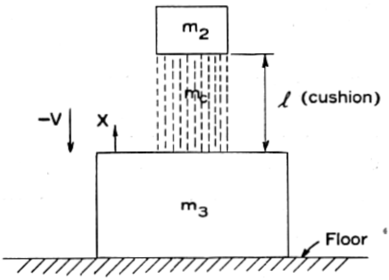


Fig. 4.2.1—Packaged article of mass m_2, supported on distributed cushioning of depth ℓ and mass m_c, depicted at the instant of first contact of the outer container (m_3) and the floor.

displacement of the plane x is designated by u. The undamped motion of the cushioning after contact is governed by the one-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \tag{4.2.1}$$

in which a is the velocity of propagation of longitudinal waves in the cushioning. If the cushioning is continuous,

$$a^2 = \frac{E}{\rho}, \tag{4.2.2}$$

where E is the modulus of elasticity and ρ is the density of the cushioning. If the cushioning is a helical spring,

$$a^2 = \frac{k\ell^2}{m_c}, \tag{4.2.3}$$

where k is the spring rate.

The initial conditions of the system are

$$[u]_{t=0} = 0, \tag{4.2.4}$$

$$\left[\frac{\partial u}{\partial t}\right]_{t=0} = -v. \tag{4.2.5}$$

The boundary conditions are

$$[u]_{x=0} = 0, \tag{4.2.6}$$

$$k\ell \left[\frac{\partial u}{\partial x}\right]_{x=\ell} = -m_2 \left[\frac{\partial^2 u}{\partial t^2}\right]_{x=\ell}. \tag{4.2.7}$$

Equation (4.2.7) expresses the requirement that the force on the upper end of the cushioning must balance the inertia force of the packaged article. For continuous cushioning $k\ell$ should be replaced by EA, where A is the cross-sectional area of the cushioning.

A solution of (4.2.1) satisfying conditions (4.2.4) and (4.2.6) is

$$u = \sum_{n=1}^{\infty} A_n \sin \frac{\omega_n x}{a} \sin \omega_n t, \tag{4.2.8}$$

where ω_n is the n^{th} root of a transcendental equation to be obtained from (4.2.7) and A_n is a constant to be determined by (4.2.5). Substituting (4.2.8) in (4.2.7) and equating coefficients of like terms of the series, we obtain the transcendental equation

$$\frac{\omega_n \ell}{a} \tan \frac{\omega_n \ell}{a} = \frac{m_c}{m_2}. \tag{4.2.9}$$

Substituting (4.2.8) in (4.2.5) we obtain, by the usual methods of expansion into trigonometric series,

$$A_n = -\frac{2v}{\omega_n \left(\frac{\omega_n \ell}{a} + \frac{1}{2} \sin \frac{2\omega_n \ell}{a}\right)}. \tag{4.2.10}$$

Hence the complete solution of the problem is

$$u = -\sum_{n=1}^{\infty} \frac{2v \sin \frac{\omega_n x}{a} \sin \omega_n t}{\omega_n \left(\frac{\omega_n \ell}{a} + \frac{1}{2} \sin \frac{2\omega_n \ell}{a}\right)}. \tag{4.2.11}$$

Our chief interest is in the acceleration of m_2. Making use of (4.2.7) and (4.2.9) we find, from (4.2.11), that this acceleration is

$$\left[\frac{\partial^2 u}{\partial t^2}\right]_{x=\ell} = v\omega_0 \sum_{n=1}^{\infty} B_n \sin \omega_n t, \tag{4.2.12}$$

where

$$\omega_0^2 = \frac{k}{m_2} \tag{4.2.13}$$

and

$$B_n = \frac{2\sqrt{\dfrac{m_c}{m_2}\left(\dfrac{m_c^2}{m_2^2} + \dfrac{\omega_n^2 \ell^2}{a^2}\right)}}{\dfrac{m_c}{m_2} + \dfrac{m_c^2}{m_2^2} + \dfrac{\omega_n^2 \ell^2}{a^2}}. \tag{4.2.14}$$

The acceleration of m_2 is, therefore, a sum of sinusoids of frequency ω_n and amplitude $v\omega_0 B_n$. Now, $v\omega_0$ is the maximum acceleration that m_2 would attain if the mass of the cushioning were negligible. Calling G_n the maximum acceleration in the n^{th} mode and G_0 the maximum acceleration neglecting the mass of the cushioning, as in Part I, we have

$$\frac{G_n}{G_0} = B_n. \tag{4.2.15}$$

But B_n depends only on the ratio m_c/m_2, as may be seen from equations (4.2.9) and (4.2.14). Similarly the ratio of the frequency (ω_n) of any mode to the frequency (ω_0) with massless cushioning depends only on m_c/m_2, as may be seen from equations (4.2.3), (4.2.9) and (4.2.13). *Hence, both the amplitude and frequency ratios for the acceleration in any mode depend only on the ratio of the mass of the cushioning to the mass of the packaged article.* The ratios G_n/G_0 and ω_n/ω_0 are plotted against m_c/m_2 in Figs. 4.2.2 and 4.2.3 for the first five modes. *It may be seen from these figures that the accelerations in the higher modes can be very important.* For example, if the cushioning weighs half as much as the packaged article the maximum acceleration in the second mode is about 40% of the acceleration in the first mode and the latter is about the same as found by the elementary method of Part I. This could have a disastrous effect on an element of the packaged article if the latter had a fundamental frequency near that of the second mode of the cushioning, the latter being found, from Fig. 4.2.3, to be about five times the fundamental frequency of the package.

It must be remembered that damping has been neglected in the above investigation and damping in the cushioning will serve to mitigate the severity of the higher mode accelerations to a great extent. However, the danger is always present at the start of a design and the possibilities of unfavorable combinations should be studied in every case.

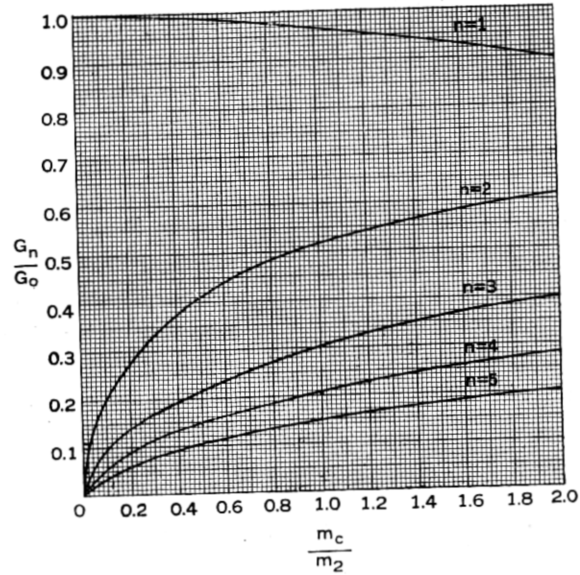


Fig. 4.2.2—Influence of ratio of mass of cushioning (m_c) to mass of packaged article (m_2) on acceleration ratio. The numerator of the acceleration ratio is the maximum acceleration (G_n) in the n^{th} mode of vibration transmitted through the cushioning. The denominator of the acceleration ratio is the maximum acceleration ($G_0 = \sqrt{2hk_2/m_2g}$) that the mass m_2 would experience if the mass of the cushioning were negligible. See equations (4.2.15), (4.2.14), (4.2.9).

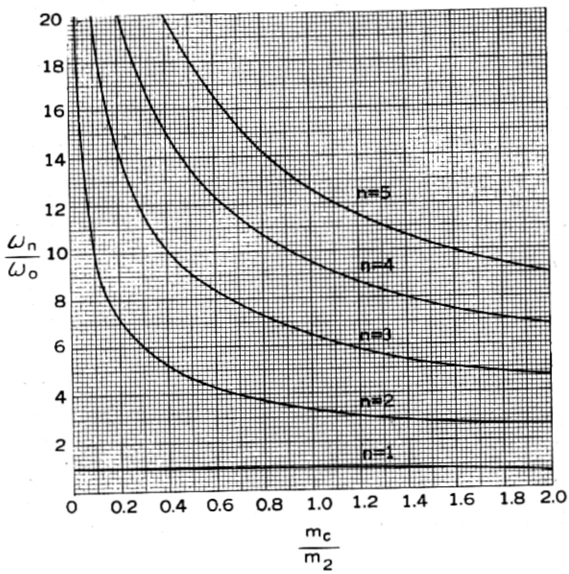


Fig. 4.2.3—Influence of ratio of mass of cushioning (m_c) to mass of packaged article on frequency ratio. The numerator of the frequency ratio is the frequency (ω_n) of the n^{th} mode of vibration transmitted through the cushioning. The denominator of the frequency ratio is the frequency ($\omega_o = \sqrt{k_2/m_2}$) of vibration of the mass m_2 neglecting

4.3 Effect of Distributed Mass and Elasticity, of an Element of the Packaged Article, on the Amplification Factor for a Half-Sine-Wave Pulse Acceleration

In this section we shall determine the contribution of the higher modes of vibration of a structural element to its total response to a half-sine-wave pulse acceleration. For the shape of the element, we choose a prismatic bar because this leads to the simplest mathematical formulation of the problem and such a bar is also a common structural element. Other considerations influence the choice of direction of acceleration with respect to the axis of the bar. The transverse direction (cantilever) is the most practical from a physical standpoint, but, for purposes of comparison with the one-degree-of-freedom system, the parallel (axial) direction of acceleration is the more logical. Both problems lead to solutions in the form of infinite series, but, in the latter case, the expression for the strain at a fixed

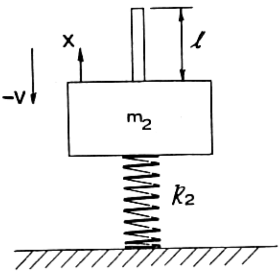


Fig. 4.3.1—The system studied in Section 4.3 depicted at the instant of contact with the floor.

end can be summed in terms of elementary functions without difficulty. Since it is necessary to determine maximum values of strain over a wide range of frequency ratios for the plotting of an amplification factor curve, an enormous reduction in the time required for accurate computations is obtained by choosing the axial case. Furthermore, the axial case appears to contain the essential features which might result in differences between the response of a one-degree-of-freedom system and a continuous one.

The complete system to be studied is illustrated in Fig. 4.3.1. To the mass m_2, supported on massless cushioning of constant spring rate k_2, is attached one end of an elastic prismatic bar, of length ℓ, cross sectional area A, modulus of elasticity E, and density ρ, with its axis oriented vertically. The system is dropped from a height h so that its velocity is v at the instant of contact of the cushioning with the floor. The mass of the bar is supposed to be small in comparison with m_2 and perfect rebound is assumed, so that the motion of m_2 during contact is a half-sine wave of frequency

$$\omega_2 = \sqrt{\frac{k_2}{m_2}}. \tag{4.3.1}$$

The maximum acceleration of m_2 is thus $v\omega_2$. If this magnitude of acceleration were reached very slowly, so as not to excite transient longitudinal waves in the bar, the maximum force between the bar and m_2 would be the product of the acceleration and the mass of the bar:

$$F = v\omega_2 \rho A \ell. \tag{4.3.2}$$

Hence the strain at the end of the bar attached to m_2 would be

$$\epsilon_0 = \frac{v\omega_2 \rho \ell}{E}. \tag{4.3.3}$$

Our problem is to find the ratio of the maximum transient strain to ϵ_0.

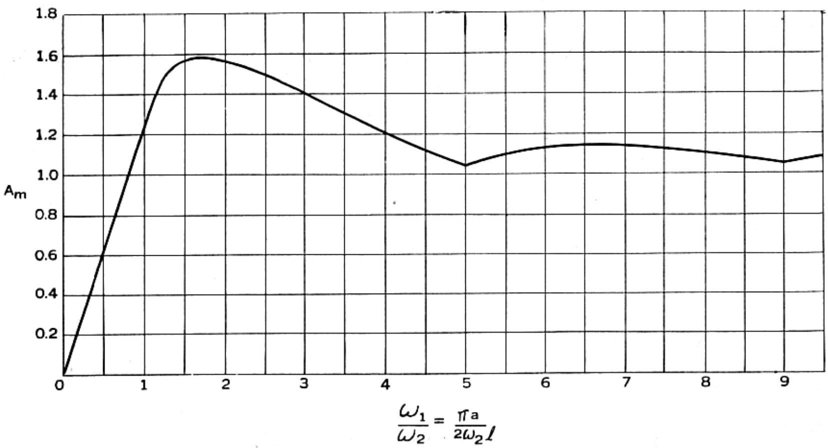


Fig. 4.3.2—Amplification factors for an element of the packaged article having distributed mass and elasticity. The package has linear undamped cushioning and perfect rebound. See equations (4.3.15), and (4.3.14).

Let u be the displacement of a transverse plane section of the bar distant x from the end attached to m_2. Then, the equation of motion of the bar is

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \tag{4.3.4}$$

where a is the velocity of propagation of longitudinal waves in the bar:

$$a^2 = \frac{E}{\rho}. \tag{4.3.5}$$

Taking the instant of first contact of the cushioning with the floor to be $t = 0$, we know, from Part II, that the system will leave the floor when $t =$

π/ω_2. We shall therefore treat separately, as in Part III, the motion during contact

$$0 \lesseqgtr t \lesseqgtr \frac{\pi}{\omega_2}$$

and after rebound

$$t \gtreqless \frac{\pi}{\omega_2}.$$

During the first interval, the initial and boundary conditions are

$$[u]_{t=0} = 0, \tag{4.3.6}$$

$$\left[\frac{\partial u}{\partial t}\right]_{t=0} = -v, \tag{4.3.7}$$

$$[u]_{x=0} = \frac{-v}{\omega_2} \sin \omega_2 t, \tag{4.3.8}$$

$$\left[\frac{\partial u}{\partial x}\right]_{x=l} = 0. \tag{4.3.9}$$

The first and second conditions state that, at the instant of contact, all points in the bar are moving with the approach velocity v, without relative displacement. The third condition prescribes the half-sine wave motion of the end of the bar that is attached to m_2. The fourth condition states that the strain at the free end of the bar is always zero.

By the usual methods, a solution of (4.3.4) satisfying conditions (4.3.6) to (4.3.9) is found to be

$$u = -\frac{v \cos \frac{\omega_2}{a}(\ell - x) \sin \omega_2 t}{\omega_2 \cos \frac{\omega_2 \ell}{a}} + \frac{8v\ell}{\pi^2 a} \sum_{n=1,3,5\cdots}^{\infty} \frac{\sin \frac{n\pi x}{2\ell} \sin \frac{n\pi a t}{2\ell}}{n^2\left[\left(\frac{n\pi a}{2\omega_2 \ell}\right)^2 - 1\right]} \tag{4.3.10}$$

$$\left(0 \lesseqgtr t \lesseqgtr \frac{\pi}{\omega_2}, \frac{n\pi a}{2\omega_2\ell} \neq 1\right).$$

The displacement is seen to be a forced vibration at the frequency (ω_2) of the applied acceleration, on which are superposed the free vibrations of the bar given by the series expression. The frequency of the fundamental mode of vibration of the bar is $\pi a/2\ell$ and the frequencies of the higher modes are the odd integral multiples of the fundamental.

The strain at the attached end of the bar is

$$\epsilon = \left[\frac{\partial u}{\partial x}\right]_{x=0}$$

$$= -\frac{v}{a}\left\{\tan\frac{\omega_2\ell}{a}\sin\omega_2 t - \frac{4}{\pi}\sum_{n=1,3,5\cdots}^{\infty}\frac{\sin\dfrac{n\pi a t}{2\ell}}{n\left[\left(\dfrac{n\pi a}{2\omega_2\ell}\right)^2 - 1\right]}\right\} \tag{4.3.11}$$

$$\left(0 \gtreqless t \gtreqless \frac{\pi}{\omega_2}, \frac{n\pi a}{2\omega_2\ell} \neq 1\right).$$

It may be verified that the sum of the series in (4.3.11) is given by

$$\frac{4}{\pi}\sum_{n=1,3,5\cdots}^{\infty}\frac{\sin\dfrac{n\pi a t}{2\ell}}{n\left[\left(\dfrac{n\pi a}{2\omega_2\ell}\right)^2 - 1\right]} = \tan\frac{\omega_2\ell}{a}\sin\omega_2 t + \cos\omega_2 t - 1, \tag{4.3.12}$$

$$\left(0 < t < \frac{2\ell}{a}\right).$$

It should be observed that the summation is valid only in the interval $0 < t < 2\ell/a$. However, the series is periodic with half period $2\ell/a$ and includes only the odd terms, so that the function repeats itself with reversed sign after each interval $2\ell/a$. Hence the summation, valid for all t, can be written

$$\frac{4}{\pi}\sum_{n=1,3,5\cdots}^{\infty}\frac{\sin\dfrac{n\pi a t}{2\ell}}{n\left[\left(\dfrac{n\pi a}{2\omega_2\ell}\right)^2 - 1\right]}$$

$$= (-1)^k\left[\tan\frac{\omega_2\ell}{a}\sin\omega_2\left(t - \frac{2k\ell}{a}\right) + \cos\omega_2\left(t - \frac{2k\ell}{a}\right) - 1\right] \tag{4.3.13}$$

$$k = m \quad \text{when} \quad \frac{2m\ell}{a} < t < \frac{2(m+1)\ell}{a}$$

$$m = 0, 1, 2, 3, \cdots.$$

We may, therefore, rewrite (4.3.11) in the form

$$\frac{\epsilon a}{v} = -\tan\frac{\omega_2\ell}{a}\sin\omega_2 t + (-1)^k\left[\tan\frac{\omega_2\ell}{a}\sin\omega_2\left(t - \frac{2k\ell}{a}\right) + \cos\omega_2\left(t - \frac{2k\ell}{a}\right) - 1\right] \tag{4.3.14}$$

$$0 \lesseqgtr t \lesseqgtr \frac{\pi}{\omega_2}$$

$$k = m \text{ when } \frac{2m\ell}{a} < t < \frac{2(m+1)\ell}{a}$$

$$m = 0, 1, 2, 3 \cdots .$$

The expression (4.3.14) is simple enough so that the maximum value (ϵ_m) of the strain at the attached end can be obtained without difficulty for any ratio of the fundamental frequency ($\omega_1 = \pi a/2\ell$) of the bar to the frequency (ω_2) of the disturbing acceleration. The amplification factor

$$A_m = \left| \frac{\epsilon_m}{\epsilon_0} \right| = \frac{|\epsilon_m| E}{v\omega_2 \rho \ell} = \frac{2\omega_1 a}{\pi \omega_2 v} |\epsilon_m|, \tag{4.3.15}$$

may then be calculated. The results of these calculations are plotted in Fig. 4.3.2. The important feature of this curve is that the amplification factor is everywhere less than the corresponding amplification factor for the one-degree-of-freedom system (Fig. 3.2.2, $\beta_1 = 0$). *Hence the assumption of lumped parameters is on the side of safety as regards amplification factor.*

It is interesting to observe that the curve of A_m vs. ω_1/ω_2, for this case, is a straight line between $\omega_1/\omega_2 = 0$ and $\omega_1/\omega_2 = 1$. This arises from the fact that, for $\omega_1/\omega_2 \lesseqgtr 1$, equation (4.3.14) reduces to

$$\frac{\epsilon a}{v} = \cos \omega_2 t - 1, \qquad \left(\frac{\omega_1}{\omega_2} \lesseqgtr 1 \right). \tag{4.3.16}$$

Hence, when the duration of shock is less than the half period of the fundamental mode of vibration, the maximum value of strain occurs at the end of impact and is equal to twice the ratio of the approach velocity to the velocity of wave propagation in the bar.

The whole solution of the problem is not yet completed; for, although it is fairly evident from the fact that there is at least one maximum in the interval $0 \lesseqgtr t \lesseqgtr \pi/\omega_2$ for all values of ω_1/ω_2, it must be verified that the maximum strain (and, therefore, the amplification factor) is never greater after $t = \pi/\omega_2$ than before. Defining a new time coordinate

$$t' = t - \frac{\pi}{\omega_2}, \tag{4.3.17}$$

we have, for the initial and boundary conditions of equation (4.3.4) for $t \gtreqless \pi/\omega_2$,

$$[u]_{t'=0} = \frac{8v\ell}{\pi^2 a} \sum_{n=1,3,5\cdots}^{\infty} \frac{\sin \dfrac{na\pi^2}{2\omega_2\ell} \sin \dfrac{n\pi x}{2\ell}}{n^2\left[\left(\dfrac{n\pi a}{2\omega_2\ell}\right)^2 - 1\right]} \tag{4.3.18}$$

$$\left[\frac{\partial u}{\partial t}\right]_{t'=0} = \frac{v \cos \dfrac{\omega_2}{a}(\ell - x)}{\cos \dfrac{\omega_2 \ell}{a}} + \frac{4v}{\pi} \sum_{n=1,3,5\cdots}^{\infty} \frac{\cos \dfrac{na\pi^2}{2\omega_2\ell} \sin \dfrac{n\pi x}{2\ell}}{n\left[\left(\dfrac{n\pi a}{2\omega_2\ell}\right)^2 - 1\right]} \tag{4.3.19}$$

$$[u]_{x=0} = vt', \tag{4.3.20}$$

$$\left[\frac{\partial u}{\partial x}\right]_{x=l} = 0. \tag{4.3.21}$$

The first and second conditions state that the displacement and velocity of every point in the bar must be the same at the beginning of the second interval as at the end of the first interval; the expressions in (4.3.18) and (4.3.19) are obtained from (4.3.10). The third condition prescribes the constant velocity of departure from the floor of the mass m_2 and, therefore, of the end of the bar attached to it. The last condition states, again, that the strain at the free end of the bar is zero.

It may be verified that a solution of (4.3.4) satisfying conditions (4.3.18) to (4.3.21) is

$$u = vt' + \sum_{n=1,3,5\cdots}^{\infty} C_n \sin \frac{n\pi x}{\ell} \sin\left(\frac{n\pi a t'}{2\ell} + \gamma_n\right) \tag{4.3.22}$$

$$(t' \gtreqless 0,\ t \gtreqless \pi/\omega_2),$$

where

$$C_n \sin \gamma_n = \frac{8v\ell \sin \dfrac{na\pi^2}{2\omega_2\ell}}{\pi^2 a n^2\left[\left(\dfrac{n\pi a}{2\omega_2\ell}\right)^2 - 1\right]} \tag{4.3.23}$$

$$C_n \cos \gamma_n = \frac{8v\ell \left(1 + \cos \dfrac{na\pi^2}{2\omega_2\ell}\right)}{\pi^2 a n^2\left[\left(\dfrac{n\pi a}{2\omega_2\ell}\right)^2 - 1\right]} \tag{4.3.24}$$

$$\frac{n\pi a}{2\omega_2\ell} \neq 1.$$

Hence, the strain at the attached end of the bar is

$$\epsilon = \left[\frac{\partial u}{\partial x}\right]_{x=0} = \frac{4v}{\pi a} \sum_{n=1,3,5\cdots}^{\infty} \frac{\sin \frac{n\pi a t}{2\ell}}{n\left[\left(\frac{n\pi a}{2\omega_2 \ell}\right)^2 - 1\right]} + \frac{4v}{\pi a} \sum_{n=1,3,5\cdots}^{\infty} \frac{\sin \frac{n\pi a t'}{2\ell}}{n\left[\left(\frac{n\pi a}{2\omega_2 \ell}\right)^2 - 1\right]}. \tag{4.3.25}$$

The two series may be summed, as before, with the result

$$\frac{\epsilon a}{v} = (-1)^k \left[\tan \frac{\omega_2 \ell}{a} \sin \omega_2\left(t - \frac{2k\ell}{a}\right) + \cos \omega_2\left(t - \frac{2k\ell}{a}\right) - 1\right]$$
$$+ (-1)^{k'} \left[\tan \frac{\omega_2 \ell}{a} \sin \omega_2\left(t' - \frac{2k'\ell}{a}\right) + \cos \omega_2\left(t' - \frac{2k'\ell}{a}\right) - 1\right] \tag{4.3.26}$$

$$t \geqq \pi/\omega_2, \qquad t' = t - \pi/\omega_2$$

$$k = m \quad \text{when} \quad \frac{2m\ell}{a} < t < \frac{2(m+1)\ell}{a},$$

$$k' = m' \quad \text{when} \quad \frac{2m'\ell}{a} < t' < \frac{2(m'+1)\ell}{a}$$

$$m = 0, 1, 2, 3 \cdots \qquad m' = 0, 1, 2, 3 \cdots .$$

Once more, the expression for the strain at the attached end of the bar is in a form suitable for rapid calculation and it can be shown the ϵ in equation (4.3.26) for $t \geqq \pi/\omega_2$ is never greater than the ϵ in equation (4.3.14) for $0 \leqq t \leqq \pi/\omega_2$ for the same ω_1/ω_2. Hence, Fig. 4.3.2 and the conclusions following equations (4.3.15) and (4.3.16) need not be modified.

NOTATIONS

A	Cross sectional area of a bar element of the packaged article. Also, a constant of integration.
A_0	Amplification factor when the reference acceleration is G_0. Ratio of maximum dynamic response to the response to a slowly applied acceleration of magnitude $G_0 g$.
A_m	Amplification factor when the reference acceleration is G_m. Ratio of maximum dynamic response to the response to a slowly applied acceleration of magnitude $G_m g$.
A_n	In Section 1.15, the sum of all the trapezoidal areas from $x_2 = 0$ to $x_2 = (x_2)_n$. Also, in Section 4.2, the coefficient of the nth term of a series.
ΔA_n	The area of a trapezoid with altitude $\Delta(x_2)_n$ and sides P_{n-1} and P_n.
a	x_0/l in the tension spring package. Also, in Part IV, the velocity of propagation of longitudinal waves.

B	A parameter of cushioning with cubic elasticity defined in equation (1.5.3). Also, a constant of integration.
B_n	Coefficient in the n^{th} term of a series.
b	f/l in the tension spring package.
C	A constant of integration.
C_n	Coefficient of the n^{th} term of a series.
c	A constant defined in equation (1.7.11)
c_1	Damping coefficient of an element of a packaged article.
c_2	Damping coefficient of linear cushioning.
cn	The elliptic cosine function.
d_0	Hypothetical displacement that would result if initial spring rate were maintained.
d_b	Maximum possible displacement of packaged article in cushioning with tangent elasticity.
d_m	Maximum displacement of packaged article.
d'_m	Value of d_m when $k_0 = k'_0$.
d_s	Displacement of bi-linear cushioning at which the spring rate changes from k_0 to k_b.
E	Modulus of elasticity.
e	In the tension spring package the stretch of a spring when the displacement is d_m.
exp ( )	$e^{(\,)}$, where e is the Naperian base $2.718\cdots$
F	In section 2.7, a frictional force.
F_m	In the tension spring package, the maximum force on a spring.
f	In the tension spring package, the difference between l and the distance between hooks of an unstretched spring.
f_1	Frequency of vibration of an element of the packaged article.
f_2	Frequency of vibration of the packaged article on its cushioning.
G_0	Hypothetical maximum acceleration (in number of times g) that would result if initial spring rate were maintained.
G_e	$A_m G_m$ or $A_0 G_0$, i.e. the slowly applied acceleration (in number of times g) that will produce the same maximum response as a transient acceleration of maximum value G_m or G_0.
G_F	Maximum acceleration (in number of times g) in cushioning with friction and spring rate k_F.
G_m	Absolute value of maximum acceleration of packaged article in units of "number of times gravitational acceleration."
G_m'	Value of G_m when $k_0 = k'_o$.

G_n — In section 1.15, the maximum acceleration (in number of times g) experienced by the suspended mass when dropped from a height h_n. In Part IV, the maximum acceleration (in number of times g) of the n^th mode of vibration.

G_r — Maximum acceleration (in number of times g) after rebound.

G_s — Safe value of G_e.

g — Gravitational acceleration.

h — Height of drop.

h_n — In Section 1.15, the height of fall that will cause the cushioning to displace an amount $(x_2)_n$.

K — In the tension spring package, the initial spring rate of the suspension. In Section 2.8, the complete elliptic integral of the first kind.

K_1, K_2, K_3 — The initial spring rates in the three mutually perpendicular directions normal to the faces of the package frame.

k — In the tension spring package, the spring rate of a spring. In Section 2.8, the modulus of an elliptic integral.

k, k' — In Section 4.3, 0, 1, 2, 3, $\cdots$.

k_0 — Initial spring rate of non-linear cushioning.

k_0' — Optimum value of initial spring rate k_0.

k_1 — Spring rate of lumped elasticity of element of packaged article.

k_2 — Spring rate of linear cushioning.

k_b — Spring rate of bilinear cushioning after bottoming.

k_F — Spring rate defined in equation (2.7.7).

L — Constant defined in equation (1.8.2).

l — In the tension spring, the projection of l_i on a horizontal plane. In Section 4.2, length of cushioning. In Section 4.3, length of element of packaged article.

l_i — In the tension spring package, the distance between the two support points of a spring when the suspended article is in the equilibrium position.

M — Constant defined in equation (1.8.4), equal to G_m/G_0.

m — Reduced mass defined in equation (2.4.5).

m, m' — In Section 4.3, 0, 1, 2, 3, $\cdots$.

m_1 — Lumped mass of element of packaged article.

m_2 — Lumped mass of packaged article.

m_3 — Lumped mass of outer container.

m_c — Mass of cushioning.

N — M^2.

n	0, 1, 2, 3, $\cdots$.
P	Force transmitted through cushioning.
P_0	Asymptotic value of force transmissible through cushioning with hyperbolic tangent elasticity.
P_m	Maximum force exerted on packaged article by cushioning.
P_n	In Section 1.15, the load that produces displacement $(x_2)_n$.
R	Force between package and floor.
r	Coefficient of cubic term in load-displacement function for cushioning with cubic elasticity.
s, t, u	The direction cosines of the acceleration direction with respect to the normals to the faces of the package frame.
sn	The elliptic sine function.
T_2	The period of vibration of the packaged article on its cushioning.
t	Time coordinate.
t'	$t - \frac{\pi}{\omega_2}$
t_0	Time of first contact of package with floor.
t_m	Time at which maximum displacement or acceleration occurs.
t_r	Time at which package leaves floor on rebound.
t_s	Time at which the displacement reaches d_s.
u	Displacement in x direction.
v	Approach velocity.
W_2	Weight of packaged article.
W_3	Weight of outer container.
x	$x_1 - x_2$; relative displacement of m_1 with respect to m_2.
$\dot{x}$	$\dot{x}_1 - \dot{x}_2$.
$\ddot{x}$	$\ddot{x}_1 - \ddot{x}_2$.
x'	Relative displacement of m_1 with respect to m_2 at time t'.
x_0	In the tension spring package, the perpendicular distance from an inner spring support point to the nearest plane, perpendicular to the displacement direction and containing four outer spring support points.
x_1	Displacement of m_1.
$\dot{x}_1$	Velocity of m_1.
$\ddot{x}_1$	Acceleration of m_1.
x_2	Displacement of m_2.
$\dot{x}_2$	Velocity of m_2.

$\ddot{x}_2$	Acceleration of m_2.
$x_{\max}$	Maximum value of x.
$(x_2)_n$	In Section 1.15, the displacement associated with the n^{th} point.
x_{st}	The value x would have if the acceleration reached its maximum value in a very long time.
$(\Delta x_2)_n$	In Section 1.15, equals $(x_2)_n - (x_2)_{n-1}$.
y	$x_2 - x_3$.
z	x_2/l (tension spring package).
$\alpha, \gamma, \delta, \zeta, \eta$	Phase angles.
β_1	Fraction of critical damping of an element of the packaged article.
β_2	Fraction of critical damping of package cushioning.
γ_n	Phase angle of n^{th} term of series (equation (4.3.22)).
ϵ	Strain at attached end of element under transient conditions.
ϵ_0	Strain at attached end of element under non-transient conditions.
ϵ_m	Maximum strain at attached end of element under transient conditions.
θ	Angle between the displacement direction and the acceleration direction.
π	$3.14159\cdots$.
ρ	Density (mass per unit of volume)
τ_0	Pulse duration of a half-sine-wave acceleration.
τ_2	Pulse duration associated with non-linear cushioning.
τ_B	Duration of bottoming of cushioning with bi-linear elasticity.
τ_P	Time required to reach peak value of a triangular acceleration pulse.
ω	Radian frequency defined in equation (2.4.6).
ω_1	Radian frequency of vibration of an element of the packaged article.
ω_1'	Radian frequency of vibration of damped element of packaged article.
ω_2	Radian frequency of vibration of the packaged article on its cushioning.
ω_2'	Radian frequency of vibration of the packaged article on damped cushioning.
ω_b	A frequency defined in equation (2.10.10).
ω_c	A frequency defined in equation (2.8.8).
ω_n	Radian frequency of nth mode.