Electrical and Mechanical Analogies

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Introduction

During the past few years, apparatus which transfers electrical into acoustical or mechanical energy has received wide application. This came about through the popular use of radios, phonographs, public address systems, and sound motion pictures. While the fundamental principles of such electro-mechanical or electro-acoustic transducers have been known for decades, it is safe to say that the rapid progress and excellent design obtained have been due in a large part to the knowledge derived from the related subject of electrical network theory.

Two examples may be cited to show the nature and extent of the improvement. Barton in his "Theory of Sound" (1914) cites measurements on the efficiency of acoustic foghorns operated from an electrical source of power and finds that the efficiency of conversion from electrical into acoustic energy is less than one per cent. Today large loud speakers have been developed which can be used for similar purposes and these have efficiencies of conversion greater than 50 per cent. Another and more striking example is the mechanical phonograph. From the days of its invention by Edison, mechanical phonograph reproducers had been constructed from such mechanical units as needles, diaphragms, horns, and their connecting mechanical elements. As late as 1925 the best of such units was capable of reproducing only three octaves. About this time, another mechanical phonograph[1] was constructed from the same sort of elements, but with their dimensions and relationships designed according to relations developed in electrical network theory, and the resulting structure was able to reproduce a frequency band corresponding to five octaves with greater uniformity and an increase in the efficiency of conversion.

The type of electrical network which has received the greatest application in the design of mechanical and acoustical systems is the electrical wave filter invented by Dr. G. A. Campbell. This may seen surprising at first sight, since the filter is usually regarded as a device for attenuating unwanted frequency bands while passing other frequency bands which it is desired to receive. The filter has two properties which make it of interest in electro-acoustic transducer systems. These are: first, the filter is able to coordinate

[1] Maxfield and Harrison, *Bell Sys. Tech. Jour.*, Vol. 5, No. 3, p. 493, 1926.

the action of several resonant elements to produce a device with a uniform transmission over a wide frequency range; and second, the dissipationless filter, with matched impedance terminations, is a device which delivers to its output all of the energy impressed upon it over the widest possible frequency range consistent with the elements composing it. These properties of the filter have been made use of in purely electrical networks to determine the largest band width a vacuum tube with known characteristics can have and still deliver a specified gain at a specified impedance level. Applied to electro-mechanical transducer systems, the filter theory shows how to combine resonant mechanical or electro-mechanical elements to produce a uniform conversion of electrical to mechanical energy, or vice versa, over a wide frequency range. Also, it is able to determine the greatest band width that can be obtained without loss of efficiency for any type of conversion element.

This transfer of knowledge from one branch of science, electrical network theory, to another branch of science dealing with mechanical and electro-mechanical structures is one example of a long line of such interchanges that have been going on for over a hundred years. These interchanges are made possible by the fundamental analogies which exist between electrical and mechanical systems and which rest finally on the fact that electrical motions and mechanical motions satisfy the same type of differential equations. Since such analogies have been very productive in the past and are likely to continue to be so in the future, it seems worthwhile to examine their foundation and development.

Early Borrowings of Electrical from Mechanical Theory

The equations of motion of mechanical bodies and mechanical media were developed and studied long before the equations for electrical wave propagation were derived. Under these circumstances it is natural that attempts should have been made to explain electrical wave propagation as a mechanical phenomenon. The view that electrical actions are ultimately dynamic was one whose development in the hands of Maxwell led to notable advances in the science, and it was the view toward which most of the early authorities leaned. In support of this point of view Maxwell showed that the forces on any system of charged bodies could be attributed to a system of stresses in the medium in which they are embedded. Since magnetic energy is associated with the presence of charge in motion while electro-static energy is present for charges at rest, an identification was made between kinetic and magnetic energy and between electro-static and potential energy. Applied to a concentrated system, this point of view indicates that an inductance is the analogue of a mass, while a capacitance is the analogue of a spring.

A case for which this point of view bore useful results was the case of anomalous dispersion in optics. It was found experimentally that when light was sent through certain substances the velocity of propagation depended markedly on the frequency in the neighborhood of a certain critical frequency. Below this critical frequency the velocity decreased as the frequency approached it, going rapidly toward zero as the critical frequency was approached. Above this critical frequency the phase velocity was greater than the velocity of light in the material and gradually approached that value for high frequencies. At the critical frequency a large absorption of light occurred. This was first explained by Sellmeier as being due to some element in the medium having a resonant frequency at the critical frequency In obtaining his equations he used a mechanical model in which the resonant elements were spaced at equal intervals and excited by waves propagated by virtue of the mass and elasticity of the substance.

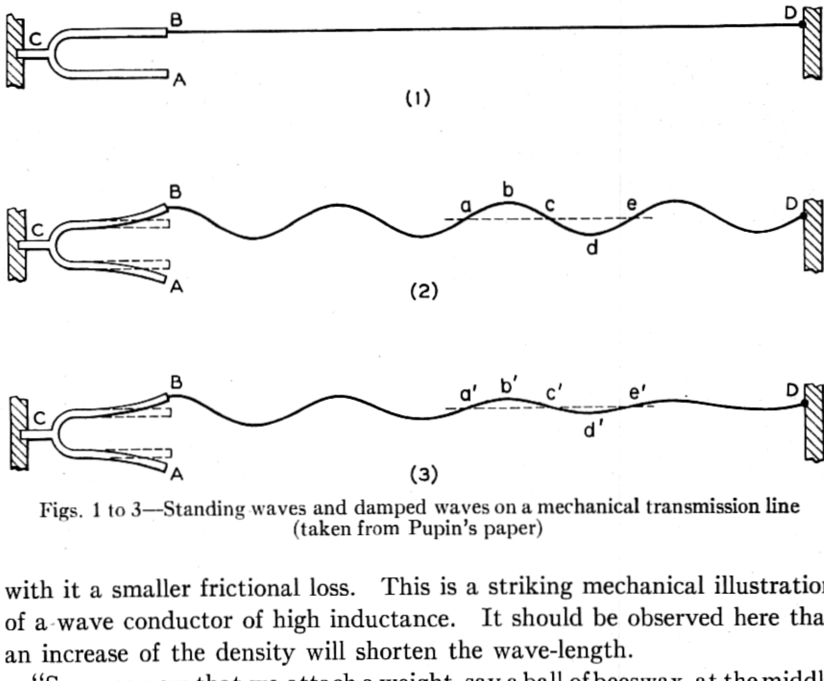
The case of greatest interest from the communication viewpoint is the influence of mechanical theory on the theory of the loaded transmission line. Wave propagation in a mechanical bar or stretched string has similar characteristics to that of a dissipationless electrical line, but when the effect of series resistances and shunt leakances were taken account of, effects appeared for the electric line which had not previously been studied in mechanical systems. These were high attenuation, which cut down the amount of power delivered to the output, and distortion, which caused the shape of the signal received at the end of the line to be different from that sent into the line. Heaviside showed that the distortion could be removed by having a certain relationship between the inductance, capacitance, resistance and leakance, and moreover that a smaller attenuation and a lower distortion would result, if an inductance were uniformly distributed along the line.

It was not a practical matter, however, to put in extra inductance at every point of the line so Heaviside suggested and tested out the effect of placing inductances at discrete points along the line, and found no beneficial results. It was not until Campbell and Pupin independently showed that the inductances had to have discrete values and be placed at definite separations that any progress was made in approaching the desired conditions.

Pupin's method of arriving at the solution is well illustrated by the following extract from his paper.[2] "The main features of the theory are extremely simple and can be explained by a simple mechanical illustration. Consider the arrangement of Fig. 1. A tuning fork has its handle C rigidly fixed. To one of its prongs is attached a flexible inextensible cord BD. One terminal of the cord is fixed at D. Let the fork vibrate steadily, the vibration being maintained electromagnetically or otherwise. The motion of the cord will be a wave motion If the frictional resistances opposing the motion

[2] "Wave Transmission Over Non-uniform Cables and Long Distance Air Lines," M. I. Pupin, *Trans. A. I. E. E.*, Vol. XVII, May 19, 1900.

of the cord are negligibly small the wave motion will be approximately that of stationary waves as in Fig. 2. The direct waves coming from the tuning fork and the reflected waves coming from the fixed point D will have nearly equal amplitudes and by their interference form approximately stationary waves. If, however, the frictional resistances are not negligibly small, then there will be dissipation of the propagated wave energy. Hence, the direct and the reflected waves will not result in stationary waves. The attenuation of the wave is represented graphically in Fig. 3. Experiment will show that, other things being equal, increased density of the string will diminish attenuation, because a larger wave requires a smaller velocity in order to store up a given quantity of kinetic energy and a smaller velocity brings



Figs. 1 to 3—Standing waves and damped waves on a mechanical transmission line (taken from Pupin's paper)

with it a smaller frictional loss. This is a striking mechanical illustration of a wave conductor of high inductance. It should be observed here that an increase of the density will shorten the wave-length.

"Suppose now that we attach a weight, say a ball of beeswax, at the middle point of the string, in order to increase the vibrating mass. This weight will become a source of reflection and less wave energy will reach the point D than before. The efficiency of transmission will be smaller now than before the weight was attached. Subdivide now the beeswax into three equal parts and place them at three equidistant points along the cord. The efficiency of wave transmission will be better now than it was when all the wax was concentrated at a single point. By subdividing still further the efficiency will be still more improved; but a point is soon reached when further subdivision produces an inappreciable improvement only. This point is reached when the cord thus loaded vibrates very nearly like a uniform cord of the same mass, tension, and frictional resistance."

Campbell, who first arrived at the design formulae for the coil loaded line, was guided by the solution, given by Lagrange over 100 years earlier, of the propagation of a wave along a string loaded with masses at discrete intervals, and a generalization of it made by Charles Godfrey,[3] for he states[4] "For the method of treatment which I first employed I am indebted to an interesting article by Mr. C. Godfrey on the 'Discontinuities of Wave Motion Along a Periodically Loaded String.'" The spacing of the coils arrived at is the same as the spacing of the massive beads along a string, namely, that π coils or beads occur per actual wave-length of the highest frequency to be transmitted. The added result not given in the mechanical case was that the addition of such coils reduced the attenuation and decreased the distortion.

The different points of view of the two inventors are well illustrated by the quotations. Pupin was attempting to obtain a system which approached a uniform line while Campbell was investigating the propagation characteristics of the structure without particular regard as to whether the transmission was the same as that which would be provided by an equal amount of inductance uniformly distributed along the line. It was the broader point of view of Campbell which proved of wide significance and which in particular led to the invention of the electrical wave filter.

Development of Electrical Network Theory

The first structures capable of transmitting bands of frequencies and attenuating all other frequencies were mechanical structures, although this was not generally realized at the time or made use of. The first structure of this sort was the string loaded with massive beads, which was first studied by Lagrange. Introducing the approximation that the mass of the string could be neglected, Lagrange showed that all of the natural frequencies of the device came below a certain critical frequency f_c. Routh[5] after discussing Lagrange's solution, points out that there may be a period of excitation of the string which is "so short that no motion of the nature of a wave is transmitted along the string."[6] An acoustic forerunner of the wave filter was the combination of two tubes of different lengths first proposed by

[3] *Phil. Mag.* 45, 456 (1898).
[4] See "Collected Papers of George A. Campbell," page 16.
[5] "Advanced Rigid Dynamics," p. 260, Paragraph 411 (1892).
[6] It should be noted that all mechanical filter theory of this time had to do with the natural resonances of unterminated filters or the transmission through misterminated sections. The idea of matched impedance terminations to introduce power into and absorb power from a filter is a development of electrical network theory.

Herschel. This structure passes low frequencies but attenuates strongly frequencies for which the difference in path length of the two tubes is an odd number of half wave-lengths.

It is interesting to note that all of the fore-runners of the filter were of the continuously distributed type which had their elements distributed uniformly along the length of the device. All such devices have an infinite number of pass bands, usually harmonically related. This is true also of the dissipationless loaded line considered as a filter structure. It was only by such abstractions as neglecting the weight of the loaded string that single pass bands were obtainable.

The low-pass electrical filter grew out of a network to simulate the operation of a long cable. By using series coils to simulate the loading and the distributed inductance of the line, condensers to simulate the distributed capacitance of the line, series resistances to simulate the ohmic resistance of the line and the resistance of the coils, and shunt resistances to simulate the leakance of the line, Campbell was able to obtain in a small space, a device which had the same propagation characteristics up to the cut-off frequency as a long section of loaded cable. Furthermore, by making the resistance small, he was able to obtain a frequency range from zero frequency up to a cut-off frequency f_c with small attenuation, and a high attenuation at higher frequencies, and thus obtained the first true low-pass filter. He also put his filter to practical use for he says[7] "I have made use of these results by employing artificial loaded lines for cutting out harmonics in generator circuits. The harmonics may all be cut down as far as desired by the use of a sufficient number of sections, while the attenuation of the fundamental can be reduced at pleasure by decreasing the resistance."

Having developed the fundamental idea of a filter as a device for transmitting without loss one frequency range and attenuating all other frequencies, he went on to extend the idea to other types of filters in which different frequency ranges were passed. The band-pass filter was already realized in 1903 for Campbell says[7] "Combining condensers and inductances, we may make a system which will not only cut out higher frequencies, but also all frequencies below a certain limit." The high-pass and band elimination filters followed shortly after. With the invention of the electrical wave filter, electrical network theory can be considered as well started.

The science of electrical networks did not progress much farther for a number of years. This was due primarily to the lack of application for any of the structures developed. However, with the development of carrier current transmission on telephone lines, the necessary stimulus was received. Carrier current systems were an adaptation of radio communication systems

[7] "On Loaded Lines in Telephonic Transmission," *Philosophical Magazine*, Ser. 6, Vol. 5, pp. 313–330, March 1903.

to wire lines, with the line taking the place of the ether as the wave transmitting medium. Previously, radio systems had been developed which would transmit messages in definite frequency ranges. These transmitting ranges were selected from other frequency ranges by means of electrical tuned circuits, which were themselves a borrowing from the acoustic resonators of Helmholtz devised many years earlier. Tuned circuits are not advantageous for selecting out channels in a carrier system, because with them it is not possible to regulate the band width received or to get the necessary discrimination between the pass band and the attenuated region. It was found, however, that filters could meet these requirements and consequently they were applied in separating the channels of the first carrier systems.

This use stimulated the further development of electrical network theory. Filters with sharper discriminating properties, composite filters containing sections of like image impedances but different attenuation characteristics, transforming filters, impedance corrected filters for reducing reflections, filters using mutual inductances, attenuation and phase correcting networks are among the later developments. These investigations were carried out by a large number of individuals among whom may be mentioned Bartlett, Bode, Carson, Cauer, Foster, Fry, Guillemin, Johnson, Norton, Shea, Wagner, and Zobel. Electrical network theory has progressed to such an extent that it is now possible to select substantially any desired frequency range, with very little of the frequency range wasted in obtaining the desired selectivity, and to control the amplitude and phase of the currents received over long distance lines so that a high degree of fidelity of the received signal can be maintained.

Borrowings of Mechanical Theory from Electrical Network Theory

While this development of electrical theory was progressing, very little development of a parallel nature was being carried out for mechanical theory, due probably to the lack of a corresponding stimulus. With the advent of the vacuum tube, public address system, and radio broadcasting, however, a demand developed for loud speakers and related equipment. It was shortly realized that the parallel developments of electrical network theory provided a base for the design of such equipment. One of the first to recognize this possibility was Professor A. G. Webster,[8] who pointed out the usefulness of the concept of impedance in mechanical systems. He applied the concept to the phonograph and developed the first theory of the action of acoustic horns. After this occurred the widespread application of the

[8] A. G. Webster, "Acoustic Impedance, and The Theory of Horns and of the Phonograph," *Nat'l. Acad. of Science,* Vol. 5, p. 275, 1919.

electrical network theory to the design of electro-mechanical systems mentioned in the introduction.

Aside from this electro-mechanical field special applications have been made in acoustic and mechanical apparatus where problems occur similar to those solved by electrical means. In all of these applications it is the filter type structure that is applied.

One of the first of these applications was the acoustic filter. In ventilating ducts, automobile, and other types of engines, and for many other uses, it is desirable to pass a steady or slowly varying stream of air, and attenuate the more rapid vibrations which constitute the undesired noise. Furthermore, it is desirable to pass the low-frequency variations with little or no loss, since such loss increases the back pressure on the engine or blower and greatly decreases their efficiency. For this purpose the low-pass filter type structure is well suited since it passes a low-frequency band with little or no attenuation and strongly suppresses higher frequency components.

The rudimentary idea of the acoustic filter probably dates back to Herschel (1833) who suggested the use of combinations of tubes capable of suppressing certain frequencies. Following the development of electrical wave filters, Professor G. W. Stewart[9] showed that combinations of tubes and resonators could be devised which would give transmission characteristics at low frequencies similar to electrical filters. This theory worked well as long as the structure was small or the frequency low, but broke down for large structures and high frequencies due to the essentially distributed nature of the elements. A theory of acoustic filters was given by the writer in 1927,[10] which took account of wave motion in the elements, and this theory could account for the properties of the filters to much higher frequencies. Since then, Lindsay[11] and his collaborators have discussed a number of acoustic type filters with various types of side branches and obstructions.

Mufflers existed long before the theory of acoustic filters was worked out but they were designed as a series of baffles, which introduced considerable back pressure on the engine. Most recently designed mufflers have a straight conducting path with side-branches in conformance with acoustic filter theory and are proportioned to attenuate most of the frequencies above 100 cycles. As a result they are considerably more effective than

[9] *Phys. Rev.* 20, 528 (1922); 23, 520 (1924); 25, 590 (1925). See also Stewart and Lindsay "Acoustics," Chap. VII. D. Van Nostrand.

[10] A Study of the Regular Combination of Acoustic Elements, with Applications to Recurrent Acoustic Filters, Tapered Acoustic Filters, and Horns. *B.S.T.J.* Vol. VI, pp. 258–294, April 1927.

[11] An excellent review and resumé of the literature on gaseous and solid acoustic filters is given by Lindsay, "The Filtration of Sound I," *Jour. App. Phys.* 9, 612 (1938); "The Filtration of Sound II," *Jour. App. Phys.* 10, 620 (1939).

early mufflers and introduce considerably less back pressure on the engine or blower.

Other uses to which mechanical filters have been put are in obtaining shockproof mountings and vibration damping devices, in obtaining vibration and noiseproof walls and floors, and in obtaining constant speed motors in which the effects of gear irregularities are removed by the use of a low-pass mechanical filter.

Mechanical and Electro-Mechanical Counterparts of Electrical Filters

Although combinations of electrical elements were first studied and applied in wave filters and other structures, it does not follow that they have any inherent advantages over analogous combinations of mechanical or electro-mechanical elements which can be used as filters. In fact, elements which depend on mechanical motion have the great advantage that they have very little energy dissipation associated with their motion and, hence, the equivalent mechanical elements have a higher ratio of reactance to resistance, or "Q," than do their electrical counterparts. The result is that considerably more selective filters can be made from mechanical or electro-mechanical elements than can be obtained by employing electrical coils and condensers.

The first attempts[12,13] along this line were made in substituting masses for coils and springs for condensers in standard electrical filter configurations. This work resulted in usable filters up to several thousand cycles in frequency, which have been used for certain special purposes.

More recently, electro-mechanical elements have been used to take the place of some or all of the electrical elements of a filter and this work has resulted in filters with markedly superior characteristics to those obtained with filters using only electrical elements. The type of electro-mechanical element which has been used most extensively in selective filters is the piezo-electric crystal and particularly the quartz crystal. This element has the advantage of an electro-mechanical converting system in the piezo-electric effect and a very high mechanical Q. Moreover, a quartz crystal is very stable mechanically and can be cut so that its frequency changes very little over a wide temperature range. For these reasons, quartz crystals have been applied extensively when it is desirable to obtain a narrow band filter or a very selective filter.

[12] This work was carried out principally by Messers. Hartley, Lane and Wegel.

[13] The use of a mechanical filter in visually studying the properties of a wave filter is described in a paper "A Mechanical Demonstration of the Properties of Wave Filters," C. E. Lane, *S.M.P.E. Jour.* Vol. 24, pp. 206–220, March 1935.

Such filters have received a wide variety of uses. Very narrow band filters have been used in carrier systems as pilot channel filters for separating out the pilot or control frequency from the other frequencies present; in radio systems for separating the carrier frequency from the sideband frequencies; and in heterodyne sound analyzing devices for analyzing the frequencies present in industrial noises, speech, and music. Wider band filters employing coils as well as crystals have provided very selective devices which are able to separate one band of speech frequencies from another band different by only a small percentage frequency range from the desired band. This property makes it possible to space channels close together with only a small frequency separation up to a high frequency, and such filters have had a wide use in the high frequency carrier systems and in the coaxial system which transmits up to 480 conversations over one pair of conductors. In radio systems such filters have been used extensively in separating one sideband from the other in single sideband systems.