The Fundamental Equations of Electron Motion (Dynamics of High Speed Particles)

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I. Introduction

In work relating to the motion of electrons and other particles it is fairly common to assume that the particles obey the laws of Newtonian dynamics. That is, briefly, it is assumed that the rectangular coordinates (x, y, z) of the particle under consideration satisfy the differential equations

$$m\ddot{x} = X, \qquad m\ddot{y} = Y, \qquad m\ddot{z} = Z,$$

where m is the mass of the particle (assumed constant), X, Y, and Z are the components of the applied force, and the dots indicate differentiation with respect to the time t.

However, it is well recognized now that the above equations are not strictly correct, and that they merely represent an approximation which is adequate when the speed of the particle is sufficiently small compared with the speed of light. The system of dynamics based upon the correct equations[1] (which will be exhibited presently) is commonly called *relativistic dynamics*, not because any knowledge of the theory of relativity is essential to its understanding and use[2], but because it is in agreement with the theory of relativity (which Newtonian dynamics is not), because it was first developed in connection with work on the theory of relativity, and because even yet virtually all of the expositions of the subject are to be found in books and papers dealing primarily with the theory of relativity.

Just where the dividing line should be set between cases in which Newtonian dynamics is an adequate approximation and cases in which it is necessary to use relativistic dynamics is, of course, a rather vague question which cannot be answered simply and definitely. We may note, however,

[1] It is not the purpose of this article to discuss questions of fundamental physics, or the physical validity of any particular equations. For purposes of discussion, we assume outright that relativistic dynamics is at least more nearly correct than is Newtonian dynamics.

[2] The theory of relativity can be described briefly as a theory of the relations between the descriptions of phenomena in terms of different systems of reference. We shall *not* be concerned with this theory, because we shall be employing the same reference system throughout most of our discussion. In the final section of the paper we shall consider purely geometrical transformations of the coordinate system. These transformations, however, involve nothing that is really characteristic of the theory of relativity in the usual sense.

that according to relativistic dynamics the mass of a five thousand volt electron is about one per cent greater than the mass of an electron at rest. From this we can infer that, while Newtonian dynamics may be adequate for many purposes in our studies of electron motion, we do not have any great amount of margin, and that it will be necessary to use relativistic dynamics whenever we wish to obtain really good results concerning the motion of even moderately high speed electrons.

This article is purely expository. Its purpose is to set forth the fundamental equations and theorems of relativistic particle dynamics in a clear and concise form, unencumbered with any material relating to the theory of relativity proper. Almost all of the material is to be regarded as already known, but apparently it is only to be found in an inconvenient and scattered form. The incomplete bibliography at the end of the paper gives references to some of the more accessible sources of this and other related material.

II. The Elementary Differential Equations of Motion

Our discussion might be begun in any one of a number of ways, and no doubt the different approaches would appeal unequally to different readers. Considering the nature and purposes of this article, the author has deemed it best to begin by writing down at once the differential equations of motion of a particle (according to relativistic dynamics) in their most elementary form. Then, for the purposes of this discussion, these equations will have the status of a fundamental assumption. It need hardly be said that the equations are not written down arbitrarily. On the contrary, they represent the consensus of modern opinion as to the laws under which particles really do move.[3] The grounds, experimental and theoretical, for this opinion are set forth in various of the works cited in the bibliography.

For the time being, until the contrary is stated in the final section, we employ a fixed rectangular coordinate system. Instead of denoting the coordinates of the particle by x, y, and z, as we have done provisionally in the Introduction, we shall denote them by x_1, x_2, and x_3. Then $\dot{x}_1$, $\dot{x}_2$, and $\dot{x}_3$ denote the components of the velocity of the particle. The components of the force acting on the particle will be denoted by X_1, X_2, and X_3. For the time being we need only note that the force may depend upon the coordinates, the velocity, and the time; later on we shall introduce some more explicit assumptions about the force. The symbol c will be used to denote the speed of light in vacuo.

[3] The validity of these laws is not unrestricted. It is limited on the one hand by the quantum phenomena which become appreciable on the atomic scale, and on the other hand by certain phenomena revealed by the general theory of relativity which become appreciable on the cosmic scale.

We assume that the particle moves, under the influence of the force (X_1, X_2, X_3), so that its coordinates satisfy the system of differential equations

$$\frac{d}{dt}\frac{m_0\,\dot{x}_n}{\sqrt{1-(v^2/c^2)}} = X_n, \qquad (n = 1, 2, 3), \tag{1}$$

where m_0 is a positive constant characteristic of the particle, and v^2 is an abbreviation for the expression $\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2$.* The positive value of the square root is the significant one; and wherever square roots appear in the subsequent work it will be understood, unless the contrary is stated, that the positive values are intended.

A few remarks may help bring out the significance of the foregoing assumption and its relations to the corresponding fundamental assumption of Newtonian dynamics.

We call the constant m_0 the *rest-mass* of the particle, and we assume (in accordance with the experimental evidence) that m_0 is identical with the mass of the particle which is used in Newtonian dynamics. In relativistic dynamics the quantity m defined by the equation

$$m = \frac{m_0}{\sqrt{1-(v^2/c^2)}}$$

is called the *mass* of the particle. We note that as v/c approaches zero the mass approaches the rest-mass (whence the appropriateness of the latter term), and that as v/c approaches unity the mass increases without limit.

Consider the vector having the components p_1, p_2, p_3 defined by the formulae

$$p_n = \frac{m_0\,\dot{x}_n}{\sqrt{1-(v^2/c^2)}}. \tag{2}$$

We call this vector the *momentum* of the particle. The momentum is equal to the velocity of the particle multiplied by the mass.

Now equations (1) assert that the time-rate of change of the momentum of the particle is equal to the applied force.

We have already observed that as v/c approaches zero the relativistic mass of a particle approaches the Newtonian mass. We now note that as v/c approaches zero the components of the relativistic momentum approach the values

$$p_n = m_0\,\dot{x}_n, \tag{2'}$$

* We might merely say that v is the speed of the particle. However, for our immediate purposes, it is important not to lose sight of the fact that v is a certain particular function of the components of velocity.

which are precisely the components of the momentum according to the Newtonian theory.

Finally, as v/c approaches zero, the differential equations of motion (1) approach the forms[4]

$$\frac{d}{dt}(m_0\,\dot{x}_n) = X_n, \tag{1'}$$

which are the Newtonian differential equations of motion.

Thus we see that Newtonian dynamics is in effect a simplified approximate form of relativistic dynamics which is valid when the speed of the particle under consideration is sufficiently small compared with the speed of light.

Let us carry out the indicated differentiations in equations (1), and then solve the resulting equations for the quantities $m_0\ddot{x}_1$, $m_0\ddot{x}_2$, $m_0\ddot{x}_3$. The work is straightforward, and need not be given here. We obtain the following set of formulae:

$$\begin{aligned}
m_0\,\ddot{x}_1 &= (1 - v^2c^{-2})^{-1/2}\begin{vmatrix} X_1 & \dot{x}_1\dot{x}_2c^{-2} & \dot{x}_1\dot{x}_3c^{-2} \\ X_2 & 1-(\dot{x}_1^2+\dot{x}_3^2)c^{-2} & \dot{x}_2\dot{x}_3c^{-2} \\ X_3 & \dot{x}_2\dot{x}_3c^{-2} & 1-(\dot{x}_1^2+\dot{x}_2^2)c^{-2} \end{vmatrix}, \\
m_0\,\ddot{x}_2 &= (1 - v^2c^{-2})^{-1/2}\begin{vmatrix} 1-(\dot{x}_2^2+\dot{x}_3^2)c^{-2} & X_1 & \dot{x}_1\dot{x}_3c^{-2} \\ \dot{x}_1\dot{x}_2c^{-2} & X_2 & \dot{x}_2\dot{x}_3c^{-2} \\ \dot{x}_1\dot{x}_3c^{-2} & X_3 & 1-(\dot{x}_1^2+\dot{x}_2^2)c^{-2} \end{vmatrix}, \qquad (3) \\
m_0\,\ddot{x}_3 &= (1 - v^2c^{-2})^{-1/2}\begin{vmatrix} 1-(\dot{x}_2^2+\dot{x}_3^2)c^{-2} & \dot{x}_1\dot{x}_2c^{-2} & X_1 \\ \dot{x}_1\dot{x}_2c^{-2} & 1-(\dot{x}_1^2+\dot{x}_3^2)c^{-2} & X_2 \\ \dot{x}_1\dot{x}_3c^{-2} & \dot{x}_2\dot{x}_3c^{-2} & X_3 \end{vmatrix}.
\end{aligned}$$

These equations are, of course, the differential equations of motion (1) written in a new, but equivalent, form.

If, at some particular instant, the particle is moving parallel to the x_1-axis, so that $\dot{x}_2 = \dot{x}_3 = 0$, the equations (3) reduce *at that instant* to the forms:

$$\frac{m_0\,\ddot{x}_1}{(1 - v^2c^{-2})^{3/2}} = X_1, \qquad \frac{m_0\,\ddot{x}_2}{(1 - v^2c^{-2})^{1/2}} = X_2, \qquad \frac{m_0\,\ddot{x}_3}{(1 - v^2c^{-2})^{1/2}} = X_3.$$

These equations show that a particle of rest-mass m_0, moving with speed v, responds to a force parallel to the velocity as would a Newtonian particle[5] of mass

$$m_\ell = \frac{m_0}{(1 - v^2c^{-2})^{3/2}},$$

[4] If this conclusion is not entirely evident, the reader is referred to equations (3), from which the conclusion follows at once.

[5] I.e. an ideal particle which obeys the laws of Newtonian dynamics.

and that the particle responds to a force perpendicular to the velocity as would a Newtonian particle of mass

$$m_t = \frac{m_0}{(1 - v^2c^{-2})^{1/2}}.$$

For this reason, it was usual in the early work on relativistic dynamics to ascribe two masses to a particle: the *longitudinal mass* m_ℓ, and the *transverse mass* m_t. However, in general this procedure leads only to inconveniences, and it has been almost entirely abandoned.

This concludes our discussion of the elementary differential equations of motion. Without any further general theory of relativistic dynamics it is possible to solve many interesting and important problems. For instance, it can be shown easily that the trajectory of a particle subjected to a force which is constant in magnitude and direction is a catenary (rather than a parabola, which is the curve predicted by Newtonian dynamics).[6] In the following sections we shall discuss some of the less elementary parts of the subject.

III. The Lagrangian Equations

In the foregoing the components of the applied force have been any functions of the coordinates, the components of the velocity, and the time. However, in problems concerning the motion of electrons, and for that matter in many other physical problems also, we are usually concerned with forces of a somewhat special kind. Throughout the remainder of the article we shall assume that the force belongs to this special class.

We consider four given functions of the coordinates and time, namely

$$V(x_1, x_2, x_3, t), \qquad A_n(x_1, x_2, x_3, t), \qquad (n = 1, 2, 3),$$

and we assume that the components of the force are given by the formulae

$$\begin{aligned} X_1 &= -\frac{\partial V}{\partial x_1} - \frac{\partial A_1}{\partial t} + \dot{x}_2\left[\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right] - \dot{x}_3\left[\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right], \\ X_2 &= -\frac{\partial V}{\partial x_2} - \frac{\partial A_2}{\partial t} + \dot{x}_3\left[\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right] - \dot{x}_1\left[\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right], \\ X_3 &= -\frac{\partial V}{\partial x_3} - \frac{\partial A_3}{\partial t} + \dot{x}_1\left[\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right] - \dot{x}_2\left[\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right]. \end{aligned} \tag{4}$$

Let us suppose, for purposes of illustration, that we are considering the motion of an electron. Then the physical interpretation of our assumption

[6] L. A. MacColl, *American Mathematical Monthly*, Vol. 45 (1938), pp. 669–676.

concerning the force is the following. $V(x_1, x_2, x_3, t)$ is the potential energy of the electron in an electromagnetic field; that is

$$V(x_1, x_2, x_3, t) = -\epsilon\varphi(x_1, x_2, x_3, t),$$

where ϵ is the absolute value of the electronic charge, and $\varphi(x_1, x_2, x_3, t)$ is the scalar potential of the field. The functions $A_n(x_1, x_2, x_3, t)$ are related to the components $a_n(x_1, x_2, x_3, t)$ of the vector potential of the field by the equations

$$A_n(x_1, x_2, x_3, t) = -\epsilon a_n(x_1, x_2, x_3, t).$$

The terms $-\partial A_n/\partial t$ are $-\epsilon$ times the contributions of the vector potential to the components of the electric force. The quantity $\partial A_3/\partial x_2 - \partial A_2/\partial x_3$ is $-\epsilon B_1$, where B_1 is the x_1-component of the magnetic induction; and similarly for the quantities $\partial A_1/\partial x_3 - \partial A_3/\partial x_1$ and $\partial A_2/\partial x_1 - \partial A_1/\partial x_2$.* In other cases also, equations (4), which may degenerate considerably, can be interpreted without difficulty.

Now we define a function $L(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3, t)$ of the coordinates, the components of the velocity, and the time, as follows:

$$L = -m_0c^2(1 - v^2c^{-2})^{1/2} - V + \dot{x}_1A_1 + \dot{x}_2A_2 + \dot{x}_3A_3. \qquad (5)$$

We call this the Lagrangian function.

We write the equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_n} - \frac{\partial L}{\partial x_n} = 0, \qquad (n = 1, 2, 3), \qquad (6)$$

carry out the indicated differentiations, and readily verify that the resulting equations are identical with those obtained by substituting the expressions (4) in equations (1). Hence, equations (6) are merely a form of the differential equations of motion. We call equations (6) the Lagrangian equations. The chief importance of these equations is due to the ease with which they enable us to use coordinate systems which are not rectangular. This will be discussed in the final section.

In the Newtonian case, i.e. the case in which the speed of the particle is small compared with the speed of light, the Lagrangian function reduces approximately to the form

$$L = -m_0c^2 + \frac{m_0}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - V + \dot{x}_1A_1 + \dot{x}_2A_2 + \dot{x}_3A_3. \qquad (5')$$

* These relations between the A's and the components of the vector potential, and between the partial derivatives of the A's and the components of the magnetic induction, are based upon the use of the M.K.S. system of units. If we measure the electromagnetic quantities in other units, certain constant proportionality factors may appear in the relations.

If we employ the function (5′) in equations (6), we do indeed get the Newtonian differential equations. Since the constant term $-m_0c^2$ is of no effect in the formation of the differential equations of motion, it is ordinarily omitted in writing the Newtonian form of the Lagrangian function.

IV. Hamilton's Canonical Equations

Let us write

$$p_n + A_n = \pi_n. \tag{7}$$

Solving equations (2) for $\dot{x}_1$, $\dot{x}_2$, $\dot{x}_3$, we get the result

$$\begin{aligned} \dot{x}_n &= cp_n[m_0^2c^2 + p_1^2 + p_2^2 + p_3^2]^{-1/2} \\ &= c(\pi_n - A_n)[m_0^2c^2 + (\pi_1 - A_1)^2 + (\pi_2 - A_2)^2 + (\pi_3 - A_3)^2]^{-1/2}. \end{aligned} \tag{8}$$

Also, it is readily seen that the differential equations (1) can be written, with the aid of equations (7) and (8), in the form

$$\begin{aligned} \dot{\pi}_n &= -\frac{\partial V}{\partial x_n} + \dot{x}_1\frac{\partial A_1}{\partial x_n} + \dot{x}_2\frac{\partial A_2}{\partial x_n} + \dot{x}_3\frac{\partial A_3}{\partial x_n} \\ &= -\frac{\partial V}{\partial x_n} - c\frac{\partial}{\partial x_n}[m_0^2c^2 + (\pi_1 - A_1)^2 + (\pi_2 - A_2)^2 + (\pi_3 - A_3)^2]^{1/2}. \end{aligned} \tag{9}$$

Now let us define a function $H(x_1, x_2, x_3, \pi_1, \pi_2, \pi_3, t)$ as follows:

$$H = c[m_0^2c^2 + (\pi_1 - A_1)^2 + (\pi_2 - A_2)^2 + (\pi_3 - A_3)^2]^{1/2} + V. \tag{10}$$

Then equations (8) take the forms

$$\dot{x}_n = \frac{\partial H}{\partial \pi_n}, \tag{11}$$

and equations (9) take the forms

$$\dot{\pi}_n = -\frac{\partial H}{\partial x_n}. \tag{12}$$

The function H is called the Hamiltonian function. The six equations (11) and (12), which are equivalent to the three equations (1), are called Hamilton's canonical equations of motion. These equations are of great importance in all of the deeper theoretical work in dynamics.

An easy calculation shows that we have the identity

$$H + L = \pi_1\dot{x}_1 + \pi_2\dot{x}_2 + \pi_3\dot{x}_3. \tag{13}$$

In the Newtonian case the Hamiltonian function given by (10) reduces approximately to the form

$$H = m_0 c^2 + \frac{1}{2m_0} [(\pi_1 - A_1)^2 + (\pi_2 - A_2)^2 + (\pi_3 - A_3)^2] + V. \quad (10')$$

The equations (11) and (12), with H given by (10′), are equivalent to the Newtonian differential equations of motion (1′). Here again the constant term m_0c^2 is of no effect, and it is ordinarily omitted in writing the Newtonian form of the function H. The Newtonian forms of the functions H and L satisfy the identity (13), whether or not the constant terms m_0c^2 and $-m_0c^2$ are included.

V. Static Fields of Force: The Energy Integral; Natural Families of Trajectories

By equations (11) and (12), we have the relation

$$\begin{aligned} \frac{dH}{dt} &= \frac{\partial H}{\partial t} + \sum_{n=1}^{3} \left[\frac{\partial H}{\partial x_n} \dot{x}_n + \frac{\partial H}{\partial \pi_n} \dot{\pi}_n \right] \\ &= \frac{\partial H}{\partial t} + \sum_{n=1}^{3} \left[\frac{\partial H}{\partial x_n} \frac{\partial H}{\partial \pi_n} - \frac{\partial H}{\partial \pi_n} \frac{\partial H}{\partial x_n} \right] = \frac{\partial H}{\partial t} . \end{aligned} \quad (14)$$

In particular, if no one of the functions V, A_1, A_2, A_3 involves the time explicity, we have $dH/dt = 0$, so that the value of H remains constant during the motion of the particle. That is, under the condition stated we have

$$m_0 c^2 [1 - v^2 c^{-2}]^{-1/2} + V(x_1, x_2, x_3) = \text{constant}. \quad (15)$$

In the Newtonian case equation (15) reduces approximately to the form

$$m_0 c^2 + \frac{m_0}{2} v^2 + V(x_1, x_2, x_3) = \text{constant},$$

which is equivalent to the equation

$$\frac{m_0}{2} v^2 + V(x_1, x_2, x_3) = \text{constant}. \quad (15')$$

It is well known that this equation is a consequence of the Newtonian differential equations of motion.

The left-hand member of equation (15′) is the energy of the particle in

Newtonian dynamics, the first and second terms being the kinetic energy and the potential energy, respectively. The equation itself is called the energy integral.[7] Similarly, we call (15) the energy integral in relativistic dynamics, and we call the expression

$$m_0c^2[1 - v^2c^{-2}]^{-1/2} + V$$

the *relativistic energy*. This energy is the sum of three parts: the *proper energy* m_0c^2, the *relativistic kinetic energy*

$$m_0c^2[1 - v^2c^{-2}]^{-1/2} - m_0c^2,$$

and the *potential energy* V.

The totality of possible trajectories of a particle in a static field of force forms a five-parameter family. We now see that if the field of force is static and of the kind we are considering now, the five-parameter family of curves consists of ∞^1 four-parameter subfamilies, each of which corresponds to a different value of the energy of the particle. Each of these four-parameter subfamilies is called a natural family of trajectories. We proceed to derive the differential equations defining a natural family.

If the constant in the right-hand member of equation (15) is denoted by the symbol E, we have the relation

$$\dot{x}_1[1 + x_2'^2 + x_3'^2]^{1/2} = c[1 - m_0^2c^4(E - V)^{-2}]^{1/2}, \tag{16}$$

where

$$x_2' = dx_2/dx_1, \qquad x_3' = dx_3/dx_1.$$

Hence,

$$dt = c^{-1}[1 + x_2'^2 + x_3'^2]^{1/2}[1 - m_0^2c^4(E - V)^{-2}]^{-1/2}\,dx_1.$$

From this, and the two equations

$$\frac{d}{dt}\frac{m_0\dot{x}_2}{(1 - v^2c^{-2})^{1/2}} = -\frac{\partial V}{\partial x_2} + \dot{x}_3\left[\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right] - \dot{x}_1\left[\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right],$$

$$\frac{d}{dt}\frac{m_0\dot{x}_3}{(1 - v^2c^{-2})^{1/2}} = -\frac{\partial V}{\partial x_3} + \dot{x}_1\left[\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right] - \dot{x}_2\left[\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right],$$

it follows that we have the following system of differential equations defining the natural family of trajectories corresponding to the total energy E:

[7] In the theory of differential equations, an equation relating the unknowns involved in a system of differential equations, their derivatives of orders less than the highest orders appearing in the system, the independent variable, and one or more arbitrary constants, is called an integral of the system of differential equations.

$$\begin{aligned}[1 + x_2'^2 + x_3'^2]^{-1/2} \frac{d}{dx_1}\left(x_2'\left[\frac{(E - V)^2 - m_0^2 c^4}{1 + x_2'^2 + x_3'^2}\right]^{1/2}\right) \\ = \frac{\partial}{\partial x_2}[(E - V)^2 - m_0^2 c^4]^{1/2} \\ + c[1 + x_2'^2 + x_3'^2]^{-1/2}\left(x_3'\left[\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right] - \left[\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right]\right), \\ [1 + x_2'^2 + x_3'^2]^{-1/2} \frac{d}{dx_1}\left(x_3'\left[\frac{(E - V)^2 - m_0^2 c^4}{1 + x_2'^2 + x_3'^2}\right]^{1/2}\right) \\ = \frac{\partial}{\partial x_3}[(E - V)^2 - m_0^2 c^4]^{1/2} \\ + c[1 + x_2'^2 + x_3'^2]^{-1/2}\left(\left[\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right] - x_2'\left[\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right]\right).\end{aligned} \tag{17}$$

The equations which correspond to (17) in the Newtonian case are most readily obtained by going back to the Newtonian differential equations of motion and employing the integral

$$m_0 v^2/2 + V = E.$$

An easy calculation, which is entirely parallel to the foregoing, gives us the following system of equations:

$$\begin{aligned}[1 + x_2'^2 + x_3'^2]^{-1/2} \frac{d}{dx_1}\left(x_2'\left[\frac{E - V}{1 + x_2'^2 + x_3'^2}\right]^{1/2}\right) = \frac{\partial}{\partial x_2}(E - V)^{1/2} \\ + [2m_0(1 + x_2'^2 + x_3'^2)]^{-1/2}\left(x_3'\left[\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right] - \left[\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right]\right), \\ [1 + x_2'^2 + x_3'^2]^{-1/2} \frac{d}{dx_1}\left(x_3'\left[\frac{E - V}{1 + x_2'^2 + x_3'^2}\right]^{1/2}\right) = \frac{\partial}{\partial x_3}(E - V)^{1/2} \\ + [2m_0(1 + x_2'^2 + x_3'^2)]^{-1/2}\left(\left[\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right] - x_2'\left[\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right]\right).\end{aligned} \tag{17'}$$

On comparing the systems of equations (17) and (17′), we get the following useful theorem.

If the constants E, E^*, m_0, m_0^*, k, *and the functions (of* x_1, x_2, x_3,*)* V, A_1, A_2, A_3, V^*, A_1^*, A_2^*, A_3^* *are such that we have identically*

$$(E - V)^2 - m_0^2 c^4 = k^2(E^* - V^*),$$

$$\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} = \frac{k}{c(2m_0^*)^{1/2}}\left[\frac{\partial A_3^*}{\partial x_2} - \frac{\partial A_2^*}{\partial x_3}\right],$$

$$\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} = \frac{k}{c(2m_0^*)^{1/2}} \left[\frac{\partial A_2^*}{\partial x_1} - \frac{\partial A_1^*}{\partial x_2} \right],$$

$$\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} = \frac{k}{c(2m_0^*)^{1/2}} \left[\frac{\partial A_1^*}{\partial x_3} - \frac{\partial A_3^*}{\partial x_1} \right],$$

the natural family of trajectories of a relativistic particle[8] *(of rest-mass m_0) moving with relativistic total energy E in the field of force derived from the functions V, A_1, A_2, A_3 is identical with the natural family of trajectories of a Newtonian particle (of mass m_0^*) moving with Newtonian energy E^* in the field of force derived from the functions V^*, A_1^*, A_2^*, A_3^*.*

In particular, the conditions of the theorem are satisfied if

$$k = c(2m_0)^{1/2}, \qquad E^* = c^{-2}(2m_0)^{-1}(E^2 - m_0^2c^4), \qquad m_0^* = m_0,$$

$$V = V^* = 0, \qquad A_1^* = A_1, \qquad A_2^* = A_2, \qquad A_3^* = A_3.$$

Hence, we have the corollary:

In the case of an electrified particle moving in any static magnetic field the natural family of trajectories corresponding to any value of the energy given by relativistic (Newtonian) dynamics is identical with the natural family of trajectories corresponding to a certain other value of the energy given by Newtonian (relativistic) dynamics.

The equation

$$E^* = c^{-2}(2m_0)^{-1}(E^2 - m_0^2c^4)$$

establishes a one-to-one correspondence between the physically significant ($E \geqq m_0c^2$ and $E^* \geqq 0$) values of the relativistic energy E and the Newtonian energy E^*. From this fact and the preceding corollary we get the following further result:

In the case of an electrified particle moving in any static magnetic field the total five-parameter family of trajectories given by relativistic dynamics is identical with that given by Newtonian dynamics.

Of course, these peculiar properties of motion of an electrified particle moving in a static magnetic field are explained physically by the fact that the magnetic forces do no work, so that the speed of the particle, and consequently also its mass, remain constant during the motion.

VI. Some Formulae from the Calculus of Variations

This section is devoted to the derivation of some formulae from the Calculus of Variations which will be needed in the further discussion of the

[8] I.e. a particle obeying the laws of relativistic dynamics.

dynamics of a particle. All constants, variables, and functions considered here are understood to be real.[9]

Let $F(t, x, y, z, p, q, r)$ be a function of the seven arguments indicated,[10] which, together with all of its partial derivatives of the first three orders, is continuous in a region R defined as follows:

$$R:\quad \begin{aligned} &a_1 < t < a_2, \\ &b_1 < x < b_2, \\ &c_1 < y < c_2, \\ &d_1 < z < d_2, \\ &p,\ q, \text{ and } r \text{ unrestricted,} \end{aligned}$$

the a's, b's, c's, and d's, being constants.

Let $x(t)$, $y(t)$, $z(t)$, $\varphi(t)$, $\psi(t)$, and $\omega(t)$ be continuous functions with continuous first derivatives, and let ϵ, η, and θ be parameters, independent of t, such that we have the relations

$$\begin{aligned} b_1 &< x(t) + \epsilon\varphi(t) < b_2, \\ c_1 &< y(t) + \eta\psi(t) < c_2, \qquad (a_1 < t < a_2). \\ d_1 &< z(t) + \theta\omega(t) < d_2, \end{aligned}$$

Let T_1 and T_2 be constants, and let t_1 and t_2 be parameters, such that

$$a_1 < T_1 + t_1 < T_2 + t_2 < a_2.$$

We now consider the integral

$$I(\epsilon, \eta, \theta, t_1, t_2) = \int_{T_1+t_1}^{T_2+t_2} F(t, x + \epsilon\varphi, y + \eta\psi, z + \theta\omega, x' + \epsilon\varphi', y' + \eta\psi', z' + \theta\omega')\, dt.$$

It can be shown without difficulty that the integral exists and is a differentiable function of ϵ, η, θ, t_1, t_2. We are interested in formulae giving the values of $\partial I/\partial\epsilon$, $\partial I/\partial\eta$, $\partial I/\partial\theta$, $\partial I/\partial t_1$, $\partial I/\partial t_2$ at the point $\epsilon = \eta = \theta = t_1 = t_2 = 0$.

[9] Since this section is purely mathematical, the constants, variables, and functions do not necessarily have any special physical significance.

[10] We treat the case of a function of seven arguments in order to fix the ideas, and because this is a case we shall meet in Section VII. However, the discussion applies essentially to other cases as well. In particular, in Section VII we shall also deal with a case in which F has only five arguments, z and r being absent.

By a well known theorem concerning the differentiation of definite integrals with respect to parameters,[11] we have

$$\frac{\partial I}{\partial \epsilon} = \int_{T_1+t_1}^{T_2+t_2} \left[\varphi \frac{\partial}{\partial(x+\epsilon\varphi)} + \varphi' \frac{\partial}{\partial(x'+\epsilon\varphi')}\right] F(t, x+\epsilon\varphi, \cdots, z'+\theta\omega')\, dt,$$

$$\frac{\partial I}{\partial t_1} = -F[T_1+t_1, x(T_1+t_1)+\epsilon\varphi(T_1+t_1), \cdots, z'(T_1+t_1)+\theta\omega'(T_1+t_1)],$$

$$\frac{\partial I}{\partial t_2} = F[T_2+t_2, x(T_2+t_2)+\epsilon\varphi(T_2+t_2), \cdots, z'(T_2+t_2)+\theta\omega'(T_2+t_2)].$$

The formulae for $\partial I/\partial\eta$ and $\partial I/\partial\theta$ are similar to that for $\partial I/\partial\epsilon$, and need not be written down.

In particular, if $[\partial I/\partial\epsilon]_0$, etc. denote the values of the derivatives at the point $\epsilon = \eta = \theta = t_1 = t_2 = 0$, we have

$$\begin{aligned}
\left[\frac{\partial I}{\partial \epsilon}\right]_0 &= \int_{T_1}^{T_2} \left[\varphi \frac{\partial}{\partial x} + \varphi' \frac{\partial}{\partial x'}\right] F(t, x, \cdots, z')\, dt,\\
\left[\frac{\partial I}{\partial \eta}\right]_0 &= \int_{T_1}^{T_2} \left[\psi \frac{\partial}{\partial y} + \psi' \frac{\partial}{\partial y'}\right] F(t, x, \cdots, z')\, dt,\\
\left[\frac{\partial I}{\partial \theta}\right]_0 &= \int_{T_1}^{T_2} \left[\omega \frac{\partial}{\partial z} + \omega' \frac{\partial}{\partial z'}\right] F(t, x, \cdots, z')\, dt, \qquad (18)\\
\left[\frac{\partial I}{\partial t_1}\right]_0 &= -F[T_1, x(T_1), \cdots, z'(T_1)],\\
\left[\frac{\partial I}{\partial t_2}\right]_0 &= F[T_2, x(T_2), \cdots, z'(T_2)].
\end{aligned}$$

The first three of equations (18) can be transformed to advantage, as follows. Integrating by parts, we obtain the formula

$$\int_{T_1}^{T_2} \varphi' \frac{\partial}{\partial x'} F(t, x, \cdots, z')dt = \left[\varphi \frac{\partial}{\partial x'} F(t, x, \cdots, z')\right]_{T_1}^{T_2} - \int_{T_1}^{T_2} \varphi \frac{d}{dt}\frac{\partial}{\partial x'} F(t, x, \cdots, z')\, dt,$$

and similar formulae for the integrals

$$\int_{T_1}^{T_2} \psi' \frac{\partial}{\partial y'} F(t, x, \cdots, z')\, dt$$

and

$$\int_{T_1}^{T_2} \omega' \frac{\partial}{\partial z'} F(t, x, \cdots, z')\, dt.$$

[11] The theorem is given, often in the form of two separate theorems, in most works on Advanced Calculus and the Theory of Functions of Real Variables. See the bibliography.

It follows, therefore, that we have

$$\left[\frac{\partial I}{\partial \epsilon}\right]_0 = \left[\varphi \frac{\partial}{\partial x'} F(t, x, \cdots, z')\right]_{T_1}^{T_2} + \int_{T_1}^{T_2} \varphi(t) \left[\frac{\partial}{\partial x} - \frac{d}{dt}\frac{\partial}{\partial x'}\right] F(t, x, \cdots, z')\, dt,$$

$$\left[\frac{\partial I}{\partial \eta}\right]_0 = \left[\psi \frac{\partial}{\partial y'} F(t, x, \cdots, z')\right]_{T_1}^{T_2} + \int_{T_1}^{T_2} \psi(t) \left[\frac{\partial}{\partial y} - \frac{d}{dt}\frac{\partial}{\partial y'}\right] F(t, x, \cdots, z')\, dt, \quad (19)$$

$$\left[\frac{\partial I}{d \theta}\right]_0 = \left[\omega \frac{\partial}{\partial z'} F(t, x, \cdots, z')\right]_{T_1}^{T_2} + \int_{T_1}^{T_2} \omega(t) \left[\frac{\partial}{\partial z} - \frac{d}{dt}\frac{\partial}{\partial z'}\right] F(t, \dot{x}, \cdots, z')\, dt.$$

An important special case is that in which t_1 and t_2 are zero (so that the limits of integration are fixed), and

$$\varphi(T_1) = \varphi(T_2) = \psi(T_1) = \psi(T_2) = \omega(T_1) = \omega(T_2) = 0.$$

In this case we have in general

$$I(\epsilon, \eta, \theta, 0, 0) - I(0, 0, 0, 0, 0) = \epsilon \int_{T_1}^{T_2} \varphi(t) \left[\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial x'}\right] dt + \eta \int_{T_1}^{T_2} \psi(t) \left[\frac{\partial F}{\partial y} - \frac{d}{dt}\frac{\partial F}{\partial y'}\right] dt + \theta \int_{T_1}^{T_2} \omega(t) \left[\frac{\partial F}{\partial z} - \frac{d}{dt}\frac{\partial F}{\partial z'}\right] dt + o(\epsilon, \eta, \theta),$$

where $o(\epsilon, \eta, \theta)$ denotes a term, the exact form of which is unimportant, *which is such that the expression*

$$\frac{o(\epsilon, \eta, \theta)}{|\epsilon| + |\eta| + |\theta|}$$

approaches the limit zero as ϵ, η, *and* θ *tend simultaneously toward zero.*

In particular, if the functions $x(t)$, $y(t)$, $z(t)$ satisfy the system of differential equations

$$\frac{d}{dt}\frac{\partial F}{dx'} - \frac{\partial F}{\partial x} = 0, \qquad \frac{d}{dt}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0, \qquad \frac{d}{dt}\frac{\partial F}{dz'} - \frac{\partial F}{\partial z} = 0, \quad (20)$$

we have (for all choices of the functions φ, ψ, ω subject to the conditions stated)

$$I(\epsilon, \eta, \theta, 0, 0) - I(0, 0, 0, 0, 0) = o(\epsilon, \eta, \theta). \quad (21)$$

Also, it can be shown without difficulty that in order that we have (21), for all such choices of φ, ψ, ω, it is necessary that x, y, and z, satisfy the equations (20).*

The last result can be stated in the following summary, and not quite explicit, form: If, and only if, the functions $x(t)$, $y(t)$, $z(t)$ satisfy equations (20), the integral

$$\int_{T_1}^{T_2} F(t, x, y, z, x', y', z')\, dt \tag{22}$$

is stationary with respect to infinitesimal variations of the functions $x(t)$, $y(t)$, $z(t)$ which leave the terminal values unaltered.

The problem of finding functions which render the values of definite integrals stationary is the chief subject of the Calculus of Variations.

The equations (20) are called the Eulerian equations of the Calculus of Variations problem of making the value of the integral (22) stationary, or, as we usually say, of maximizing or minimizing the integral.

VII. Hamilton's Principle and the Principle of Least Action

We immediately recognize equations (6) as the Eulerian equations of a problem in the Calculus of Variations. Thus we have the following principle (*Hamilton's principle*):

The particle moves, under forces of the type (4), *so that the value of the integral*

$$\int_{t_1}^{t_2} L\, dt,$$

with t_1 and t_2 held fixed, is stationary with respect to infinitesimal variations of the functions $x_n(t)$ which leave the initial and final points unaltered.

The precise meaning of this is determined by the discussion given in Section VI.

Hamilton's principle leads to the relativistic or Newtonian differential equations of motion, according as we use in it the function L given by (5) or by (5′).

A little inspection suffices to show that the system of equations (17) is also the system of Eulerian equations of a problem in the Calculus of

* In brief, suppose that $\frac{d}{dt}\frac{\partial F}{\partial x'} - \frac{\partial F}{\partial x}$ were not zero for some value of t. Then if we should choose a function $\varphi(t)$ which was (say) positive in the neighborhood of that value, and zero elsewhere, the integral

$$\int_{T_1}^{T_2} \varphi(t) \left[\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial x'}\right] dt$$

would have a value other than zero. We shall not give the actual proof here; it is to be found in the works on the Calculus of Variations cited in the bibliography.

Variations. Thus we get the so-called *principle of least action*, which can be stated as follows:

The particle moves, in a static field of force of the type (4), *and with the prescribed total energy E, in such a curve that the value of the integral*

$$\int_{(x_1)_1}^{(x_1)_2} ([1 + x_2'^2 + x_3'^2]^{1/2}[(E - V)^2 c^{-2} - m_0^2 c^2]^{1/2} + A_1 + A_2 x_2' + A_3 x_3')\, dx_1,$$

with the limits of integration held fixed, is stationary with respect to infinitesimal variations of the trajectory which leave the end points unaltered.

We have a precisely similar principle in Newtonian dynamics, but here the integral in question is

$$\int_{(x_1)_1}^{(x_1)_2} ([1 + x_2'^2 + x_3'^2]^{1/2}[2m_0(E - V)]^{1/2} + A_1 + A_2 x_2' + A_3 x_3')\, dx_1.$$

The last two integrals can be written more symmetrically, but not quite so explicitly, as follows:

$$\int_{P_1}^{P_2} \left([(E - V)^2 c^{-2} - m_0^2 c^2]^{1/2} + A_1 \frac{dx_1}{ds} + A_2 \frac{dx_2}{ds} + A_3 \frac{dx_3}{ds}\right) ds,$$

$$\int_{P_1}^{P_2} \left([2m_0(E - V)]^{1/2} + A_1 \frac{dx_1}{ds} + A_2 \frac{dx_2}{ds} + A_3 \frac{dx_3}{ds}\right) ds,$$

where P_1 and P_2 denote the end points of the trajectory, and $ds^2 = dx_1^2 + dx_2^2 + dx_3^2$.

VIII. The Hamilton-Jacobi Theory

Let us write

$$W = \int_{t_1}^{t_2} L[x_1(t), x_2(t), x_3(t), x_1'(t), x_2'(t), x_3'(t), t]\, dt. \tag{23}$$

We have already studied the variation of W when t_1 and t_2 are held fixed, and the functions $x_n(t)$ are varied in such a way that the terminal values are unaltered; and we have shown that under these circumstances the variation of W vanishes, to the first order of small quantities, in the natural motion.[12] In the following we shall study the variation of W under some other conditions.

Specifically, we shall study the quantity ΔW defined by equation (23) and the equation

$$W + \Delta W = \int_{t_1+\Delta t_1}^{t_2+\Delta t_2} L[x_1(t) + \xi_1(t), \cdots, x_3'(t) + \xi_3'(t), t]\, dt,$$

[12] I.e. a motion satisfying equations (1).

where the functions $x_n(t)$ represent a natural motion, the $\xi_n(t)$ are small functions, and Δt_1 and Δt_2 are small parameters.

It follows from the results of Section VI that we have (to within terms of the second order in small quantities)[13]

$$\begin{aligned}\Delta W = \quad & \Delta t_2\, L[x_1(t_2), \cdots, x_3'(t_2), t_2] \\ & - \Delta t_1\, L[x_1(t_1), \cdots, x_3'(t_1), t_1] \\ & + \sum_{n=1}^{3} \left[\frac{\partial L}{\partial x_n'}\right]_{t=t_2} \xi_n(t_2) - \sum_{n=1}^{3} \left[\frac{\partial L}{\partial x_n'}\right]_{t=t_1} \xi_n(t_1) \\ = \quad & \Delta t_2\, L[x_1(t_2), \cdots, x_3'(t_2), t_2] \\ & - \Delta t_1\, L[x_1(t_1), \cdots, x_3'(t_1), t_1] \\ & + \sum_{n=1}^{3} [\pi_n(t_2)\xi_n(t_2) - \pi_n(t_1)\xi_n(t_1)].\end{aligned}$$

Let us write

$$(\Delta x_n)_2 = x_n(t_2 + \Delta t_2) + \xi_n(t_2 + \Delta t_2) - x_n(t_2) = \xi_n(t_2) + x_n'(t_2)\,\Delta t_2,$$
$$(\Delta x_n)_1 = x_n(t_1 + \Delta t_1) + \xi_n(t_1 + \Delta t_1) - x_n(t_1) = \xi_n(t_1) + x_n'(t_1)\Delta t_1,$$

so that $(\Delta x_1)_2$, $(\Delta x_2)_2$, $(\Delta x_3)_2$ are the coordinate differences of the terminal points of the varied and unvaried curves, and similarly $(\Delta x_1)_1$, $(\Delta x_2)_1$, $(\Delta x_3)_1$ are the coordinate differences of the initial points. Then we have the formula

$$\begin{aligned}\Delta W = \quad & \left(L[x_1(t_2), \cdots] - \sum_{n=1}^{3} \pi_n(t_2)x_n'(t_2)\right) \Delta t_2 \\ & - \left(L[x_1(t_1), \cdots] - \sum_{n=1}^{3} \pi_n(t_1)x_n'(t_1)\right) \Delta t_1 \\ & + \sum_{n=1}^{3} [\pi_n(t_2)(\Delta x_n)_2 - \pi_n(t_1)(\Delta x_n)_1],\end{aligned}$$

which, by equation (13), can be written in the form

$$\Delta W = -H[x_1(t_2), \cdots]\,\Delta t_2 + H[x_1(t_1), \cdots]\,\Delta t_1 + \sum_{n=1}^{3} [\pi_n(t_2)(\Delta x_n)_2 - \pi_n(t_1)(\Delta x_n)_1]. \quad (24)$$

Now, the integration in (23) being taken over a natural motion of the particle, the value of W depends upon the initial instant, the initial coordi-

[13] This is also the sense in which the following equations are to be understood.

nates, the initial components of velocity, and the final instant. It is necessary now to consider W as depending upon the following equivalent set of eight variables: the initial and final instants t_1 and t_2, the coordinates (x_{11}, x_{21}, x_{31}) of the initial point, and the coordinates (x_{12}, x_{22}, x_{32}) of the final point. Regarding W in this manner, we at once obtain the following relations from equation (24)

$$\frac{\partial W}{\partial t_2} = -H_2, \qquad \frac{\partial W}{\partial x_{n2}} = \pi_{n2}, \tag{25}$$

$$\frac{\partial W}{\partial t_1} = H_1, \qquad \frac{\partial W}{\partial x_{n1}} = -\pi_{n1}, \tag{26}$$

where H_2 denotes $H[x_1(t_2), \cdots, \pi_1(t_2), \cdots, t_2]$ and H_1 denotes $H[x_1(t_1), \cdots, \pi_1(t_1), \cdots, t_1]$.

Let us now consider the partial differential equation

$$\frac{\partial W}{\partial t} + H(x_1, x_2, x_3, \partial W/\partial x_1 \ \ \partial W/\partial x_2, \partial W/\partial x_3, t) = 0. \tag{27}$$

The preceding work shows that the function W we have been considering (with x_{11}, x_{21}, x_{31}, t_1 regarded as parameters, and with the symbols x_{12}, x_{22}, x_{32}, t_2 replaced by x_1, x_2, x_3, t respectively) is a *particular* solution of this equation. We shall show that the *complete* solution of this equation possesses remarkable properties in connection with dynamical problems.

The complete solution of equation (27) is a function of x_1, x_2, x_3, t, and of four arbitrary constants, of which one is merely additive, and can be neglected for our purposes. Let the solution be written

$$W = W(x_1, x_2, x_3, t, \alpha_1, \alpha_2, \alpha_3),$$

where the α's are the three essential arbitrary constants.

We write the equations

$$\frac{\partial W}{\partial \alpha_n} = \beta_n, \tag{28}$$

where the β's are further arbitrary constants. These equations implicitly determine the x's as functions of t and the six arbitrary constants $\alpha_1, \cdots, \beta_3$.

We also write the equations

$$\frac{\partial W}{\partial x_n} = \pi_n. \tag{29}$$

These equations determine three functions π_n of the x's, the α's, and t. In virtue of equations (28), the π's are ultimately functions of t, the α's, and the β's.

There is no reason to foresee *a priori* that the functions $x_1(t, \alpha_1, \cdots, \beta_3)$, $\cdots, \pi_3(t, \alpha_1, \cdots, \beta_3)$ determined in this way, by means of the complete solution of equation (27), satisfy the differential equations of motion (11) and (12). Nevertheless, they actually do satisfy those equations, as we proceed to show.

By equations (28), we have the relations

$$0 = \frac{d\beta_n}{dt} = \frac{\partial^2 W}{\partial \alpha_n \, \partial t} + \sum_{m=1}^{3} \frac{\partial^2 W}{\partial \alpha_n \, \partial x_m} \dot{x}_m . \tag{30}$$

On the other hand, by (27) and (29), we have[14]

$$\begin{aligned} 0 &= \frac{\partial}{\partial \alpha_n} \left[\frac{\partial W}{\partial t} + H(x_1, x_2, x_3, \pi_1, \pi_2, \pi_3, t) \right] \\ &= \frac{\partial^2 W}{\partial \alpha_n \, \partial t} + \sum_{m=1}^{3} \frac{\partial H}{\partial \pi_m} \frac{\partial \pi_m}{\partial \alpha_n} = \frac{\partial^2 W}{\partial \alpha_n \, \partial t} + \sum_{m=1}^{3} \frac{\partial H}{\partial \pi_m} \frac{\partial^2 W}{\partial \alpha_n \, \partial x_m} . \end{aligned} \tag{31}$$

The determinant

$$\begin{vmatrix} \dfrac{\partial^2 W}{\partial \alpha_1 \, \partial x_1} & \cdots & \dfrac{\partial^2 W}{\partial \alpha_1 \, \partial x_3} \\ \cdots\cdots & \cdots & \cdots\cdots \\ \dfrac{\partial^2 W}{\partial \alpha_3 \, \partial x_1} & \cdots & \dfrac{\partial^2 W}{\partial \alpha_3 \, \partial x_3} \end{vmatrix}$$

is not zero. For if it were, we would have a relation of the form

$$\Phi[\partial W/\partial x_1, \partial W/\partial x_2, \partial W/\partial x_3, x_1, x_2, x_3, t] = 0, \tag{32}$$

independent of the α's. Now equation (32) is obviously distinct from (27), since it does not involve $\partial W/\partial t$. Hence, the vanishing of the determinant would imply that the function $W(x_1, x_2, x_3, t, \alpha_1, \alpha_2, \alpha_3)$ satisfies two distinct partial differential equations of the first order. This, however, is impossible when W is the complete solution of (27); for an essential part of the concept of the complete solution of a differential equation is that the elimination of the arbitrary constants, from the solution and the equations obtained by differentiation, shall result in the given differential equation and no other.

It follows, therefore, from (30) and (31) that

$$\dot{x}_m = \frac{\partial H}{\partial \pi_m} .$$

We also have, by (29),

$$\dot{\pi}_n = \frac{\partial^2 W}{\partial x_n \, \partial t} + \sum_{m=1}^{3} \frac{\partial^2 W}{\partial x_m \, \partial x_n} \dot{x}_m = \frac{\partial^2 W}{\partial x_n \, \partial t} + \sum_{m=1}^{3} \frac{\partial^2 W}{\partial x_m \, \partial x_n} \frac{\partial H}{\partial \pi_m} . \tag{33}$$

[14] Since the function $W(x_1, x_2, x_3, t, \alpha_1, \alpha_2, \alpha_3)$ satisfies equation (27) *identically* in the x's, t, and the α's. This remark applies also in the case of equation (34).

On the other hand, we have

$$0 = \frac{\partial}{\partial x_n}\left[\frac{\partial W}{\partial t} + H\right] = \frac{\partial^2 W}{\partial x_n \partial t} + \frac{\partial H}{\partial x_n} + \sum_{m=1}^{3} \frac{\partial H}{\partial \pi_m} \frac{\partial^2 W}{\partial x_m \partial x_n}. \tag{34}$$

By (33) and (34), we have the second set of canonical equations

$$\dot{\pi}_n = -\frac{\partial H}{\partial x_n}.$$

This completes the demonstration.

If H does not involve the time explicitly, we can write

$$W = S - Et, \tag{35}$$

where E is an arbitrary parameter, and S is a solution of the differential equation

$$H[x_1, x_2, x_3, \partial S/\partial x_1, \partial S/\partial x_2, \partial S/\partial x_3] = E. \tag{36}$$

The complete solution of (36) contains three arbitrary constants (besides the parameter E), of which one is merely additive, and can be neglected. It is easily seen that the solution of the canonical equations determined in the way described above, using the function W given by (35), and treating E as one of the α's, represents a motion of the particle with the total energy E.

All of this theory holds both for the relativistic case and for the Newtonian case, the only difference being in the forms of the differential equations (27) and (36) in the two cases.

IX. Curvilinear Coordinates

In all of the foregoing we have employed rectangular coordinates, because they afford the simplest and most direct expression of the basic physical facts. However, in the solution of particular problems it is often more convenient to use other systems of coordinates. For this reason, we shall now formulate the more important equations in terms of general curvilinear coordinates. In this work, as in all work with general coordinate systems, we shall encounter concepts and relations which can be handled most perspicuously by means of the modern tensor calculus. Actually, the amount of tensor calculus we shall use is very slight, and no extended preliminary discussion is necessary in order to make the formulae intelligible. It will suffice to give occasional explanations of the notation, and of some of the concepts, as we proceed. Further information is to be found in the works cited in the bibliography.

First consider the Lagrangian equations, which, as we have seen, are merely the Eulerian equations which follow from Hamilton's principle.

Now Hamilton's principle expresses a fact concerning the motion of a particle which is, by its very nature, independent of the choice of coordinates. Hence, the Lagrangian equations (6) hold in any coordinate system. However, the form of the function L depends upon the particular coordinate system, and we must discuss the change of the form of the function resulting from a transformation of the coordinate system.

In accordance with the common practice in the tensor calculus, we shall now denote the coordinates by the symbols x^1, x^2, x^3, instead of by the symbols x_1, x_2, x_3.

In rectangular coordinates the differential distance ds between the points (x^1, x^2, x^3) and $(x^1 + dx^1, x^2 + dx^2, x^3 + dx^3)$ is given by the simple formula

$$ds^2 = dx^{1^2} + dx^{2^2} + dx^{3^2},$$

but this is highly special; in general coordinates we have

$$ds^2 = \sum_{m=1}^{3} \sum_{n=1}^{3} g_{mn}(x^1, x^2, x^3)\, dx^m\, dx^n, \tag{37}$$

where the g's are functions which depend upon the particular coordinate system under consideration. It is understood that $g_{mn} = g_{nm}$. Henceforth, we shall write (37) in the form

$$ds^2 = g_{mn}(x^1, x^2, x^3) dx^m dx^n, \tag{38}$$

and we shall observe this general rule throughout: *When the same literal index occurs twice in a term, once as a subscript and once as a superscript, that term is understood to be summed for the three values of the index.*

We now have the result

$$v^2 = [ds/dt]^2 = g_{mn}(x^1, x^2, x^3)\dot{x}^m\dot{x}^n,$$

and

$$m_0c^2(1 - v^2c^{-2})^{1/2} = m_0c^2[1 - c^{-2}g_{mn}\dot{x}^m\dot{x}^n]^{1/2}.$$

The function $V(x^1, x^2, x^3, t)$ is a scalar. That is to say, when the coordinate system is changed, the first three arguments of the function are replaced by their expressions in terms of the new coordinates, and so we obtain a function which is of a new analytical form, but which has the same value as the original function at each point of space.

Now we consider the expression

$$A_1\dot{x}^1 + A_2\dot{x}^2 + A_3\dot{x}^3.$$

In rectangular coordinates this is the scalar product of the vectors (A_1, A_2, A_3) and $(\dot{x}^1, \dot{x}^2, \dot{x}^3)$. The expression retains its form and interpretation under changes of the coordinate system, provided (as the notation implies) (A_1, A_2, A_3) is treated as a covariant vector.[15]

With these understandings as to the significance of the symbolism, we can now write down the following general expressions for the Lagrangian function L in the relativistic and Newtonian cases, respectively,

$$L = -m_0 c^2 [1 - c^{-2} g_{mn} \dot{x}^m \dot{x}^n]^{1/2} - V + A_m \dot{x}^m ,$$

$$L = -m_0 c^2 + \frac{m_0}{2} g_{mn} \dot{x}^m \dot{x}^n - V + A_m \dot{x}^m .$$

These hold for any coordinate system; and from the appropriate one of these, and the Lagrangian equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}^n} - \frac{\partial L}{\partial x^n} = 0,$$

we obtain the relativistic or Newtonian differential equations of motion in any coordinates.

Now let us consider the Hamiltonian canonical equations.

We have already agreed to consider (A_1, A_2, A_3) as a covariant vector. We now make the same convention in regard to (π_1, π_2, π_3). Then it readily follows that the equations

$$\frac{\partial L}{\partial \dot{x}^n} = \pi_n \tag{40}$$

[15] Suppose that with a point P (which may be either a special point or a typical point), and with each coordinate system, we have associated an ordered triple of numbers.

If the triples of number (a_1, a_2, a_3) and (a_1', a_2', a_3') associated, respectively, with any two coordinate systems (x^1, x^2, x^3) and $x^{1'}, x^{2'}, x^{3'})$ satisfy the relations

$$a_{m'} = \frac{\partial x^n}{\partial x^{m'}} a_n,$$

the numbers (a_1, a_2, a_3) are said to be the components of a covariant vector in the coordinate system (x^1, x^2, x^3). (It is understood, of course, that the partial derivatives are evaluated at the point P.)

On the other hand, if the triples of numbers (a^1, a^2, a^3) and $(a^{1'}, a^{2'}, a^{3'})$ associated with the typical coordinate systems (x^1, x^2, x^3) and $(x^{1'}, x^{2'}, x^{3'})$ satisfy the relations

$$a^{m'} = \frac{\partial x^{m'}}{\partial x^n} a^n,$$

the numbers (a^1, a^2, a^3) are said to be the components of a contravariant vector in the coordinate system (x^1, x^2, x^3).

These concepts agree only in part with the ones used in the elementary theory of vectors. From our present standpoint, the only vectors used in the elementary theories are those which are defined with reference to rectangular coordinate systems. When other coordinate systems are used (e.g. cylindrical coordinates), the vectors, defined in terms of rectangular coordinates, are merely resolved along the tangents to the coordinate curves. The components obtained in this way are not the same as the components considered in the tensor calculus, which we are using here.

are tensor equations; and since they hold when the coordinates are rectangular, they hold for all coordinate systems.[16]

We let g^{mn} denote g^{-1} times the cofactor of the element g_{mn} in the determinant

$$g = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix}.$$

Now we write

$$H = c[m_0^2 c^2 + g^{mn}(\pi_m - A_m)(\pi_n - A_n)]^{1/2} + V \tag{41}$$

for the relativistic case, and

$$H = m_0c^2 + (2m_0)^{-1}g^{mn}(\pi_m - A_m)(\pi_n - A_n) + V \tag{41'}$$

for the Newtonian case. We see that these expressions specialize into the ones given earlier for the Hamiltonian function when the coordinates are rectangular.

H, L, and $\pi_n\dot{x}^n$ are all scalars. Consequently, the equation

$$H + L = \pi_n\dot{x}^n \tag{42}$$

is a tensor relation; and since it holds when the coordinates are rectangular, it holds for all coordinate systems.

The Lagrangian equations can be written in the form

$$\dot{\pi}_n = \frac{\partial L}{\partial x^n}. \tag{43}$$

Let us consider the variation of the function L resulting from small variations of the x's and $\dot{x}$'s. By (40) and (43), we have the relation

$$\begin{aligned} \delta L &= \frac{\partial L}{\partial x^n}\,\delta x^n + \frac{\partial L}{\partial \dot{x}^n}\,\delta \dot{x}^n \\ &= \dot{\pi}_n\,\delta x^n + \pi_n\,\delta\dot{x}^n \\ &= \delta(\pi_n\,\dot{x}^n) + (\dot{\pi}_n\,\delta x^n - \dot{x}^n\,\delta\pi_n). \end{aligned} \tag{44}$$

It follows from (42) and (44) that the variation of H resulting from small variations of the x's and the π's is given by the formula

$$\delta H = \dot{x}^n\delta\pi_n - \dot{\pi}_n\delta x^n.$$

[16] The argument is explained in detail in the works on the tensor calculus cited in the bibliography.

From this it follows that we have the Hamiltonian canonical equations

$$\dot{x}^n = \frac{\partial H}{\partial \pi_n}, \qquad \dot{\pi}_n = -\frac{\partial H}{\partial x^n},$$

in any coordinate system.

We have already seen how to state Hamilton's principle in terms of general coordinates.

In the relativistic case the principle of least action takes the form: *The particle moves, in a static field of force of the type* (4), *and with the prescribed total energy E, in such a curve that the value of the integral*

$$\int_{(x^1)_1}^{(x^1)_2} \left(\left[g_{mn} \frac{dx^m}{dx^1} \frac{dx^n}{dx^1} \right]^{1/2} [(E - V)^2 c^{-2} - m_0^2 c^2]^{1/2} + A_m \frac{dx^m}{dx^1} \right) dx^1,$$

with the limits of integration held fixed, is stationary with respect to infinitesimal variations of the trajectory which leave the end points unaltered. The corresponding form of the principle for the Newtonian case is obvious.

We are now in a position to dispose very quickly of the problem of formulating the Hamilton-Jacobi theory in terms of general curvilinear coordinates.

The general form of the Hamiltonian function being given by (41) (for the relativistic case) or (41′) (for the Newtonian case), we can at once write down the partial differential equation

$$\frac{\partial W}{\partial t} + H(x^1, x^2, x^3, \partial W/\partial x^1, \partial W/\partial x^2, \partial W/\partial x^3, t) = 0. \qquad (45)$$

Let

$$W = W(x^1, x^2, x^3, t, \alpha^1, \alpha^2, \alpha^3)$$

represent the complete solution of (45), without the irrelevant additive constant of integration.

Our chief problem is that of proving that the functions $x^n(t, \alpha^1, \alpha^2, \alpha^3, \beta_1, \beta_2, \beta_3)$, $\pi_n(t, \alpha^1, \alpha^2, \alpha^3, \beta_1, \beta_2, \beta_3)$ determined by the equations

$$\frac{\partial W}{\partial \alpha^n} = \beta_n, \qquad \frac{\partial W}{\partial x^n} = \pi_n,$$

where the β's are further arbitrary constants, satisfy the canonical equations

$$\dot{x}^n = \frac{\partial H}{\partial \pi_n}, \qquad \dot{\pi}_n = -\frac{\partial H}{\partial x^n}.$$

Now, referring to the proof given in Section VIII for the special case of rectangular coordinates, we see at once that nothing in the proof depends upon the special forms which the Hamiltonian function and equation (45) assume in those coordinates. Hence the proof already given applies immediately to the present general case.

Similar remarks apply also to the case in which H does not involve the time explicitly, and in which we write

$$W = S - Et,$$

where S is the complete solution (without the additive arbitrary constant) of the equation

$$H(x^1, x^2, x^3, \partial S/\partial x^1, \partial S/\partial x^2, \partial S/\partial x^3) = E.$$

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