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Industrial Mathematics*

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The report consists of three major sections. The first discusses mathematical specialists in industry, calls attention to the essentially consultative character of their work, and makes some observations regarding the education, employment and supervision of this type of personnel.

The second section deals, not with the work of these specialists, but with the uses to which mathematics is put at the hands of industrial workers in general, the various ways in which it contributes to the economy and effectiveness of research, and the kinds of mathematics that are most used. A number of illustrations are given, together with brief surveys of the utilization of mathematics in four important industries: communications, electrical manufacturing, petroleum and aircraft.

The third section is devoted to statistics, which touches industrial life at rather different points, and hence could not conveniently be included in the general discussion.

Introduction

MATHEMATICAL technique is used in some form in most research and development activities, but the men who use these techniques would not usually be called mathematicians.

Mathematicians also play an important role in industrial research, but their services are of a special character and do not touch the development program at nearly so many points.

Because of this contrast between the ubiquity of mathematics and the fewness of the mathematicians, this report is divided into sharply differentiated parts. Under "Mathematicians in Industry" an attempt is made to explain what sort of service may be expected of industrial mathematicians, and to develop some principles of primary importance in employing and managing them. An attempt is also made to appraise future demand for men of this type, and to discuss the sources from which they can be drawn. Under "Mathematics in Industry" appear brief surveys of the extent and character of the utilization of mathematics in a few special industries, and examples of specific problems in the solution of which mathematical methods have been necessary or advantageous.

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* This discussion of the part which Mathematics might play and to a certain extent is playing in industry was prepared for the National Research Council *Survey of Industrial Research*, a survey undertaken at the request of the National Resources Planning Board. The document which the survey produced has been published as "Research—A National Resource, Part II, Industrial Research" and is available through the Government Printing Office.

In these two sections mathematics is interpreted broadly to include not only the fundamental subjects, algebra, geometry, analysis, etc., but also their manifestations in applied form as mechanics, elasticity, electromagnetic theory, hydrodynamics, etc. Statistics, however, touches industrial activity in a rather different way, and is therefore discussed separately under a third heading, "Industrial Statistics and Statisticians."

One observation which will be made in more detail later is worthy of mention here, because of the present and prospective scarcity of suitably trained industrial mathematicians. Though the United States holds a position of outstanding leadership in pure mathematics, there is no school which provides an adequate mathematical training for the student who wishes to use the subject in the field of industrial applications rather than to cultivate it as an end in itself. Both science generally, and its industrial applications in particular, would be advanced if a group of suitable teachers were brought together in an institution where there was also a strong interest in the basic sciences and in engineering.

Mathematicians in Industry

What is a Mathematician?

If every man who now and then computes the average of a set of instrumental readings or solves a differential equation is a mathematician, there are few research workers who are not. If, on the other hand, only those who are primarily engaged in making additions to mathematical knowledge are mathematicians, there are almost none in industry. Neither definition is sound. The first is absurd; the second not closely related to the essential nature of mathematical thought. This report adopts a definition based upon the character of the man's thinking rather than the ultimate use to which his thinking is put.

Some men would be called mathematicians in any man's language; others physicists or engineers. These *typical* men are differentiated in certain essential respects:

The typical mathematician feels great confidence in a conclusion reached by careful reasoning. He is not convinced to the same degree by experimental evidence. For the typical engineer these statements may be reversed. Confronted by a carefully thought-out theory which predicts a certain result, and a carefully performed experiment which fails to produce it, the typical mathematician asks first, "What is wrong with the experiment?" and the typical engineer, "What is wrong with the argument?" Because of this confidence in thought processes the mathematician turns naturally to paper and pencil in many situations in which the engineer or physicist would resort to the laboratory. For the same reason the mathematician in his "pure" form delights in building logical structures, such as topology or abstract algebra, which have no apparent connection with the world of physical reality and which would not interest the typical engineer; while conversely the engineer or physicist in his "pure" form takes great interest in such useful information as a table of hardness data which may, so far as he is aware, be totally unrelated to any theory, and which the typical mathematician would find quite boring.

A second characteristic of the typical mathematician is his highly critical attitude toward the details of a demonstration. For almost any other class of men an argument may be good enough, even though some minor question remains open. For the mathematician an argument is either perfect in every detail, in form as well as in substance, or else it is wrong. There are no intermediate classes. He calls this "rigorous thinking," and says it is necessary if his conclusions are to be of permanent value. The typical engineer calls it "hair splitting," and says that if he indulged in it he would never get anything done.

The mathematician also tends to idealize any situation with which he is confronted. His gases are "ideal," his conductors "perfect," his surfaces "smooth." He admires this process and calls it "getting down to essentials"; the engineer or physicist is likely to dub it somewhat contemptuously "ignoring the facts."

A fourth and closely related characteristic is the desire for generality. Confronted with the problem of solving the simple equation $x^3 - 1 = 0$, he solves $x^n - 1 = 0$ instead. Or asked about the torsional vibration of a galvanometer suspension, he studies a fiber loaded with any number of mirrors at arbitrary points along its length. He calls this "conserving his energy"; he is solving a whole class of problems at once instead of dealing with them piecemeal. The engineer calls it "wasting his time"; of what use is a galvanometer with more than one mirror?

In the vast army of scientific workers who cannot be tagged so easily with the badge of some one profession, those may properly be called "mathematicians" whose work is dominated by these four characteristics of greater confidence in logical than experimental proof, severe criticism of details, idealization, and generalization. The boundaries of the profession are perhaps not made sharper by this definition, but it has the merit of being based upon type of mind, which is an attribute of the man himself, and not upon such superficial and frequently accidental matters as the courses he took in college or the sort of job he holds.

It is, moreover, a more fundamental distinction than can be drawn between, say, physicist, chemist and astronomer. That is why the mathematician holds toward industry a different relationship than other scientists, a relationship which must be clearly understood by management if his services are to be successfully exploited.

The Place of the Mathematician in Industrial Research

The typical mathematician described above is not the sort of man to carry on an industrial project. He is a dreamer, not much interested in things or the dollars they can be sold for. He is a perfectionist, unwilling to compromise; idealizes to the point of impracticality; is so concerned with the broad horizon that he cannot keep his eye on the ball. These traits are not weaknesses; they are, on the contrary, of the highest importance in the job of finding a system of thought which will harmonize the complex phenomena of the physical world, that is in reducing nature to a science. The job of industry, however, is not the advancement of natural science, but the development, production and sale of marketable goods. The physicist, the chemist, and especially the engineer, with their interest in facts, things and money are obviously better adapted to contribute directly to these ends. To the extent that the mathematician takes on project responsibility, he is forced to compromise; he must specialize instead of generalize; he must deal with concrete detail instead of abstract principles. Some mathematicians cannot do these things at all; some by diligence and self-restraint can do them very well. To the extent, however, that they succeed along these lines they are functioning not as mathematicians but as engineers. As mathematicians their place in industry is not to supply the infinite attention to practical detail by which good products, convenient services, and efficient processes are devised; their function is to give counsel and assistance to those who do supply these things, to appraise their everyday problems in the light of scientific thought, and conversely to translate the abstract language of science into terms more suitable for concrete exploitation.

In other words, the mathematician in industry, to the extent to which he functions as a mathematician, is a consultant, not a project man.

Qualifications Necessary for Success as an Industrial Mathematician

The successful industrial mathematician must not only be competent as a mathematician; he must also have the other qualities which a consultant requires:

First, though his major interests will necessarily be abstract, he must have sufficient interest in practical affairs to provide stimuli for useful work and to reconcile him to the compromises and approximations which are necessary even in the theoretical treatment of practical problems. This usually means that the type of mathematician who could not do a good engineering job if he turned his hand to it will not get on very well in an industrial career.

Second, he must be gregarious and sympathetic. If he shuts himself off from his associates, much of his thinking will have no bearing on their needs and that which does will exert less influence than it might. If he does not translate his thoughts into their language, they will miss the significance of much of his work and he will have but a limited clientele.

Third, he must be cooperative and unselfish. A man cannot be at once consultant and competitor to his associates. Self-seeking attempts to gain credit for his contributions to the industry will inevitably alienate his clientele. There are two reasons for this: In the first place a mathematician's appraisal of mathematical work, even if made from a detached point of view, is heavily weighted on the side of its fundamental scientific significance, whereas its industrial value should be judged on very different grounds and can best be appraised by the engineer. In the second place, the engineer in charge of a project can give credit without embarrassment for help received; it is to his credit to have known where help was to be had. The same story told by another, and particularly by the consultant himself, has an entirely different flavor.

Fourth, he must be versatile. Jobs change, and even the same job may give rise to questions which require very different mathematical techniques.

Fifth, he must be a man of outstanding ability. No one wants the advice of mediocrity. Among industrial mathematicians there is no place for the average man.

Employment and Supervision

Perhaps the greatest hazard in hiring mathematicians for industry arises from the fact that the employment officer is not often a judge of mathematical ability. Paradoxically, however, his mistakes are not usually made in judging mathematical aptitude, since general scholastic rating is an unusually trustworthy index of mathematical ability. But because of a feeling of incompetence bred by his lack of mathematical lore, he spreads the mantle of charity over other characteristics with regard to which he should trust his own judgment. If, for example, the applicant gives an incoherent account of the problems on which he has been working, the interviewer excuses it on the ground of his own lack of mathematical training, an excuse which would be quite adequate if the circumstances demanded that he meet the applicant on the applicant's ground. What he overlooks is that the applicant has failed to meet him on his own ground; has failed, in other words, to display the essential ability to translate his thoughts into the language of his hearer. Or perhaps a personality defect is excused on the ground that "after all, he will be working by himself and won't have to meet people," whereas in fact the real value of a consultant comes not in what he does at his desk, but in how much of it gets through to his associates. The applicant who is boastful or pushing or querulous should not be hired on the general theory that "all mathematicians are queer."

High standards in all such matters, and an interest in practical things as well, are as important as technical mathematical ability. These are stiff specifications, and the men to fill them are not to be found in every market place. They are, however, the requirements implicit in the nature of the job and no good can come from failing to recognize them.

After the right man is hired, he is not a difficult person to supervise if his function as a consultant to the rest of the staff is kept clearly in mind. The broad objectives must be to avoid barriers which would tend to deter his associates from seeking his services, and to assure that his work is justly appraised and fairly compensated.

The three barriers most likely to arise between him and his associates are jealousy, red tape and unavailability.

Jealousy is unavoidable if the man himself is self-seeking; once such a man is hired trouble is inevitable. But the man is not always to blame. A generous and cooperative recruit will be spoiled by an atmosphere too highly charged with progress reports, or by a salary policy which bases revisions upon the dollar value of the last year's work. Actually the "progress" which is significant to management will be far more accurately appraised by his colleagues than by himself, hence his reports have little value except as they give him an opportunity to review and criticize his own activities. If too much emphasis is placed upon them, even this value will be lost and they will be written in the spirit of making a case for himself, which is exactly the spirit most certain to breed jealousy. Similarly, a salary policy based on dollar returns is essentially unjust, for the money value of various bits of theoretical work has almost no correlation with the scientific acumen which they require. This does not mean that a mathematician's pay should, in the long run, be independent of the dollar value of his services. It means only that whether he gets a raise this year, and how big it shall be, should properly be based on the size, character and satisfaction of his clientele, and not upon the commercial importance of the questions they saw fit to bring him last year.

Red tape is easily avoided by avoiding it. No engineer, whatever his rank in the organization, ought ever need permission to consult a mathematician in the company's employ, and the mathematician in turn ought not need a specific work order or expense allowance before giving his advice. In this respect he should be on the same basis as the free-lance investigators who are to be found in most large research laboratories, and who are generally known as staff engineers.

Unavailability is a more serious matter. It is well recognized that in industrial research the urgent job always tends to take precedence over the important one. Left to themselves, fundamental studies give way to the detailed development "which ought to go into production next month." Mathematical studies are no more susceptible than other fundamental research to such interruptions, but the effect upon the career of the mathematician may be more far-reaching, for as soon as he is assigned an urgent project of special character his availability as a consultant ceases or at best is temporarily impaired. If his value to the industry is greater as a project man than as a consultant this need not be a cause for regret; but to turn a good mathematician into a poor engineer, or an irreplaceable mathematician into a replaceable engineer, is unfortunate for both employer and employee.

The Mathematical Research Department of the Bell Telephone Laboratories

In the Bell Telephone Laboratories men of this type have been grouped together as a separate organization unit. They have no more specific function than to be helpful to their associates in other parts of the Laboratories. No engineer is obliged to consult them about any phase of his work; no particular jobs come to them by reason of prerogative; conversely, there is no sort of help which an engineer or physicist may not seek from them if he so desires. No routine need be complied with in advance in order to secure their services, and no report is required afterwards, though written reports are frequently prepared when needed for scientific record. The expense of the group is distributed broadly over the activities of the Laboratories, not charged to specific jobs. Every effort is made to maintain a spirit of service among the members of this group, and though responsibility for engineering projects occasionally descends upon them, it is regarded as an undesirable necessity to be avoided whenever possible and liquidated at the earliest opportunity.

The group has functioned successfully for a number of years. Its members are respected by their engineering associates, and like their jobs. Information regarding their activities reaches management almost entirely through spontaneous acknowledgments made by the engineers they assist. These expressions of appreciation are generous, but rather erratic in that they concentrate attention first on one man, then on another, as the genius and training of the individual happen to click with the important job of the moment. This has not affected the morale of the group adversely, probably because a serious effort is made to avoid erratic salary revisions in which the man who is at the moment in the limelight benefits at the expense of others who are doing equally good but less conspicuous work.

From the standpoint of the men, the principal advantages of being associated together instead of distributed through the engineering departments, is the stimulus of contact with men of like interests. From the standpoint of management, the advantages are wider availability, greater flexibility in matching the talents of the man with the requirements of the job, and a more uniform appraisal of ability because of supervision by a man of adequate mathematical background.

So far as is known, mathematicians have not been organized into separate administrative groups in other industries. In most laboratories their numbers have been thought too small to make such an arrangement feasible, and they have been treated as staff engineers distributed throughout the various general departments. It is believed, however, that there are a few industries in which this arrangement could be introduced with profit at this time, and that it has sufficient merit to justify its adoption wherever possible.

The Mathematician in the Small Laboratory

What has been said above relates primarily to conditions in large industries. The qualifications for success in the small industry are not dissimilar, though the relative emphasis to be placed upon them is somewhat different. Matters of personality (gregariousness, unselfishness, etc.) are not quite so important, because they are offset to some extent by the friendly coherence of the small group. On the other hand, a strong interest in things as well as ideas, and the ability to translate from the language of concrete experience to that of abstract thought and conversely, take on even greater importance. As Dr. H. M. Evjen, himself a worker in a small laboratory, says:

> "In order to be of optimum value, the mathematician must keep in close touch with realities. In a sufficiently large organization, employing both theoretical and experimental men, the best results, therefore, can be obtained only by the closest cooperation between the two groups. In smaller organizations, employing —for instance—only one scientifically qualified man, it is difficult to say whether this man should be of the theoretical or the experimental type. If he is a theoretical man, no success can be expected unless he is willing to roll up his sleeves

and get his feet firmly planted on the ground. In fact, even if he has highly qualified experimental assistants, he should not feel averse to 'getting down in the dirt.' Secondhand information is always of inferior quality. . . .

The mathematician not only is useful as an auxiliary to whom the practical man can turn with special problems. A properly trained mathematician, with a sufficiently broad vision, can be very much more useful as an active participant in the industrial problems. Due to his training in exact thinking he should be better able to see through the maze of intricate details and discover the fundamental problems involved."

Number Employed

The number of mathematicians employed in communications, electrical manufacturing, petroleum and aircraft is estimated at about 100. The number employed in other places is no doubt somewhat less, but it is probably not an insignificant part of the whole, since mathematicians are found here and there in some very small industries. For example, the Brush Development Company with a total engineering force of only 17, has found it desirable to supplement this group with a man hired specifically as a consultant in mathematics.

It is perhaps not too wide of the mark to estimate the total number at 150, not including actuaries and statisticians.

This number can be checked in another way. The membership list of the American Mathematical Society lists 202 men with industrial addresses. Of these, 102 are in financial and insurance firms and are presumably statisticians. The remaining 100 names are those of industrial employees with mathematical interests strong enough to belong to an organization devoted exclusively to the promotion of mathematical research. Some of these are not mathematicians by the definition adopted in this report. On the other hand, there are also 158 names for which only street addresses are given, some of whom are known to be industrial mathematicians. Balancing these uncertainties against one another, and remembering that many industrial mathematicians find little profit in belonging to an association devoted primarily to pure mathematics, the estimate given above does not appear unreasonable.

Future Demand

The appraisal of future demand is even more speculative than the estimation of present personnel. Two statements, however, seem warranted: (1) The demand for mathematicians will never be comparable to that for physicists, chemists or engineers. (2) It will certainly increase beyond the number at present employed.

The first statement is justified by the fact that physicists, chemists, and other experimental workers deal directly with the natural laws and natural resources which it is the business of industry to exploit, whereas mathematicians touch these things only in a secondary way.

The second statement would perhaps be granted on the general ground that throughout the whole of industry, research is becoming more complex and theoretical, and hence the value of consultants in general, and of mathematical consultants in particular, must increase. It is not necessary, however, to rely solely on such general considerations. Direct evidence exists in certain industries, notably aircraft,[1] where many of the major research problems are generally recognized to be more readily accessible to theoretical than experimental study, and in certain others, such as industrial chemistry,[2] where one may reasonably assume that modern molecular physics will soon begin to play an important part in determining speeds of reaction. There is also the general alertness of executives to the dollar value of a theoretical framework in planning expensive experiments, and the gradually changing attitude toward mathematics that stems from it. As Dr. W. R. Burwell, Chairman of the Brush Development Company, writes:

> "There is a definite trend toward a greater use of mathematics in industry which is somewhat commensurate with the trend toward the acceptance of research and development departments as necessary adjuncts to successful businesses. It is becoming more and more generally recognized that mathematics is not only a necessary tool for all engineers, physicists and chemists who make any pretense of going beyond strictly observational methods and experimental solutions to their problems but that it is also performing an important function as the recording medium for those generalizations which lay the foundation for the advances of scientific knowledge. . . .
>
> Even in an organization as small as ours, the use as a consultant is really important and we are constantly having instances where the mathematician because of his training is serving as an interpreter of mathematical and physical theories, sometimes influencing the direction of experimental work and sometimes eliminating the need for it."

If, therefore, the estimate of 150 mathematicians in industry at present is realistic, it may not be too wide of the mark to forecast several times that number a decade or so hence.

Source of Supply

Based on these estimates, a demand for new personnel of the order of 10 a year may be predicted. This number sounds small; but if we reiterate that mediocrity has no place in the consulting field, and that

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[1] See pages 31–34.
[2] See pages 30–31.

these 10 must be *exceptional* men, it does not seem unreasonable to ask where they may be found.

Most mathematicians now in industry were trained as physicists or as electrical or mechanical engineers, and gravitated into their present work because of a strong interest in mathematics. Few came from the mathematical departments of universities. As scientists they are university trained, but as mathematicians they are self-educated.

Their training has not been ideal. Industrial mathematics is being carried on by graduates of engineering or physics not so much because of the value of that training as because of the weakness of mathematical education in America. The properly trained industrial mathematician should have, beyond the usual courses of college grade, a good working background of algebra (matrices, tensor theory, etc.), some geometry, particularly the analytic sort, and as much analysis as he can absorb (function theory, theory of differential and integral equations, orthogonal functions, calculus of variations, etc.). These should have been taught with an attitude sympathetic to their applications, and reinforced by theoretical courses in sound, heat, light and electricity, and by heavy emphasis upon mechanics, elasticity, hydrodynamics, thermodynamics and electromagnetic field theory. He should understand what rigor is so that he will not unwittingly indulge in unsound argument, but he should also gain experience in such useful but sometimes treacherous practices as the use of divergent series or the modification of terms in differential equations. He should have enough basic physics and chemistry of the experimental sort to give him a realistic outlook on the power as well as the perils of experimental technique. By the time he has acquired this training he will usually also have acquired a Ph.D. degree, but the degree itself is not now, and is not likely to become, the almost indispensable prerequisite to employment that it is in university life.

There is nowhere in America a school where this training can be acquired. No school has attempted to build a faculty of mathematics with such training in mind. Hence industry has had to make such shift as might be with *ersatz* mathematicians culled from departments of physics and engineering. To make matters worse, a student with strong theoretical interests who enrolls in physics these days is almost certain to spend most of his time on modern mathematical physics, which insists almost as little upon fidelity to experience and experiment as does "pure" mathematics, from which it differs more essentially in matters of language and rigor than of general philosophic attitude. At the moment, therefore, engineering schools must be looked upon as the most hopeful sources of industrial mathematicians.

Historically it is easy to explain how this situation came about. Fifty years ago America was so backward in the field of mathematics that there was not even a national association of mathematicians. A quarter of a century later it was just coming of age in mathematics and was properly, if not indeed necessarily, devoting its entire attention to improving the quality of instruction in the "pure" field. The first faint indications that industrial mathematics might someday become a career had indeed begun to appear, but they were not impressive enough to attract the attention of university executives.

Today we lead the world in pure mathematics, and perhaps also in that other field of mathematics which has somehow come to be known as modern physics. We have strong centers of actuarial and statistical training. But in the field of applied mathematics which is the particular subject of this report, we stand no further forward than at the turn of the century, and far behind most European countries.

A quarter of a century ago it would have been difficult to find suitable teachers. Just now it could be done, primarily because a number of European scholars of the right type have been forced to come here, and a few others have developed spontaneously within our own borders. There are perhaps half a dozen of them, but they are so scattered, sometimes in such unpropitious places, as to have little influence on the development of industrial personnel.

It is unfortunate that no university with strong engineering and science departments has seen fit to bring this group together and establish a center of training in industrial mathematics. We have estimated a demand of about 10 *exceptional* graduates per year. If that estimate is even remotely related to the facts, such a department would have a most important job to do.

MATHEMATICS IN INDUSTRY

Subjects Used

As Dr. H. M. Evjen, Research Physicist of the Geophysical Section of the Shell Oil Company, remarks:

> "Higher mathematics, of course, means simply those branches of the science which have not as yet found a wide field of application and hence have not as yet, so to speak, emerged from obscurity. It is, therefore, a temporal and subjective term."

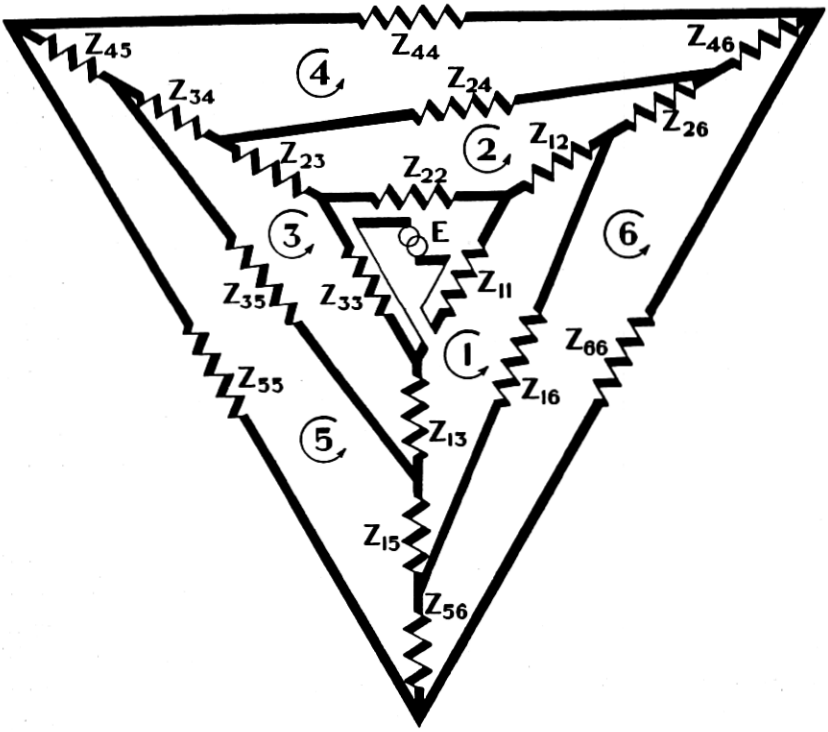
If this is accepted as a definition of higher mathematics—and it is a valid one for the pure science as well as for its applications—it follows automatically that industry relies principally upon the lower branches. What it uses much, ceases by the very muchness of its use to be high. The theory of linear differential equations, for example, is a subject by which the average well-trained engineer of 1890 would have been completely baffled. The well-trained engineer of 1940 takes it in his stride and regards it as almost commonplace. The well-trained engineer of 1990 will certainly regard as equally commonplace the theory of analytic functions, matrices and the characteristic numbers (Eigenwerte) of differential equations, which today are thought of as quite advanced.

With this as a background, there need be no apology associated with the statement that such simple processes as algebra, trigonometry and the elements of calculus are the most common and the most productive in modern industrial research. They frequently lead to results of the greatest practical importance. The single sideband system of carrier transmission, for example, was a mathematical invention. It virtually doubled the number of long distance calls that could be handled simultaneously over a given line. Yet the only mathematics involved in its development was a single trigonometric equation, the formula for the sine of the sum of two angles.

Next in order of usefulness come such subjects as linear differential equations (e.g., in studying the reaction of mechanical and electrical systems to applied forces, the strains in elastic bodies, heat flow, stability of electric circuits and of coupled mechanical systems, etc.); the theory of functions of a complex variable (particularly in dealing with potential theory and wave transmission, propagation of radio waves and of currents in wires, gravitational and electric fields as used in prospecting for oil, design of filters and equalizers for communication systems, etc.); Fourier, Bessel, and other orthogonal series (in problems of heat flow, flow of currents in transmission lines, deformation and vibration of gases, liquids and elastic solids, etc.); the theory of determinants (particularly in solving complicated linear differential equations, especially in the study of coupled dynamical systems); and the like.

Less frequently we meet such subjects as integral equations, which has been made the basis of one version of the Heaviside operational calculus, and which has also been used in studying the seismic and electric methods of prospecting for oil; matrix algebra, which has been applied to the study of rotating electric machinery, to the vibration of aircraft wings, and in the equivalence problem in electric circuit theory; the calculus of variations, in improving the efficiency of relays; and even such abstract subjects as Boolean algebra, in designing relay circuits; the theory of numbers, in the

DETERMINANTS



$$D = \begin{vmatrix} Z_1 & -Z_{12} & -Z_{13} & 0 & -Z_{15} & -Z_{16} \\ -Z_{12} & Z_2 & -Z_{23} & -Z_{24} & 0 & -Z_{26} \\ -Z_{13} & -Z_{23} & Z_3 & -Z_{34} & -Z_{35} & 0 \\ 0 & -Z_{24} & -Z_{34} & Z_4 & -Z_{45} & -Z_{46} \\ -Z_{15} & 0 & -Z_{35} & -Z_{45} & Z_5 & -Z_{56} \\ -Z_{16} & -Z_{26} & 0 & -Z_{46} & -Z_{56} & Z_6 \end{vmatrix} ; Z_j = \sum_{k=1}^{6} Z_{jk}$$

Driving point impedance in mesh j = $Z_{(jj)} = \frac{D}{D_{jj}}$

Transfer impedance between mesh j and mesh k = $Z_{(jk)} = \frac{D}{D_{jk}}$

(D_{jk} = the first minor of the element Z_{jk} in D)

Many properties of the complicated networks studied at Bell Telephone Laboratories are most conveniently expressed by means of determinants. Above are shown a six-mesh network; its "circuit discriminant", D; and some formulae which illustrate how simply the properties of the system can be found from D. Note that, since $Z_{jk} = Z_{kj}$, D is symmetrical.

design of reduction gears, and in developing a systematic method for splicing telephone cables; and analysis situs, in the classification of electric networks.

Least frequently of all, but by no means never, the industrial mathematician is forced to invent techniques which the pure mathematician has overlooked. The method of symmetric coordinates for the study of polyphase power systems; the Heaviside[3] calculus for the study of transients in linear dynamical systems; the method of matrix iteration in aerodynamic theory;[4] much of the technique used in the design of electric filters and equalizers—these may stand as illustrative examples.

The student of modern mathematics will be impressed at once by two aspects of this review: first, by the heavy emphasis on algebra and analysis, and the almost complete absence of geometry beyond the elementary grade; second, the complete absence of the specific techniques which play such a large role in modern physics and astrophysics. It is not easy to say just why advanced geometry plays no larger part in industrial research; however, the fact remains that it does not.[5] As regards modern physics, one may perhaps extrapolate from past history and infer that what is now being found useful in interatomic physics will soon be needed in industrial chemistry. In making this extrapolation, however, it is well to bear in mind that the physics in question is for the most part a mental discipline, its connection with the world of reality still ill-defined and incompletely understood. Therefore it may not prove to be as quickly assimilable into technology as have other disciplines whose symbols could be more immediately identified with experience.[6]

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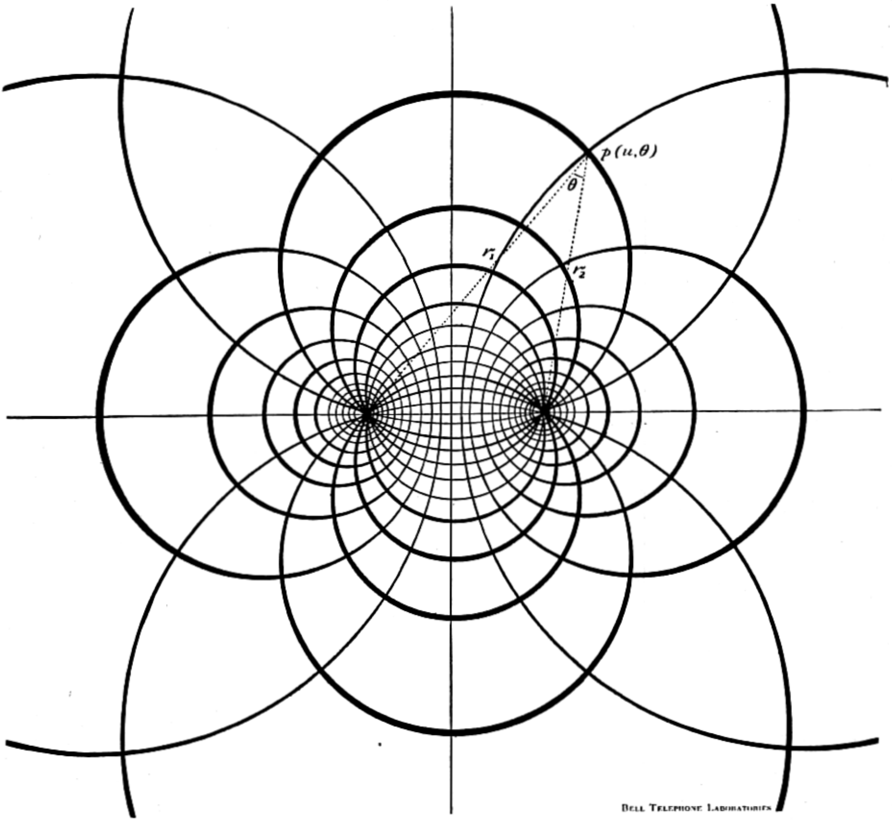
[3] Heaviside was not himself an industrial employee, but the reformulation of his work in terms of integral equations, and its interpretation in terms of Fourier transforms were both carried out in America by industrial mathematicians.

[4] This method was developed in The National Physical Laboratory of England, in the course of studies which in America would probably have been undertaken by a government or industrial laboratory.

[5] Mr. Hall C. Hibbard of the Lockheed Aircraft Corporation comments on this remark as follows: "It is possible that the usefulness of this principle of mathematics has been overlooked to a large extent in certain fields where it might be applied to advantage. In particular, that phase of engineering known as 'lofting,' which deals with the development of smooth curved surfaces, might offer an interesting field for certain types of advanced geometry. Practically all of this work is now done by 'cut and try' methods and the application of mathematics would no doubt save a great deal of time. The same thing is true in the field of stress analysis, where a great deal of time is absorbed in determining the location and direction of certain structural members. It is even possible that the application of vector analysis technique would greatly simplify certain forms of structural analysis, particularly space frameworks. The lack of application of geometry in these fields is probably due to the wide gap that exists between the mathematician and the 'practical' designer and draftsman. Advanced geometry might also turn out to be a very useful tool in connection with problems that we are now encountering in the forming of flat sheet into surfaces with double curvature, an operation that is extensively employed in aircraft manufacture."

[6] In this connection, see the quotation from Dr. E. C. Williams on pages 30–31.

Bicircular Coordinates



$$(x + \coth u)^2 + y^2 = \operatorname{csch}^2 u;\ x^2 + (y - \cot\Theta)^2 = \csc^2\Theta$$
$$u = \log\,(r_2/r_1)$$

Using the bicircular system of coordinates facilitates finding the distribution of electric charge on two parallel conductors, and thence their capacity. Rotating the bicircular system about the vertical axis generates a toroidal coordinate system which facilitates determining the capacity of a torus.

Finally, we must remark upon two facts: (1) that approximate solutions of problems, and hence methods of iteration (successive approximation), play a much more conspicuous role in applied mathematics than in the pure science; (2) that the highly convenient assumption that linear approximations to natural laws (such as Hooke's law and Ohm's law) are sufficiently exact for practical purposes is less often true than formerly was the case, so that nonlinear differential equations are of great importance to the modern engineer.

Types of Service Performed by Mathematics

Leaving aside the important but rather trite observation that mathematics is a language which simplifies the process of thinking and makes it more reliable, and that this is its principal service to industry, we may distinguish certain less inclusive, but perhaps for that reason more illuminating, categories of usefulness.

First: It provides a basis for interpreting data in terms of a preconceived theory, thus making it possible to draw deductions from them regarding things which could not be observed conveniently, if at all.

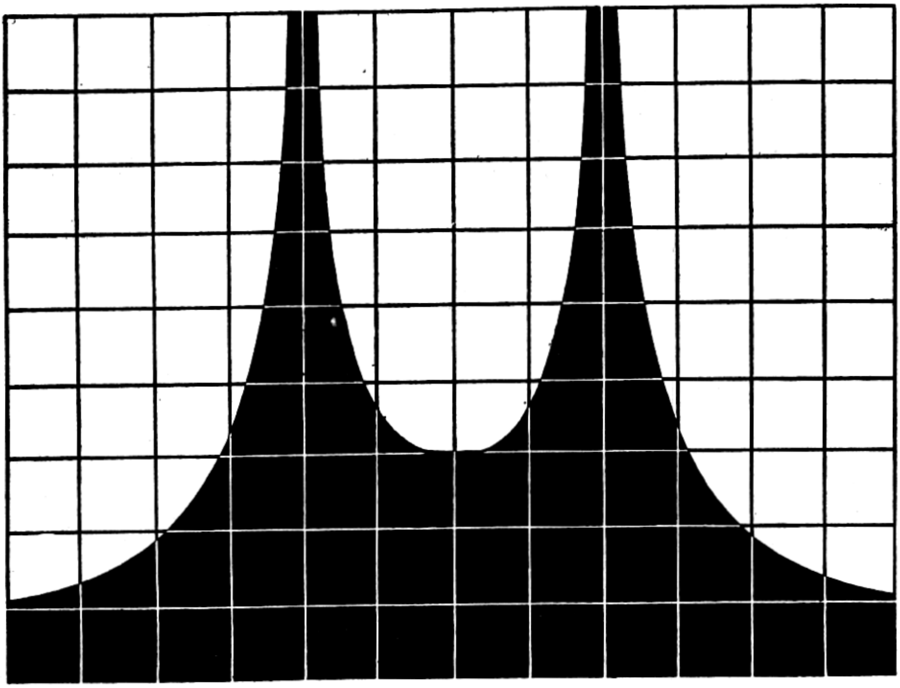
(*a*) An illustration is the standard method for locating faults on telephone lines. Mathematical theory shows that a fault will affect the impedance of the line in a way which varies with frequency, and that the distance from the place of measurement to the fault can be deduced at once from the frequencies at which the impedance is most conspicuously affected. This is obviously much more convenient than hunting the fault directly.

(*b*) A second illustration is the mapping of geological strata by means of measurements made upon the surface of the earth. One method extensively employed uses a large number of seismographs, each of which records the miniature earthquake shock produced at its location by a charge of dynamite set off at a known place. A theory of reflection and refraction similar to that used in geometrical optics shows that certain observable characteristics of these records are related to the depth and tilt of the underground layers, and hence enables the situation of these layers to be plotted. By this means the location of the highest point of an oil-bearing stratum can be found, and the most favorable position for drilling determined.

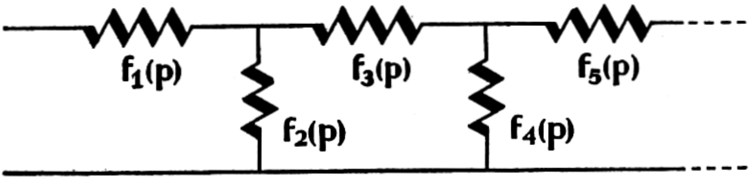
Underground geology is also studied by means of gravity, electrical or magnetic measurements upon the surface. In this case the basic theory is that of the Newtonian potential field, and the interpretation of the data leads into the subject of inverse boundary value problems, which is still insufficiently understood. Enough progress has been made in several geophysical laboratories, however, so that the gravity method is now being widely used, and the electrical methods appear promising for some applications.

Second: When data are incompatible with the preconceived theory, a mathematical study frequently aids in perfecting the theory itself. The

CONTINUED FRACTIONS



$$Z = f_1(p) + \cfrac{1}{\cfrac{1}{f_2(p)} + \cfrac{1}{f_3(p) + \cfrac{1}{\cfrac{1}{f_4(p)} + \cdots\cdots}}}$$



A mathematical method of systematically designing a circuit of predetermined impedance has been developed in Bell Telephone Laboratories. The given impedance, as a function of frequency, is expanded in a Stieltjes continued fraction, whose terms give the electrical constants of the desired network.

classical illustration in pure science is the discovery of the planet Neptune. The motion of the planet Uranus was found to be inconsistent with the predictions of the Newtonian theory of gravitation, if the solar system consisted only of the seven planets then known. Mathematical investigation indicated, however, that if an eighth planet of a certain size was assumed to be moving in a certain orbit, these discrepancies disappeared. Upon turning a telescope to the spot predicted, the new planet was found.

An illustration comes from the aircraft industry. I quote it from a report sent me by Mr. C. T. Reid, Director of Education of the Douglas Aircraft Company:

(*c*) "The behaviour of airplanes with 'power on' did not check closely enough with stability predictions which had been made without consideration of the effects of the application of power; therefore, a purely mathematical analysis of the longitudinal motion of an airplane was carried out, involving the solution of three simultaneous linear first-degree differential equations. The results led to the development of equations for dynamic longitudinal stability with 'power on' which enable the aerodynamicist more accurately to predict the stability characteristics of a given design. 'Power-on' dynamic longitudinal stability is an important design criterion in aircraft construction."

(*d*) Another illustration arises in communication engineering. Theoretical studies had established the fact that vacuum tubes would spontaneously generate noise because of the discrete character of the electrons of which the space current is composed. The theory predicted how loud this noise would be in any particular type of vacuum tube, a most significant result since it established a limit to the weakness of signals which could be amplified by this type of tube. The predictions of the theory were supported by experimental data so long as the tubes were operating without appreciable space charge. But it was found that when space charge was present the noise level fell far below the predicted minimum. In this case the missing factor in the theory was immediately obvious, but an understanding of the mechanism by which the reduction was affected and its incorporation into the theory in a workable form, required an extensive and difficult mathematical attack.

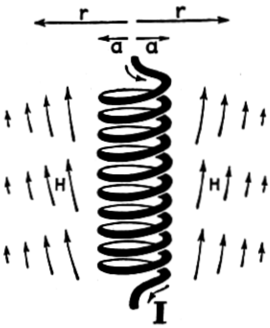
Third: It is frequently necessary in practice to extrapolate test data from one set of dimensions to a widely different set, and in such cases some sort of mathematical background is almost essential.

An example of this kind of service, concerned with the theory of arcs in various gases, is furnished me by Mr. P. L. Alger, Staff Assistant to the Vice President in Charge of Engineering, of the General Electric Company:

(*e*) "An example of this kind of problem is that of the theory of arcs in various gases. It has been experimentally known that the duration, stability and voltage characteristics of electric arcs in different gases and under different pressures vary very widely. The behaviour of such arcs is of great importance, both in

Elliptic Integrals

$$H=\frac{4I}{r}\left[\frac{1}{1-k^2}\int_0^{\pi/2}\sqrt{1-k^2\sin^2\lambda}\,d\lambda-\int_0^{\pi/2}\frac{d\lambda}{\sqrt{1-k^2\sin^2\lambda}}\right]$$



Some simple engineering problems require advanced mathematics in their solution. This is true, for example, in the computation of the magnetic field outside the spiral grid of a vacuum tube, a problem of interest to Bell Telephone Laboratories. If the grid is closely coiled, the current can be treated as a continuous cylindrical sheet, of radius a. Then the component of the magnetic field parallel to the axis of the grid at a distance r from the axis is given by the above function of two Elliptic Integrals whose "modulus" is $k=a/r$.

> welding and in the design of circuit breakers and other protective devices. Recently a mathematical theory has been developed which relates the arc phenomena to the heat transfer characteristics of different gases. This theory has given excellent correlation between the known experimental results, and has enabled very useful predictions of performance under new conditions to be made. The theory has been applied in the design of high voltage air circuit breakers, which are of important commercial value, and it is also greatly curtailing the time and expense necessary to develop many other devices in which arc phenomena are of importance."

A second example, furnished me by Mr. Reid, has to do with the interpretation of wind-tunnel data in aerodynamics:

> (*f*) "Here it is obviously impracticable to perform full-scale tests of such parts as wings or fuselage, much less of entire aircraft, and the extrapolation from the results of wind-tunnel measurements to the full-scale characteristics of airplanes must be based on theoretical considerations."

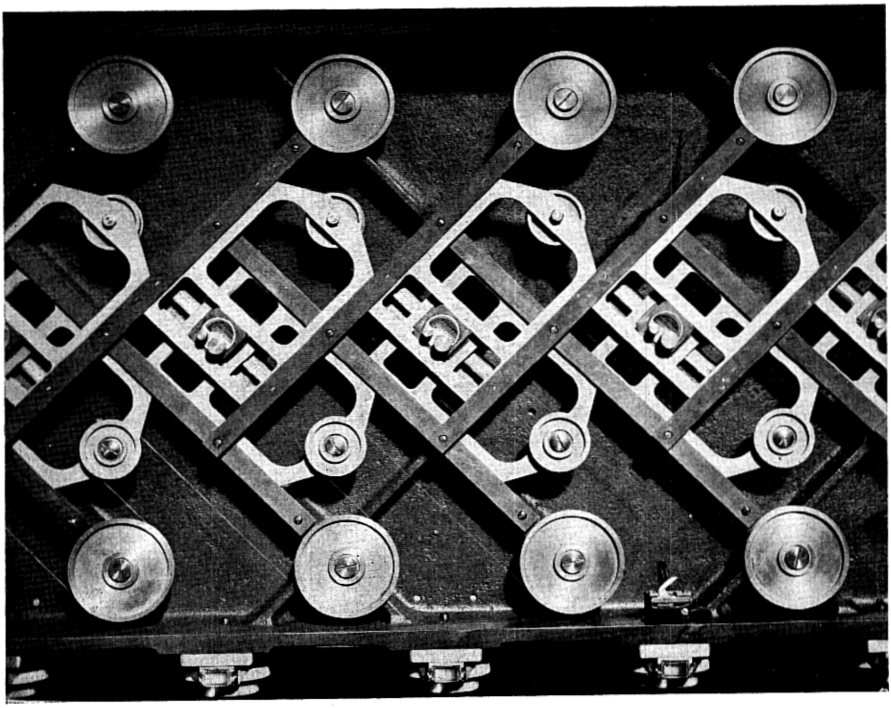
Fourth: Mathematics frequently aids in promoting economy either by reducing the amount of experimentation required, or by replacing it entirely. Instances of this kind are met everywhere in industry, not only in research activities, but in perfecting the design of apparatus and in its subsequent manufacture as well.

Mr. Alger describes in general terms one situation frequently met in research activities as follows:

> "The first type of problem is one in which there are so many different independent dimensions of a proposed shape to be chosen, or in general so many independent variables, that it is hopeless to find the optimum proportions by experiment. The truth of this can readily be seen when it is realized that the number of test observations to be made increases exponentially with the number of variables. If 10 points are required to establish a performance curve for one variable, 1,000 observations will be required if there are 3 independent variables, and a million if there are 6 variables."

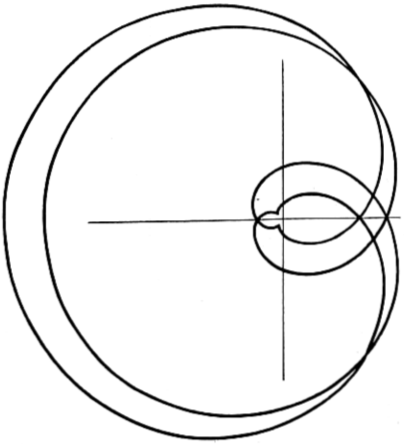
As an illustration he cites the following problem:

> (*g*) "An example of this kind of problem is that of designing a *T* dovetail to hold the salient poles in place on a high speed synchronous generator. A large machine of this type may have 10 or more laminated poles carrying heavy copper field coils, each assembled pole weighing several tons and traveling at a surface peripheral speed of 3 miles a minute. The centrifugal force on each pound of the pole then amounts to approximately 500 pounds. The problem of designing dovetails to hold these poles in place, even at over speed, is, therefore, one of great importance and technical difficulty. For each such dovetail, there are 7 different dimensions which may be independently chosen. While empirical methods have enabled satisfactory results to be obtained in some cases, application of mathematics has recently enabled marked improvements in dovetail



THE ISOGRAPH

The Isograph was developed in Bell Telephone Laboratories to find mechanically the complex roots of polynomials of high degree. Let the polynomial to be factored be $p(z) = \sum_{0}^{n} a_j z^j$ or $\sum_{0}^{n} a_j r^j \cos j\Theta + i \sum_{i}^{n} a_j r^j \sin j\Theta$ if $z = r(\cos\Theta + i\sin\Theta)$. The isograph maps the complex values of $p(z)$ as the variable describes the circle $|z| = r$. This graph loops the origin once for each root smaller in absolute value than r. The number of roots between trial values of r is determined by counting loops, and by interpolation a value of r is found for which the graph passes through the origin. This value of r and the corresponding value of Θ define the real and imaginary parts of a root.



designs to be made. Generally speaking, these improvements have permitted an overall strength increase of 20 per cent to be obtained under steady stresses, and much higher gains to be made under fatigue stress conditions; while at the same time the certainty of obtaining the desired results on new designs has been very greatly enhanced."

A second example was brought to my attention by Mr. L. W. Wallace, Director of the Engineering and Research Division of the Crane Company:

(*h*) "A pipe fitting weighing several hundred pounds and intended for high pressure service had a neck of elliptical cross-section. As originally designed, the thickness of the casting was intentionally not uniform, the variations having been introduced empirically to strengthen it where strength was supposed to be most needed. A redesign carried out on the basis of the theory of elasticity showed the distribution of metal to be inefficient and resulted in a new casting in which the weight was reduced by half, while at the same time the bursting strength was doubled. The method used in arriving at this result is an interesting illustration of sensible mathematical idealization. The casting was regarded as an elliptical cylinder under hydrostatic pressure. As the stresses for this idealized structure were already known, the design problem reduced at once to the simple matter of establishing thicknesses sufficient to withstand these stresses."

Another example from the field of geophysical prospecting is furnished by Mr. Eugene McDermott, President of Geophysical Service, Inc.:

(*i*) "A specific case of mathematical research in instrument design was recently encountered. The instrument in question was intended for the measurement of gravity. After the machine had been completely built it was found to be unexplainably inaccurate. After weeks of trial and error it was turned over to a mathematician to try to find the trouble. He soon showed by simple trigonometry that the axis of the instrument would have to be located on its pivot with an accuracy which is not attainable. He also pointed out a means of avoiding this feature by a relatively simple change in design, and this appears to have remedied the trouble."

Another illustration from the petroleum industry, but this time concerned with the production of oil rather than prospecting for it, comes from Dr. E. C. Williams, Vice President in Charge of Research of the Shell Development Company:

(*j*) "The petroleum industry has one important problem not found in other fields; it has to do with oil production from the ground. A mathematical problem arising from this subject is the following: The oil-gas mixture underground flows under pressure through porous media; with a certain spacing of wells, determine the most economical way to recover this mixture. This is sometimes equivalent to asking: 'In what way can the largest fraction of the oil be obtained over a certain period of time?' Simplified problems of this kind have been solved by potential theory methods, since classical hydrodynamics becomes too involved, and in the general problems where the flow constants vary with liquid-gas composition, etc., partial differential equations are found which can be solved by approximate methods. On the basis of the solution of this mathematical problem, aided by extensive laboratory determinations of the required constants, one is able to find the best of several ways of producing from a given oil field."

As a final example under the heading of economy, we may mention the flight testing requirements imposed upon the aircraft industry by the Civil Aeronautics Authority. Of these, Mr. E. T. Allen, Director of Flight and Research of the Boeing Aircraft Company, says:

(*k*) "It was formerly required that each type of transport plane must be tested at all the altitudes at which it was intended to be flown, and at all flying fields where it was expected to be used. The cost of such testing was extremely high. A mathematical study of steady flight performance has, however, identified the basic parameters and established their relations to one another. This has made possible a scientific interpretation of flight test data taken at any suitable location convenient to the aircraft factory, and a reliable conclusion therefrom as to the performance to be expected under other conditions. This has greatly reduced both the cost and the time necessary to establish performance figures."

Fifth: Sometimes experiments are virtually impossible and mathematics must fill the breach. An example comes to me from Mr. Hall C. Hibbard, Vice President and Chief Engineer of the Lockheed Aircraft Corporation:

(*l*) "An unfortunate phenomenon that must be dealt with in aircraft design is a type of violent vibration which may be set up in the wings if the plane is flown too fast. It is known as flutter, and is highly dangerous, since the vibrations may be of such intense character as to cause loss of control or even structural failure. The technical problem is therefore to be sure that the critical speed at which flutter would occur is higher than any at which the craft would ever be flown. It is a phenomenon with respect to which wind tunnel experimentation is difficult, and flight testing very dangerous. It has been the subject of a number of mathematical investigations, the results of which have reached a sufficiently advanced stage that they are now being used to predict the critical speeds and flutter frequencies of aircraft while still in the design stage. Even more important, the mathematical investigation of this problem points the way to modifications of design which will insure that flutter cannot occur in the usable speed range."

Telephony provides a second example:

(*m*) The equipment in an automatic telephone exchange must be capable of connecting any calling subscriber with any called subscriber. It consists of several stages of switches, each of which can be caused to make connection with a number of trunks which lead in turn to switches in the next succeeding stage. Enough switches must be provided so that only a very small proportion of subscribers' calls will fail to be served immediately. Since the demands made by the subscribers fluctuate from moment to moment, the number of switches required depends in part upon the height to which the crests occasionally rise in this fluctuating load. It is also influenced, however, by the way the trunks are arranged, by the order in which the switches choose them, and by many other factors. Experimental appraisal of the effect of these various factors is impossible, both because it would be very costly, and because it would be exceedingly slow. Mathematically, however, they have been studied by the theory of *a priori* probability,[7] which is used not only in determining how much apparatus to install in a working exchange, but also in comparing the relative merits of alternative arrangements while in the development stage.

Sixth: Mathematics is frequently useful in devising so-called crucial experiments to distinguish once for all between rival theories. A famous example in the field of physics was the study of the refraction of starlight near the sun's disc, which afforded a means of deciding between Newtonian and relativistic mechanics. In this case, mathematical investigation showed that the result to be expected was different according to the two theories, and astronomical observations confirmed the prediction of relativistic mechanics. In the industrial field, an example of this kind comes to me from Dr. Joseph A. Sharpe, Chief Physicist in the Geophysical Laboratory of the Stanolind Oil and Gas Company:

(*n*) "As an example of the second sort of use of analysis there is the case of our study of 'ground-roll,' the large amplitude, low frequency surficial wave which caused so much grief in the early days of seismic reflection prospecting when filters were not used as extensively as at present. We hope to use our study of this wave motion as an aid to a better understanding of the properties of the surficial layers of soil and their effects on the reflected waves in which we are primarily interested.

Two views on the ground-roll are current, although neither is based on very much observation, and this of an uncontrolled sort. One view states that the ground-roll is an elastic wave. Analysis predicts that this wave will have a certain velocity in relation to the velocities of other waves, that it will have a certain direction of particle motion and relation of maximum horizontal to maximum vertical component of displacement, that it will attenuate with distance according to a certain law, that it will attenuate with depth in a certain way, and that its velocity will follow a certain dispersion law. The second view maintains that the 'ground-roll' is a wave in a viscous fluid, and analysis predicts a behavior which is similar in certain cases, and different in others, to that of the elastic wave. Having the predictions of the analysis at hand, we are enabled to devise a group of observations, and the special equipment for their prosecution, which will provide crucial tests of the two hypotheses."

Seventh: Mathematics also frequently performs a negative service, but one which is sometimes of very great importance, in forestalling the search

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[7] *Not* statistics, which is *a posteriori* probability. This is one of the few cases in industry where the *a priori* theory finds application.

for the impossible; for many desirable objectives in industry are as unattainable as perpetual motion machines, and frequently the only way to recognize the fact is by means of a mathematical argument.

(*o*) A certain type of electric wave filter which is usually referred to as an "ideal" filter would be very useful if it could be produced. However, it has been shown mathematically that such a structure would respond to a signal before the signal reached it; in other words, that it would have the gift of prophecy. Since this is absurd, it follows that no such filter can be built, and consequently no one tries to build it.

Still another example from the field of communication deals with the design of feedback amplifiers.

(*p*) In practice, any amplifier is intended to handle signals in a given frequency band. For various reasons, it is preferable not to have it amplify disturbances outside this band, and hence its gain characteristic is made to drop off as rapidly as possible outside the limits of the useful band. It has been shown theoretically, however, that the gain cannot decrease at more than a certain rate, which can easily be computed, without causing the amplifier to become unstable. As a matter of fact, the allowable rate at which the gain may fall is often surprisingly low, and a great deal of design effort would be wasted in the attempt to obtain an impossible degree of discrimination if the theoretical limitations were unknown.

Eighth: Finally, mathematics frequently plays an important part in reducing complicated theoretical results and complicated methods of calculation to readily available working form. So many and so varied are the services falling in this category that it is difficult to illustrate them by means of examples. We arbitrarily restrict ourselves to two, chosen primarily for the sake of variety. The first comes from Mr. Hibbard:

(*q*) "In aircraft design the metal skin, though thin, contributes a large part of the structural strength. Nevertheless, such thin metallic plates will buckle or wrinkle after a certain critical load is exceeded. Beyond this point the usual structural theories can not be applied directly and it is therefore necessary to introduce new methods of attack to predict the ultimate strength of the structure. These stiffened plates are difficult to deal with theoretically, but by interpreting the effect of the stiffeners as equivalent to an increase in plate thickness or a decrease in plate width, the calculations can be brought within useful bounds."

The reduction of electric transducers to equivalent T or Π configurations, the interpretation of the elastic reaction of air upon a microphone as equivalent to an increase in the mass of its diaphragm, the postulation of an "image current" as a substitute for the currents induced in a conducting ground by a transmission line above it, and a host of other common procedures could be cited as similar instances of simplification based upon more or less valid mathematical reasoning.

The second example is furnished by Dr. E. U. Condon, Associate Director of the Research Laboratories of the Westinghouse Electric and Manufacturing Company:

> (r) "In the manufacture of rotating machinery it is of extreme importance to have the rotating parts dynamically balanced, in order to reduce to a minimum the vibration reaction on the bearings which unbalance produces. Theory shows the phases and amplitudes of the bearing vibrations produced by excess masses located at various places on the rotor; conversely, by solving backward from observed vibration data, one can compute what correction is needed to eliminate the unbalance. Recently a most valuable machine has been developed which not only measures the unbalance, but also automatically shows what correction should be made, thus eliminating the necessity for these calculations.
>
> The rotor to be balanced is whirled in bearings on which are mounted microphones that generate alternating voltages corresponding to the vibrations of the bearings. These voltages are fed into an analyzing network, which automatically indicates the correction needed in order to achieve dynamic balance. In some cases the output of the balancing machine has been arranged to set up a drilling machine so it will automatically remove the right amount of metal at the right place. These machines are finding application in the manufacture of small motors, of automobile crankshafts, and in the heavy rotors of power machines."

In the same class would come the isograph, by means of which the complex roots of polynomials can be located; the tensor gauge which registers the principal components of strain in a stressed membrane without advance knowledge of the principal axes; and slide rules for a great variety of special purposes such as computations with complex numbers, the calculation of aircraft performance, aircraft weight and balance, and the like. Perhaps we ought also include in the same category the use of soap-bubble films for the study of elastic stresses in beams, the use of current flow in tanks of electrolyte for the study of potential fields, and the use of steel balls rolling on rubber membranes stretched over irregular supports as a means of studying the trajectories of electrons in complicated electric fields. These are all mechanical methods for saving mathematical labor, but they are more than that, for they all rest upon a foundation of mathematical theory. They are, in fact, examples of the use of mathematics to avoid the use of mathematics.

Mathematics in Some Particular Industries

Communications

The communication field is the one in which mathematical methods of research have been most freely used. This is due partly to the fact that the transmission of electric waves along wires and through the ether follows laws which are particularly amenable to mathematical study; partly also to the fact that so much of the research has been centralized in a single laboratory, thus bringing together a large number of engineers into a single compact group, and justifying the employment of consultative specialists. Most important of all, however, is the fact that there are two devices—vacuum tubes and electrical networks—without which modern long-distance telephony would be impossible; and one of these, the electrical network, is and has been since its earliest days almost entirely a product of mathematical research. Mathematics has thus been as essential to the development of nation-wide telephony as copper wire or carbon microphones.

Number of Mathematicians. The Mathematical Research Department of the Bell Telephone Laboratories contains 14 mathematicians. Perhaps an equal number of men scattered through various engineering departments should also be classified as mathematicians according to the definition adopted for this report. Say a total of 25 or 30 for the Bell Laboratories, a few more for the Bell System as a whole, and perhaps 40 or 50 for the entire communication field including the companies interested in radio and television. A few of these men carry on a considerable amount of experimentation, but their significant work is theoretical.

In addition, there is a much larger number of men who use mathematical methods extensively in their daily work, but whose mental type is not that which we have described as mathematical, and who are therefore not included in the numbers quoted above. This is true in particular of the engineers who have the responsibility for designing networks.

Uses of Mathematics. Mathematical activity is most intense: (1) in designing wave filters and equalizers, (2) in studying transmission by wire and ether, the concomitant problems of antenna radiation and reception, inductive interference between lines, etc., (3) in studying various problems related to the standard of service in telephone exchanges, such as the amount of equipment required, the probability of delays and double connections, the hunting time of switches, etc., (4) in providing a rational basis for the design of instruments, such as transmitters and receivers, vacuum tubes, television scanning devices, etc., (5) in developing efficient statistical methods for the planning and interpretation of experiments, and for controlling the quality of manufactured apparatus.

Future Prospects. During the last 20 years the number of men employed in communication research has increased with great rapidity, but this rapid expansion appears to be about over. A large increase in the mathematical personnel of the industry therefore appears unlikely. It seems inevitable that the problems will increase in complexity, and that theoretical methods will become increasingly important, but it is believed that this trend will be matched by progressively better trained engineering personnel, rather than by an increased number of mathematicians. Indeed, unless the qualifications of the mathematicians rise progressively with those of the engineers, it may turn out that less rather than more will be employed.

Electrical Manufacturing

Substantially all the research in the power fields is carried on by a few electrical manufacturers. The power companies usually accept and exploit such equipment as the manufacturers supply, and contribute to improved design principally through their criticisms of past performance. Many of their engineers, however, are individually active in the invention and development of improved equipment.

Number of Mathematicians. The number of mathematicians in the industry is smaller than in communications, and is not easy to estimate because their work is less segregated from other activities. The total number who would here be rated as mathematicians is probably about 20.

As in communications, some are engaged partly in experimental work. There are some, however, whose relationship as consultants is clearly recognized, and there is evidence that management is becoming increasingly conscious of the nature and value of their services.

Uses of Mathematics. Mathematical activity is most intense: (1) in studying structural and dynamic problems, such as the strain, creep and fatigue in machine parts, vibration and instability in turbines and other rotating machinery, etc., (2) in appraising the evil effects of suddenly applied loads, lightning or faults upon power lines, and their associated sources of power, and devising methods to minimize these effects, (3) in studying system performance, particularly the most effective or economical location of proposed new equipment, and the evaluation of performances of alternative transmission or distribution systems, (4) in refining the design of generators, motors, transformers and the like, so as to improve their electrical efficiency and reliability, and in similar improvement of the thermal efficiency of turbines, (5) in the design of miscellaneous instruments and apparatus.

Statistical methods are being introduced into manufacturing and research, but are not yet utilized to the same extent as in telephony.

Future Prospects. The amount of money spent on development in these industries is gradually increasing, and as in other fields the problems are becoming more complex. Hence a slow increase in the number of mathematicians seems probable, with rising standards in the qualifications required, not only as to mathematical training, but as to temperament and personality as well.

The Petroleum Industry

The petroleum industry consists of many producing units of various sizes, highly competitive in character, and surrounded by a number of consulting service organizations, all of which are small. The larger producing companies—and within their resources, the service units also—maintain research laboratories. They tend to be secretive about the developments which take place in these, sometimes to a surprising degree. Hence there is much duplication of effort, particularly in such matters as the design of instruments for geophysical prospecting, and in methods of interpreting the data derived from them.

Number of Mathematicians. The industry employs more mathematicians than is generally appreciated, some of them men of very considerable ability. The total of first-rank men is perhaps 15 or 20. Due to the small size of the individual research staffs, however, most of these men carry considerable project responsibility along with their theoretical work. This is the normal state of affairs in small groups: the abnormality is the lack of contact with, and stimulus from, similar men in other companies.

Uses of Mathematics. Petroleum research extends in three directions: prospecting for oil, producing it, and refining it.

There are five recognized methods of prospecting: gravity, seismic, electric, magnetic and chemical. In the first four, important mathematical problems arise in designing sufficiently sensitive instruments and in interpreting data. The fifth requires the use of statistical methods.

Research on methods of producing a field has led to a few mathematical studies of underground flow, and would undoubtedly give rise to others if the results of these studies could be profitably applied. However, since the rate at which oil is brought to the surface is almost entirely determined by law, and the same is indirectly true of well location also, mathematical consideration of the subject is largely sterile, at least so far as American oil fields are concerned.

The third activity—refining—is essentially a chemical industry. Hence the following remarks by Dr. E. C. Williams, Vice President in Charge of Research of the Shell Development Company, presumably apply not only to the petroleum business, but to manufacturing chemistry in general:

> "The two chief problems in chemistry are (aside from the identification of substances): The calculation of chemical equilibrium and the calculation of the rates of attainment of these equilibria. The first problem, involving thermo-

dynamics and statistical mechanics, is rather well understood and usually, by very simple computations, information sufficiently accurate for industrial application, at least, can be found. Frequently, when several equilibria are possible simultaneously, complicated equations arise, but we rarely solve them directly, but rather set up tables of the dependent variable (the per cent conversion possible) as a function of the independent variables (temperature, pressure concentration). The sources of these data, however, are numerous and at times require complicated mathematics, as in the calculation of thermodynamic properties from spectroscopic data via quantum statistics.

The situation is much less favorable in the calculation of the rates of chemical reactions. A semi-empirical method, based on quantum mechanics, has been applied with a little success to some of the simplest reactions taking place in the gas phase, but virtually no progress has been made in the more important field of heterogeneous reactions (reactions of gases on surfaces, for example). We may say that no satisfactory mathematical theory for such calculation exists at the present time. Some progress is being made, but we are far from being able to predict a suitable catalyst for any desired reaction. For the present we are happy to be able to account for observations made on some simple reactions."

Future Prospects. It is inconceivable that research in the industry will not continue at at least its present level. Hence more, rather than less mathematical work will probably be undertaken in prospecting and in refining. A demand of moderate proportions should exist for able mathematicians with a suitable background of geology and classical physics for the geophysical work, and of physical chemistry and molecular physics in the chemical field.

Aircraft Manufacture

The aircraft industry also consists of a number of independent units, and is highly competitive. It is a new industry in which rapid technical development and rapid increase in size has been the rule. It has depended primarily upon government-supported laboratories and, to a lesser extent, upon the universities for its research, and has busied itself with the exploitation of that research in the advancement of aircraft design. No unit of the industry has had or, for that matter, now has a research laboratory, in the sense in which the words would be used in older and larger businesses, but the beginnings of research departments have appeared, and individual researchers and research projects are clearly recognizable.

Numbers of Mathematicians. Some men in the engineering departments of these companies should undoubtedly be classed as mathematicians, but it is impossible to make even an approximate estimate of their number. It is possible, however, to cite pertinent information which bears on the importance of mathematics to the industry.

The design of a modern four-engine transport plane requires about 600,000 hours of engineering time up to the point where complete working drawings have been prepared. About 100,000 hours are spent on mathematical analysis of structures, performance, lift distribution and stability. Most of this work is routine, but some is fundamental in character, as is evident from several of the examples mentioned earlier in this report.

Of 670 men in the engineering department of one of the larger companies, about 25 have mathematical training beyond that usually obtained by engineers, and 10 or so of these are using this advanced training to a significant extent.

Uses of Mathematics. In designing an airplane, five factors are of particular importance. These may be used to indicate the directions in which mathematical research may be expected.

(1) Performance (that is, pay-load, range, speed, climbing rate, etc.)

In the past, forecasts of performance have been based almost entirely on empirical data. Mathematical methods of estimation are now being developed from hydrodynamic theory, however, and are being used to an increasingly greater extent.

(2) Lift and Drag (i.e., the force variation over the wings)

This is the principal objective in the aerodynamic design of the wing. The technique of prediction rests on two supports: wind tunnel experiments and airfoil theory, by means of which experimental data are interpreted and applied. For example, airfoil theory suggests the shape of airfoil to avoid unfavorable pressure distributions and is leading to improved wing sections. This part of aircraft design is already highly mathematical, but a number of fundamental problems still remain unsolved. For example, the theory is still unable to predict stall, and too little is known about optimum shapes or about turbulence, though the recently developed statistical theory of turbulence has contributed to the understanding of the airflow over an airplane and resulted directly in a decrease in airplane drag and consequent improvement in performance.

(3) Stability (inherent steadiness of motion)

The stability of an airplane in flight is inherent in its aerodynamic design and quite distinct from its control or maneuverability. The theory of "small oscillations" has been successfully applied to rectilinear flight. More recently the problem of predicting the response of an airplane to control maneuvers has used the Heaviside operational calculus. Current problems of dynamical stability in which applied mathematicians are interested are the behavior of an airplane when running on the ground and the behavior of seaplanes when running on the water (porpoising).

(4) Structural Safety

Very precise appraisal of structural strength is required in aircraft design. In most industries inaccuracy can be compensated by increased factors of safety, but the pay-load of an airplane is so small a proportion of its total weight that slight increases in factors of safety would seriously reduce its carrying power or even make it unable to get off the ground. Mathematical methods have always been used in this phase of aircraft design in so far as they were available. The standard technique is first to design a part on the basis of calculated strength, then build and test it, and if the tests do not agree with predictions, revise the design and build and test the modified part. This process is continued as many times as necessary to attain a satisfactory result. It is slow and expensive. Theoretical methods are now reliable enough that the majority of structural tests confirm predictions with sufficient accuracy to require no revision. However, new problems constantly present themselves—the introduction of pressurized cabins recently gave rise to several—and hence continual mathematical study is required. A beginning has also been made in the use of the principles of probability in setting up structural loading factors.

(5) Flutter

We have already commented upon the impracticability of studying this phenomenon by any means other than the mathematical. The general equations are complicated, and have only been solved by making important simplifying assumptions. The results are serviceable for check purposes, but need further elaboration. The importance of the problem increases progressively as more efficient planes are designed, and the necessity for an adequate mathematical theory is becoming critical.

Future Prospects. It appears inevitable that from motives of economy the industry will rely increasingly upon theoretical methods of design, and that mathematics will play a larger part in the future than at present. It is also probable that for competitive reasons the various companies will supplement government research by fundamental studies of their own. Furthermore, in view of the present fragmentary state of aerodynamic theory, it would not be surprising if part of the research effort was devoted to the improvement of the basic theory itself.

The reliability of these predictions is, of course, conditioned by the financial prospects of the industry. Just now, war orders are causing abnormal inflation of earnings; when these cease, retrenchment will be inevitable. The industry is not highly mechanized, however, and hence its present cycle of inflation does not imply so large an expenditure for plant as would be true in most manufacturing fields. For this reason, the period of deflation may prove to be one of large war profits in the bank, but insufficient orders to occupy the time of many competent technical men whom the management would be reluctant to let go. If this should occur, an almost explosive development of research may take place.

Whether the development is explosive or not, however, it is probable that the industry will soon become one of the largest employers of industrial mathematicians.

INDUSTRIAL STATISTICS AND STATISTICIANS

The subject of statistics enters the business world at points quite distinct from those touched by the rest of mathematics. Moreover, the types of business activity to which it most frequently applies—insurance and finance, economic forecasting, market surveys, elasticity of demand against price, benefit and pension plans, etc.—belong to the field of economics which is the subject of a separate report, and need not be touched on here.

There are certain other respects in which statistical theory could be of great service in industry, but they have been exploited to only a limited extent. This report must therefore point out these hopeful fields rather than record achievements in them.

Statisticians in Industry

By "statistician" we mean a person versed in and using the mathematical theory of statistics, not one who collects, charts and scrutinizes factual data. In the business world the word is more often used in the latter sense.

There is a very great difference between the number of statisticians in industry, and the number of men interested in some form of statistics. How great the discrepancy is will be clear from a comparison of the membership of the American Statistical Association, which devotes itself to the application of statistics in its broadest sense, and of the American Institute of Mathematical Statistics, which confines itself narrowly to the development of statistical technique. The former lists 277 names with industrial addresses; the latter only 10.

Statistics in Industry

Dr. W. A. Shewhart, Research Statistician of the Bell Telephone Laboratories, has delineated broadly and succinctly the field in which statistics may be expected to find application as follows:

> "Since inductive inferences are only probable, or, in other words, since repetitions of any operation under the same essential conditions cannot be expected to give identical results, we need a scientific method that will indicate the degree of observed variability that should not be left to chance. Hence it appears that the use of mathematical statistics is essential to the development of an adequate scientific method, and that mathematical statistics may be expected to be of potential use wherever scientific method can be used to advantage."

More specifically, there are five recognizable types of industrial engineering activity in which statistical theory either is, or should be used.

(*a*) In studying experimental data to determine whether the observed variations should be regarded as accidental or significant. An example is found in the field of geochemical prospecting. The surface soil overlying regions in which there is oil contains a higher proportion of hydrocarbons and waxes than occur in other locations. Chemical analysis of surface soil therefore affords a means of prospecting for oil. Mr. Eugene McDermott writes:

> "In the geochemical method, it was found necessary to determine between samples showing significantly high analysis values, and those which were normal values. These normal sample values, of course, had considerable variation between themselves, due to analysis and in larger part sampling errors. After examining these data for a long period of time, it was decided to approach the problem statistically. This disclosed at once that areas surveyed could be divided into positive (having significant values, and hence favorable from the standpoint of petroleum possibilities), negative (no significant values and unfavorable for petroleum) and marginal (indeterminate). The latter case is always the most difficult one in surveying, and while we are now able to recognize it, further work is needed to fully interpret it. This kind of mathematics is being applied at the present moment, and bids fair to solve the problem."

(*b*) In planning the kind of experiments from which such data arise. Whether variations are or are not significant depends in no small degree upon the fashion in which the data were taken. Consideration of the experiment in advance from a statistical point of view often results in economy of procedure, or even points the difference between a trustworthy and a meaningless result.

The following example is quoted from an address by Dr. R. H. Pickard, Director of the British Cotton Research Association:

> "To illustrate the advantage of good experimental design I may refer to some experiments carried out at the Shirley Institute to find the effect of various treatments on a quality of cloth. This quality varies considerably at different parts

> of the same piece of cloth, and in order to measure the effect of the treatments the tests are repeated systematically so that the variations are 'averaged out.' Some of the natural variation, however, is systematic, and by adopting a 'Latin Square' arrangement of treatments on the cloth (such as is much used in agricultural yield trials), these systematic variations are eliminated from the comparison, and in the instance quoted the result was to reduce by one-half the number of tests necessary for a given significance as compared with a random arrangement."[8]

To the extent to which biology becomes an important element in industrial research—and it would appear to be on the point of doing so in such fields as food manufacturing—it can be expected that the type of statistical work listed under (*a*) and (*b*) will rapidly increase.

(*c*) In laying out an inspection routine. Manufacturing inspection frequently yields data which are best interpreted statistically, either because only spot-checks are taken, or because the method of inspection gives measurements which are themselves subject to accidental fluctuation. In such cases statistical theory is of great advantage in setting up an effective and economical inspection program. It is being so used in certain industries, notably in electrical manufacturing and textiles, but the potential field of usefulness is far from covered.

The following example is quoted from an address by Mr. Warner Eustis, Staff Officer on Research of the Kendall Company:

> "Surgical sutures are twisted strands of sheep intestine, which has been slit lengthwise. . . . After a stated number of days a sewing with such material, implanted in the body during a surgical operation, will be digested and disappear as the healing processes progressively take up the load originally held by the suture. . . . Here is a product which it is impossible to test in any way without destroying the product, especially as each suture is sealed in an individual, sterilized tube. Our final product tests must all be conducted by breaking open a sterile tube and testing the product therein. The quality appraisal of such a product naturally rests upon probability, rather than upon an actual testing of each item. Due to the nature of such a product, in which a single failure may destroy human life, the need for accurate quality appraisal is superlative."[9]

(*d*) In the control of manufacturing processes. Inspection is not merely a means of discarding bad product; it is also a means of detecting trouble in the factory. This is obvious in the extreme cases when the product is

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[8] "The Application of Statistical Methods to Production and Research in Industry," by R. H. Pickard, Supplement to the *Journal of the Royal Statistical Society*, Vol. 1, No. 2, 1934, pp. 9-10.

[9] "Why the Kendall Company is interested in Statistical Methods," by Warner Eustis, *Proceedings of the Industrial Statistics Conference* held at M. I. T., Cambridge, Mass., Sept. 8–9, 1938, pp. CXLIII–CXLIV, published by Pitman Publishing Corporation.

unusually bad. By the use of suitable routines set up in accordance with statistical theory, the day-to-day results of inspection can be used to detect incipient degradation in the process of manufacture which might otherwise escape notice. This procedure is used extensively by the Western Electric Company in assuring uniform quality in many items of manufacture, and to a lesser extent in other industries. Of it, Mr. J. M. Juran, Manufacturing Engineer of the Western Electric Company, says:

> "Too frequently we have seen an inspection group grow lax in vigilance until a complaint from the customer wakes them up. They promptly swing the pendulum a full stroke in the opposite direction, and the factory groans in its effort to meet the now unreasonable demands. A sound and steady control, like a sound currency in commercial relations, gives factory foremen a feeling of confidence and gives the consumer a feeling that control is being exercised before the product reaches him."[10]

(*e*) In writing rational specifications. Obviously, if such a procedure helps the manufacturer to assure uniform quality, it is also of value to the purchaser of his products. Hence the subject of statistics enters into the writing of the buyer's specifications. It has been so used to a limited extent in the Bell System in connection with telephone apparatus, and by the United States Government in the purchase of munitions. However, it must still be rated as a relatively undeveloped field. Of it, Captain Leslie E. Simon, Ordnance Department of the United States Army, says:

> "Statistical methods have proved to be a powerful tool in the critical examination of some ammunition specifications prior to final approval. Their use, either directly or indirectly, is almost essential in determining a reasonable and economic standard of quality through the method of comparing the quality desired with that which can be reasonably expected under good manufacturing practice. In like manner, the statistical technique renders a valuable service in framing the acceptance specification. Through its use the quantity and kind of evidence which will be accepted as proof that the product will meet the standard of quality can be clearly expressed in a fair, unequivocal and operationally verifiable way."

CONCLUSION

It is perhaps unusual to conclude a survey of this sort by stating the impressions which it has made upon its writer. In the present instance, however, the element of self-education has been so large that these impressions may summarize the report better than any more formal recapitulation. They are:

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[10] "Inspectors' Errors in Quality Control," by J. M. Juran, *Mechanical Engineering*, Oct., 1935, pp. 643–644.

(1) Because of its general significance as the language of natural science, mathematics already pervades the whole of industrial research.

(2) Its field of usefulness is nevertheless growing, partly through the development of new industries such as the aircraft business, and partly through the incorporation of new scientific developments into industrial research, as in the application of quantum physics in chemical manufacturing and statistical theory in the control of manufacturing processes.

(3) The need for professional mathematicians in industry will grow as the complexity of industrial research increases, though their number will never be comparable to that of physicists or chemists.

(4) There is a serious lack of university courses for the graduate training of industrial mathematicians.

(5) Management, which is already keenly alive to the importance of mathematics, is also rapidly awakening to the value of mathematicians and the peculiar relationship which they bear to other scientific personnel.

This last observation is not trivial. There was a day when, in engineering circles, mathematicians were rather contemptuously characterized as queer and incompetent. That day is about over. Just now, an attitude more commonly met is one of amazed pride in pointing to some employee who "isn't like most mathematicians; he gives you an answer you can use, and isn't afraid to make approximations." As the proper function of the industrial mathematician becomes better understood, these proud remarks will no doubt cease. Those who are adapted to the job will be taken for granted; the others will be recognized as personnel errors and not mistaken for the professional type. Perhaps the present report may speed this day. If so, it will have been a service to the profession and to industry.