A Mathematical Theory of Linear Arrays

By

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A MATHEMATICAL theory, suitable for appraising and controlling directive properties of linear antenna arrays, can be based upon a simple modification of the usual expression for the radiation intensity of a system of radiating sources. The first step in this modification is closely analogous to the passage from the representation of instantaneous values of harmonically varying quantities by real numbers to a symbolic representation of these quantities by complex numbers. The second step consists in a substitution which identifies the radiation intensity with the norm[1] of a polynomial in a complex variable. The complex variable itself represents a typical direction in space. This mathematical device permits tapping the resources of algebra and leads to a pictorial representation of the radiation intensity.

An *antenna array* is a spatial distribution of antennas in which the individual antennas are geometrically identical, similarly oriented, and energized at similarly situated points. The first and the last properties insure that the *form* of the current distribution is the same in all the elements of the array and that consequently the array is composed of antennas with the same radiation patterns. The difference between individual elements consists merely in the relative phases and intensities of their radiation fields. The second property means that the radiation patterns of the individual elements are similarly oriented and that consequently the *radiation pattern of the array is the product of the radiation patterns of its typical element and the "space factor"*. The *space factor* of an array is defined as the *radiation pattern* of a similar array of *non-directive* elements. Hence in studying the effect of spatial arrangement of antennas, we may confine ourselves to non-directive elements and thus materially simplify the analysis.

An array is *linear* if points, similarly situated on the elements, are colinear. In this paper we are concerned mostly with linear arrays of *equispaced* sources although in conclusion we shall have an occasion to say a few words about more general types.

[1] The norm of a complex number is the square of its absolute value.

Radiation Intensity and Field Strength

Consider a linear array of n equispaced nondirective sources (Fig. 1). Apart from the inverse distance factor, the instantaneous field strength of the array in the direction making an angle θ with the line of sources may be expressed as follows

$$\sqrt{\Phi_i} = A_0 \cos(\omega t + \vartheta_0) + A_1 \cos(\omega t + \psi + \vartheta_1) + A_2 \cos(\omega t + 2\psi + \vartheta_2) + \cdots + A_{n-2} \cos(\omega t + \overline{n-2}\,\psi + \vartheta_{n-2}) + \cos(\omega t + \overline{n-1}\,\psi), \quad (1)$$

$$\psi = \beta\ell \cos\theta - \vartheta, \qquad \beta = \frac{2\pi}{\lambda}.$$

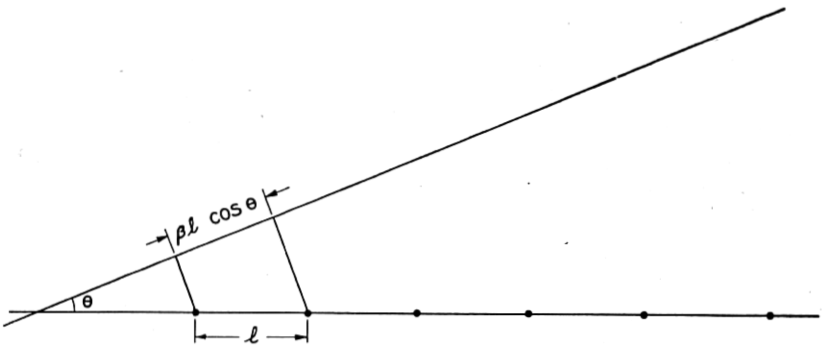


Fig. 1—A linear array of equispaced non-directive sources. If two sources are of equal intensity and in phase, their fields at a distant point are substantially equal in intensity but differ in phase by $\beta\ell\cos\theta$ where $\ell\cos\theta$ is the projection of the distance between the sources upon the particular spatial direction under consideration. If the sources are unequal, an allowance must be made for the relative field intensities in proportion to magnitudes of the sources and the phases must be adjusted for the phase difference between the sources.

In this equation: $A_0, A_1, \cdots A_{n-1} = 1$ are the relative amplitudes of the elements of the array; ϑ is a progressive phase delay, from left to right, between the successive elements of the array; $\vartheta_1, \vartheta_2, \cdots \vartheta_{n-2}, \vartheta_{n-1} = 0$ represent the phase deviations from the above progressive phase delay; $\beta = 2\pi/\lambda$ is the phase constant, where λ is the wavelength. The *radiation intensity*, that is the power radiated per unit solid angle, is proportional to the square of the amplitude of $\sqrt{\Phi_i}$.

Forming another expression similar to (1) but with sines in the place of cosines, multiplying the result by $i = \sqrt{-1}$ and adding it to (1), we have

$$\sqrt{\Phi_i'} = [A_0 e^{i\vartheta_0} + A_1 e^{i\psi + i\vartheta_1} + A_2 e^{2i\psi + i\vartheta_2} + \cdots + A_{n-2} e^{i\overline{n-2}\psi + i\vartheta_{n-2}} + e^{i\overline{n-1}\psi}]\, e^{i\omega t}. \quad (2)$$

The true instantaneous value of the field strength is the real part of (2).

Hence the amplitude $\sqrt{\Phi}$ of the field strength[2] is the absolute value of (2); thus[3]

$$\sqrt{\Phi} = \left| a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-2} z^{n-2} + z^{n-1} \right|, \qquad z = e^{i\psi}, \psi = \beta \ell \cos\theta - \vartheta, a_m = A_m e^{i\vartheta_m}. \tag{3}$$

In this equation: a_0, a_1, a_2, $\cdots$ a_{n-2}, $a_{n-1} = 1$ are complex numbers representing the relative amplitudes of the elements of the array and the phase deviations of these elements from a given progressive phasing. Thus if all the coefficients are real and positive, they represent the relative amplitudes of the elements of the array. If the algebraic sign of a particular coefficient is reversed, the phase of the corresponding element is changed by 180°; if some coefficient is multiplied by i or $-i$, the phase of the corresponding element is respectively accelerated or delayed by 90°; and in general the phase acceleration is equivalent, in our scheme, to a multiplication by a unit complex number $e^{i\delta}$. Some coefficients may be equal to zero and the corresponding elements of the array will be missing. In view of this possibility, we shall call ℓ the "apparent" separation between the elements; it is the greatest common measure of actual separations. When the elements are equispaced the apparent separation is the actual separation.

Thus we have the fundamental

Theorem I: *Every linear array with commensurable separations between the elements can be represented by a polynomial and every polynomial can be interpreted as a linear array.*[4]

The total length of the array is the product of the apparent separation between the elements and the degree of the polynomial. The degree of the polynomial is one less than the "apparent" number of elements. The actual number of elements is at most equal to the apparent number.

The above analytical representation of arrays is accomplished with the aid of the following transformation

$$z = e^{i\psi}, \tag{4}$$

in which $\psi = \beta\ell \cos\theta - \vartheta$ is a function of the angle θ made by the line of sources with a typical direction in space. Since ψ is always real, the absolute value of z equals unity and z itself *is always on the circumference of the unit circle* (Fig. 2). As θ increases from 0° (which is in a direction of the line of sources) to 180° (which is in the opposite direction), ψ decreases and

[2] For brevity's sake, we shall call $\sqrt{\Phi}$ itself the "field strength."

[3] Equation (3) could be derived directly from the physics of the situation in the same manner as (1). The foregoing method of transition from (1) to (3) serves only the purpose of showing the relationship between a less familiar formula and a very well known one.

[4] If the separations are not commensurable the arrays are represented by an *algebraic* function with incommensurable exponents.

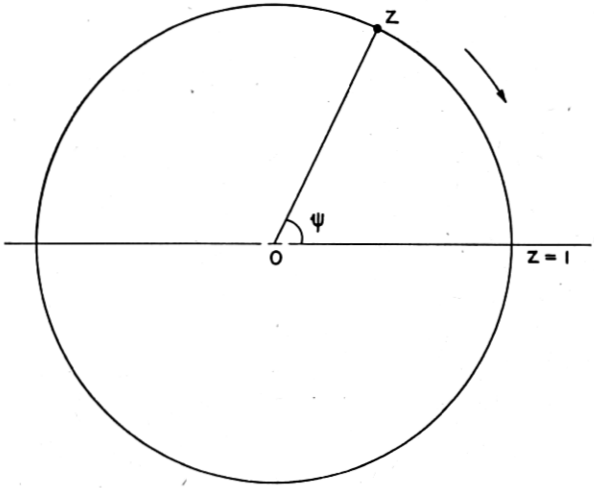


Fig. 2—A typical direction in space is represented by a complex variable which is represented in a complex plane by a point lying on the circumference of a circle of unit radius, having its center at the origin. As the angle θ made by a typical direction with the line of sources, increases from 0° to 180°, point z moves clockwise.

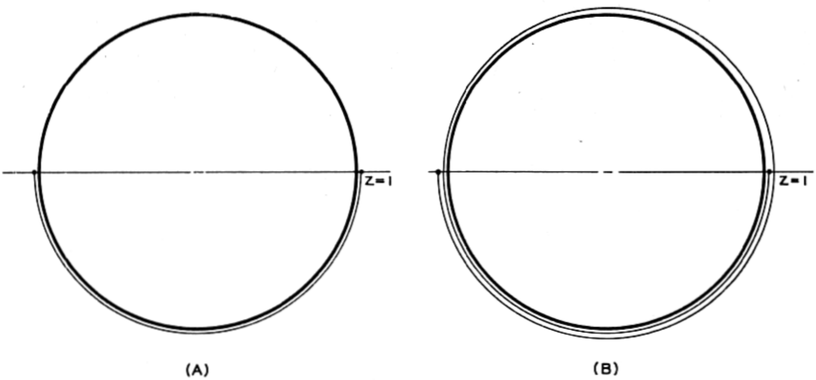


Fig. 3—(A) The active range of z, corresponding to $\vartheta = \beta\ell$ and one-quarter wave-length separation between the elements. (B) The active range of z, corresponding to $\vartheta = \beta\ell$ and $\ell = \frac{3}{4}\lambda$.

z moves in the clockwise direction. When $\theta = 0$, $\psi = \beta\ell - \vartheta$; and when $\theta = 180°$, $\psi = -\beta\ell - \vartheta$. Hence the range $\bar{\psi}$ described by z is

$$\bar{\psi} = 2\beta\ell. \tag{5}$$

When the separation ℓ between the successive elements of the array is equal to one-half wavelength, the range of $z = 2\pi$ and as θ varies from 0° to 180°, z describes a complete cycle and returns to its original position.

In this case there is a one-to-one correspondence between the points of the circumference of the unit circle and conical surfaces coaxial with the line of sources. Such conical surfaces, called radiation cones, are loci of directions in which the radiation intensities are equal. If the separation between the elements $< \lambda/2$, the range of z is smaller than 2π and z describes only a portion of the unit circle (Fig. 3A). Finally, if $\ell > \lambda/2$, then the path of z overlaps itself (Fig. 3B). Such a path, winding upon itself, will be called a *Riemann circle.* In this instance, one and the same point on the circle may correspond to several radiation cones; but if we regard different positions of z along its path as distinct points on the Riemann circle, then there will be a one-to-one correspondence between the points on the circle and the radiation cones.

Since the radiation intensity is a periodic function of ψ, the space factor of a given array will repeat itself if the separation between the elements is greater than one-half wavelength.

Composition of Space Factors

Since the product of two polynomials is a polynomial, we obtain the following corollary to Theorem I

Theorem II: *There exists a linear array with a space factor equal to the product of the space factors of any two linear arrays.*

In other words, there is a linear array such that its radiation intensity in any given direction is the product of the radiation intensities in this direction of any two given arrays. Thus we have

$$\begin{aligned} \sqrt{\Phi_1} &= | a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} |, \\ \sqrt{\Phi_2} &= | b_0 + b_1 z + b_2 z^2 + \cdots + b_{m-1} z^{m-1} |, \\ \sqrt{\Phi_1}\sqrt{\Phi_2} &= | (a_0 + a_1 z + \cdots + a_{n-1} z^{n-1})(b_0 + b_1 z + \cdots + b_{m-1} z^{m-1}) | \\ &= | a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \cdots |. \end{aligned} \tag{6}$$

The coefficients of the expanded product represent the amplitudes and the phases of the derived array.

Naturally the process may be repeated and a linear array can be constructed with its space factor equal to the product of the space factors of any number of linear arrays or to any power of the space factor of any array.

For example, let us start with a pair of equal sources, represented by

$$\sqrt{\Phi} = | 1 + z |, \tag{7}$$

and construct a linear array with the space factor equal to the square of (7). The field strength of the required array will be

$$\sqrt{\Phi} = | 1 + z |^2 = | 1 + 2z + z^2 |. \tag{8}$$

This array consists of three elements with amplitudes proportional to 1, 2, 1. If the elements of the original couplet are one-quarter wavelength apart and 90° out of phase, the couplet is "unidirectional". The space factor of such a couplet is depicted by Curve A in Fig. 4. The space factor of the triplet represented by (8) is shown by Curve B. In the directions in which the couplet radiates half as much or a third as much power as in the principal direction, the triplet radiates correspondingly only a quarter or a ninth of the power radiated in the principal direction.

The above considerations suggest a simple method for suppressing subsidiary radiation lobes. It is well known that in a uniform linear array[5]

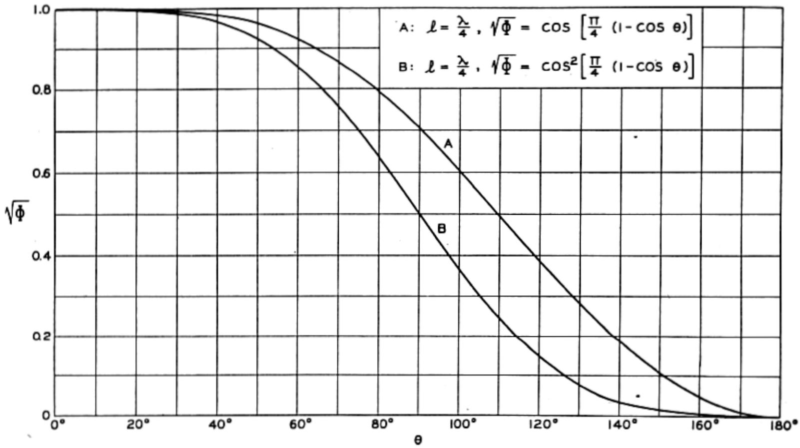


Fig. 4—Space Factors—Curve A is the space factor of a unidirectional couplet in which $\ell = \lambda/4$. Curve (B) represents the space factor of an array with amplitudes proportional to 1, 2, 1.

the difference in levels of the principal maximum of radiation and the first subsidiary is substantially independent of the number of elements, provided this number is sufficiently large. Thus in the limit, the first subordinate maximum is 13.5 decibels below the principal maximum. Consequently for the array with its space factor equal to the square of the space factor of the uniform array, the limiting difference in levels must be 26.9 decibels.

Since the uniform array is represented by

$$\sqrt{\Phi} = |\, 1 + z + z^2 + \cdots + z^{n-1} \,|, \tag{9}$$

[5] A "uniform" array is an array made up of sources of equal strength with a uniform progressive phase delay.

the other array is given by

$$\begin{aligned}\sqrt{\Phi} &= |\,1 + z + \cdots + z^{n-1}\,|^2 \\ &= |\,1 + 2z + 3z^2 + \cdots + nz^{n-1} + (n-1)z^n \\ &\qquad + \cdots + 2z^{2n-3} + z^{2n-2}\,|.\end{aligned} \tag{10}$$

Thus the amplitudes of the individual sources are proportional to 1, 2, 3, $\cdots$ $n-1$, n, $n-1$, $\cdots$ 3, 2, 1. Figure 5 depicts the effect of such "triangular" amplitude distribution.

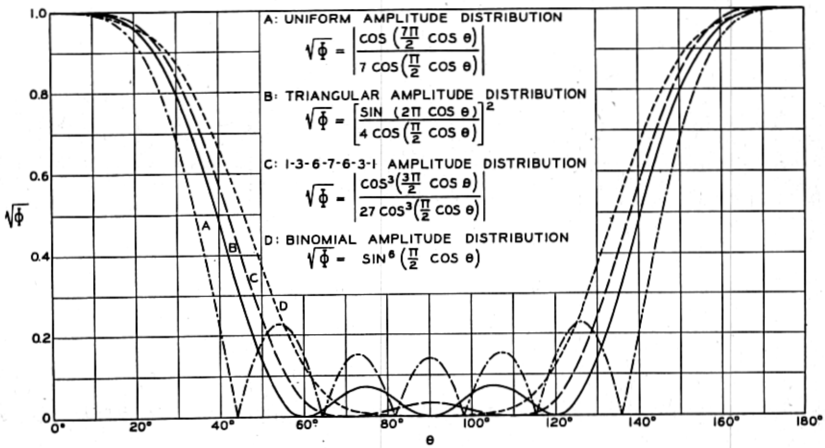


Fig. 5—Space Factors—(A) is for a uniform array and (B) for an array with "triangular" amplitude distribution.

Evidently we could raise (9) to any given power

$$\sqrt{\Phi} = |\,1 + z + z^2 + \cdots + z^{n-1}\,|^m. \tag{11}$$

This process does not change the number of separate radiation lobes. The so-called "binomial" distribution of amplitudes was first suggested by John Stone Stone.[6] His scheme is a special case of (11) if we let $n = 2$. For the effect of the binomial amplitude distribution see Fig. 5.

The relative merits of two forms for the radiation intensity as given by (1) and (3) can now be appraised in the light of the foregoing examples. Using (1), we have for the instantaneous radiation intensity of the unidirectional couplet

[6] U. S. Patents 1,643,323 and 1,715,433.

$$\sqrt{\Phi_i} = \cos \omega t + \cos\left(\omega t + \frac{\pi}{2} \cos \theta - \frac{\pi}{2}\right)$$
$$= \cos \omega t + \sin\left(\omega t + \frac{\pi}{2} \cos \theta\right). \tag{12}$$

By just inspecting this equation, we find no evidence for existence of a linear array with a space factor equal to the square of the space factor of the couplet. Still less obvious is the method of obtaining proper amplitude ratios.

Arrays of Arrays

The foregoing method of composition of space factors is in reality an analytical expression of geometric construction of "arrays of arrays". Consider, for instance, a pair of equiphase sources of equal strengths

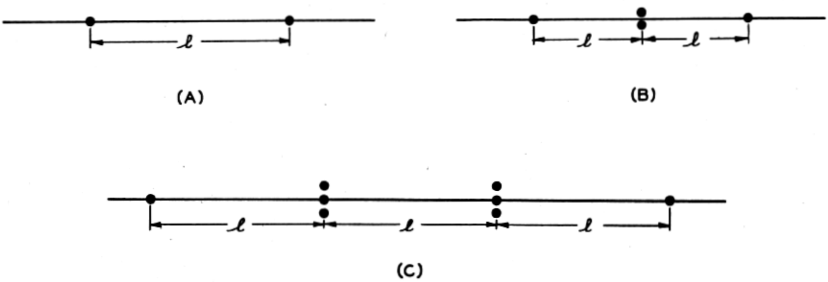


Fig. 6

(Fig. 6A). Take two such pairs as elements of an array of the same type (Fig. 6B). The middle sources add up to a single source of strength two. If the operation is repeated by taking (B) as elements of (A) or by taking (A) as elements of (B), then (C) is obtained; the amplitudes of (C) are proportional to 1, 3, 3, 1.

Each shift of a source to the right through distance ℓ is represented analytically as multiplication by z. An algebraic identity

$$(a_0 + a_1 z + a_2 z^2)z = a_0 z + a_1 z^2 + a_2 z^3 \tag{13}$$

is an expression of an obvious fact that each element of an array is shifted through the same distance as the entire array. Similarly a given change in the strength and the phase of the array is achieved by making the same change in all its elements; this fact is expressed by the identity

$$b(a_0 + a_1 z + a_2 z^2) = ba_0 + ba_1 z + ba_2 z^2. \tag{14}$$

In general, if an array represented by

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} \tag{15}$$

is taken as the element of an array given by

$$F(z) = b_0 + b_1 z + b_2 z^2 + \cdots + b_{m-1} z^{m-1}, \tag{16}$$

then the resulting array of arrays is represented by

$$f(z)F(z) = b_0 f(z) + b_1 z f(z) + b_2 z^2 f(z) + \cdots + b_{m-1} z^{m-1} f(z). \tag{17}$$

Decomposition Theorem

Consider now a pair of non-directive sources with strengths proportional to 1, $-t$; then

$$\sqrt{\Phi} = | z - t |. \tag{18}$$

Geometrically, the complex number $z - t$ is represented by a line drawn from point t to point z (Fig. 7A). Accordingly, the *radiation intensity*

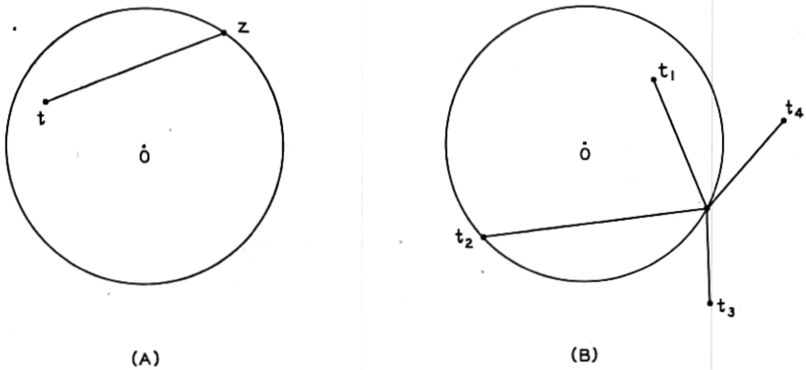


Fig. 7—The radiation intensity of a linear array is represented by the square of the product of the lines joining the null points of $\sqrt{\Phi}$ to a point z on the unit circle.

of the pair of sources *is represented by the distance between t and z.* If $\sqrt{\Phi}$ vanishes for some particular direction in space, it vanishes for all directions making the same angle with the line of sources; these directions form a *cone of silence* of the radiation system. Obviously, a radiating couplet has a cone of silence if and only if the null point of $\sqrt{\Phi}$ is in the range of z; in particular, there can be no cone of silence unless the null is on the unit circle.

By the fundamental theorem of algebra a polynomial of degree $(n - 1)$ has $(n - 1)$ zeros (some of which may be multiple zeros) and can be factored into $(n - 1)$ binomials; thus

$$\sqrt{\Phi} = | (z - t_1)(z - t_2) \cdots (z - t_{n-1}) |. \tag{19}$$

Each binomial represents the directive pattern of a pair of elements separated by distance ℓ. Hence

Theorem III: *The space factor of a linear array of n apparent elements is the product of the space factors of* $(n - 1)$ *virtual couplets with their null points at the zeros of* $\sqrt{\Phi}$: $t_1, t_2, \cdots t_{n-1}$.

Accordingly the radiation intensity of an array is equal to the square of the product of the distances from the null points of the array to that point z on the unit circle which corresponds to the chosen direction (Fig. 7B). To each null point lying in the range of z, there corresponds one and only one *cone of silence* provided each null point is counted as many times as z happens to pass it in describing the complete range.

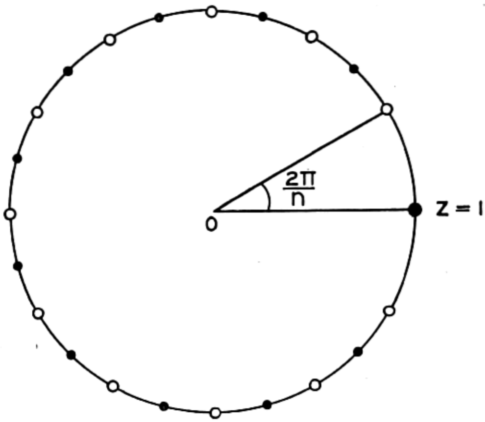


Fig. 8—The null points of a uniform linear array and the point $z = 1$ representing the direction of the greatest radiation divide the unit circle into equal parts. The hollow circles represent the null points and the solid circles the points of maximum radiation.

By summing the geometric progression (9) the radiation intensity of a uniform array can be represented as follows

$$\sqrt{\Phi} = \left| \frac{z^n - 1}{z - 1} \right|. \tag{20}$$

Hence the null points of such an array are the n-th roots of unity, excluding $z = 1$. Since z is a unit complex number,[7] any power of it is also a unit complex number. Moreover, each multiplication by $z = e^{i\psi}$ represents a displacement through an arc of ψ radians. Hence the n-th roots

[7] A *unit complex number* is a complex number whose absolute value is equal to unity.

of unity divide the circle into n equal parts (Fig. 8). Analytically we have

$$z^n - 1 = 0, \qquad t_m = e^{-\frac{2m\pi i}{n}}, \qquad m = 1, 2, 3, \cdots n - 1,$$
$$\psi_m = -\frac{2m\pi}{n}, \qquad \cos\theta_m = \frac{\vartheta}{\beta\ell} - \frac{2m\pi}{n\beta\ell}. \tag{21}$$

When $z = 1$, $\sqrt{\Phi}$ is evidently a *principal* maximum. Other maxima of smaller magnitude, the so-called *subordinate* or *subsidiary* maxima, occur approximately half way between the null points. The general

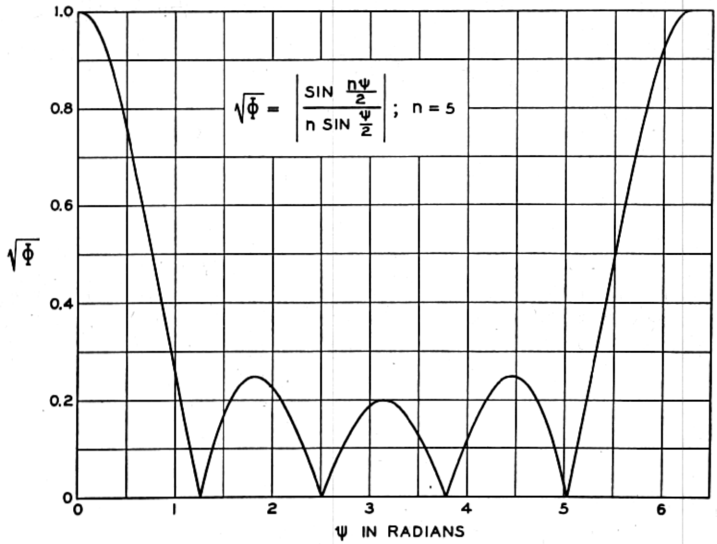


Fig. 9—The field strength $\sqrt{\Phi}$ as a function of ψ for $n = 5$. The principal maximum is reduced to unity.

behavior of the field strength can readily be understood if we follow z around the unit circle. When plotted against ψ, $\sqrt{\Phi}$ has the shape shown in Fig. 9. This is a universal radiation characteristic which can be interpreted for any particular spacing and phasing between the elements with the aid of the curve for $\psi + \vartheta = \beta\ell \cos\theta$ (Fig. 10).

It is easy to estimate the relative level of the first subordinate maximum. For a fairly large number of elements, the difference in levels is determined largely by the distances of the maximum points from the *nearest* null points. The distances are approximately equal to the circular arcs joining the corresponding points. Since the arcs joining the first subordinate maximum with the nearest null points are nearly half as long as those for the principal maximum, the first subordinate maximum of the field strength is about one-quarter of the principal maximum. In other words, the subordinate maximum is approximately 12 decibels below the principal maximum.

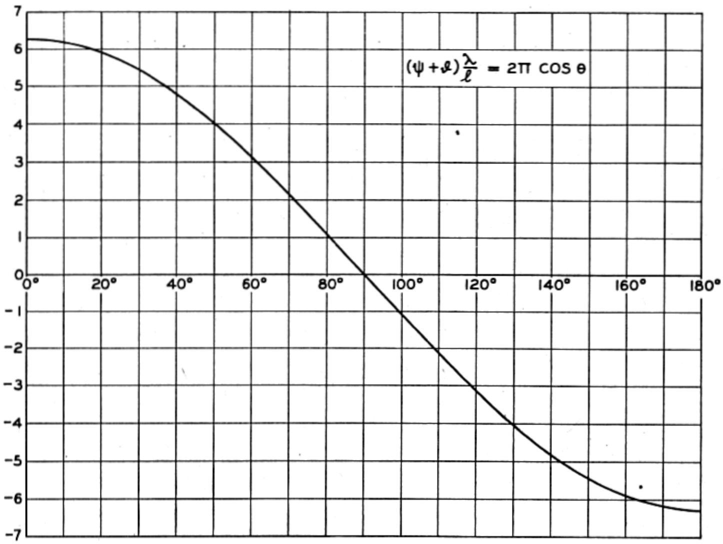


Fig. 10

A more accurate value for this difference in levels can be obtained by first rewriting (20) in the form

$$\sqrt{\Phi} = \left| \frac{z^{\frac{n}{2}} - z^{-\frac{n}{2}}}{z^{\frac{1}{2}} - z^{-\frac{1}{2}}} \right| = \frac{\sin \dfrac{n\psi}{2}}{\sin \dfrac{\psi}{2}} \tag{22}$$

and then substituting successively $\psi = 0$ and $\psi = \dfrac{3\pi}{n}$, one for the principal maximum and the other for the first subordinate. Accordingly we obtain

$$\frac{\sqrt{\Phi(0)}}{\sqrt{\Phi\left(\dfrac{3\pi}{n}\right)}} = n \sin \frac{3\pi}{2n}. \tag{23}$$

If n is large $\sin \dfrac{3\pi}{2n}$ is approximately equal to $\dfrac{3\pi}{2n}$ and the field strength ratio becomes $\dfrac{3\pi}{2} = 4.71$. This ratio corresponds to the difference in levels equal to 13.5 decibels.

Directivity of Arrays

The "decomposition theorem" of the preceding section throws considerable light on directive properties of arrays. The number of elements in the array is one greater than the number of virtual couplets. Hence to secure the greatest possible directivity with a given number of elements, the virtual couplets must be properly combined.

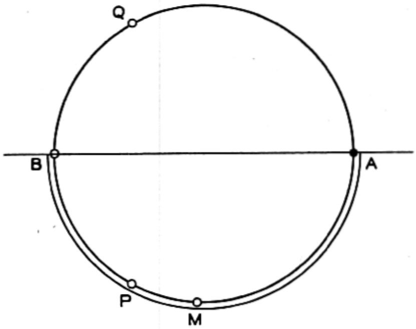


Fig. 11—The null points of several three-element arrays. The spacing between the elements is $\lambda/4$ and the progressive phase delay is $T/4$ (T equals period).

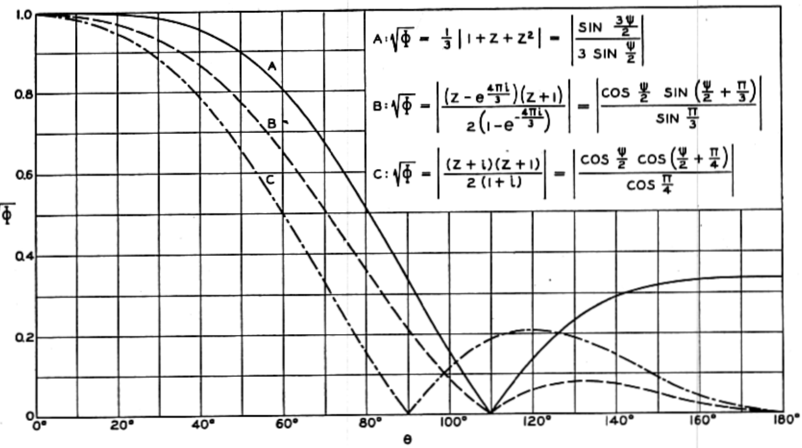


Fig. 12—Comparison of directivity of several three-element arrays. The spacing between the elements is $\lambda/4$; the direction of principal radiation is $\theta = 0°$. Curve (A) refers to the uniform array, (B) to an array with nulls at P and B (see Fig. 11), and Curve (C) refers to an array with its nulls at B and M.

For example, the null points of a uniform array of three elements, one-quarter wavelength apart, are at P and Q (Fig. 11). If $\vartheta = \pi/2$, the range of z consists of the lower half of the unit circle and principal radiation takes place in the direction $\theta = 0$. Evidently, the virtual couplet with its null at Q is comparatively nondirective. Substituting for this couplet another couplet with a null at B should improve the directivity of the array. This is indeed the case: In Fig. 12, Curve A depicts directive properties of the uniform array and Curve B depicts those of an array with its nulls at P and B. The field strength of the second array is

$$\begin{aligned}\sqrt{\Phi} &= \left|(z - e^{-\frac{2\pi i}{3}})(z+1)\right| = \left|z^2 + (1 - e^{-\frac{2\pi i}{3}})z - e^{-\frac{2\pi i}{3}}\right| \\ &= \left|1 + z\sqrt{3}\, e^{-\frac{i\pi}{6}} + z^2 e^{-\frac{i\pi}{3}}\right|, \quad z = \frac{\pi}{2}(\cos\theta - 1);\end{aligned} \tag{24}$$

hence the amplitudes of the elements are proportional to 1, $\sqrt{3}$, 1 and the total progressive phase delay in the direction of maximum radiation is $\frac{\pi}{2} + \frac{\pi}{6} = \frac{2\pi}{3}$ radians.

The minor lobe of the second array is substantially smaller than that of the first array. The major lobes, however, are equally "wide"[8] although one lobe is somewhat sharper than the other. The width of the major lobe can be reduced at the expense of increasing the minor lobe by moving the null from P to M (Fig. 11). The effect of this change is shown by Curve C (Fig. 12). The corresponding field strength is[9]

$$\begin{aligned}\sqrt{\Phi} &= |(z+i)(z+1)| = |z^2 + (1+i)z + i| \\ &= |1 - i(1+i)z - iz^2| = \left|1 + \sqrt{2}\, e^{-\frac{i\pi}{4}} z + e^{-\frac{i\pi}{2}} z^2\right|;\end{aligned} \tag{25}$$

hence the amplitudes are proportional to 1, $\sqrt{2}$, 1 and the total progressive phase delay is $\frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$.

For arrays of six elements, one-quarter wavelength apart and with $\vartheta = \pi/2$, we have Fig. 13. Curve A represents the directive characteristic of a uniform array, with its nulls as shown in Fig. 14A, and Curve B shows the directive properties of an array with its nulls equispaced on the lower half of the unit circle as shown in Fig. 14B.

If the spacing between the elements is $\ell = \lambda/8$ and if the phase delay $\vartheta = \pi/2$, then the effect of distribution of the null points is even more pronounced (Figs. 15 and 16). This time z is confined to the fourth quadrant of the unit circle. In Fig. 15, $n = 3$; Curve A corresponds to an array with equal amplitudes in which case the nulls are equispaced on the complete unit circle (Fig. 17A) and Curve B corresponds to an array with its nulls equi-

[8] If the "width" of a lobe is measured by the angle of the cone of silence enclosing the lobe.

[9] When transforming the expressions for $\sqrt{\Phi}$, it is well to remember that the absolute value of a complex quantity does not change if this quantity is multiplied by a unit complex number.

spaced within the range of z (Fig. 17B). In Fig. 16, $n = 6$; Curve A represents an array with nulls distributed evenly on the complete circle and Curve B represents an array with nulls evenly spaced in the range of z.

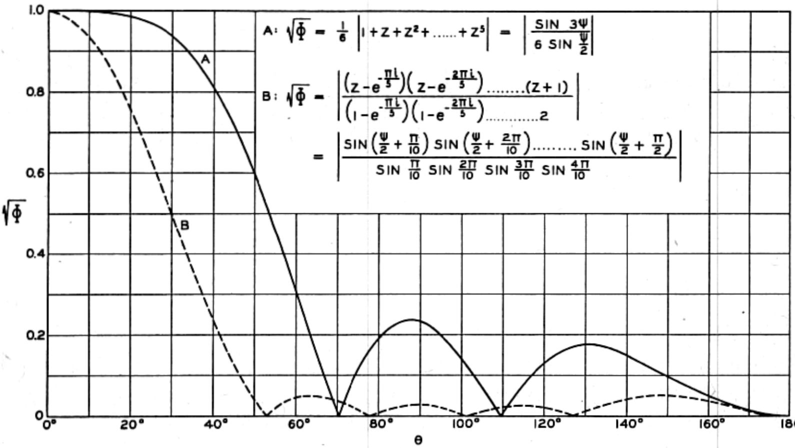


Fig. 13—Directive properties of 2 six-element arrays with $\ell = \lambda/4$. Curve (A) refers to a uniform array and Curve (B) refers to an array with its nulls equispaced in the range of z.

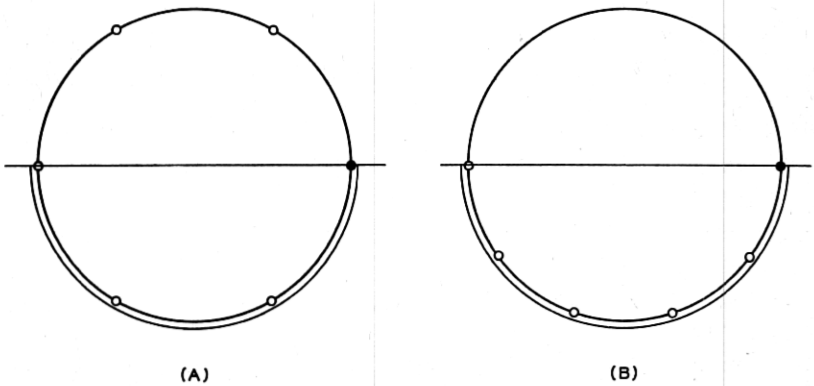


Fig. 14—Disposition of null points for the arrays with directive characteristics as shown in Fig. 13.

If the total length of an array is kept constant but the number of elements is increased, the array may be made more directive; Figure 18 illustrates this point. This increase in directivity can be secured only if the null points of the array are properly distributed within the range of z; in Fig. 18

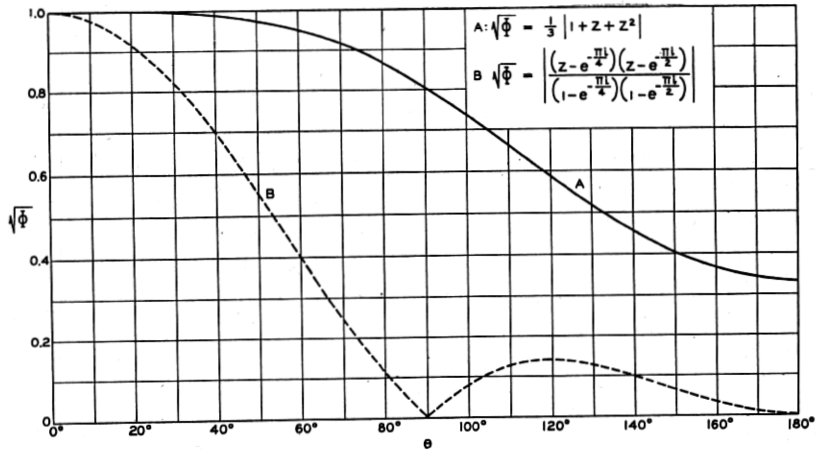


Fig. 15—Directive properties of three-element linear arrays with $\ell = \lambda/8$. Curve (A) refers to a uniform array and Curve (B) to an array with its nulls equispaced in the range of z.

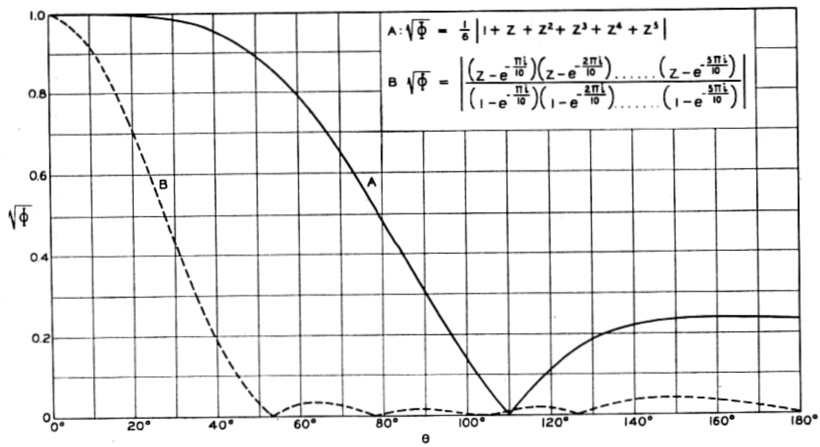
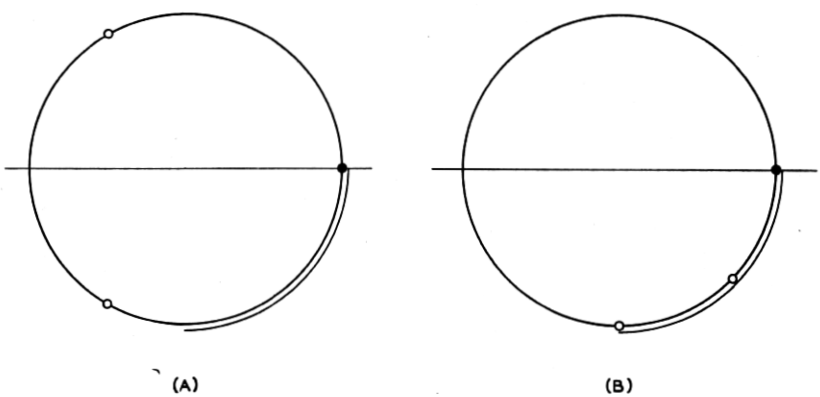


Fig. 16—Directive properties of six-element linear arrays with $\ell = \lambda/8$. Curve (A) refers to a uniform array and Curve (B) to an array with its nulls equispaced in the range of z.



(A) (B)

Fig. 17—Disposition of nulls for the arrays whose directive properties are shown in Fig. 15.

the null points are evenly spaced in the range of z, appropriate to each separation between the elements.

If the elements of the array are directive, the null points should be distributed with due reference to the directive pattern of the elements in order that a further increase in directivity could be secured.

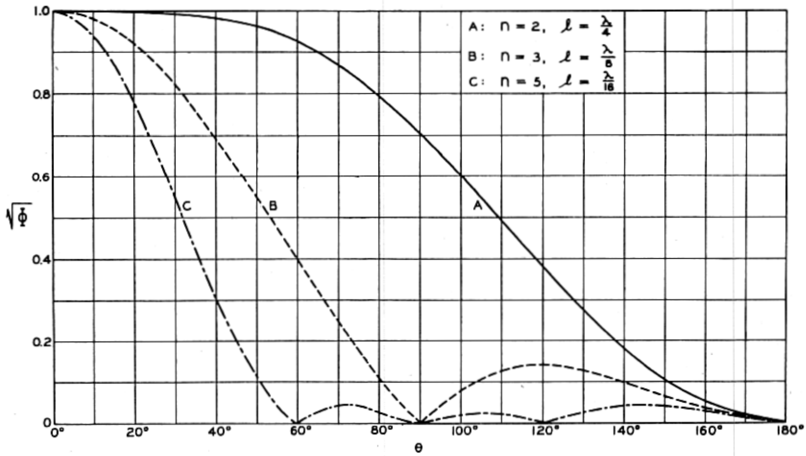


Fig. 18—Directive properties of linear arrays with total length equal to $\lambda/4$. (A), $n = 2$, $\ell = \lambda/4$; (B), $n = 3$, $\ell = \lambda/8$; (C), $n = 5$, $\ell = \lambda/16$.

Multi-Dimensional Arrays

The simplest method of constructing multi-dimensional arrays is to take a linear array as an element of another linear array. The axis of the second array may be chosen to make any angle with the axis of the first array. In this way only a special class of multi-dimensional arrays can be formed. Analytical expressions for the radiation intensities of more general arrays can be formulated in terms of two or more complex variables. These variables, however, will not be independent and a given direction in space will be represented by a group of related points, one point on each circle representing the particular complex variable. At this time we shall not be concerned with any developments applicable to such general multi-dimensional arrays.

Arrays with Prescribed Space Factors

If the minimum separation between the elements does not exceed $\lambda/2$, it is theoretically possible to design a linear array with a space factor given by an arbitrary function $f(\psi)$ or $F(\theta)$ of direction of radiation. Naturally the number of required elements will be usually infinite; with a finite number of elements the space factor may only be approximate.

Consider an array with an *odd* number of elements $n = 2m + 1$. Since the modulus of z is unity the polynomial (3) can be divided by z^m without affecting $\sqrt{\Phi}$; thus

$$\sqrt{\Phi} = | a_0 z^{-m} + a_1 z^{-m+1} + a_2 z^{-m+2} + \cdots + a_m + a_{m+1} z + \cdots + a_{2m} z^m |. \tag{26}$$

Let us now assume that the coefficients equidistant from the ends of the polynomial are conjugate complex; then the polynomial is real and we can drop the bars. Thus setting

$$a_m = A_0, \qquad a_{m+k} = A_k - iB_k, \qquad k > 0, \qquad a_{m-k} = \overset{*}{a}_{m+k}, \tag{27}$$

where the A's and B's are real; we have

$$\begin{aligned} a_{m+k} z^k + a_{m-k} z^{-k} &= (A_k - iB_k)e^{ik\psi} + (A_k + iB_k)e^{-ik\psi} \\ &= 2A_k \cos k\psi + 2B_k \sin k\psi. \end{aligned} \tag{28}$$

Consequently, (26) becomes

$$\sqrt{\Phi} = \sum_{k=0}^{m} \epsilon_k (A_k \cos k\psi + B_k \sin k\psi), \tag{29}$$

where ϵ_k is the Neumann number.[10]

If now we wish $\sqrt{\Phi}$ to be a prescribed function $f(\psi)$ of the variable ψ, we need only expand this function in a Fourier series

$$\sqrt{\Phi} = f(\psi) = \sum_{k=0}^{\infty} \epsilon_k (p_k \cos k\psi + q_k \sin k\psi), \tag{30}$$

and approximate it with any desired accuracy by means of a finite series (29). Once the A's and B's are known, we calculate the a's from (27).

It must be remembered that the real independent variable is not ψ but θ and the directive pattern is to be assigned as a function of θ. Besides being dependent on θ, ψ is a function of the distance ℓ between the successive elements of the array. Since θ varies from 0° to 180°, the range of ψ is $\bar{\psi} = 2\beta\ell$. The function $f(\psi)$ is prescribed within this range. On the other hand the period of the expressions (29) and (30) is 2π. This means that if $\bar{\psi} > 2\pi$, that is if $\ell > \lambda/2$, it is impossible to obtain the desired directive pattern with our scheme, because the pattern repeats itself automatically as ψ increases or decreases by 2π. But if $\ell < \lambda/2$, we have a considerable latitude in the design; outside the range of ψ, we can supplement $f(\psi)$ by an arbitrary function of ψ. It is only when $\ell = \lambda/2$ that there is a unique class of linear arrays that will produce a directive pattern given by the first

[10] $\epsilon_0 = 1$, $\epsilon_k = 2$ when $k \neq 0$.

$(m + 1)$ terms of (30). Dr. T. C. Fry of these Laboratories has suggested that leaving ℓ undetermined and fixing the number of elements, an array could be designed which would have the best fit to the prescribed pattern. In this connection, the "best fit" means the least mean square deviation of the approximating pattern from the given pattern.

If $\sqrt{\Phi}$ is given as a function $F(\theta)$ of θ, then by virtue of the definition of ψ we can write

$$F(\theta) = F\left(\cos^{-1}\frac{\psi + \vartheta}{\beta\ell}\right) = f(\psi). \tag{31}$$

Let us now consider a simple example for the sake of illustrating the method. Let $f(\psi)$ be defined by

$$\begin{aligned} f(\psi) &= 0, \qquad 0 < \psi < \pi, \\ &= 1, \qquad \pi < \psi < 2\pi. \end{aligned} \tag{32}$$

We shall assume that the separation between the elements is one-half wavelength. This makes the range of ψ equal to 2π. It is also seen that regarded as a function of θ, $f(\psi)$ retains its essential characteristic: being equal to zero over one-half of the range and to unity over the remaining half.

Expanding (32) into a Fourier series we have

$$f(\psi) = \frac{1}{2} - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2k-1)\psi}{2k-1}. \tag{33}$$

Consequently

$$\begin{aligned} A_0 &= \tfrac{1}{2},\ A_k = 0 \text{ if } k \neq 0; \\ B_k &= 0, \text{ if } k \text{ is even}; \\ B_k &= -\frac{1}{k\pi}, \text{ if } k \text{ is odd}. \end{aligned} \tag{34}$$

Figure 19 shows several approximations to $f(\psi)$ by means of a finite number of elements. The curve S_m corresponds to an approximation by the finite series (29). If S_9 is deemed to be a sufficiently good approximation to the given directive pattern, then

$$\begin{aligned} \sqrt{\Phi} = \frac{1}{\pi}\left|\frac{1}{9} + \frac{1}{7}z^2 + \frac{1}{5}z^4 + \frac{1}{3}z^6 + z^8 + \frac{i\pi}{2}z^9\right. \\ \left. - z^{10} - \frac{1}{3}z^{12} - \frac{1}{5}z^{14} - \frac{1}{7}z^{16} - \frac{1}{9}z^{18}\right|. \end{aligned} \tag{35}$$

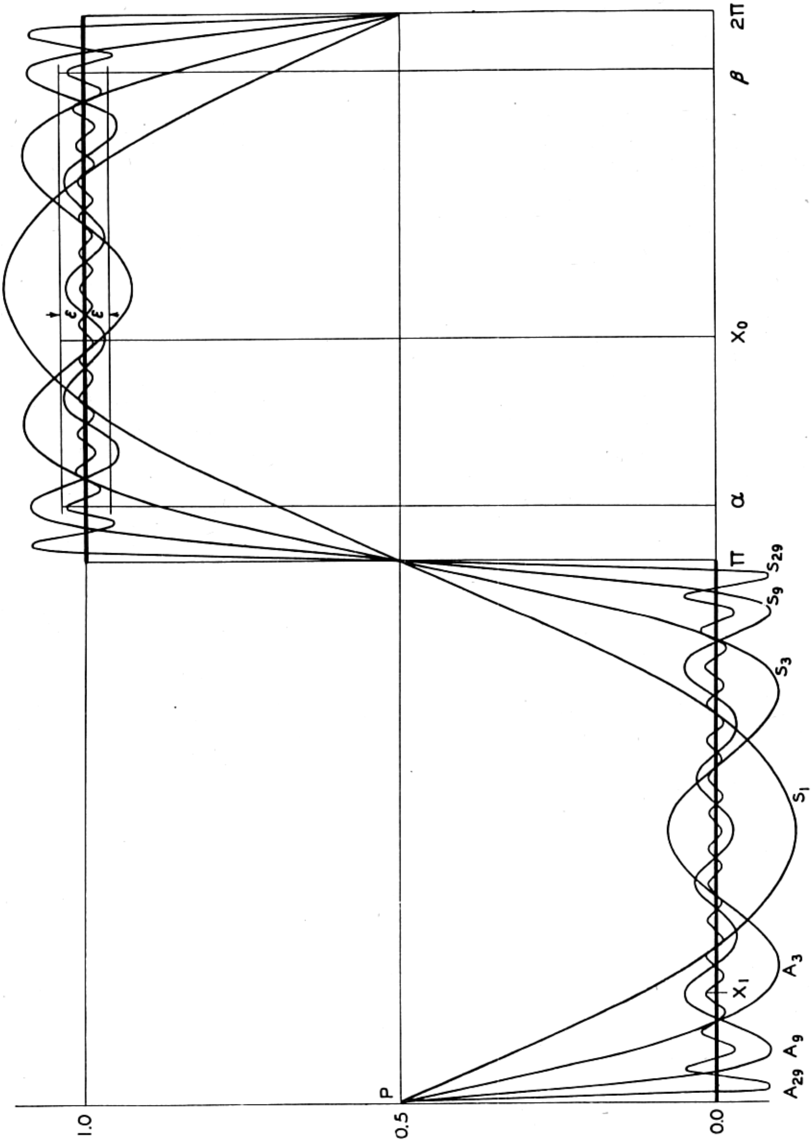


Fig. 19

The total length of this array is $(n - 1)\frac{\lambda}{2} = 2m\frac{\lambda}{2} = 9\lambda$. All elements except the three central ones are separated by one wavelength since the odd powers of z except z^9 are missing.

End-On Arrays with Equispaced Null Points

We now pass to a more detailed analysis of end-on arrays with null points equispaced on a given circular arc.

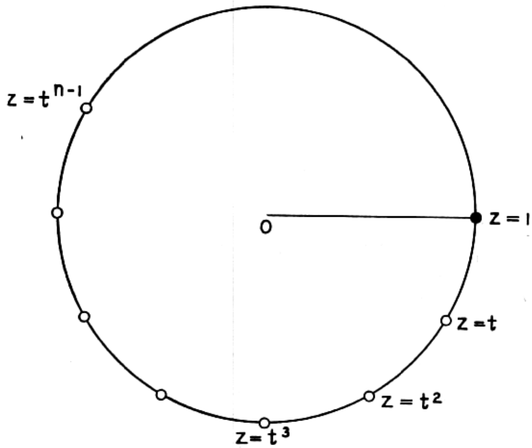


Fig. 20

For an end-on array $\vartheta = \beta\ell$ and

$$z = e^{i\psi}, \qquad \psi = \beta\ell(\cos\theta - 1). \tag{36}$$

The range of z begins at $z = 1$ and extends clockwise to a point determined by $\psi = -2\beta\ell$. Let $n - 1$ null points be equispaced on an arc of length $\bar{\psi}$ as shown in Fig. 20; the field strength is then

$$\sqrt{\Phi} = \left| (z - t)(z - t^2)\cdots(z - t^{n-1}) \right|, \qquad t = e^{-\frac{i\bar{\psi}}{n-1}}. \tag{37}$$

This can be expressed as

$$\sqrt{\Phi} = 2^{n-1}\left| \sin\frac{1}{2}\left(\psi + \frac{\bar{\psi}}{n-1}\right) \times \sin\frac{1}{2}\left(\psi + \frac{2\bar{\psi}}{n-1}\right)\cdots \sin\frac{1}{2}\left(\psi + \frac{\overline{n-1}\,\bar{\psi}}{n-1}\right)\right|. \tag{38}$$

The angle of the cone of silence enclosing the major radiation lobe is determined from

$$\beta\ell(\cos\theta_1 - 1) = -\frac{\bar{\psi}}{n-1}; \tag{39}$$

thus

$$1 - \cos\theta_1 = \frac{\bar{\psi}}{(n-1)\beta\ell}, \qquad \sin\frac{\theta_1}{2} = \sqrt{\frac{\bar{\psi}}{2(n-1)\beta\ell}}. \tag{40}$$

If the arc $\bar{\psi}$ is equal to the range of z, then (40) becomes

$$1 - \cos\theta_1 = \frac{2}{n-1}, \qquad \sin\frac{\theta_1}{2} = \frac{1}{\sqrt{n-1}}. \tag{41}$$

In this case, the size of the first cone of silence is determined *solely by the number of elements.* On the other hand, if $\bar{\psi} = 2\pi - 2\pi/n$, the nulls are equispaced on the unit circle and we have an ordinary uniform array; then

$$1 - \cos\theta_1 = \frac{2\pi}{n\beta\ell} = \frac{\lambda}{n\ell}, \qquad \sin\frac{\theta_1}{2} = \sqrt{\frac{\lambda}{2n\ell}}. \tag{42}$$

This time the size of the first cone of silence depends upon the total length $L = (n-1)\ell$ of the array measured in wavelengths.

When the number of elements in the first case and the total length of the array in the second are large, then we have approximately

$$\theta_1' = \frac{2}{\sqrt{n-1}}, \qquad \theta_1'' = 2\sqrt{\frac{\lambda}{2n\ell}}. \tag{43}$$

For a large n the ratio of the two cone angles is approximately

$$\frac{\theta_1'}{\theta_1''} = \sqrt{\frac{2\ell}{\lambda}}. \tag{44}$$

For example, if $\ell = \lambda/8$, the angle of the major lobe in the first case is one-half of that in the second case or one-quarter if we are to compare the solid angles.

Equispacing the null points in the range of z not only makes the major lobe narrower but it also makes it sharper. Thus at the point lying halfway between the point of maximum radiation and the first null point, the field strength relative to the principal maximum is

$$\chi = \frac{\sin\dfrac{\bar{\psi}}{4(n-1)}\sin\dfrac{3\bar{\psi}}{4(n-1)}\sin\dfrac{5\bar{\psi}}{4(n-1)}\cdots\sin\dfrac{(2n-3)\bar{\psi}}{4(n-1)}}{\sin\dfrac{\bar{\psi}}{2(n-1)}\sin\dfrac{2\bar{\psi}}{2(n-1)}\sin\dfrac{3\bar{\psi}}{2(n-1)}\cdots\sin\dfrac{(n-1)\bar{\psi}}{2(n-1)}}. \tag{45}$$

For a quarter wavelength separation between the elements $\bar{\psi} = \pi$ and this ratio is equal to

$$\chi = \frac{1}{\sqrt{2(n-1)}} \tag{46}$$

so that the drop in the radiation intensity becomes [10 Log_{10} $(n-1) + 3$] decibels. On the other hand, for a long uniform array the corresponding drop is independent of n and is equal to 4 decibels.

Another consequence of equispacing the null points in the range of z consists in substantial suppression of subsidiary radiation lobes. The first subordinate maximum is situated approximately halfway between the first two null points where $\psi = -\dfrac{3\bar{\psi}}{2(n-1)}$; thus the field strength there, relative to the principal maximum, is

$$\chi = \frac{\sin\dfrac{\bar{\psi}}{4(n-1)}\sin\dfrac{\bar{\psi}}{4(n-1)}\sin\dfrac{3\bar{\psi}}{4(n-1)}\cdots\sin\dfrac{(2n-5)\bar{\psi}}{4(n-1)}}{\sin\dfrac{\bar{\psi}}{2(n-1)}\sin\dfrac{2\bar{\psi}}{2(n-1)}\sin\dfrac{3\bar{\psi}}{2(n-1)}\cdots\sin\dfrac{(n-1)\bar{\psi}}{2(n-1)}}. \tag{47}$$

For a quarter wavelength separation this field strength becomes

$$\begin{aligned}\chi &= \frac{2^{-\frac{2n-3}{2}}\sin\dfrac{\pi}{4(n-1)}}{\sqrt{2(n-1)}\,2^{-\frac{2n-3}{2}}\sin\dfrac{(2n-3)\pi}{4(n-1)}}\\ &= \frac{\sin\dfrac{\pi}{4(n-1)}}{\sqrt{2(n-1)}\sin\dfrac{2(n-3)\pi}{4(n-1)}}.\end{aligned} \tag{48}$$

When n is sufficiently large, we have approximately

$$\chi = \frac{\pi}{4(n-1)\sqrt{2(n-1)}} \tag{49}$$

and the subsidiary maximum is [30 Log_{10} $(n-1) + 5$] decibels below the principal maximum. Each time the number of elements is doubled, the level of the subsidiary maximum is diminished by about 9 decibels. Thus an array of the type (37) with $\bar{\psi} = 2\beta\ell$ has very sharp directive properties.

In order to find the relative amplitudes and phase deviations of the elements of the array represented by (37), we expand $\sqrt{\Phi}$ into a single polynomial as follows[11]

$$\sqrt{\Phi} = |(1 - t^{-1}z)(1 - t^{-2}z) \cdots (1 - t^{-n+1}z)|$$

$$= \left| 1 + \sum_{k=1}^{n-1} (-)^k \frac{(1 - t^{-n+1})(1 - t^{-n+2}) \cdots (1 - t^{-n+k})}{(1 - t^{-1})(1 - t^{-2}) \cdots (1 - t^{-k})} t^{-\frac{k(k+1)}{2}} z^k \right|$$

$$= \left| 1 + \sum_{k=1}^{n-1} (-)^k \frac{(t^{-\frac{n-1}{2}} - t^{\frac{n-1}{2}}) \cdots (t^{-\frac{n-k}{2}} - t^{\frac{n-k}{2}})}{(t^{-\frac{1}{2}} - t^{\frac{1}{2}})(t^{-1} - t) \cdots (t^{-\frac{k}{2}} - t^{\frac{k}{2}})} t^{-\frac{kn}{2}} z^k \right| \qquad (50)$$

$$= \left| 1 + \sum_{k=1}^{n-1} (-)^k \frac{\sin \frac{(n-1)\bar{\psi}}{2(n-1)} \cdots \sin \frac{(n-k)\bar{\psi}}{2(n-1)}}{\sin \frac{\bar{\psi}}{2(n-1)} \cdots \sin \frac{k\bar{\psi}}{2(n-1)}} e^{\frac{ikn\bar{\psi}}{2(n-1)}} z^k \right| .$$

Hence the progressive phase delay from one antenna to the next is equal to $\vartheta = \pi - \dfrac{n\bar{\psi}}{2(n-1)}$ and the amplitudes are in the ratio

$$1, \quad \frac{\sin \frac{(n-1)\bar{\psi}}{2(n-1)}}{\sin \frac{\bar{\psi}}{2(n-1)}}, \quad \frac{\sin \frac{(n-1)\bar{\psi}}{2(n-1)} \sin \frac{(n-2)\bar{\psi}}{2(n-1)}}{\sin \frac{\bar{\psi}}{2(n-1)} \sin \frac{2\bar{\psi}}{2(n-1)}},$$
$$\frac{\sin \frac{(n-1)\bar{\psi}}{2(n-1)} \sin \frac{(n-2)\bar{\psi}}{2(n-1)} \sin \frac{(n-3)\bar{\psi}}{2(n-1)}}{\sin \frac{\bar{\psi}}{2(n-1)} \sin \frac{2\bar{\psi}}{2(n-1)} \sin \frac{3\bar{\psi}}{2(n-1)}}, \quad \cdots, \quad 1. \qquad (51)$$

The amplitudes of the elements equidistant from the ends of the array are equal. In the special case of an end-on array with nulls equispaced in the range of z, $\bar{\psi} = 2\beta\ell$ and $\vartheta = \beta\ell$; hence the progressive phase delay from one antenna to the next is $\pi - \dfrac{\beta\ell}{n-1}$.

While (50) serves well for finding the amplitude and phase distribution in the individual elements of the array, another form is more general for calculating the directive properties. In order to obtain this form we set

[11] Chrystal's Algebra, Vol. 2, p. 340, (1926).

$$p_0 = 1, \qquad p_{n-1-k} = p_k, \qquad p_k = \frac{\sin \dfrac{(n-1)\bar{\psi}}{2(n-1)} \cdots \sin \dfrac{(n-k)\bar{\psi}}{2(n-1)}}{\sin \dfrac{\bar{\psi}}{2(n-1)} \cdots \sin \dfrac{k\bar{\psi}}{2(n-1)}} \tag{52}$$

$$\varphi = \psi + \frac{\bar{\psi}}{2} + \frac{\bar{\psi}}{2(n-1)} + \pi,$$

divide the last expression in (50) by $e^{\frac{i(n-1)\varphi}{2}}$ and combine the terms equidistant from the ends. Thus we obtain

$$\sqrt{\Phi} = 2 \cos \frac{(n-1)\varphi}{2} + 2p_1 \cos \frac{(n-3)\varphi}{2} + 2p_2 \cos \frac{(n-5)\varphi}{2} + \cdots, \tag{53}$$

where the last term is $2p_{\frac{n}{2}-1} \cos \frac{\varphi}{2}$ if n is even and $p_{\frac{n-1}{2}}$ if n is odd.

Let D be the maximum value of $\sqrt{\Phi}$; then the gain of the array over a single source is given by

$$G = 10 \operatorname{Log}_{10} \frac{4\pi D^2}{\iint \Phi \, d\Omega} = 10 \operatorname{Log}_{10} \frac{2D^2}{\int_0^\pi \Phi \sin \theta \, d\theta} \text{ decibels}, \tag{54}$$

where Ω is the solid angle and the integration is extended over a unit sphere. For an end-on array with nulls equispaced in the range of z, the maximum radiation is in the direction $\psi = 0$. Thus we shall have

$$D = 2 \cos \frac{(n-1)\varphi_0}{2} + 2p_1 \cos \frac{(n-3)\varphi_0}{2} + 2p_2 \cos \frac{(n-5)\varphi_0}{2} + \cdots, \tag{55}$$

where

$$\varphi_0 = \beta\ell + \frac{\beta\ell}{n-1} + \pi. \tag{56}$$

A convenient expression for the radiation intensity can be obtained from (50) by taking its norm

$$\Phi = [p_0 + p_1 e^{i\varphi} + p_2 e^{2i\varphi} + \cdots + p_{n-1} e^{i(n-1)\varphi}] \cdot [p_0 + p_1 e^{-i\varphi} + \cdots + p_{n-1} e^{-i(n-1)\varphi}]. \tag{57}$$

Since the set of coefficients p_0, p_1, $p_2 \cdots p_{n-1}$ is symmetric about the center, we find

$$\begin{aligned}\Phi = 2p_0^2 \cos (n-1)\varphi &+ 2(p_0p_1 + p_1p_0) \cos (n-2)\varphi \\ &+ 2(p_0p_2 + p_1p_1 + p_2p_0) \cos (n-3)\varphi \\ &+ 2(p_0p_3 + p_1p_2 + p_2p_1 + p_3p_0) \cos (n-4)\varphi + \cdots \\ &+ 2(p_0p_{n-2} + p_1p_{n-3} + p_2p_{n-4} + \cdots + p_{n-2}p_0) \cos \varphi \\ &+ (p_0p_{n-1} + p_1p_{n-2} + p_2p_{n-3} + \cdots + p_{n-1}p_0).\end{aligned} \tag{58}$$

Since

$$\int_0^\pi \Phi \sin \theta \, d\theta = \frac{1}{\beta\ell}\int_{\varphi_1}^{\varphi_0} \Phi \, d\varphi,$$
$$\varphi_0 = \beta\ell - \vartheta + \frac{\bar{\psi}}{2} + \frac{\bar{\psi}}{2(n-1)} + \pi, \qquad \varphi_1 = \varphi_0 - 2\beta\ell \tag{59}$$

we can write

$$\begin{aligned}\int_0^\pi \Phi \sin \theta \, d\theta = \frac{1}{\beta\ell}\Bigg[&\frac{2p_0^2 \sin (n-1)\varphi}{n-1} + \frac{2(p_0p_1 + p_1p_0) \sin (n-2)\varphi}{n-2} \\ &+ \cdots + (p_0p_{n-1} + p_1p_{n-2} + \cdots + p_{n-1}p_0)\varphi\Bigg]_{\varphi_1}^{\varphi_0}.\end{aligned} \tag{60}$$

For an end-on array with nulls equispaced in the range of z, (60) becomes

$$\begin{aligned}\int_0^\pi \Phi \sin \theta \, d\theta = \frac{2}{\beta\ell}\Bigg[&\frac{2(-)^{n-1}p_0^2}{n-1} \sin (n-1)\beta\ell \cos \frac{(n-1)\beta\ell}{n-1} \\ &+ \frac{2(-)^{n-2}(p_0p_1 + p_1p_0)}{n-2} \sin (n-2)\beta\ell \cos \frac{(n-2)\beta\ell}{n-1} + \cdots \\ &+ (p_0p_{n-1} + p_1p_{n-2} + \cdots + p_{n-1}p_0)\beta\ell\Bigg].\end{aligned} \tag{61}$$

Substituting in (54), we shall obtain the gain of the array.

Similar expressions can be obtained for an end-on array in which the amplitudes of the individual elements are equal. Thus we have

$$\begin{aligned}\Phi &= \frac{1}{n^2}[e^{i(n-1)\psi} + e^{i(n-2)\psi} + \cdots + e^{i\psi} + 1] \\ &\qquad \cdot [1 + e^{-i\psi} + e^{-2i\psi} + \cdots + e^{-i(n-1)\psi}] \\ &= \frac{1}{n^2}[2 \cos (n-1)\psi + 4 \cos (n-2)\psi \\ &\qquad + 6 \cos (n-3)\psi + \cdots + 2(n-1) \cos \psi + n],\end{aligned} \tag{62}$$

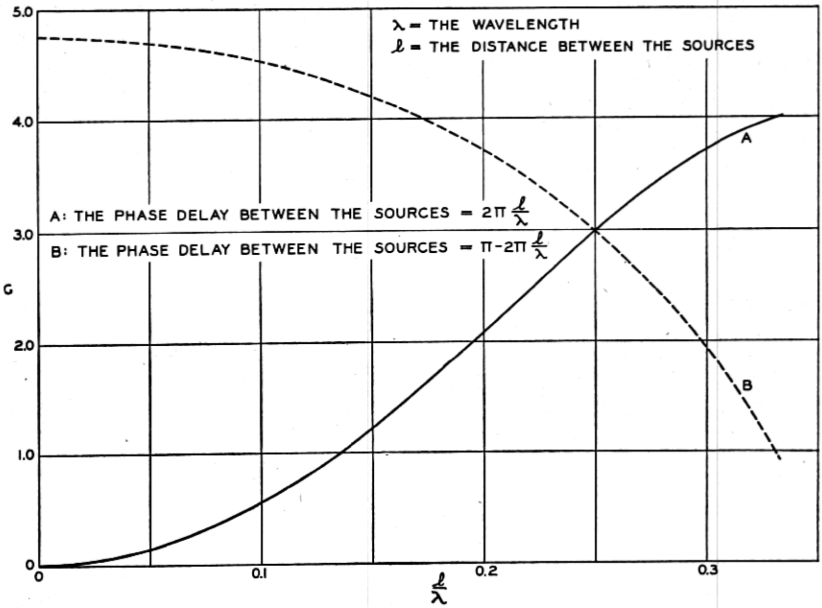


Fig. 21—The directive gain in decibels of a pair of sources with equal amplitudes. (A), the phase delay between the sources is $2\pi\ell/\lambda$; (B), the phase delay between the sources is $\pi - 2\pi\ell/\lambda$.

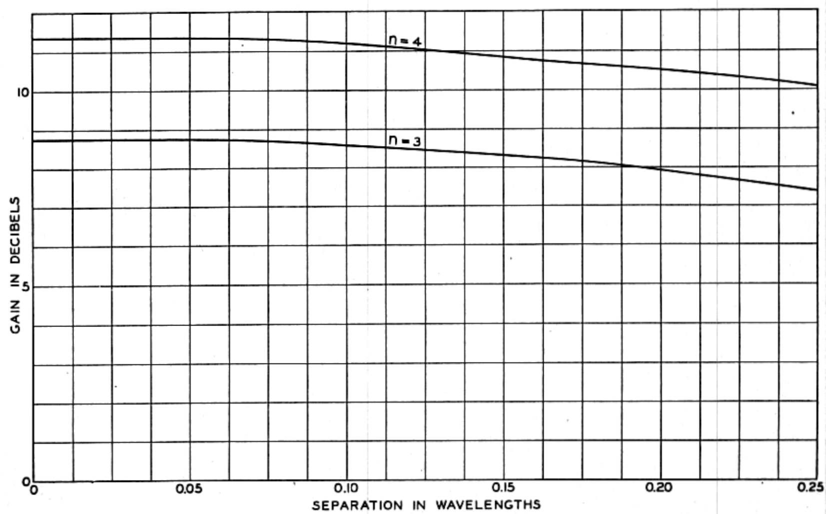


Fig. 22—The gain as a function of separation in wavelengths: n is the number of elements.

where $\psi = \beta\ell\,(\cos\theta - 1)$. In this case $D = 1$ and

$$\int_0^\pi \Phi \sin\theta\, d\theta = \frac{2}{n^2\beta\ell}\left[n\beta l + \frac{\sin 2(n-1)\beta\ell}{n-1} + \frac{2\sin 2(n-2)\beta\ell}{n-2} + \cdots + \frac{3\sin 2(n-3)\beta\ell}{n-3} + \cdots + (n-1)\sin 2\beta\ell\right]. \tag{63}$$

When the separation between the elements is exactly an integral number of quarter wavelengths, (63) becomes

$$\int_0^\pi \Phi \sin\theta\, d\theta = \frac{2}{n} \tag{64}$$

and consequently the gain is

$$G = 10\ \mathrm{Log}_{10}\, n. \tag{65}$$

Figure 21 contrasts the directive gain of a pair of sources of equal strength with the phase delay $2\pi\ell/\lambda$ (Curve A) with a directive gain of another pair of sources of equal strength but with the phase delay $\pi - 2\pi\ell/\lambda$ (Curve B). In one case the directive gain diminishes with separation between the elements and in the other it increases. Figure 22 shows the directive gain of three-element and four-element end-on arrays with nulls equispaced in the range of z.

As the separation between the elements decreases, the directive gain of an end-on array with nulls equispaced in the range of z increases but the radiation intensity per ampere-meter decreases. This circumstance would be of no importance if we had perfect conductors at our disposal to make transmitting and receiving antennas; but in reality parasitic losses in themselves cannot be removed and the efficiency of an array decreases, therefore, with the separation between the elements. This decrease in efficiency will impose an upper limit on the overall gain that can be obtained with small antenna arrays in spite of the fact that the directive gain could be made very large.

Likewise the band width diminishes as the distance between the elements decreases. This imposes another limitation on arrays of this type.