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STEADY STATE SOLUTIONS OF TRANSMISSION LINE EQUATIONS

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Methods of obtaining the steady state voltages and currents in a uniform transmission line consisting of several parallel wires are described in Part I. This line may or may not be acted upon by an externally impressed field distributed along its length. A square matrix Γ, which is a generalization of the propagation constant γ for a single circuit, is introduced. Matrix expressions obtained for the voltages and currents involve Γ in much the same way as the corresponding single circuit expressions involve γ. In Part II similar methods are described for obtaining the voltages and currents in a transmission line composed of a number of multi-terminal symmetrical sections connected in tandem. Expressions for the voltages and currents in a line composed of unsymmetrical sections are also given. These sections may or may not contain generators.

THE transmission lines considered here are of two kinds, namely the uniform transmission line, and the transmission line consisting of a number of identical sections connected in tandem. The problem discussed is that of determining the steady state electrical behavior of these lines when the terminal conditions are given. Often there arises the problem of determining the currents induced in a uniform transmission line by an arbitrary impressed field of some fixed frequency or of determining the currents produced by generators placed in the branches of the sections if the line is of the second kind. This is the type of problem with which we shall be particularly concerned.

In dealing with the uniform transmission line it is found convenient to introduce a matrix Γ, which is a generalization of the propagation constant γ for a single wire with ground return, or for a single circuit. This enables us to obtain matrix expressions for the currents and voltages which are similar in form to the single circuit expressions.

A similar situation exists for the transmission line composed of a number of symmetrical sections. However, when the sections are unsymmetrical the corresponding procedure does not appear to yield a corresponding simplification and the formulas are considerably more complicated than in the symmetrical case.

This paper is divided into two parts corresponding to the two kinds of transmission lines. The first part discusses the uniform line. After a statement of the transmission equations in matrix form, expressions for the voltages and currents are given. Two methods of evaluating these expressions are described. The first is based upon a property possessed by many transmission systems, namely that the various modes of propagation have nearly the same speed. The second method is based upon equations which may be obtained by the formal application of a theorem due to Sylvester. The first part concludes with the proof that these two methods lead to the correct results.

After a short introduction the second part discusses the difference equations which govern the transmission in a line composed of multi-terminal sections. The sections may contain generators. Expressions for the voltages and currents in a symmetrical section line, i.e. a line whose sections are symmetrical, are stated and proved in much the same order as the corresponding expressions for the uniform line. A discussion of the unsymmetrical section line concludes the second part.

A sketch of the solution of the uniform transmission line equations by the classical method is given in Appendix I. In Appendices II and III methods are described for solving the symmetrical section line difference equations. These methods are similar to the one of Appendix I. The method of Appendix III uses section constants which may be obtained from measurements made at one end of a typical section.

PART I

UNIFORM TRANSMISSION LINES

1.1 Differential Equations

For the sake of convenience in writing down equations we shall assume that the particular line under consideration consists of three parallel wires with ground return, or of three parallel circuits, denoted by the subscripts a, b, and c respectively. The differential equations for this line in an arbitrary impressed field are[1]

$$
\begin{aligned}
\frac{dv_a}{dx} &= -Z_{aa}i_a - Z_{ab}i_b - Z_{ac}i_c + l_a(x) \\
\frac{dv_b}{dx} &= -Z_{ba}i_a - Z_{bb}i_b - Z_{bc}i_c + l_b(x) \qquad (1.1) \\
\frac{dv_c}{dx} &= -Z_{ca}i_a - Z_{cb}i_b - Z_{cc}i_c + l_c(x)
\end{aligned}
$$

[1] These equations are given in substance by J. R. Carson and R. S. Hoyt, *B.S.T.J.*, Vol. 6, pp. 495–545 (1927). Equations (1.2) are equivalent to their equation (90) and equations (1.1) may be obtained by combining their equations (83), (84), and (94). We shall use the term "impressed field" to mean a field distributed along the line. According to our convention there is no impressed field when the line is energized only at the terminals.

and

$$\frac{di_a}{dx} = -Y_{aa}v_a - Y_{ab}v_b - Y_{ac}v_c + t_a(x)$$

$$\frac{di_b}{dx} = -Y_{ba}v_a - Y_{bb}v_b - Y_{bc}v_c + t_b(x) \tag{1.2}$$

$$\frac{di_c}{dx} = -Y_{ca}v_a - Y_{cb}v_b - Y_{cc}v_c + t_c(x)$$

where $Z_{ab} = Z_{ba}$, $Y_{ab} = Y_{ba}$, etc. If we are dealing with three parallel wires $l_a(x)$, $l_b(x)$, $l_c(x)$ are the longitudinal components of the electric force of the impressed field at the wire surfaces; $t_a(x)$, $t_b(x)$, $t_c(x)$ are specified by the admittance of the direct leakage paths and the values of the impressed potentials at the wires. If there are no direct leakage paths the t's are zero.

In order to put these equations in matrix form[2] we introduce the column matrices

$$v = \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix}, \quad i = \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}, \quad l(x) = \begin{bmatrix} l_a(x) \\ l_b(x) \\ l_c(x) \end{bmatrix}, \quad t(x) = \begin{bmatrix} t_a(x) \\ t_b(x) \\ t_c(x) \end{bmatrix}, \tag{1.3}$$

and the symmetrical square matrices

$$Z = \begin{bmatrix} Z_{aa} & Z_{ab} & Z_{ac} \\ Z_{ba} & Z_{bb} & Z_{bc} \\ Z_{ca} & Z_{cb} & Z_{cc} \end{bmatrix} \qquad Y = \begin{bmatrix} Y_{aa} & Y_{ab} & Y_{ac} \\ Y_{ba} & Y_{bb} & Y_{bc} \\ Y_{ca} & Y_{cb} & Y_{cc} \end{bmatrix} \tag{1.4}$$

The equations (1.1) and (1.2) may now be written as

$$\frac{dv}{dx} = -Zi + l(x)$$

$$\frac{di}{dx} = -Yv + t(x) \tag{1.5}$$

and these are the equations to be solved.

When there is no impressed field equations (1.5) give

$$\frac{d^2v}{dx^2} = ZYv$$

$$\frac{d^2i}{dx^2} = YZi \tag{1.6}$$

[2] Cf. L. A. Pipes, *Phil. Mag.*, Vol. 24 (1937), p. 97.

and the analogy with the one circuit case leads us to put

$$\Gamma^2 = ZY, \qquad \Gamma = \sqrt{ZY} \tag{1.7}$$

where Γ is a square matrix representing a generalization of the propagation constant. Putting aside for the moment the question of interpreting the square root, we note that interchanging the rows and columns in $\Gamma^2 = ZY$ gives

$$\Gamma'^2 = Y'Z' = YZ, \qquad \Gamma' = \sqrt{YZ} \tag{1.8}$$

where the primes denote transposition. Y' and Z' are equal to Y and Z respectively because of their symmetry. We thus expect Γ' to be associated with the propagation of i in the same way that Γ is associated with the propagation of v.

1.2 Statement of Results for an Infinite Line—No Impressed Field

It is shown that when there is no impressed field the voltages and currents at any point x in a transmission line extending from $x = 0$ to $x = \infty$ are given by

$$\begin{aligned} v(x) &= e^{-x\Gamma}v(o) = e^{-x\Gamma}Z_o i(o) \\ i(x) &= e^{-x\Gamma'}i(o) \\ v(x) &= Z_o i(x) \end{aligned} \tag{1.9}$$

where $e^{-x\Gamma}$ is the square matrix defined by the convergent series of matrices[3]

$$e^{-x\Gamma} = I - \frac{x\Gamma}{1!} + \frac{x^2\Gamma^2}{2!} - \frac{x^3\Gamma^3}{3!} + \cdots \tag{1.10}$$

and $e^{-x\Gamma'}$ is the transposed of $e^{-x\Gamma}$. I denotes the unit matrix. Z_o is a square matrix and is called the characteristic impedance matrix:

$$Z_o = \Gamma^{-1}Z = \Gamma Y^{-1} \tag{1.11}$$

Additional expressions of the same type for Z_o are given by equations (1.45). The matrix $e^{-x\Gamma}Z_o$, being of the nature of a transfer impedance, is symmetrical.

The matrices $e^{-x\Gamma}$ and Z_o may be computed in several ways, the choice depending upon the circumstances. The first method to be described is useful when x is not too large and when the propagation constants of the various modes of propagation are nearly equal to each other. In the case of open-wire lines these propagation constants are grouped around the value $j\omega/v$ where v is of the order of 180,000 miles per second. The second method may be used for all cases, including those for which the series in

[3] Frazer, Duncan and Collar, "Elementary Matrices," Cambridge University Press, §2.5. In the work which follows, this text will be referred to as "F.D.C."

the first method converge too slowly to be of value. However, it requires the solution of an mth degree equation and the determination of the m modes of propagation where m is the number of circuits. For $m = 2$ this is no handicap and the method is quite convenient. In this case the method is closely related to one described by John Riordan in an unpublished memorandum.

First Method: Multiply the matrices Z and Y together to obtain ZY. Choose the number γ^2 in

$$ZY = I\gamma^2 + R, \tag{1.12}$$

where I is the unit matrix, so that the elements of R are small in comparison with γ^2. For many transmission lines it is possible to do this. Γ may be obtained by using the binomial theorem to expand the square root in the formula

$$\Gamma = \sqrt{ZY} = \gamma(I + \gamma^{-2}R)^{\frac{1}{2}}, \tag{1.13}$$

where γ is that square root of γ^2 whose real and imaginary parts are non-negative. In carrying out the work it is convenient to introduce the matrix S whose elements are small in comparison with unity.

$$\Gamma = \gamma(I + S) \tag{1.14}$$

To compute S, first compute the matrix $R/2\gamma^2$ and then use the power series

$$\begin{aligned} S = \left(\frac{R}{2\gamma^2}\right) - \frac{1}{2}\left(\frac{R}{2\gamma^2}\right)^2 + \frac{1}{2}\left(\frac{R}{2\gamma^2}\right)^3 - \frac{5}{8}\left(\frac{R}{2\gamma^2}\right)^4 \\ + \frac{7}{8}\left(\frac{R}{2\gamma^2}\right)^5 - \frac{21}{16}\left(\frac{R}{2\gamma^2}\right)^6 + \cdots \end{aligned} \tag{1.15}$$

This series will usually converge rapidly. The matrix $e^{-x\Gamma}$ is given by

$$e^{-x\Gamma} = e^{-z} \cdot e^{-zS} \tag{1.16}$$

where z is a number, $z = \gamma x$, and e^{-zS} is to be computed from

$$e^{-zS} = I - \frac{zS}{1!} + \frac{(zS)^2}{2!} - \frac{(zS)^3}{3!} + \cdots \tag{1.17}$$

$e^{-x\Gamma'}$ is obtained from $e^{-x\Gamma}$ by interchanging the rows and columns. The characteristic impedance matrix may be obtained from (1.11),

$$Z_o = \Gamma Y^{-1},$$

after computing Γ from S as in (1.14).

If only $e^{-x\Gamma}$ is required the following series may be used.

$$e^{-x\Gamma} = \sum_{p=0}^{\infty} \left(\frac{Rx}{2\gamma}\right)^p \frac{b_p(z)}{p!} \tag{1.18}$$

where R, γ, and z have the same meaning as above and the coefficients are computed from

$$b_o = e^{-z}, \qquad b_1(z) = -e^{-z}, \qquad b_2(z) = e^{-z}\left(1 + \frac{1}{z}\right)$$

$$b_{p+2}(z) = b_p(z) - \frac{2p+1}{z} b_{p+1}(z)$$

In the first term of the series $\left(\frac{Rx}{2\gamma}\right)^0$ denotes I.

Second Method: Γ, $e^{-x\Gamma}$ and Z_o may be regarded as functions of the square matrix ZY. In order to express these functions in a form suitable for calculation we apply Sylvester's theorem[4]. The characteristic matrix of ZY is

$$f(\gamma^2) = \gamma^2 I - ZY \tag{1.19}$$

where now γ^2 is regarded as a variable instead of a fixed number as in the first method. We shall suppose that ZY is a square matrix of order m and that the roots $\gamma_1^2, \gamma_2^2, \cdots \gamma_m^2$ of the characteristic function, i.e. of the determinantal equation

$$| f(\gamma^2) | = 0, \tag{1.20}$$

are distinct. Let the matrix $F(\gamma^2)$ be the adjoint of $f(\gamma^2)$ and denote the derivative of the characteristic function by

$$| f(\gamma^2) |^{(1)} = \frac{d}{d(\gamma^2)} | f(\gamma^2) | \tag{1.21}$$

Since $\gamma_1^2, \gamma_2^2, \cdots \gamma_m^2$ are all different $| f(\gamma_r^2) |^{(1)}$ is unequal to zero for $r = 1, 2, \cdots m$. Sylvester's theorem says that if $P(ZY)$ is any polynomial in ZY then

$$P(ZY) = \sum_{r=1}^{m} N(\gamma_r^2) P(\gamma_r^2) \tag{1.22}$$

where $P(\gamma_r^2)$ is a scalar (and thus deviates from our convention that capital letters denote square matrices). $N(\gamma_r^2)$ is a square matrix:

$$N(\gamma_r^2) = \frac{F(\gamma_r^2)}{| f(\gamma_r^2) |^{(1)}} \tag{1.23}$$

When $m = 2$, $N(\gamma_2^2)$ is equal to $I - N(\gamma_1^2)$.

[4] F.D.C. §3.9. The u and λ of the reference are the ZY and γ^2 of the present section.

Applying (1.22) to Γ, $e^{-x\Gamma}$ and Z_o even though they are not polynomials in ZY gives results which may be verified to be true.

$$\begin{aligned} \Gamma &= \sqrt{ZY} = \sum N(\gamma_r^2)\gamma_r \\ e^{-x\Gamma} &= e^{-x\sqrt{ZY}} = \sum N(\gamma_r^2)e^{-x\gamma_r} \\ Z_o &= (ZY)^{\frac{1}{2}}Y^{-1} = \sum N(\gamma_r^2)\gamma_r Y^{-1} \\ e^{-x\Gamma}Z_o &= \sum N(\gamma_r^2)\gamma_r e^{-x\gamma_r}Y^{-1} \end{aligned} \tag{1.24}$$

where the summations extend from $r = 1$ to $r = m$ and $\gamma_1, \gamma_2, \cdots \gamma_m$ are the square roots of $\gamma_1^2, \gamma_2^2, \cdots \gamma_m^2$ respectively whose real parts are non-negative. $\gamma_1, \gamma_2, \cdots \gamma_m$ are also the propagation constants of the "normal modes" of propagation. Some light is thrown on the physical significance of the matrix $N(\gamma_r^2)$ by supposing that only the rth normal mode is being propagated on the transmission line. $N(\gamma_r^2)$ is such that it can be expressed as a column matrix times a row matrix. The voltages in circuits 1, 2, $\cdots m$ are proportional to the first, second, $\cdots m$th elements, respectively of the column matrix. The currents in circuits 1, 2, $\cdots m$ are proportional to the corresponding elements in the row matrix.

1.3 Results for Any Uniform Line—No Impressed Field

When the length of the line is finite the voltages and currents may be expressed as

$$\begin{aligned} v(x) &= \cosh x\Gamma\, v(o) - \sinh x\Gamma\, Z_o i(o) \\ i(x) &= -\sinh x\Gamma'\, Z_o^{-1}v(o) + \cosh x\Gamma'\, i(o) \end{aligned} \tag{1.25}$$

where Z_o and Γ have the same meaning as before. The matrices $\sinh x\Gamma Z_o$ and $\sinh x\Gamma' Z_o^{-1}$ are symmetrical. The square matrices $\cosh x\Gamma$ and $\sinh x\Gamma$ are defined by the series

$$\begin{aligned} \cosh x\Gamma &= I + \frac{x^2\Gamma^2}{2!} + \frac{x^4\Gamma^4}{4!} + \cdots \\ \sinh x\Gamma &= \frac{x\Gamma}{1!} + \frac{x^3\Gamma^3}{3!} + \cdots \end{aligned} \tag{1.26}$$

$\cosh x\Gamma'$ is obtained by interchanging the rows and columns of $\cosh x\Gamma$ and $\sinh x\Gamma'$ is obtained similarly from $\sinh x\Gamma$. Solving (1.25) for $v(o)$ and $i(o)$ gives

$$\begin{aligned} v(o) &= \cosh x\Gamma\, v(x) + \sinh x\Gamma\, Z_o i(x) \\ i(o) &= \sinh x\Gamma'\, Z_o^{-1}v(x) + \cosh x\Gamma'\, i(x) \end{aligned}$$

As in the case of the infinite line, we have two ways of computing the coefficients of $v(o)$ and $i(o)$ in the expressions (1.25) for $v(x)$ and $i(x)$.

First Method: Choose a number γ^2 and compute the matrices R, S, Γ, Z_o as described in the first method for the infinite line. The matrix $e^{x\Gamma}$ is given by

$$e^{x\Gamma} = e^{z} \cdot e^{zS}$$

where $z = \gamma x$ and e^{zS} is computed from the series

$$e^{zS} = I + \frac{zS}{1!} + \frac{z^2 S^2}{2!} + \cdots$$

If the elements of zS are so large that the series converges slowly it may be worthwhile to divide zS by 16, say, compute $\exp\left(\frac{zS}{16}\right)$ from the series, and then obtain e^{zS} by four matrix multiplications. When e^{zS} is known its inverse e^{-zS} can be computed and $e^{-x\Gamma}$ obtained from (1.16). The hyperbolic functions are given by

$$\begin{aligned} \cosh x\Gamma &= \tfrac{1}{2}\left(e^{x\Gamma} + e^{-x\Gamma}\right) \\ \sinh x\Gamma &= \tfrac{1}{2}\left(e^{x\Gamma} - e^{-x\Gamma}\right) \end{aligned} \tag{1.27}$$

which follow from the series definitions of the various matrices.

If only the coefficients in (1.25) are required we may choose γ^2 and compute R and powers of the matrix $Rx/2\gamma$. Then the coefficients in (1.25) are given by

$$\begin{aligned} \cosh x\Gamma &= \sum_{p=0}^{\infty} \left(\frac{Rx}{2\gamma}\right)^p \frac{a_p(z)}{p!} \\ \sinh x\Gamma\, Z_o &= \sum_{p=0}^{\infty} \left(\frac{Rx}{2\gamma}\right)^p \frac{a_{p+1}(z)}{p!\gamma}\, Z \\ \sinh x\Gamma' Z_o^{-1} &= \sum_{p=0}^{\infty} \left(\frac{R'x}{2\gamma}\right)^p \frac{a_{p+1}(z)}{p!\gamma}\, Y \end{aligned} \tag{1.28}$$

where R' is the transposed of R, and the scalar coefficient $a_p(z)$ is a function of $z = \gamma x$ given by

$$\begin{aligned} a_o(z) &= \cosh z \qquad a_1(z) = \sinh z \\ a_2(z) &= \cosh z - \frac{\sinh z}{z} \\ a_{p+2}(z) &= a_p(z) - \frac{2p+1}{z}\, a_{p+1}(z), \end{aligned} \tag{1.29}$$

and it is understood that $(Rx/2\gamma)^0 = I$.

Second Method: Compute the propagation constants $\gamma_1, \gamma_2, \cdots \gamma_m$ and the square matrices $N(\gamma_r^2)$ given by (1.23) as in the second method for the infinite line. Then

$$\begin{aligned} \cosh x\Gamma &= \Sigma\, N(\gamma_r^2) \cosh x\gamma_r \\ \sinh x\Gamma\, Z_o &= \sinh x\Gamma\, \Gamma Y^{-1} \\ &= \Sigma\, N(\gamma_r^2) \sinh x\gamma_r\, \gamma_r Y^{-1} \\ \sinh x\Gamma'\, Z_o^{-1} &= \sinh x\Gamma'\, \Gamma'^{-1} Y \\ &= \sum N'(\gamma_r^2) \frac{\sinh x\gamma_r}{\gamma_r} Y \end{aligned} \tag{1.30}$$

where $N'(\gamma_r^2)$ is the transposed of $N(\gamma_r^2)$, $N(\gamma_r^2)$ being defined by (1.23), and the summations extend from $r = 1$ to $r = m$.

When the transmission line consists of perfectly conducting wires strung on perfect insulators over a perfectly conducting earth the magnetic and electrostatic fields are related so as to make Z equal to $\gamma_o^2 Y^{-1}$ where

$$\gamma_o = j\omega/c,$$

ω being 2π times the frequency and c the speed of light.

It is interesting to apply the first method of solution to this line. Even though the proof of the first method, which is given in §1.10, does not cover this case there seems to be little doubt that the correct answer is obtained. We have

$$ZY = \gamma_o^2 I$$

Choosing $\gamma = \gamma_o$ gives $R = 0$ and therefore $S = 0$. It follows that

$$\begin{aligned} \Gamma &= \gamma_o I,\ Z_o = \Gamma^{-1} Z = \gamma_o^{-1} Z \\ \cosh x\Gamma &= \cosh (x\gamma_o I) = \cosh x\gamma_o\, I \\ \sinh x\Gamma\, Z_o &= \sinh x\gamma_o\, \gamma_o^{-1} Z \\ \sinh x\Gamma'\, Z_o^{-1} &= \sinh x\gamma_o\, \gamma_o Z^{-1} \end{aligned}$$

When these are put into equations (1.25) the expressions for $v(x)$ and $i(x)$ in a perfect transmission line are obtained:

$$\begin{aligned} v(x) &= \cosh x\gamma_o v(o) - \frac{\sinh x\gamma_o}{\gamma_o} Z i(o) \\ i(x) &= -\gamma_o \sinh x\gamma_o Z^{-1} v(o) + \cosh x\gamma_o i(o) \end{aligned} \tag{1.31}$$

1.4 Results for Any Uniform Line—Impressed Field

The differential equations to be satisfied in this case are given by (1.5). A solution which reduces to $v(o)$ and $i(o)$ at $x = 0$ is

$$\begin{aligned} v(x) &= \cosh x\Gamma\, v(o) - \sinh x\Gamma\, Z_o i(o) \\ &\quad + \int_0^x \cosh (x - \xi)\Gamma\, l(\xi)\, d\xi - \int_0^x \sinh (x - \xi)\Gamma\, Z_o\, t(\xi)\, d\xi \\ i(x) &= -\sinh x\Gamma'\, Z_o^{-1} v(o) + \cosh x\Gamma'\, i(o) \\ &\quad - \int_0^x \sinh (x - \xi)\Gamma' Z_o^{-1} l(\xi)\, d\xi + \int_0^x \cosh (x - \xi)\Gamma' t(\xi)\, d\xi \end{aligned} \tag{1.32}$$

The matrices $\cosh x\Gamma$, $\sinh x\Gamma$ and Z_o are the same as the ones discussed in §1.2 and §1.3. The elements of the integral[5] of a matrix U (U is not necessarily a square matrix) are given by the integrals of the corresponding elements of U.

In many cases of practical interest the impressed field varies exponentially with respect to x. The column matrices $l(x)$ and $t(x)$ may then be expressed as

$$l(x) = e^{-x\theta} \begin{bmatrix} \lambda_a \\ \lambda_b \\ \lambda_c \end{bmatrix} \qquad t(x) = e^{-x\theta} \begin{bmatrix} \tau_a \\ \tau_b \\ \tau_c \end{bmatrix} \tag{1.33}$$

where the λ's and τ's are constants and θ is the propagation constant of the impressed field in the direction of the line. The integrations in the expressions (1.32) may be performed with the result

$$\begin{aligned} v(x) &= \cosh x\Gamma\, v(o) - \sinh x\Gamma\, Z_o i(o) \\ &\quad + \tfrac{1}{2} (e^{x\Gamma} - e^{-x\theta} I)\, (\Gamma + \theta I)^{-1} (\lambda - Z_o \tau) \\ &\quad - \tfrac{1}{2} (e^{-x\Gamma} - e^{-x\theta} I)\, (\Gamma - \theta I)^{-1} (\lambda + Z_o \tau) \\ i(x) &= -\sinh x\Gamma'\, Z_o^{-1} v(o) + \cosh x\Gamma'\, i(o) \\ &\quad + \tfrac{1}{2} (e^{x\Gamma'} - e^{-x\theta} I)\, (\Gamma' + \theta I)^{-1} (\tau - Z_o^{-1}\lambda) \\ &\quad - \tfrac{1}{2} (e^{-x\Gamma'} - e^{-n\theta} I)\, (\Gamma' - \theta I)^{-1} (\tau + Z_o^{-1}\lambda) \end{aligned} \tag{1.34}$$

provided that the inverse matrices exist. The matrix $(e^{x\Gamma'} - e^{-x\theta} I)(\Gamma' + \theta I)^{-1}$ is the transposed of $(e^{x\Gamma} - e^{-x\theta} I)(\Gamma + \theta I)^{-1}$, etc. If one of these matrices, say $\Gamma - \theta I$, has no inverse then it is necessary to evaluate the

[5] F.D.C. §2.10.

corresponding integral in some other way. Thus it may be advantageous to use the formula

$$\begin{aligned} -(e^{-x\Gamma} - e^{-x\theta}I)(\Gamma - \theta I)^{-1} &= e^{-x\Gamma} \int_0^x e^{\xi\Gamma - \xi\theta I}\, d\xi \\ &= e^{-x\Gamma}\left[xI + \frac{x^2}{2!}(\Gamma - \theta I) + \frac{x^3}{3!}(\Gamma - \theta I)^2 + \cdots\right] \end{aligned} \tag{1.35}$$

Two special cases of (1.34) are of interest. When the line is shorted at both ends, $v(o) = v(x) = 0$, where x is the line length, and

$$\begin{aligned} i(o) = \tfrac{1}{2} Z_o^{-1} (\sinh x\Gamma)^{-1}[(e^{x\Gamma} - e^{-x\theta}I)(\Gamma + \theta I)^{-1}(\lambda - Z_o\tau) \\ - (e^{-x\Gamma} - e^{-x\theta}I)(\Gamma - \theta I)^{-1}(\lambda + Z_o\tau)] \end{aligned}$$

$$\begin{aligned} i(x) = \frac{e^{-x\theta}}{2} Z_o^{-1} (\sinh x\Gamma)^{-1}[(e^{x\Gamma} - e^{x\theta} I)(\Gamma - \theta I)^{-1}(\lambda + Z_o\tau) \\ - (e^{-x\Gamma} - e^{x\theta}I)(\Gamma + \theta I)^{-1}(\lambda - Z_o\tau)] \end{aligned}$$

When the line is terminated in its characteristic impedance at both ends, $v(o) = -Z_o i(o)$, $v(x) = Z_o i(x)$, and

$$i(o) = \tfrac{1}{2} (I - e^{-x\theta}e^{-x\Gamma'})(\Gamma' + \theta I)^{-1}(Z_o^{-1}\lambda - \tau)$$

$$i(x) = -\tfrac{1}{2} (e^{-x\Gamma'} - e^{-x\theta}I)(\Gamma' - \theta I)^{-1}(Z_o^{-1}\lambda + \tau)$$

The matrices occurring in the expressions (1.34) for $v(x)$ and $i(x)$ may be computed by the first or second method described for the uniform line in the absence of an impressed field. The second method involves the use of expansions similar to

$$\begin{aligned} &(e^{x\Gamma} - e^{-x\theta}I)(\Gamma + \theta I)^{-1}(\lambda - Z_o\tau) \\ &\qquad = \sum N(\gamma_r^2)\left(\frac{e^{x\gamma_r} - e^{-x\theta}}{\gamma_r + \theta}\right)\left(\lambda - \frac{Z}{\gamma_r}\tau\right) \\ &(e^{x\Gamma'} - e^{-x\theta} I)(\Gamma' + \theta I)^{-1}(Z_o^{-1}\lambda - \tau) \\ &\qquad = \sum N'(\gamma_r^2)\left(\frac{e^{x\gamma_r} - e^{-x\theta}}{\gamma_r + \theta}\right)\left(\frac{Y}{\gamma_r}\lambda - \tau\right) \end{aligned} \tag{1.36}$$

where the summations run from $r = 1$ to $r = m$ and $N'(\gamma_r^2)$ is the transposed of the square matrix $N(\gamma_r^2)$ given by (1.23). In obtaining these expansions by Sylvester's theorem, Z_o in the first is replaced by $\Gamma^{-1}Z$ and Z_o^{-1} in the second by $\Gamma'^{-1}Y$.

If we assume that an impressed field acts upon the perfect transmission line of equations (1.31), we see that $t(x) = 0$ because there are no direct leakage paths. We may also write

$$(e^{x\Gamma} - e^{-x\theta} I)(\Gamma + \theta I)^{-1} = (e^{x\gamma_o} I - e^{-x\theta} I)(\gamma_o I + \theta I)^{-1}$$
$$= \frac{e^{x\gamma_o} - e^{-x\theta}}{\gamma_o + \theta} I$$

From this and similar equations it follows that

$$v(x) = \cosh x\gamma_o v(o) - \frac{\sinh x\gamma_o}{\gamma_o} Zi(o) + \frac{1}{2}\left[\frac{e^{x\gamma_o} - e^{-x\theta}}{\gamma_o + \theta} - \frac{e^{-x\gamma_o} - e^{-x\theta}}{\gamma_o - \theta}\right]\lambda \tag{1.37}$$

$$i(x) = -\gamma_o \sinh x\gamma_o\, Z^{-1} v(o) + \cosh x\gamma_o i(o) - \frac{\gamma_o}{2}\left[\frac{e^{x\gamma_o} - e^{-x\theta}}{\gamma_o + \theta} + \frac{e^{-x\gamma_o} - e^{-x\theta}}{\gamma_o - \theta}\right] Z^{-1}\lambda$$

1.5 Results for Infinite Uniform Line—Impressed Field

When the line extends from $x = 0$ to $x = \infty$ and the impressed field is such that the voltages and currents remain finite at $x = \infty$, the appropriate solutions may be obtained from the results of §1.4 by a limiting process. The condition that $v(x)$ remain finite suggests that the coefficient of $e^{x\Gamma}$ be zero in the expression (1.32) for $v(x)$. This gives a relation between $v(o)$ and $i(o)$ which must be satisfied:

$$v(o) = Z_o i(o) - \int_0^\infty e^{-\xi\Gamma} [l(\xi) - Z_o t(\xi)]\, d\xi \tag{1.38}$$

If the impressed field varies exponentially with x expression (1.34) gives

$$v(o) = Z_o i(o) - (\Gamma + \theta I)^{-1}(\lambda - Z_o \tau) \tag{1.39}$$

Expressions for $v(x)$ and $i(x)$ may be obtained by using relations (1.38) and (1.39) in (1.32) and (1.34) respectively. As these are somewhat lengthy we shall state only two which follow from (1.39).

$$v(x) = e^{-x\Gamma} v(o) + \tfrac{1}{2}(e^{-x\Gamma} - e^{-x\theta} I)[(\Gamma + \theta I)^{-1}(\lambda - Z_o\tau) - (\Gamma - \theta I)^{-1}(\lambda + Z_o\tau)]$$

$$i(x) = Z_o^{-1} e^{-x\Gamma} v(o) + \tfrac{1}{2} Z_o^{-1}(e^{-x\Gamma} + e^{-x\theta} I)(\Gamma + \theta I)^{-1}(\lambda - Z_o\tau) - \tfrac{1}{2} Z_o^{-1}(e^{-x\Gamma} - e^{-x\theta} I)(\Gamma - \theta I)^{-1}(\lambda + Z_o\tau) \tag{1.40}$$

Two similar expressions may be obtained in which the initial current $i(o)$ instead of $v(o)$ appears on the right. If the line is terminated in its characteristic impedance at $x = 0$, $v(o) = -Z_o i(o)$, and the voltages and currents produced by the impressed field are

$$\begin{aligned} v(o) &= -\tfrac{1}{2}\,(\Gamma + \theta I)^{-1}(\lambda - Z_o\tau) \\ i(o) &= \tfrac{1}{2}\,Z_o^{-1}(\Gamma + \theta I)^{-1}(\lambda - Z_o\tau) \end{aligned} \tag{1.41}$$

As in §1.4 these expressions may be computed by the first and second methods described in §1.3. For example, the application of the second method to the relation (1.39) which must exist between $v(o)$ and $i(o)$ in an infinite line gives

$$v(o) = \sum_{r=1}^{m} N(\gamma_r^2)\left[\frac{Z}{\gamma_r}\,i(o) - \frac{1}{\gamma_r + \theta}\left(\lambda - \frac{Z}{\gamma_r}\,\tau\right)\right] \tag{1.42}$$

where $N(\gamma_r^2)$ is the square matrix (1.23).

1.6 Outline of Proofs

The proof of the results which have been stated is divided into three parts. In the first part it is shown that if Γ is a matrix such that (a) its square is ZY and (b) every element in the matrix $e^{-x\Gamma}$ approaches zero as $x \to \infty$, then the expressions for $v(x)$ and $i(x)$ involving Γ and Z_o satisfy the transmission line equations. In the second part of the proof it is shown that if certain requirements are met Γ as obtained by the first method satisfies the conditions (a) and (b) and hence the expressions for $v(x)$ and $i(x)$ given by the first method are correct. The third part of the proof discusses a general procedure which may be used to prove the equations which constitute the second method.

Both the second and the third parts of the proof are based upon the solution of the transmission line equations which is sketched in Appendix I. This solution assumes that the propagation constants of the normal modes of propagation are unequal, and our proofs are limited accordingly. However, considerations of continuity seem to show that the first method is valid even when two or more propagation constants are equal. Under the same circumstances the second method suggests the use of the confluent form of Sylvester's theorem.[6]

1.7 Relations Obtained by Considering An Infinite Line

We suppose that we are going to deal with transmission lines possessing the non-singular, symmetrical impedance and admittance matrices Z and Y. We further suppose that, by some means or other, we have determined a matrix Γ which satisfies the two conditions; (a) the square of Γ is

$$\Gamma^2 = ZY, \tag{1.43}$$

and (b) every element in the matrix $e^{-x\Gamma}$ approaches zero as $x \to \infty$.

[6] F.D.C. §3.10.

Consider a line extending from $x = 0$ to $x = \infty$, there being no impressed field. Viewing the line at $x = 0$ as an n terminal network shows that there is a symmetrical matrix Z_o such that $v(o) = Z_o i(o)$. Let this be taken as the definition of the characteristic impedance matrix Z_o. We shall show from the differential equations of the line that

1. The voltages and currents in the infinite line are given by

$$\begin{aligned} v(x) &= e^{-x\Gamma} v(o) \\ i(x) &= e^{-x\Gamma'} i(o) \end{aligned} \tag{1.44}$$

2. The matrix Z_o satisfies the relations

$$\begin{aligned} Z_o &= \Gamma^{-1} Z = Z\Gamma'^{-1} = \Gamma Y^{-1} = Y^{-1}\Gamma' \\ Z_o^{-1} &= Z^{-1}\Gamma = \Gamma' Z^{-1} = Y\Gamma^{-1} = \Gamma'^{-1} Y \end{aligned} \tag{1.45}$$

$$v(x) = Z_o i(x) \tag{1.46}$$

3. The matrices Z_o, Z, and Y obey the commutation rules

$$\begin{aligned} \Phi(\Gamma) Z_o &= Z_o \Phi(\Gamma') \\ \Phi(\Gamma) Z &= Z \Phi(\Gamma') \\ Y\Phi(\Gamma) &= \Phi(\Gamma') Y \end{aligned} \tag{1.47}$$

where $\Phi(\Gamma)$ is any square matrix, such as $e^{-x\Gamma}$, representable as a convergent power series in Γ with scalar coefficients. Furthermore, the matrices $\Phi(\Gamma)Z_o$, $\Phi(\Gamma)Z$, and $Y\Phi(\Gamma)$ are symmetrical.

The differential equations of the transmission line are

$$\frac{dv}{dx} = -Zi, \qquad \frac{di}{dx} = -Yv, \qquad \frac{d^2 v}{dx^2} = ZYv \tag{1.48}$$

the third following from the first two when i is eliminated. That $v(x) = e^{-x\Gamma} v(o)$ is a solution of the third equation may be verified by direct substitution and differentiation[7]. Since this expression for $v(x)$ approaches zero as $x \to \infty$ and reduces to $v(o)$ at $x = 0$, it represents the voltages in an infinite transmission line. Hence the first equation in (1.44) is true. Setting it in the first differential equation of (1.48), putting $x = 0$, replacing $v(o)$ by $Z_o i(o)$, and noting that $i(o)$ may be regarded as an arbitrary column gives

$$\Gamma Z_o = Z \tag{1.49}$$

Since Γ was assumed to be non-singular, Z_o is equal to $\Gamma^{-1} Z$. Z is symmetrical and the reciprocity theorem for electrical networks requires that Z_o

[7] The differentiation of the exponential function is discussed in F.D.C. §2.7.

be symmetrical, hence

$$Z_o = \Gamma^{-1}Z = Z\Gamma'^{-1}$$

The first group of equations in (1.45) follow from this together with the expression $\Gamma^2 Y^{-1}$ for Z obtained from (1.43). The second group in (1.45) is obtained from the first group.

The commutation rule for Z_o is obtained from (1.49) together with the equation obtained from (1.49) by transposition. Since Z is symmetrical

$$\Gamma Z_o = Z_o\Gamma', \qquad \Gamma^2 Z_o = \Gamma Z_o\Gamma' = Z_o\Gamma'^2,$$
$$\Gamma^n Z_o = Z_o\Gamma'^n$$

and the first of equations (1.47) follow from this. The second and third of equations (1.47) may be obtained similarly from the relations (1.45). The matrix $\Phi(\Gamma)Z_o$ is symmetrical since its transposed is $Z_o\,[\Phi(\Gamma)]'$ and this is equal to $Z_o\Phi(\Gamma') = \Phi(\Gamma)Z_o$. A similar argument applies to the other matrices in (1.47).

The expression for $i(x)$ in (1.44) may be obtained by Maclaurin's expansion. Setting $x = 0$ in the second differential equation of (1.48),

$$\left(\frac{di}{dx}\right)_o = -Yv(o) = -YZ_oi(o) = -\Gamma'i(o)$$

where we have used the equality between the first and last members of the first equation of (1.45) and where the subscript 0 denotes the value of the derivative at $x = 0$. Repeated differentiation gives

$$\frac{d^2 i}{dx^2} = -Y\frac{dv}{dx} = YZi = \Gamma'^2 i$$

$$\left(\frac{d^3 i}{dx^3}\right)_o = \Gamma'^2\left(\frac{di}{dx}\right)_o = -\Gamma'^3 i(o)$$

and so on. Hence

$$i(x) = \left[I - \frac{x\Gamma'}{1!} + \frac{x^2\Gamma'^2}{2!} - \cdots\right]i(o)$$
$$= e^{-x\Gamma'}i(o)$$

Equation (1.46) may now be obtained by using the commutation rule for Z_o:

$$v(x) = e^{-x\Gamma}v(o) = e^{-x\Gamma}Z_oi(o)$$
$$= Z_oe^{-x\Gamma'}i(o) = Z_oi(x)$$

This completes the proof of equations (1.44) to (1.47).

1.8 *Proof of Relations for Any Uniform Line—Impressed Field*

Here it is shown that if a matrix Γ satisfies the two conditions of §1.7 and if Z_o is the characteristic impedance matrix defined there, then the voltages and currents in any uniform line are given by the expressions (1.32). If suitable conditions are fulfilled the relation (1.38) between $v(o)$ and $i(o)$ for an infinite line may be obtained from (1.32).

First of all, $v(x)$ and $i(x)$ reduce to the required values of $v(o)$ and $i(o)$ at $x = 0$. All that remains to be shown is that $v(x)$ and $i(x)$ as given by (1.32) are solutions of the transmission line equations (1.5). By substituting (1.32) in (1.5) and using the formulas

$$\frac{d}{dx} \cosh x\Gamma = \Gamma \sinh x\Gamma = \sinh x\Gamma\, \Gamma$$

$$\frac{d}{dx} \sinh x\Gamma = \Gamma \cosh x\Gamma = \cosh x\Gamma\, \Gamma$$

which follow immediately from the series definitions (1.26) of the hyperbolic functions, we obtain two matrix equations corresponding to the two differential equations. The terms in these equations involving $v(o)$ may be canceled out provided

$$\begin{aligned} \Gamma \sinh x\Gamma &= Z \sinh x\Gamma'\, Z_o^{-1} \\ \Gamma' \cosh x\Gamma'\, Z_o^{-1} &= Y \cosh x\Gamma \end{aligned} \tag{1.50}$$

and these are seen to be true from (1.45) and (1.47). The terms involving $i(o)$ may be canceled by a similar argument. The terms involving $l(x)$ may be canceled provided

$$\int_0^x \sinh (x - \xi)\Gamma\, \Gamma l(\xi)\, d\xi = \int_0^x Z \sinh (x - \xi)\Gamma' Z_o^{-1} l(\xi)\, d\xi$$

$$\int_0^x \Gamma' \cosh (x - \xi)\Gamma' Z_o^{-1} l(\xi)\, d\xi = \int_0^x Y \cosh (x - \xi)\Gamma\, l(\xi)\, d\xi$$

and these are seen to be true when x in (1.50) is replaced by $(x - \xi)$. The terms involving $t(x)$ may be similarly canceled. Thus we have verified that $v(x)$ and $i(x)$ as given by (1.32) are solutions of the transmission line equation provided that the commutation rules (1.47) and the relations (1.45) involving Z_o of §1.7 are satisfied. This is the case when Γ is such that (a) Γ^2 is equal to ZY and also (b) every element in $e^{-\Gamma x}$ approaches zero as $x \to \infty$.

In order to establish equation (1.38) for the Γ of §1.7 several assumptions regarding the impressed field are required. Writing the hyperbolic functions in the first of equations (1.32) in exponential form and premultiplying both sides by $2e^{-\Gamma x}$ gives

$$2e^{-x\Gamma}v(x) = \left[v(o) - Z_o i(o) + \int_0^x e^{-\xi\Gamma}[l(\xi) - Z_o t(\xi)]\,d\xi\right] + e^{-2x\Gamma}[v(o) + Z_o i(o)] + e^{-x\Gamma}\int_0^x e^{-(x-\xi)\Gamma}[l(\xi) + Z_o t(\xi)]\,d\xi$$

When $x \to \infty$ equation (1.38) is obtained provided that the impressed field and the terminal conditions at the far end are such that (a) $v(x)$ remains finite, (b) the integral in (1.38) converges, and (c), the last expression on the right in the equation above approaches zero as $x \to \infty$.

1.9 Derivation of Equations (1.25)

Although equations (1.25) may be obtained by setting $l(x) = t(x) = 0$ in §1.8, it is of some interest to derive them directly. By repeated differentiation of the equations

$$\frac{dv}{dx} = -Zi, \qquad \frac{di}{dx} = -Yv \tag{1.48}$$

the second, third and higher order derivatives may be obtained. Using these in Maclaurin's expansion about $x = 0$ gives

$$\begin{aligned} v(x) &= \left[I + \frac{x^2}{2!}ZY + \frac{x^4}{4!}(ZY)^2 + \cdots\right]v(o) \\ &\qquad - \left[\frac{x}{1!}I + \frac{x^3}{3!}ZY + \frac{x^5}{5!}(ZY)^2 + \cdots\right]Zi(o) \\ i(x) &= -\left[\frac{x}{1!}I + \frac{x^3}{3!}YZ + \frac{x^5}{5!}(YZ)^2 + \cdots\right]Yv(o) \\ &\qquad + \left[I + \frac{x^2}{2!}YZ + \frac{x^4}{4!}(YZ)^2 + \cdots\right]i(o) \end{aligned} \tag{1.51}$$

These series converge for all values of x and could be used for computation were it not for the unfortunate fact that in most problems a great many terms would be required for a satisfactory answer. For the time being, let Γ be any matrix whose square is ZY. The definitions (1.26) of the hyperbolic functions enable us to write (1.51) as

$$\begin{aligned} v(x) &= \cosh x\Gamma\, v(o) - \sinh x\Gamma\, \Gamma^{-1}Zi(o) \\ i(x) &= -\sinh x\Gamma'\, \Gamma'^{-1}Yv(o) + \cosh x\Gamma'\, i(o) \end{aligned} \tag{1.52}$$

If in addition to being a matrix whose square is ZY, Γ is also such that every element in $e^{-x\Gamma}$ approaches zero as $x \to \infty$, then we may use the relations (1.45) for Z_o and obtain (1.25).

Incidentally, when we put $ZY = I\gamma^2 + R$ in (1.51) and rearrange the terms so as to get a power series in R we get the series (1.28).

1.10 Proof of the First Method

The first method consists essentially of determining Γ from the series expansion of (1.13):

$$\Gamma = \gamma \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})_n}{n!} \frac{(-R)^n}{\gamma^{2n}}, \tag{1.53}$$

where $(-\frac{1}{2})_n = (-\frac{1}{2})(\frac{1}{2})(\frac{3}{2}) \cdots (n - \frac{3}{2})$ when $n > 0$ and $(-\frac{1}{2})_0 = 1$, and then computing Z_0 and the required exponential and hyperbolic functions of $x\Gamma$. From §1.7 and §1.8 it follows that the first method gives the correct result provided that Γ as determined by (1.53) satisfies the conditions: (a) its square is equal to ZY and (b) every element of $e^{-x\Gamma}$ approaches zero as $x \to \infty$.

These two conditions are satisfied by the matrix

$$\Gamma = PGP^{-1} \tag{1.54}$$

where P and G are matrices defined by equations (A1.1) and (A1.3) of Appendix I, G being a diagonal matrix whose rth element is γ_r. For from (A1.9) the square of Γ is

$$\Gamma^2 = PG^2P^{-1} = ZY$$

Furthermore,

$$\begin{aligned} e^{-x\Gamma} &= \sum_0^{\infty} \frac{(-x)^n}{n!} (PGP^{-1})^n \\ &= P \sum_0^{\infty} \frac{(-x)^n}{n!} G^n P^{-1} \\ &= PM(x)P^{-1} \end{aligned} \tag{1.55}$$

where $M(x)$ is diagonal matrix (A1.5) whose rth element is $e^{-\gamma_r x}$. Since the real part of γ_r is positive and the elements of P are independent of x it follows that the second condition is satisfied.

It will now be shown that PGP^{-1} may be expanded in the series (1.53) provided that γ may be chosen so as to make all of the points $\zeta_r = \dfrac{\gamma_r}{\gamma}$, $r = 1, 2, \cdots m$, in the complex ζ plane lie within that loop of the lemniscate $|\zeta^2 - 1| = 1$ which contains the point $\zeta = 1$. For then we may write the rth element in G as a convergent series:

$$\begin{aligned} \gamma_r &= \gamma \left(1 + \frac{\gamma_r^2 - \gamma^2}{\gamma^2}\right)^{\frac{1}{2}} \\ &= \gamma \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})_n}{n!} \frac{(\gamma^2 - \gamma_r^2)^n}{\gamma^{2n}} \end{aligned} \tag{1.56}$$

and PGP^{-1} may be written as a convergent infinite series, the nth term of which contains the matrix (assuming only three circuits for the sake of simplicity)

$$P\begin{bmatrix} \gamma_1^2 - \gamma^2 & 0 & 0 \\ 0 & \gamma_2^2 - \gamma^2 & 0 \\ 0 & 0 & \gamma_3^2 - \gamma^2 \end{bmatrix}^n P^{-1} = R^n, \qquad (1.57)$$

where the equality follows from the definition (1.12) of R and equation (A1.9) of Appendix I. This series for PGP^{-1} is exactly the same as the series (1.53), and this completes the proof of the first method.

The equations (1.18) and (1.28) which are incidental to the first method, will now be established for the case in which the matrix Γ occurring in them is equal to PGP^{-1}. For then we have equation (1.55) and the equations

$$\cosh x\Gamma = P\begin{bmatrix} \cosh x\gamma_1 & 0 & 0 \\ 0 & \cosh x\gamma_2 & 0 \\ 0 & 0 & \cosh x\gamma_3 \end{bmatrix} P^{-1} \qquad (1.58)$$

$$\sinh x\Gamma\, Z_o = \sinh x\Gamma\, \Gamma^{-1} Z$$

where $\sinh x\Gamma\, \Gamma^{-1}$ may be expressed in the same fashion as $\cosh x\Gamma$, the rth element of the diagonal matrix being $\dfrac{\sinh x\gamma_r}{\gamma_r}$. The elements in the diagonal matrices occurring in these expressions may be expanded in series by replacing γ_r by its representation (1.56), assuming $\left|\dfrac{\gamma_r^2}{\gamma^2} - 1\right| < 1$, and using[8]

$$e^{-z\sqrt{1+r}} = \sum_0^\infty \left(\frac{rz}{2}\right)^p \frac{(-)^p}{p!} \sqrt{\frac{2z}{\pi}}\, K_{p-\frac{1}{2}}(z)$$

$$\cosh z\sqrt{1+r} = \sum_0^\infty \left(\frac{rz}{2}\right)^p \frac{1}{p!} \sqrt{\frac{\pi z}{2}}\, I_{p-\frac{1}{2}}(z)$$

$$\frac{\sinh z\sqrt{1+r}}{\sqrt{1+r}} = \sum_0^\infty \left(\frac{rz}{2}\right)^p \frac{1}{p!} \sqrt{\frac{\pi z}{2}}\, I_{p+\frac{1}{2}}(z)$$

where $I_{p-\frac{1}{2}}(z)$ and $K_{p-\frac{1}{2}}(z)$ are Bessel functions of the first and second kinds, respectively, for imaginary argument. Equations (1.18) and (1.28) are obtained when equation (1.57) and the Bessel function recurrence relations are used.

[8] These are special cases of formulas given in "Theory of Bessel Functions," by G. N. Watson, page 141.

1.11 *Proof of the Second Method*

To establish the second method we must prove the various formulas which are used. These formulas all involve the square matrix $N(\gamma_r^2)$ defined by (1.23).

Since $N(\gamma_r^2)$ is proportional to $F(\gamma_r^2)$ it follows that $N(\gamma_r^2)$ may be expressed as

$$N(\gamma_r^2) = p_r \rho_r \cdots \tag{1.59}$$

where p_r is the column matrix defined in Appendix I and ρ_r is a row matrix specified by p_r and $N(\gamma_r^2)$. Applying Sylvester's theorem to the unit matrix gives

$$I = \sum N(\gamma_r^2) = \sum p_r \rho_r = [p_1, p_2, p_3] \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix}$$

where the two matrices on the extreme right are partitioned square matrices. From the definition of P in Appendix I it follows that

$$[p_1, p_2, p_3] = P, \qquad \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix} = P^{-1} \tag{1.60}$$

These relations enable us to verify the equations (1.24) when Γ is equal to PGP^{-1}. Thus for the first of equations (1.24)

$$\Gamma = PGP^{-1} = [p_1, p_2, p_3]G \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix}$$

$$= [p_1, p_2, p_3] \begin{bmatrix} \gamma_1 \rho_1 \\ \gamma_2 \rho_2 \\ \gamma_3 \rho_3 \end{bmatrix} = \sum p_r \gamma_r \rho_r = \sum N(\gamma_r^2)\gamma_r$$

The second equation of (1.24) follows likewise from the expression (1.55) for $e^{-x\Gamma}$.

The third equation of (1.24) follows at once from the first when we use (1.45), $Z_o = \Gamma Y^{-1}$. The fourth equation is obtained by writing

$$e^{-x\Gamma} Z_o = PM(x)P^{-1}PGP^{-1}Y^{-1}$$
$$= PM(x)GP^{-1}Y^{-1}$$

and proceeding as in the case of the first equation.

All of the other equations connected with the second method may be proved in a similar way. Incidentally, the formulas obtained by the second method are closely related to the "special form of solution" described in §6.5 of F.D.C.

PART II

TRANSMISSION LINES COMPOSED OF MULTI-TERMINAL SECTIONS

2.1 *Introductory*

Some transmission systems may be regarded as consisting of a number of identical sections connected in tandem. The question of determining the steady state electrical behavior of such a system from a knowledge of the properties of a single section will be considered here.

Each section will have a certain number, say $m + 1$, terminals on its left end and an equal number on the right. The case in which there are only two terminals ($m = 1$) has been completely worked out, and some studies of more general cases have been made. The ones which most nearly approach the point of view of the present paper are those due to S. Koizumi[9].

In the present work difference equations are used to solve the general case in much the same manner as they have been used in studying the two-terminal case. This approach differs from that used by Koizumi and throws additional light on the problem.

In several lists of formulas, particularly in Appendix IV, I have included a number of results due to Koizumi for the sake of completeness.

2.2 *Transmission Equations for a Typical Section*

We consider the equations connecting the input and output currents and voltages for the nth section which is shown in Fig. 1. The directions which are assumed for positive current flow are indicated by arrows. The leads marked 0 play a special role in that all the voltages are

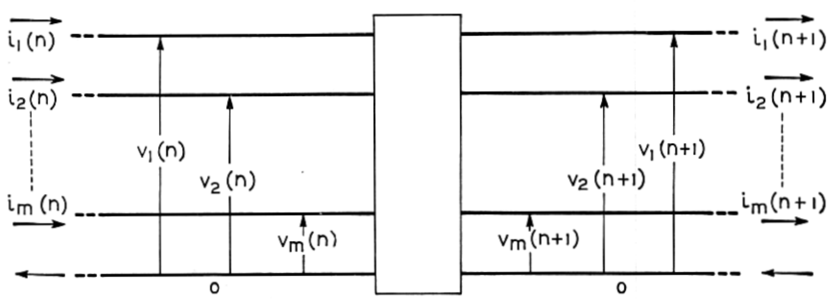

FIG. 1

measured with respect to them, and the currents which they carry are the sum of the currents flowing into or out of the remaining terminals at the end under consideration. In applications to transmission lines the terminals 0 would correspond to the ground or the cable sheath.

The currents and voltages shown in Fig. 1 are related by a number of

[9] Archiv für Electrotechnik, Vol. 33, pp. 171–188, 609–622 (1939). See also a paper by M. G. Malti and S. E. Warschawski, Trans. A.I.E.E., Vol. 56, pp. 153–158 (1937).

sets of $2m$ linear equations which may be conveniently written in matrix form. One such set is

$$\begin{aligned} v(n) &= Z_{11}i(n) - Z_{12}i(n+1) + v^{\circ}(n) \\ v(n+1) &= Z_{21}i(n) - Z_{22}i(n+1) + u^{\circ}(n) \end{aligned} \tag{2.1}$$

Z_{11}, Z_{12}, Z_{21}, Z_{22} are square matrices of order m whose elements are impedances. $v(n)$ and $i(n)$ are the column matrices

$$v(n) = \begin{bmatrix} v_1(n) \\ v_2(n) \\ \vdots \\ v_m(n) \end{bmatrix} \qquad i(n) = \begin{bmatrix} i_1(n) \\ i_2(n) \\ \vdots \\ i_m(n) \end{bmatrix}$$

The column matrices $v^{\circ}(n)$ and $u^{\circ}(n)$ arise from generators which may be acting within the nth section. If both ends of the section are open circuited so that $i(n) = i(n+1) = 0$ the equations show that $v(n) = v^{\circ}(n)$, $v(n+1) = u^{\circ}(n)$. Consequently, $v^{\circ}(n)$ and $u^{\circ}(n)$ give the open circuit voltages produced on the left and right ends of the nth section by the internal generators. If the section is a passive network then $v^{\circ}(n) = u^{\circ}(n) = 0$ and they do not appear in the equations. The subscripts on the square matrices, the Z's, are chosen so as to preserve the analogy for the simple case $m = 1$, where the left and right ends of the section are denoted by the subscripts 1 and 2, respectively.

Solving the equation (1.1) for $i(n)$ and $i(n+1)$ gives

$$\begin{aligned} i(n) &= Y_{11}v(n) + Y_{12}v(n+1) + i^{\circ}(n) \\ -i(n+1) &= Y_{21}v(n) + Y_{22}v(n+1) - j^{\circ}(n) \end{aligned} \tag{2.2}$$

where the elements of the Y's are admittances and $i^{\circ}(n)$, $j^{\circ}(n)$ are the currents produced by the internal sources when the terminals on the right and left are short-circuited so that $v(n) = v(n+1) = 0$.

A third set of equations is

$$\begin{aligned} v(n) &= Av(n+1) + Bi(n+1) - Bj^{\circ}(n) \\ i(n) &= Cv(n+1) + Di(n+1) - Cu^{\circ}(n) \end{aligned} \tag{2.3}$$

Solving these equations for $v(n+1)$ and $i(n+1)$ gives

$$\begin{aligned} v(n+1) &= D'v(n) - B'i(n) + B'i^{\circ}(n) \\ i(n+1) &= -C'v(n) + A'i(n) + C'v^{\circ}(n) \end{aligned} \tag{2.4}$$

There are a great many relations between the square matrices appearing in the equations (2.1) to (2.4). These are discussed in Appendix IV.

For symmetrical sections $Y_{21} = Y_{12}$, $Y_{22} = Y_{11}$, $Z_{21} = Z_{12}$, $Z_{22} = Z_{11}$ and equations (2.1) and (2.2) become

$$\begin{aligned} v(n) &= Z_{11}i(n) - Z_{12}i(n+1) + v^\circ(n) \\ v(n+1) &= Z_{12}i(n) - Z_{11}i(n+1) + u^\circ(n) \end{aligned} \tag{2.5}$$

$$\begin{aligned} i(n) &= Y_{11}v(n) + Y_{12}v(n+1) + i^\circ(n) \\ -i(n+1) &= Y_{12}v(n) + Y_{11}v(n+1) - j^\circ(n) \end{aligned} \tag{2.6}$$

Eliminating $i(n)$ from (2.5) and $v(n)$ from (2.6) and using, from (A4.4), $A = Z_{11}Z_{12}^{-1} = -Y_{12}^{-1}Y_{11}$ leads to the difference equations

$$v(n+1) + v(n-1) - 2Av(n) = B[i^\circ(n) - j^\circ(n-1)] \tag{2.7}$$

$$i(n+1) + i(n-1) - 2A'i(n) = C[v^\circ(n) - u^\circ(n-1)] \tag{2.8}$$

Since we also have $B' = B$, $C' = C$, $D' = A$ for symmetrical sections equations (2.4) become

$$\begin{aligned} v(n+1) &= Av(n) - Bi(n) + Bi^\circ(n) \\ i(n+1) &= -Cv(n) + A'i(n) + Cv^\circ(n) \end{aligned} \tag{2.9}$$

We assume that the distribution of the sources in the branches of a symmetrical network need not be symmetrical with respect to the two ends, even though the impedances of the branches are.

2.3 *Statement of Results for Infinite Symmetrical Section Line—Passive*

When the sections are passive the equations to be solved are, from (2.9),

$$\begin{aligned} v(n+1) &= Av(n) - Bi(n) \\ i(n+1) &= -Cv(n) + A'i(n) \end{aligned} \tag{2.10}$$

If the line extends from $n = 0$ to $n = \infty$ the solution is

$$\begin{aligned} v(n) &= e^{-n\Gamma}v(o) \\ i(n) &= e^{-n\Gamma'}i(o) \\ v(n) &= Z_o i(n) \end{aligned} \tag{2.11}$$

where the matrix $e^{-\Gamma}$ is such that (a) the equation

$$e^{-\Gamma} + e^{\Gamma} = 2A \tag{2.12}$$

is satisfied, e^{Γ} being the inverse of $e^{-\Gamma}$, and (b) all the elements of the matrix $e^{-n\Gamma}$ approach zero as $n \to \infty$. In dealing with sections we shall never have occasion to consider Γ itself but only its exponential and associated functions. The characteristic impedance matrix Z_o is defined by the relation between the initial currents and voltages in an infinite line

$$v(o) = Z_o i(o) \tag{2.13}$$

A formal solution of (2.12) may be obtained by writing it as

$$\cosh \Gamma = A \tag{2.14}$$

Then

$$e^{-\Gamma} = \cosh \Gamma - \sinh \Gamma$$

$$= A - (A^2 - I)^{\frac{1}{2}} = A - (BC)^{\frac{1}{2}}$$

where the square root is to be chosen so that condition (b) for $e^{-\Gamma}$ is satisfied. The characteristic impedance matrix Z_o is given by equations (2.34) of which the following two are representative.

$$Z_o = (\sinh \Gamma)^{-1} B = \sinh \Gamma \, C^{-1} \tag{2.15}$$

where $\sinh \Gamma$ is given by $2 \sinh \Gamma = e^{\Gamma} - e^{-\Gamma}$.

The wide variety of sections makes it appear unlikely that there is a general method of determining $e^{-\Gamma}$ analogous to the first method discussed for the uniform line. However, in some cases rapidly convergent series for $e^{-\Gamma}$ and e^{Γ} may be obtained. For example, suppose that the elements of $(2A)^{-1}$ are small compared to those of 2A. Then, from (2.12),

$$e^{\Gamma} = 2A - (2A)^{-1} - (2A)^{-3} - 2(2A)^{-5} - \cdots$$

$$e^{-\Gamma} = (2A)^{-1} + (2A)^{-3} + 2(2A)^{-5} + \cdots$$

Again, if $A^2 - I = BC$ is expressible as $I\gamma^2 + R$ where the elements of R are small in comparison with γ^2, we have (cf. equations (1.14), (1.15))

$$e^{\Gamma} = A + \gamma \left[I + \frac{R}{2\gamma^2} - \frac{1}{2}\left(\frac{R}{2\gamma^2}\right)^2 + \cdots \right]$$

$$e^{-\Gamma} = A - \gamma \left[I + \frac{R}{2\gamma^2} - \frac{1}{2}\left(\frac{R}{2\gamma^2}\right)^2 + \cdots \right]$$

Finally, it follows from a comparison of equations (2.11) and (A2.12) that a suitable $e^{-\Gamma}$ is given by

$$e^{-\Gamma} = P\Lambda P^{-1}, \qquad e^{-\Gamma'} = Q\Lambda Q^{-1} \tag{2.16}$$

where P, Q and Λ are the matrices designated by the same symbols in Appendices II and III.

The formal application of Sylvester's theorem leads to a method of solving the symmetrical section line which is analogous to the second method discussed for the uniform line. Thus, if $P(A)$ is any polynomial in A, then

$$P(A) = \sum_{r=1}^{m} N(\zeta_r)P(\zeta_r) \tag{2.17}$$

where $P(\zeta_r)$ is not a square matrix but a scalar and $N(\zeta_r)$ is the square matrix

$$N(\zeta_r) = \frac{F(\zeta_r)}{|f(\zeta_r)|^{(1)}}. \tag{2.18}$$

$F(\zeta)$ is the adjoint of the characteristic matrix

$$f(\zeta) = I\zeta - A \tag{2.19}$$

and $\zeta_1, \zeta_2, \cdots \zeta_m$ are the roots, assumed to be unequal, of the characteristic equation

$$|f(\zeta)| = 0.$$

The denominator in the expression for $N(\zeta_r)$ is the derivative of the characteristic function:

$$|f(\zeta_r)|^{(1)} = \left[\frac{d}{d\zeta}|f(\zeta)|\right]_{\zeta=\zeta_r}$$

The formal application of Sylvester's theorem then gives

$$\begin{aligned} \cosh \Gamma &= A = \Sigma N(\zeta_r)\zeta_r \\ e^{-\Gamma} &= A - (A^2 - I)^{\frac{1}{2}} = \Sigma N(\zeta_r)\lambda_r \\ e^{-n\Gamma} &= \Sigma N(\zeta_r)\lambda_r^n \\ Z_o &= (\sinh \Gamma)^{-1} B = \Sigma \frac{N(\zeta_r)\, 2}{(\lambda_r^{-1} - \lambda_r)} B \end{aligned} \tag{2.20}$$

where $N(\zeta_r)$ is given by (2.18), the summations run from $r = 1$ to $r = m$, and λ_r is related to ζ_r through

$$2\zeta_r = \lambda_r + \lambda_r^{-1}, \qquad \lambda_r = \zeta_r - \sqrt{\zeta_r^2 - 1} \tag{2.21}$$

where the sign of the square root is chosen so that $|\lambda_r| < 1$. λ_r is related to $e^{-\Gamma}$ in the same way that ζ_r is related to $\cosh \Gamma$.

2.4 *Results for Any Symmetrical Section Line—Passive*

The solutions of equations (2.10) which reduce to the given values $v(o)$, $i(o)$ at $n = 0$ are

$$\begin{aligned} v(n) &= \cosh n\Gamma\, v(o) - \sinh n\Gamma\, Z_o i(o) \\ i(n) &= -\sinh n\Gamma'\, Z_o^{-1}\, v(o) + \cosh n\Gamma'\, i(o) \end{aligned} \tag{2.22}$$

where $e^{-\Gamma}$ and Z_o are the matrices of §2.3. These may be put in slightly different form by using the relations

$$\sinh n\Gamma\, Z_o = Z_o \sinh n\Gamma'$$

$$\sinh n\Gamma'\, Z_o^{-1} = Z_o^{-1} \sinh n\Gamma$$

When (2.22) are interpreted by Sylvester's theorem we obtain

$$\begin{aligned} v(n) &= \Sigma N(\zeta_r)\left[\tfrac{1}{2}(\lambda_r^{-n} + \lambda_r^{n})v(o) - \frac{\lambda_r^{-n} - \lambda_r^{n}}{\lambda_r^{-1} - \lambda_r} Bi(o)\right] \\ i(n) &= \Sigma N'(\zeta_r)\left[-\frac{\lambda_r^{-n} - \lambda_r^{n}}{\lambda_r^{-1} - \lambda_r} Cv(o) + \tfrac{1}{2}(\lambda_r^{-n} + \lambda_r^{n})i(o)\right] \end{aligned} \tag{2.23}$$

where $N'(\zeta_r)$ is the transposed of $N(\zeta_r)$ and $N(\zeta_r)$ is given by (2.18) and the summations run from $r = 1$ to $r = m$.

2.5 Results for Any Symmetrical Section Line—Active

When the sections contain generators the equations to be solved are those of (2.9). The solutions corresponding to the initial values $v(o)$ and $i(o)$ are, for $n \geqq 1$,

$$\begin{aligned} v(n) &= \cosh n\Gamma\, v(o) - \sinh n\Gamma\, Z_o i(o) \\ &\quad + \sum_{p=1}^{n} \{\cosh (n-p)\Gamma\, Bi^\circ(p-1) - \sinh (n-p)\Gamma\, Z_o Cv^\circ(p-1)\} \\ i(n) &= -\sinh n\Gamma'\, Z_o^{-1} v(o) + \cosh n\Gamma'\, i(o) \\ &\quad + \sum_{p=1}^{n} \{\cosh (n-p)\Gamma'\, Cv^\circ(p-1) - \sinh (n-p)\Gamma'\, Z_o^{-1} Bi^\circ(p-1)\} \end{aligned} \tag{2.24}$$

These may be simplified somewhat by replacing Z_oC and $Z_o^{-1}B$ by $\sinh \Gamma$ and $\sinh \Gamma'$, respectively.

The series in the above expressions may be summed when the generators are such that

$$v^\circ(n) = e^{-n\theta} i^\circ, \qquad i^\circ(n) = e^{-n\theta} v^\circ \tag{2.25}$$

where θ is a scalar and i° and v° are column matrices whose elements are independent of n. Thus

$$\begin{aligned} v(n) &= \cosh n\Gamma\, v(o) - \sinh n\Gamma\, Z_o i(o) \\ &\quad + \tfrac{1}{2}(e^{n\Gamma} - e^{-n\theta}I)(e^{\Gamma} - e^{-\theta}I)^{-1}(Bi^\circ - Z_oCv^\circ) \\ &\quad + \tfrac{1}{2}(e^{-n\Gamma} - e^{-n\theta}I)(e^{-\Gamma} - e^{-\theta}I)^{-1}(Bi^\circ + Z_oCv^\circ) \\ i(n) &= -\sinh n\Gamma'\, Z_o^{-1} v(o) + \cosh n\Gamma'\, i(o) \\ &\quad + \tfrac{1}{2}(e^{n\Gamma'} - e^{-n\theta}I)(e^{\Gamma'} - e^{-\theta}I)^{-1}(Cv^\circ - Z_o^{-1}Bi^\circ) \\ &\quad + \tfrac{1}{2}(e^{-n\Gamma'} - e^{-n\theta}I)(e^{-\Gamma'} - e^{-\theta}I)^{-1}(Cv^\circ + Z_o^{-1}Bi^\circ) \end{aligned} \tag{2.26}$$

provided that the inverse matrices exist.

We may interpret these expressions by Sylvester's theorem. For example,

$$v(n) = \sum_{r=1}^{m} N(\zeta_r)\Bigg[\tfrac{1}{2}(\lambda_r^{-n} + \lambda_r^{n})v(o) - \frac{\lambda_r^{-n} - \lambda_r^{n}}{\lambda_r^{-1} - \lambda_r} Bi(o) + \frac{1}{2}\frac{\lambda_r^{-n} - e^{-n\theta}}{\lambda_r^{-1} - e^{-\theta}}\left(Bi^\circ - \frac{2BCv_o}{\lambda_r^{-1} - \lambda_r}\right) + \frac{1}{2}\frac{\lambda_r^{n} - e^{-n\theta}}{\lambda_r - e^{-\theta}}\left(Bi^\circ + \frac{2BCv_o}{\lambda_r^{-1} - \lambda_r}\right)\Bigg] \tag{2.27}$$

where $N(\zeta_r)$ is given by (2.18).

When the line extends to $n = \infty$ and the sources and end conditions satisfy suitable conditions we have the relation

$$v(o) = Z_o i(o) - \sum_{p=1}^{\infty} e^{-p\Gamma}[Bi^\circ(p-1) - Z_o Cv^\circ(p-1)] \tag{2.28}$$

When the impressed field is of the form (2.25) this becomes

$$v(o) = Z_o i(o) - (e^{\Gamma} - e^{-\theta}I)^{-1}(Bi^\circ - Z_o Cv^\circ) \tag{2.29}$$

provided that the inverse matrix exists. Expressions for $v(n)$ and $i(n)$ in such an infinite line may be obtained by using (2.28) or (2.29) in (2.24) or (2.26).

Applying Sylvester's theorem to (2.29) gives

$$v(o) = \sum_{r=1}^{m} N(\zeta_r)\left(\frac{2Bi(o)}{\lambda_r^{-1} - \lambda_r} - \frac{Bi^\circ}{\lambda_r^{-1} - e^{-\theta}} + \frac{\lambda_r^{-1} - \lambda_r}{2(\lambda_r^{-1} - e^{-\theta})} v^\circ\right) \tag{2.30}$$

The last term within the braces may be replaced by

$$\frac{2BCv^\circ}{(\lambda_r^{-1} - \lambda_r)(\lambda_r^{-1} - e^{-\theta})}$$

2.6 *Derivation of the Properties of an Infinite Line*

We shall consider a symmetrical section line which is specified by the equations

$$\begin{aligned} v(n+1) &= Av(n) - Bi(n) \\ i(n+1) &= -Cv(n) + A'i(n) \end{aligned} \tag{2.10}$$

From these equations and the relations $A^2 - BC = I$, $AB = BA'$, $A'C = CA$ of (A4.6) it follows that

$$\begin{aligned} v(n+1) + v(n-1) &= 2Av(n) \\ i(n+1) + i(n-1) &= 2A'i(n) \end{aligned} \tag{2.31}$$

If $e^{-\Gamma}$ is a matrix satisfying the conditions of §2.3, namely, (a) $e^{-\Gamma}$ satisfies the equation

$$2 \cosh \Gamma = e^{\Gamma} + e^{-\Gamma} = 2A \tag{2.14}$$

and (b), every element in $e^{-n\Gamma}$ approaches zero as $n \to \infty$, and if Z_o is defined by $v(o)$ and $i(o)$ for an infinite line as in (2.13), then

1. In an infinite line

$$\begin{aligned} v(n) &= e^{-n\Gamma} v(o), \\ i(n) &= e^{-n\Gamma'} i(o), \end{aligned} \tag{2.32}$$

$$v(n) = Z_o i(n) \tag{2.33}$$

2. The characteristic impedance matrix Z_o is given by

$$\begin{aligned} Z_o &= (\sinh \Gamma)^{-1} B = B(\sinh \Gamma')^{-1} = C^{-1} \sinh \Gamma' = \sinh \Gamma \, C^{-1} \\ Z_o^{-1} &= B^{-1} \sinh \Gamma = \sinh \Gamma' \, B^{-1} = (\sinh \Gamma')^{-1} C = C(\sinh \Gamma)^{-1} \end{aligned} \tag{2.34}$$

3. The matrices Z_o, B and C obey the commutation rules

$$\begin{aligned} \Phi(e^{\Gamma}) Z_o &= Z_o \Phi(e^{\Gamma'}) \\ \Phi(e^{\Gamma}) B &= B \Phi(e^{\Gamma'}) \\ C \Phi(e^{\Gamma}) &= \Phi(e^{\Gamma'}) C \end{aligned} \tag{2.35}$$

where $\Phi(e^{\Gamma})$ is a square matrix representable as a sum of powers of $e^{\pm\Gamma}$. The matrices $\Phi(e^{\Gamma}) Z_0$, $\Phi(e^{\Gamma}) B$, and $C\Phi(e^{\Gamma})$ are symmetrical.

To prove these statements we proceed as follows: By direct substitution into (2.31) it is seen that $v(n) = e^{-n\Gamma} v(o)$ is a solution by virtue of condition (a) satisfied by $e^{-\Gamma}$. Since, by condition (b), $v(n) \to o$ as $n \to \infty$ it follows that $v(n)$ is the voltage in an infinite line. Similarly, $i(n) = e^{-n\Gamma'} i(o)$ is the current in such a line. Substituting the expressions (2.32) for $v(n)$ and $i(n)$ into the difference equations (2.10), setting $n = 0$, using the definition of Z_o, and regarding $v(o)$ and $i(o)$ as arbitrary columns gives

$$\begin{aligned} e^{-\Gamma} &= A - B Z_o^{-1} \\ e^{-\Gamma'} &= -C Z_o + A' \end{aligned} \tag{2.36}$$

Applying condition (a) in the form of (2.14) to these equations gives

$$B Z_o^{-1} = \sinh \Gamma, \qquad C Z_o = \sinh \Gamma' \tag{2.37}$$

Since the sections are symmetrical, B and C are symmetrical matrices, and from the reciprocal theorem for networks it follows that Z_o is also symmetrical. These remarks and (2.37) lead to (2.34). Setting the expressions (2.32) for $v(n)$ and $i(n)$ in the second of the difference equations (2.10) using the definition of Z_o, and regarding $i(o)$ as an arbitrary column gives

$$e^{-(n+1)\Gamma'} = -Ce^{-n\Gamma}Z_o + A'e^{-n\Gamma'}$$

$$(A' - e^{-\Gamma'})e^{-n\Gamma'} = Ce^{-n\Gamma}Z_o$$

Replacing $A' - e^{-\Gamma'}$ by CZ_o, as follows from the case $n = 0$, and pre-multiplying by C^{-1} gives

$$Z_o e^{-n\Gamma'} = e^{-n\Gamma}Z_o$$

and this leads to the first of equations (2.35). From (2.34) and the relations $AB = BA'$, $CA = A'C$ we have

$$\sinh \Gamma\, B = B \sinh \Gamma' \qquad \cosh \Gamma\, B = B \cosh \Gamma'$$

$$C \sinh \Gamma = \sinh \Gamma'\, C \qquad C \cosh \Gamma = \cosh \Gamma'\, C$$

Addition and subtraction leads to

$$e^{\pm\Gamma}B = Be^{\pm\Gamma'} \qquad Ce^{\pm\Gamma} = e^{\pm\Gamma'}C$$

from which the last two of equations (2.35) follow. Since each of equations (2.35) expresses the equality of a matrix and its transposed, it follows that the matrices are symmetrical.

Equation (2.33), which is almost self-evident on physical grounds, follows from

$$v(n) = e^{-n\Gamma}v(o) = e^{-n\Gamma}Z_o i(o)$$

$$= Z_o e^{-n\Gamma'}i(o) = Z_o i(n).$$

2.7 *Proof of Relations for Any Symmetrical Section Line*

The expressions (2.24) for $v(n)$ and $i(n)$ in a line whose sections contain generators may be verified to satisfy the difference equations (2.9). The expressions (2.34) for Z_o and the commutation rules (2.35) for B and C are used in the verification. Setting $n = 1$ in the expressions for $v(n)$ and $i(n)$ gives the difference equations (2.9) and hence $v(n)$ and $i(n)$ are the solutions which correspond to the initial values $v(o)$ and $i(o)$.

In order to derive the relation (2.28) between $v(o)$ and $i(o)$ for an infinite line we put the hyperbolic functions in the expression (2.24) for $v(n)$ in exponential form and multiply through by $2e^{-n\Gamma}$

$$2e^{-n\Gamma}v(n) = v(o) - Z_o i(o) + \sum_{p=1}^{n} e^{-p\Gamma}[Bi^\circ(p-1) - Z_o Cv^\circ(p-1)]$$

$$+ e^{-2n\Gamma}[v(o) + Z_o i(o)]$$

$$+ e^{-n\Gamma}\sum_{p=1}^{n} e^{-(n-p)\Gamma}[Bi^\circ(p-1) + Z_o Cv^\circ(p-1)]$$

Hence, letting $n \to \infty$ and using condition (b) satisfied by $e^{-\Gamma}$, equation (2.28) is obtained provided that (i) the terminal conditions at the far end are such that $v(n)$ remains finite, (ii) the sum in (2.28) converges, and (iii) the expression in the last line in the equation just above approaches zero as $n \to \infty$.

The results obtained by the formal application of Sylvester's theorem may be verified by using the results of Appendix II and writing $N(\zeta_r)$ as the product of a column matrix and a row matrix. They may also be verified more directly. For example, setting $n = 0$ in the expressions (2.23) for $v(n)$ and $i(n)$ in any passive symmetrical section line and using

$$\sum_{r=1}^{m} N(\zeta_r) = I, \tag{2.38}$$

which follows from Sylvester's theorem, we see that $v(n)$ and $i(n)$ reduce to the appropriate values $v(o)$ and $i(o)$ at $n = 0$. Substituting $v(n)$ and $i(n)$ into the difference equations (2.10) and using

$$\begin{aligned} BC &= A^2 - I \\ (I\zeta_r - A)N(\zeta_r) &= N(\zeta_r)(I\zeta_r - A) = 0 \\ BN'(\zeta) &= N(\zeta)B \\ CN(\zeta) &= N'(\zeta)C, \end{aligned} \tag{2.39}$$

shows that they are solutions. The second of the relations (2.39) follows from the fact that $N(\zeta_r)$ is proportional to the adjoint $F(\zeta_r)$ of $f(\zeta_r)$. In the third and fourth relations

$$N(\zeta) = \frac{F(\zeta)}{|f(\zeta)|^{(1)}}$$

which is in agreement with the definition (2.18) of $N(\zeta_r)$. To establish the third relation we start from,[10]

$$\begin{aligned} (\zeta I - A)\, F(\zeta) &= I\, |f(\zeta)| \\ (\zeta I - A)\, N(\zeta) &= I\, |f(\zeta)|/|f(\zeta)|^{(1)} \end{aligned}$$

Postmultiplication by B gives

$$(\zeta I - A)\, N(\zeta)\, B = B\, |f(\zeta)|/|f(\zeta)|^{(1)}$$

We also have

$$\begin{aligned} (\zeta I - A')F'(\zeta) &= I\, |f(\zeta)| \\ (\zeta I - A')N'(\zeta) &= I\, |f(\zeta)|/|f(\zeta)|^{(1)} \end{aligned}$$

[10] F.D.C. §3.5.

Premultiplication by B and use of $BA' = AB$ gives

$$(\zeta I - A)BN'(\zeta) = B \mid f(\zeta) \mid / \mid f(\zeta) \mid^{(1)}$$

Hence, the third equation in (2.39) holds except possibly for $\zeta = \zeta_r$, and from the concept of continuity it holds there also. The fourth equation in (2.39) may be proved in the same manner.

The expression (2.20) for Z_o may be obtained by letting n become very large in the expression (2.23) for $v(n)$. $v(o)$ and $i(o)$ must be related so that $v(n)$ remains finite. Since $\mid \lambda_r \mid < 1$ and the λ_r's are unequal the coefficients of λ_r^{-n} must vanish. This requires

$$N(\zeta_r)v(o) = \frac{2N(\zeta_r)Bi(o)}{\lambda_r^{-1} - \lambda r}$$

Summing r from 1 to m and using (2.38) gives the required expression for Z_o.

2.8 The Unsymmetrical Section Line

The method used here is analogous to those described in Appendices I and II for the uniform line and the symmetrical section line. The other methods apparently do not lead to the simplification which occurs in the symmetrical case.

Equations (2.2) and (2.1) lead to the difference equations

$$Y_{12}v(n + 2) + [Y_{11} + Y_{22}]v(n + 1) + Y_{21}v(n) = -i^{\circ}(n + 1) + j^{\circ}(n) \quad (2.40)$$

$$Z_{12}i(n + 2) - [Z_{11} + Z_{22}]i(n + 1) + Z_{21}i(n) = v^{\circ}(n + 1) - u^{\circ}(n) \quad (2.41)$$

Both of these equations are of the form

$$Gx(n + 2) + Hx(n + 1) + G'x(n) = g(n) \quad (2.42)$$

in which G and H are square matrices of order m, H being symmetrical and G' being the transposed of G. When the sections are passive equations (2.40) and (2.41) become

$$Y_{12}v(n + 2) + [Y_{11} + Y_{22}]v(n + 1) + Y_{21}v(n) = 0 \quad (2.43)$$

$$Z_{12}i(n + 2) - [Z_{11} + Z_{22}]i(n + 1) + Z_{21}i(n) = 0 \quad (2.44)$$

In the passive, unsymmetrical case the expressions for $v(n)$ and $i(n)$ are of the form

$$\begin{aligned} v(n) &= P\Lambda^n a + \bar{P}\Lambda^{-n}\bar{a} \\ i(n) &= Q\Lambda^n a - \bar{Q}\Lambda^{-n}\bar{a} \end{aligned} \quad (2.45)$$

Comparison with (A2.8) shows that in the symmetrical case $\bar{P} = P$ and $\bar{Q} = Q$. The minus signs over $\bar{P}$, $\bar{Q}$, and $\bar{a}$ indicate that they are associated with propagation in the negative direction. The propagation constants of the m modes of propagation are the same in the positive as in the negative direction, as indicated by the appearance of Λ^n and Λ^{-n} in (2.45). Corresponding to any given propagation constant say λ_r, there are two modes of propagation, one in a positive direction and the other in the negative direction. The distribution of the voltages corresponding to these two modes are given by the rth columns in P and $\bar{P}$, respectively. The fact that P and $\bar{P}$ differ shows that the distributions differ according to the direction of propagation even though the propagation constant is the same. Λ is still the diagonal matrix defined in (A2.3) but now the computation of the elements λ_r is more difficult than when the section is symmetrical. They are defined as the roots of the equation

$$| G\lambda^2 + H\lambda + G' | = 0 \tag{2.46}$$

which are less than unity in absolute value. The second of the equations (A4.5) shows that the roots of (2.46) are the same whether the Z's or the Y's are used in place of G and H. Of course, this is to be expected on physical grounds. The third of the equations (A4.5) may be used to show that the roots of (2.46) are also the roots of

$$\begin{vmatrix} \lambda A - I & \lambda B \\ \lambda C & \lambda D - I \end{vmatrix} = 0 \tag{2.47}$$

From the form of (2.46) it follows that if λ_r is a root so is λ_r^{-1}. This fact may be used to simplify the determination of λ_r. When the substitution

$$2\zeta = \lambda + \lambda^{-1}, \qquad \lambda = \zeta - \sqrt{\zeta^2 - 1} \tag{A2.4}$$

is made equation (2.46) may be written as

$$0 = | (G + G')\zeta + H + (G' - G)\sqrt{\zeta^2 - 1} |$$

$$0 = | (G + G')\zeta + H |$$

$+ (\zeta^2 - 1)$ times the sum of $\dfrac{m(m-1)}{2!}$ determinants each obtained by replacing two columns of $|(G + G')\zeta + H|$ by the corresponding columns of $(G - G')$

$+ (\zeta^2 - 1)^2$ times the sum of $\dfrac{m(m-1)(m-2)(m-3)}{4!}$ determinants each obtained by replacing four columns of $|(G + G')\zeta + H|$ by the corresponding columns of $(G - G')$

$+ \cdots$

The last equation is a polynomial of degree m in ζ which is to be solved for its roots $\zeta_1, \zeta_2, \cdots \zeta_m$. For simplicity we assume that these roots are distinct. λ_r is then determined from ζ_r by the relations (A2.4), the sign of the radical being chosen so that $|\lambda_r| < 1$ as in the symmetrical case. In his second paper Koizumi has given a procedure which amounts to an alternative method of determining Λ.

We shall first assume that the Y's are known and that our equations are

$$\begin{aligned} i(n) &= Y_{11}v(n) + Y_{12}v(n+1) \\ -i(n+1) &= Y_{21}v(n) + Y_{22}v(n+1) \end{aligned} \tag{2.48}$$

As described above Λ may be computed from the determinantal equation

$$|f(\lambda)| = 0$$

where $f(\lambda)$ represents the matrix

$$f(\lambda) = Y_{12}\lambda^2 + (Y_{11} + Y_{22})\lambda + Y_{21} \tag{2.49}$$

Let p_r be proportional to any non-zero column in $F(\lambda_r)$ where $F(\lambda)$ is the adjoint of $f(\lambda)$ and let $\bar{p}_r'$ be proportional to any non-zero row of $F(\lambda_r)$. Then the matrices P and $\bar{P}$ in the expressions (2.45) for $v(n)$ and $i(n)$ are given by

$$\begin{aligned} P &= [p_1, p_2, \cdots p_m] \\ \bar{P} &= [\bar{p}_1, \bar{p}_2, \cdots \bar{p}_m] \end{aligned} \tag{2.50}$$

where $\bar{p}_r$ is the column obtained by transposing the row $\bar{p}_r'$. The matrices Q and $\bar{Q}$ are obtained from P and $\bar{P}$ by means of the equations

$$\begin{aligned} Q &= Y_{11}P + Y_{12}P\Lambda = -Y_{22}P - Y_{21}P\Lambda^{-1} \\ \bar{Q} &= -Y_{11}\bar{P} - Y_{12}\bar{P}\Lambda^{-1} = Y_{22}\bar{P} + Y_{21}\bar{P}\Lambda \end{aligned} \tag{2.51}$$

which are derived from (2.45) and (2.48).

The properties of the individual columns of P and $\bar{P}$ lead to the relations

$$\begin{aligned} Y_{12}P\Lambda^2 + (Y_{11} + Y_{22})P\Lambda + Y_{21}P &= 0 \\ Y_{12}\bar{P}\Lambda^{-2} + (Y_{11} + Y_{22})\bar{P}\Lambda^{-1} + Y_{21}\bar{P} &= 0 \end{aligned} \tag{2.52}$$

and these guarantee that the difference equations (2.48) will be satisfied when the expressions (2.45) for $v(n)$ and $i(n)$ are used.

When the Z's are known instead of the Y's the procedure is much the same. The difference equations are

$$\begin{aligned} v(n) &= Z_{11}i(n) - Z_{12}i(n+1) \\ v(n+1) &= Z_{21}i(n) - Z_{22}i(n+1) \end{aligned} \tag{2.53}$$

and the equation to determine the λ_r's is

$$|f(\lambda)| = 0$$

where now $f(\lambda)$ represents the matrix

$$f(\lambda) = Z_{12}\lambda^2 - (Z_{11} + Z_{22})\lambda + Z_{21} \tag{2.54}$$

Let q_r be proportional to any non-zero column in $F(\lambda_r)$ where $F(\lambda)$ is the adjoint of $f(\lambda)$ and let $\bar{q}_r'$ be proportional to any non-zero row of $F(\lambda_r)$. The matrices Q and $\bar{Q}$ in the expressions (2.45) for $v(n)$ and $i(n)$ are given by

$$\begin{aligned} Q &= [q_1, q_2, \cdots q_m] \\ \bar{Q} &= [\bar{q}_1, \bar{q}_2, \cdots \bar{q}_m] \end{aligned} \tag{2.55}$$

where $\bar{q}_r$ is the column obtained by transposing the row $\bar{q}_r'$. From (2.45) and (2.53) equations for P and $\bar{P}$ in terms of Q and $\bar{Q}$ are obtained:

$$\begin{aligned} P &= Z_{11}Q - Z_{12}Q\Lambda = -Z_{22}Q + Z_{21}Q\Lambda^{-1} \\ \bar{P} &= -Z_{11}\bar{Q} + Z_{12}\bar{Q}\Lambda^{-1} = Z_{22}\bar{Q} - Z_{21}\bar{Q}\Lambda \end{aligned} \tag{2.56}$$

The difference equations (2.53) are satisfied by our expressions for $v(n)$ and $i(n)$ by virtue of the relations

$$\begin{aligned} Z_{12}Q\Lambda^2 - (Z_{11} + Z_{22})Q\Lambda + Z_{21}Q = 0 \\ Z_{12}\bar{Q}\Lambda^{-2} - (Z_{11} + Z_{22})\bar{Q}\Lambda^{-1} + Z_{21}\bar{Q} = 0 \end{aligned} \tag{2.57}$$

which are a consequence of the properties of the individual columns of Q and $\bar{Q}$.

If the system extends to $n = +\infty$ and if the voltages and currents are to remain finite at $n = \infty$ the elements of $\bar{a}$ must be zero and the expressions (2.45) for $v(n)$ and $i(n)$ become

$$\begin{aligned} v(n) &= P\Lambda^n a = P\Lambda^n P^{-1} v(o) \\ i(n) &= Q\Lambda^n a = Q\Lambda^n Q^{-1} i(o) \\ v(n) &= PQ^{-1} i(n), \qquad i(n) = QP^{-1} v(n) \end{aligned} \tag{2.58}$$

where we have assumed that P^{-1} and Q^{-1} exist. We accordingly introduce the characteristic impedance and admittance matrices Z_o and Y_o associated with propagation in the positive direction, i.e., in the direction of n increasing.

$$\begin{aligned} v(n) = Z_o i(n), \qquad i(n) &= Y_o v(n), \qquad Z_o = Y_o^{-1} \\ Z_o = PQ^{-1} = Z_{11} - Z_{12}Q\Lambda Q^{-1} &= -Z_{22} + Z_{21}Q\Lambda^{-1}Q^{-1} \\ Y_o = QP^{-1} = Y_{11} + Y_{12}P\Lambda P^{-1} &= -Y_{22} - Y_{21}P\Lambda^{-1}P^{-1} \end{aligned} \tag{2.59}$$

Incidentally, since Z_o must be a symmetrical matrix the above equations show that $Z_{12}Q\Lambda Q^{-1}$ and $Z_{21}Q\Lambda^{-1}Q^{-1}$ are symmetrical. Z_o and Y_o satisfy the relations

$$Z_oCZ_o + Z_oD - AZ_o - B = 0 \qquad Y_oBY_o + Y_oA - DY_o - C = 0$$
$$(Z_{22} + Z_o)Z_{12}^{-1}(Z_{11} - Z_o) = Z_{21}, \qquad (Y_{22} + Y_o)Y_{12}^{-1}(Y_{11} - Y_o) = Y_{21} \quad (2.60)$$
$$Z_oQ\Lambda Q^{-1} = P\Lambda P^{-1}Z_o \qquad Y_oP\Lambda P^{-1} = Q\Lambda Q^{-1}Y_o$$

The characteristic and admittance matrices $\bar{Z}_o$ and $\bar{Y}_o$ associated with propagation in the negative direction are introduced in a similar way. Suppose the system extends to $n = -\infty$. Then $a = o$ and

$$\begin{aligned} v(n) &= \bar{P}\Lambda^{-n}\bar{a} = \bar{P}\Lambda^{-n}\bar{P}^{-1}v(o) \\ i(n) &= \bar{Q}\Lambda^{-n}\bar{a} = -\bar{Q}\Lambda^{-n}\bar{Q}^{-1}i(o) \\ v(n) &= -\bar{P}\bar{Q}^{-1}i(n), \qquad i(n) = -\bar{Q}\bar{P}^{-1}v(n) \end{aligned} \quad (2.61)$$

Hence we write

$$v(n) = -\bar{Z}_oi(n), \qquad i(n) = -\bar{Y}_ov(n)$$
$$\bar{Z}_o = \bar{P}\bar{Q}^{-1} = -Z_{11} + Z_{12}\bar{Q}\Lambda^{-1}\bar{Q}^{-1} = Z_{22} - Z_{21}\bar{Q}\Lambda\bar{Q}^{-1} \quad (2.62)$$
$$\bar{Y} = \bar{Q}\bar{P}^{-1} = -Y_{11} - Y_{12}\bar{P}\Lambda^{-1}\bar{P}^{-1} = Y_{22} + Y_{21}\bar{P}\Lambda\bar{P}^{-1}$$

$\bar{Z}_o$ and $\bar{Y}_o$ satisfy the relations

$$\begin{aligned} &\bar{Z}_oC\bar{Z}_o - \bar{Z}_oD + A\bar{Z}_o - B = 0 \qquad \bar{Y}_oB\bar{Y}_o - \bar{Y}_oA + D\bar{Y}_o - C = 0 \\ &(Z_{11} + \bar{Z}_o)Z_{21}^{-1}(Z_{22} - \bar{Z}_o) = Z_{12} \qquad (Y_{11} + \bar{Y}_o)Y_{21}^{-1}(Y_{22} - \bar{Y}_o) = Y_{12} \end{aligned} \quad (2.63)$$

The fact that $Q'(Z_o + \bar{Z}_o)\bar{Q} = P'(Y_o + \bar{Y}_o)\bar{P}$ is a diagonal matrix may be used as a check on computations.

When the expressions (2.45) for $v(n)$ and $i(n)$ are placed in (2.3), $j^\circ(n)$ and $u^\circ(n)$ being zero, we obtain the relations

$$\begin{aligned} P\Lambda^{-1} &= AP + BQ \qquad & P\bar{\Lambda} &= A\bar{P} - B\bar{Q} \\ Q\Lambda^{-1} &= CP + DQ \qquad & \bar{Q}\Lambda &= -C\bar{P} + D\bar{Q} \end{aligned} \quad (2.64)$$

When the typical section contains generators the difference equation to be solved is of the form (2.42)

$$Gx(n + 2) + Hx(n + 1) + G'x(n) = g(n) \quad (2.42)$$

This is true for symmetrical as well as unsymmetrical sections, G being a symmetrical matrix in the former case so that $G' = G$. The expressions for $v(n)$ and $i(n)$ are those of (2.45) with the particular solutions added:

$$\begin{aligned} v(n) &= P\Lambda^n a + \bar{P}\Lambda^{-n}\bar{a} + u(n) \\ i(n) &= Q\Lambda^n a - \bar{Q}\Lambda^{-n}\bar{a} + j(n) \end{aligned} \quad (2.65)$$

where P, $\overline{P}$, Q, $\bar{Q}$ are determined as before and $u(n)$ and $j(n)$ depend upon the generators.

Here we shall consider only the physically important case in which the voltages of the generators in the nth section are proportional to $e^{-n\theta}$ where θ is a constant. In this case $g(n)$ may be expressed as

$$g(n) = ge^{-n\theta} \tag{2.66}$$

where g is a column matrix whose elements are independent of n. A particular solution is obtained by assuming

$$x(n) = ye^{-n\theta}$$

Setting this in (2.42) gives

$$(Ge^{-2\theta} + He^{-\theta} + G')y = g$$

Hence a particular solution is

$$x(n) = (Ge^{-2\theta} + He^{-\theta} + G')^{-1}ge^{-n\theta} \tag{2.67}$$

This method fails when θ is equal to one of the roots $\lambda_1, \cdots \lambda_m, \lambda_1^{-1}, \cdots \lambda_m^{-1}$. In this case, a particular integral may be obtained by a method similar to one described in §5.11 of F.D.C.

APPENDIX I

Classical Solution of Uniform Transmission Line Equations

For the sake of convenience we again assume that there are three circuits in the transmission line. The equations to be solved are:

$$\frac{dv}{dx} = -Zi, \qquad \frac{di}{dx} = -Yi \tag{1.48}$$

We adopt here the notation associated with equations (1.19) and (1.20), $f(\gamma^2)$ being the characteristic matrix of ZY, $F(\gamma^2)$ the adjoint of $f(\gamma^2)$, and $\gamma_1^2, \gamma_2^2, \gamma_3^2 (m = 3)$ being the roots, supposed distinct, of $|f(\gamma^2)| = 0$. The propagation constants $\gamma_1, \gamma_2, \gamma_3$ are those square roots of $\gamma_1^2, \gamma_2^2, \gamma_3^2$ which in physical systems have a positive real part.

The solution may be constructed[11] as follows: Let the column p_r be proportional (with any convenient constant of proportionality) to any non-zero column of $F(\gamma_r^2)$. The non-zero columns of $F(\gamma_r^2)$ are proportional to each other according to a theorem in matrix algebra.[12] Construct the square matrix P from the columns p_1, p_2, p_3:

$$P = [p_1, p_2, p_3] \tag{A1.1}$$

[11] The method is that described in F.D.C. §5.7(a) and §5.10

[12] F.D.C. §3.5 Theorem D.

and obtain the square matrix Q from P:

$$Q = Z^{-1}PG = YPG^{-1} \tag{A1.2}$$

where G is the diagonal matrix

$$G = \begin{bmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{bmatrix} \tag{A1.3}$$

The voltages and currents at any point x are

$$\begin{aligned} v(x) &= PM(x)a + PM(-x)\bar{a} \\ i(x) &= QM(x)a - QM(-x)\bar{a} \end{aligned} \tag{A1.4}$$

where a and $\bar{a}$ are arbitrary column matrices associated with propagation in the positive and negative directions of x and $M(x)$ is the diagonal matrix

$$M(x) = \begin{bmatrix} e^{-\gamma_1 x} & 0 & 0 \\ 0 & e^{-\gamma_2 x} & 0 \\ 0 & 0 & e^{-\gamma_3 x} \end{bmatrix} \tag{A1.5}$$

The values of a and $\bar{a}$ are to be determined from the boundary conditions. When the line extends to $x = \infty$

$$\begin{aligned} v(x) &= PM(x)P^{-1}v(o) = Z_0 i(x) \\ i(x) &= QM(x)Q^{-1}i(o) \end{aligned} \tag{A1.6}$$

where the characteristic impedance matrix Z_0 is given by

$$\begin{aligned} Z_0 = PQ^{-1} &= PG^{-1}P^{-1}Z = PGP^{-1}Y^{-1} \\ &= ZQG^{-1}Q^{-1} = Y^{-1}QGQ^{-1} \end{aligned} \tag{A1.7}$$

Since $v = p_r e^{\gamma_r x}$ and $i = q_r e^{\gamma_r x}$, where q_r is the rth column of Q, are solutions the differential equations give

$$(I\gamma_r^2 - ZY)p_r = 0, \qquad (I\gamma_r^2 - YZ)q_r = 0 \tag{A1.8}$$

and from these it follows that

$$P^{-1}ZYP = Q^{-1}YZQ = G^2 \tag{A1.9}$$

The relations (A1.8) may be used to prove the following:

1. The elements in the rth column of Q are proportional to those in the non-zero rows of $F(\gamma_r^2)$.
2. The matrix $P'Q$ is a diagonal matrix and from this it follows that if ψ is any diagonal matrix

$$(P\psi P^{-1})' = Q\psi Q^{-1} \tag{A1.10}$$

3. The characteristic impedance matrix Z_0 satisfies the relation

$$Z = Z_0 Y Z_0 \tag{A1.11}$$

4. The inverse matrices P^{-1} and Q^{-1} always exist if $\gamma_1, \gamma_2, \gamma_3$ are distinct.

APPENDIX II

Classical Solution of Symmetrical Section Line Equations—I

The method of this section is very similar to that of Appendix I. The equations to be solved are (2.10) or one of the sets

$$\begin{aligned} v(n) &= Z_{11}i(n) - Z_{12}i(n+1) \\ v(n+1) &= Z_{12}i(n) - Z_{11}i(n+1) \end{aligned} \tag{A2.1}$$

$$\begin{aligned} i(n) &= Y_{11}v(n) + Y_{12}v(n+1) \\ -i(n+1) &= Y_{12}v(n) + Y_{11}v(n+1) \end{aligned} \tag{A2.2}$$

which are obtained from (2.5) and (2.6). We shall use the notation associated with equation (2.19), $f(\zeta)$ being the characteristic matrix of A, $F(\zeta)$ the adjoint of $f(\zeta)$, and $\zeta_1, \zeta_2, \cdots \zeta_m$ the roots, assumed unequal, of the characteristic equation $|f(\zeta)| = 0$. The diagonal matrices Λ and Σ are defined by

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \\ 0 & & & \lambda_m \end{bmatrix},$$

$$\Sigma = \begin{bmatrix} \sqrt{\zeta_1^2 - 1} & 0 & \cdots\cdots & 0 \\ 0 & \sqrt{\zeta_2^2 - 1} & & \\ 0 & & & \sqrt{\zeta_m^2 - 1} \end{bmatrix} \tag{A2.3}$$

where

$$2\zeta_r = \lambda_r + \lambda_r^{-1}, \qquad \lambda_r = \zeta_r - \sqrt{\zeta_r^2 - 1} = \frac{1}{\zeta_r + \sqrt{\zeta_r^2 - 1}} \tag{A2.4}$$

In general, electrical energy will be dissipated in the typical section and from the physical significance of λ_r, as seen from equations (A2.8) below, it follows that the sign of the radical in (A2.4) may be chosen so that $|\lambda_r| < 1$. Since $\sqrt{\zeta_r^2 - 1} = \zeta_r - \lambda_r = \frac{1}{2}(\lambda_r^{-1} - \lambda_r)$ it follows that

$$\Sigma = \tfrac{1}{2}(\Lambda^{-1} - \Lambda) \tag{A2.5}$$

Let the column matrix s_r be proportional to any non-zero column in $F(\zeta_r)$ where $F(\zeta)$ is the adjoint of $f(\zeta)$. (It follows from the theory of matrices that the non-zero columns of $F(\zeta_r)$ differ from each other only by a multiplying factor.) The matrix S is then formed by taking s_1 to be the first column, s_2 the second and so on.

$$S = [s_1 , s_2 , \cdots s_m] \tag{A2.6}$$

Similarly let the row matrix t_r' be proportional to any nonvanishing row of $F(\zeta_r)$ and form the matrix T where

$$T = [t_1 , t_2 , \cdots t_m] \tag{A2.7}$$

in which t_r is the column matrix obtained by transposing t_r'.[13]

Solving our difference equations for the passive case by the customary method gives the expressions

$$\begin{aligned} v(n) &= P\Lambda^n a + P\Lambda^{-n}\bar{a} \\ i(n) &= Q\Lambda^n a - Q\Lambda^{-n}\bar{a} \end{aligned} \tag{A2.8}$$

for the voltages and the currents. P and Q are square matrices and a and $\bar{a}$ are column matrices whose elements are determined by the boundary conditions. a and $\bar{a}$ are of the same nature as constants of integration. The minus sign over $\bar{a}$ indicates that it is associated with propagation in the negative direction, i.e., in the direction of n decreasing.

P and Q may be chosen in a number of ways, each choice requiring different values of a and $\bar{a}$ to represent the same system. In all cases, however, the rth column of P may be expressed as $\alpha_r s_r$ where α_r is a scalar multiplier which may depend upon r. Similarly the rth column of Q may be expressed as $\beta_r t_r$. When either P or Q has been chosen the other one is fixed since equations (A2.2) and (A2.1) require

$$\begin{aligned} Q &= Y_{11}P + Y_{12}P\Lambda = -Y_{11}P - Y_{12}P\Lambda^{-1} \\ P &= Z_{11}Q - Z_{12}Q\Lambda = -Z_{11}Q + Z_{12}Q\Lambda^{-1} \end{aligned} \tag{A2.9}$$

Some useful choices are,

$$\begin{aligned} &1.\quad P = S, && Q = -Y_{12}S\Sigma = B^{-1}S\Sigma \\ &2.\quad P = S\Sigma && Q = Z_{12}^{-1}S = CS \\ &3.\quad Q = T, && P = Z_{12}T\Sigma = C^{-1}T\Sigma \\ &4.\quad Q = T\Sigma && P = -Y_{12}^{-1}T = BT \end{aligned} \tag{A2.10}$$

The particular choice to be made depends upon the system of difference equations which is being used. In choices 1 and 2, T is not required and in 3 and 4, S is not required. However, if both S and T are known some of

[13] Methods of determining s_r and t_r' are available. A description will be found in F.D.C. §4.12.

the matrix multiplication may be avoided. Taking choice 1 as an example, we may determine the rth row of Q from the expression $\beta_r t_r$. To determine β_r only one element in the rth column of $-Y_{12}S\Sigma$ need be known, for β_r is the quotient obtained by dividing this element by the corresponding element in t_r. The product $P'Q$ must be a diagonal matrix, and the same is true of $S'T$. This may serve to check computations.

That the expressions for $v(n)$ and $i(n)$ given by (A2.8) and (A2.10) satisfy the transmission equations (A2.1), (A2.6) and (2.10) may be verified by direct substitution and use of

$$S(\Lambda + \Lambda^{-1}) = 2AS \qquad T(\Lambda + \Lambda^{-1}) = 2A'T \tag{A2.11}$$

These relations follow from the properties of the individual columns of S and T.

When the system extends to $n = \infty$ $\bar{a}$ must be zero in order that the voltages and currents may remain finite. This is true because λ_r is chosen so that $|\lambda_r| < 1$. From equations (A2.8) it follows that

$$\begin{aligned} v(n) &= P\Lambda^n a = P\Lambda^n P^{-1} v(o) \\ i(n) &= Q\Lambda^n a = Q\Lambda^n Q^{-1} i(o) \\ v(n) = PQ^{-1} i(n) &\qquad i(n) = QP^{-1} v(n) \end{aligned} \tag{A2.12}$$

the reciprocal matrices Q^{-1} and P^{-1} always exist when the sections are symmetrical and the roots $\zeta_1, \zeta_2, \cdots \zeta_m$ distinct. The last equations in (A2.12) suggest the introduction of the characteristic impedance and admittance matrices Z_o and Y_o:

$$\begin{aligned} v(n) &= Z_o i(n), \qquad i(n) = Y_o v(n), \qquad Z_o = Y_o^{-1}. \\ Z_o &= PQ^{-1} = Z_{11} - Z_{12}Q\Lambda Q^{-1} = -Z_{11} + Z_{12}Q\Lambda^{-1}Q^{-1} \\ &= S\Sigma^{-1}S^{-1}B = S\Sigma S^{-1}Z_{12} \\ &= Z_{12}T\Sigma T^{-1} = BT\Sigma^{-1}T^{-1} \\ Y_o &= QP^{-1} = Y_{11} + Y_{12}P\Lambda P^{-1} = -Y_{11} - Y_{12}P\Lambda^{-1}P^{-1} \\ &= -Y_{12}S\Sigma S^{-1} = CS\Sigma^{-1}S^{-1} \\ &= T\Sigma^{-1}T^{-1}C = -T\Sigma T^{-1}Y_{12} \end{aligned} \tag{A2.13}$$

Not all of the expressions for Z_o and Y_o obtainable from (A2.10) have been included in (A2.13). Z_o and Y_o are symmetrical matrices. Although P and Q are arbitrary to some extent the same is not true of Z_o and Y_o. Computed values of Z_o and Y_o may be checked by use of the relations

$$\begin{aligned} A^2 - I &= (Z_o Z_{12}^{-1})^2 = (Y_{12}^{-1} Y_o)^2 \\ Z_o C Z_o &= B, \qquad Y_o B Y_o = C \\ Y_o Z_{12} &= -Y_{12} Z_o \end{aligned} \tag{A2.14}$$

Sometimes it is desirable to terminate a line consisting of a finite number of sections by a network which simulates an infinite line. As is known, the elements in one such network may be obtained from Y_o. Every terminal is joined to every other terminal, including the return terminal (denoted by the subscript o), by the branches of this network. The admittance of the branch connecting terminal i to terminal j, $i \neq 0, j \neq 0$, is $-y_{ij}$ where y_{ij} is the element in the ith row and jth column of Y_o. The admittance of the branch connecting terminal i to terminal o is $y_{i1} + y_{i2} + \cdots + y_{ii} + \cdots + y_{im}$, i.e., it is the sum of all the elements whose first subscript is i.

APPENDIX III

Classical Solution of Symmetrical Section Line Equations—II

When the electrical properties of a typical symmetrical section are to be determined by measurement, equations (A2.1) and (A2.2) show that Z_{11} and Y_{11} may be obtained by measurements at one end. In order to obtain Y_{12} and Z_{12} measurements have to be made at both ends. Expressions for $v(n)$ and $i(n)$ will now be given which depend only upon Z_{11} and Y_{11} and hence are useful in case the measurements are restricted to one end.

The method is based upon the equations

$$\begin{aligned} v(n+2) + v(n) &= Z_{11}[i(n) - i(n+2)] \\ i(n+2) + i(n) &= Y_{11}[v(n) - v(n+2)] \end{aligned} \qquad \text{(A3.1)}$$

which may be derived from (A2.1) and (A2.2). Combining these equations leads to

$$[I - Z_{11}Y_{11}][v(n+2) + v(n-2)] + 2\,[I + Z_{11}Y_{11}]v(n) = 0$$

$$[I - Y_{11}Z_{11}][i(n+2) + i(n-2)] + 2\,[I + Y_{11}Z_{11}]i(n) = 0$$

The first step in the solution is to solve the equation

$$| \mu I - Z_{11}Y_{11} | = 0 \qquad \text{(A3.2)}$$

for its roots $\mu_1, \mu_2, \cdots \mu_m$ which we shall suppose are distinct. The diagonal matrices M and $M^{\frac{1}{2}}$ are defined by

$$M = \begin{bmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & & \\ \vdots & & & \\ 0 & \cdots & \cdots & \mu_m \end{bmatrix}, \qquad M^{\frac{1}{2}} = \begin{bmatrix} \mu_1^{\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \mu_2^{\frac{1}{2}} & & \\ \vdots & & & \\ 0 & \cdots & \cdots & \mu_m^{\frac{1}{2}} \end{bmatrix} \qquad \text{(A3.3)}$$

and Λ is defined as in (A2.3) where λ_r is given by

$$\lambda_r = \sqrt{\frac{\mu_r^{\frac{1}{2}} - 1}{\mu_r^{\frac{1}{2}} + 1}}, \qquad \mu_r = \left[\frac{1 + \lambda_r^2}{1 - \lambda_r^2}\right]^2 \qquad \text{(A3.4)}$$

The sign of $\mu_r^{\frac{1}{2}}$ is chosen so that $|\lambda_r| < 1$, and this is the value to be used in $M^{\frac{1}{2}}$. However, there is an ambiguity in the sign of λ_r which is inherent in this method. A relation between Λ and $M^{\frac{1}{2}}$ is

$$M^{\frac{1}{2}} = (I + \Lambda^2)(I - \Lambda^2)^{-1} \tag{A3.5}$$

Let u_r be proportional to any non-zero column and w_r' be proportional to any non-zero row of the matrix adjoint to $[\mu_r I - Z_{11}Y_{11}]$ and form the matrices

$$U = [u_1, u_2, \cdots u_m]$$

$$W = [w_1, w_2, \cdots w_m]$$

(cf. equations (A2.6) and (A2.7) for S and T) where w_r is the column obtained from w_r'.

The voltages and currents are given, as before, by

$$\begin{aligned} v(n) &= P\Lambda^n a + P\Lambda^{-n}\bar{a} \\ i(n) &= Q\Lambda^n a - Q\Lambda^{-n}\bar{a} \end{aligned} \tag{A2.8}$$

and there is again a number of ways in which P and Q may be chosen. In all cases the rth column of P may be expressed as $\alpha_r u_r$ and the rth column of Q as $\beta_r w_r$. The equations fixing Q when P is chosen and vice versa are, from equations (A3.1)

$$\begin{aligned} Q &= Y_{11}PM^{-\frac{1}{2}} \\ P &= Z_{11}QM^{-\frac{1}{2}} \end{aligned} \tag{A3.6}$$

where $M^{-\frac{1}{2}}$ is the inverse of $M^{\frac{1}{2}}$. Equations (A3.6) may also be obtained from (A2.10).

Suitable choices for P and Q are

$$\begin{aligned} &1.\ P = U, && Q = Y_{11}UM^{-\frac{1}{2}} = Z_{11}^{-1}UM^{\frac{1}{2}} \\ &2.\ P = UM^{\frac{1}{2}}, && Q = Y_{11}U = Z_{11}^{-1}UM \\ &3.\ Q = W, && P = Z_{11}WM^{-\frac{1}{2}} = Y_{11}^{-1}WM^{\frac{1}{2}} \\ &4.\ Q = WM^{\frac{1}{2}}, && P = Z_{11}W = Y_{11}^{-1}WM \end{aligned} \tag{A3.7}$$

$P'Q$ and $U'W$ must be diagonal matrices. That the expressions for $v(n)$ and $i(n)$ just derived satisfy the difference equations (A3.1) may be verified by making use of

$$UM = Z_{11}Y_{11}U, \qquad WM = Y_{11}Z_{11}W \tag{A3.8}$$

Equations (A3.8) follow from the properties of the individual columns of U and W. The characteristic impedance and admittance matrices are given by

$$
\begin{aligned}
Z_o &= PQ^{-1} = Z_{11}QM^{-\frac{1}{2}}Q^{-1} = PM^{\frac{1}{2}}P^{-1}Y_{11}^{-1} \\
&= UM^{-\frac{1}{2}}U^{-1}Z_{11} = UM^{\frac{1}{2}}U^{-1}Y_{11}^{-1} \\
&= Z_{11}WM^{-\frac{1}{2}}W^{-1} = Y_{11}^{-1}WM^{\frac{1}{2}}W^{-1} \\
Y_o &= QP^{-1} = Y_{11}PM^{-\frac{1}{2}}P^{-1} = QM^{\frac{1}{2}}Q^{-1}Z_{11}^{-1} \\
&= Y_{11}UM^{-\frac{1}{2}}U^{-1} = Z_{11}^{-1}UM^{\frac{1}{2}}U^{-1} \\
&= WM^{-\frac{1}{2}}W^{-1}Y_{11} = WM^{\frac{1}{2}}W^{-1}Z_{11}^{-1}
\end{aligned}
\tag{A3.9}
$$

The matrices Z_o and Y_o may be checked by means of the relations

$$Z_oY_{11} = Z_{11}Y_o, \qquad Z_oY_{11}Z_o = Z_{11}, \qquad Y_oZ_{11}Y_o = Y_{11} \tag{A3.10}$$

Another set of solutions may be based upon the equations

$$
\begin{aligned}
2v(n) &= -Z_{12}[i(n+1) - i(n-1)] \\
2i(n) &= Y_{12}[v(n+1) - v(n-1)]
\end{aligned}
\tag{A3.11}
$$

which are derivable from (A2.1) and (A2.2). Combining these equations gives, upon using $Y_{12}^{-1}Z_{12}^{-1} = -BC$,

$$
\begin{aligned}
v(n+2) - 2v(n) + v(n-2) &= 4BCv(n) \\
i(n+2) - 2i(n) + i(n-2) &= 4CBi(n)
\end{aligned}
\tag{A3.12}
$$

However, we shall not consider these equations here beyond pointing out that they lead to

$$
\begin{aligned}
P &= Z_{12}Q\Sigma, \qquad & Q &= -Y_{12}P\Sigma \\
P\Sigma^2 &= BCP, \qquad & Q\Sigma^2 &= CBQ
\end{aligned}
$$

which may also be derived from (A2.10).

APPENDIX IV

Relations Between the Square Matrices of a Multi-Terminal Section

When the reciprocal theorems of network theory are applied to equations (2.1) and (2.2) it is found that Z_{11}, Z_{22}, Y_{11}, Y_{22} are symmetrical and

$$Z_{21} = Z_{12}', \qquad Y_{21} = Y_{12}' \tag{A4.1}$$

i.e., Z_{21} and Y_{21} are the transposed matrices of Z_{12} and Y_{12}, respectively.

Solving equations (2.1) for the currents and comparing the result with (2.2) shows that

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}^{-1} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}, \qquad \begin{bmatrix} i^\circ(n) \\ -j^\circ(n) \end{bmatrix} = -\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}\begin{bmatrix} v^\circ(n) \\ u^\circ(n) \end{bmatrix}$$

These are partitioned matrices. The square matrices have $2m$ rows and columns and the column matrices have $2m$ elements. The first of these relations may be written as

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{A4.2}$$

where I denotes the unit diagonal matrix of order m. Multiplying the two matrices on the left together and equating the elements of the product to the elements on the right gives

$$\begin{aligned} Z_{11}Y_{11} + Z_{12}Y_{21} &= I \\ Z_{11}Y_{12} + Z_{12}Y_{22} &= 0 \\ Z_{21}Y_{11} + Z_{22}Y_{21} &= 0 \\ Z_{21}Y_{12} + Z_{22}Y_{22} &= I \end{aligned} \tag{A4.3}$$

Transposing the matrices in these equations leads to other relations. Thus, from the first we obtain $Y_{11}Z_{11} + Y_{12}Z_{21} = I$. These equations also yield expressions for the Y's in terms of the Z's and vice versa.

A somewhat similar treatment involving equations (2.1) and (2.3) leads to expressions for the Z's in terms of A, B, C and D. The Y's may be likewise expressed. These relations are given in the following table.

$$\begin{aligned} Y_{11} &= DB^{-1} & Y_{11}^{-1} &= Z_{11} - Z_{12}Z_{22}^{-1}Z_{21} \\ Y_{12} &= C - DB^{-1}A = -B'^{-1} & Y_{12}^{-1} &= Z_{21} - Z_{22}Z_{12}^{-1}Z_{11} \\ Y_{21} &= -B^{-1} & Y_{21}^{-1} &= Z_{12} - Z_{11}Z_{21}^{-1}Z_{22} \\ Y_{22} &= B^{-1}A & Y_{22}^{-1} &= Z_{22} - Z_{21}Z_{11}^{-1}Z_{12} \end{aligned}$$

$$\begin{aligned} Z_{11} &= AC^{-1} & Z_{11}^{-1} &= Y_{11} - Y_{12}Y_{22}^{-1}Y_{21} \\ Z_{12} &= AC^{-1}D - B = C'^{-1} & Z_{12}^{-1} &= Y_{21} - Y_{22}Y_{12}^{-1}Y_{11} \\ Z_{21} &= C^{-1} & Z_{21}^{-1} &= Y_{12} - Y_{11}Y_{21}^{-1}Y_{22} \\ Z_{22} &= C^{-1}D & Z_{22}^{-1} &= Y_{22} - Y_{21}Y_{11}^{-1}Y_{12} \end{aligned} \tag{A4.4}$$

$$\begin{aligned} A &= Z_{11}Z_{21}^{-1} = -Y_{21}^{-1}Y_{22} \\ B &= Z_{11}Z_{21}^{-1}Z_{22} - Z_{12} = -Y_{21}^{-1} \\ C &= Z_{21}^{-1} = Y_{12} - Y_{11}Y_{21}^{-1}Y_{22} \\ D &= Z_{21}^{-1}Z_{22} = -Y_{11}Y_{21}^{-1} \end{aligned}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} D' & -B' \\ -C' & A' \end{bmatrix} \qquad \begin{aligned} AD' - BC' &= I \\ CD' &= DC' \end{aligned} \qquad \begin{aligned} AB' &= BA' \\ DA' - CB' &= I \end{aligned}$$

$$\begin{bmatrix} v^{\circ}(n) \\ -i^{\circ}(n) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} u^{\circ}(n) \\ -j^{\circ}(n) \end{bmatrix}$$

The following equations in which λ is an arbitrary scalar multiplier may be verified by equating coefficients of powers of λ and using the relations just given.

$$(Z_{21} - \lambda Z_{11})(Y_{11} + \lambda Y_{12}) = (\lambda Z_{12} - Z_{22})(\lambda Y_{22} + Y_{21})$$

$$(\lambda^2 Z_{12} - \lambda Z_{11} - \lambda Z_{22} + Z_{21})(Y_{11} + \lambda Y_{12})$$
$$= (\lambda Z_{12} - Z_{22})(\lambda^2 Y_{12} + \lambda Y_{11} + \lambda Y_{22} + Y_{21}) \qquad \text{(A4.5)}$$

$$\begin{bmatrix} -Y_{21} & 0 \\ 0 & Z_{21} \end{bmatrix} \begin{bmatrix} \lambda A - I & \lambda B \\ \lambda C & \lambda D - I \end{bmatrix} = \begin{bmatrix} \lambda Y_{22} + Y_{21} & \lambda I \\ \lambda I & \lambda Z_{22} - Z_{21} \end{bmatrix}$$

Sometimes it is of interest to obtain the elements of Y_{12}, say, when Z_{11}, Z_{22}, Y_{11}, Y_{22} are known. Relations helpful in studying this problem are

$$Y_{11}Z_{11}Y_{12} = Y_{12}Z_{22}Y_{22}, \qquad Y_{11}Z_{11}Y_{11} - Y_{11} = Y_{12}Z_{22}Y_{21}$$

$$Y_{12}Y_{22}^{-1}Y_{21} = Y_{11} - Z_{11}^{-1} \qquad Z_{12} = -Z_{11}Y_{12}Y_{22}^{-1}$$

$$Y_{21}Y_{11}^{-1}Y_{12} = Y_{22} - Z_{22}^{-1} \qquad Z_{21} = -Y_{12}^{-1}(Y_{11}Z_{11} - I)$$

When the typical section is symmetrical some simplification takes place and we have

$$Y_{11} = Y_{22} \qquad Z_{11} = Z_{22} \qquad A = D' \qquad AB = BA'$$

$$Y_{12} = Y_{21} \qquad Z_{12} = Z_{21} \qquad B = B' \qquad A'C = CA$$

$$C = C' \qquad A^2 - BC = I \qquad \text{(A4.6)}$$

$$Z_{11}Y_{11} + Z_{12}Y_{12} = I \qquad A'B^{-1}A - C = B^{-1}$$

$$Z_{11}Y_{12} + Z_{12}Y_{11} = 0$$

APPENDIX V

Properties of the Matrix $G\lambda^2 + H\lambda + G'$

The matrix

$$f(\lambda) = G\lambda^2 + H\lambda + G' \qquad \text{(A5.1)}$$

which entered the discussion of the case of unsymmetrical sections has a number of interesting properties which are given below. G and H are square matrices with m rows each, and H is required to be symmetrical. As before, we shall denote by $\lambda_1, \cdots \lambda_m, \lambda_1^{-1}, \cdots \lambda_m^{-1}$ the 2 m roots of the determinantal equation

$$|f(\lambda)| = 0$$

and we shall suppose these roots to be distinct. Let the column k_r and the row l_r be such that

$$k_r l_r = F(\lambda_r) \tag{A5.2}$$

where $F(\lambda)$ is the matrix adjoint to $f(\lambda)$, and let the square matrices K and L be defined by

$$K = [k_1, k_2, \cdots k_m], \qquad L = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_m \end{bmatrix} \tag{A5.3}$$

Comparison of (A5.3) and (2.50) suggests that when G and H are expressed in terms of the Y's we have the relations

$$K = P, \qquad L = \bar{P}' \tag{A5.4}$$

The method of choosing the column p_r and the row $\bar{p}_r'$ shows that they are related by

$$p_r \bar{p}_r' = \gamma_r F(\lambda_r)$$

instead of (A5.2) where γ_r may turn out to be any non-zero constant, and consequently equations (A5.4) are not satisfied in general. Nevertheless K and L may be regarded as particular choices for P and $\bar{P}'$. In the same way K and L may be regarded as particular choices for Q and $\bar{Q}'$ when G and H are expressed in terms of the Z's. There is still some arbitrariness connected with K and L since the product $k_r l_r$ is unchanged when the k_r is multiplied by some number and l_r is divided by the same number.

The relations which correspond to (2.52) and (2.57) are

$$\begin{aligned} GK\Lambda^2 + HK\Lambda + G'K &= 0 \\ GL'\Lambda^{-2} + HL'\Lambda^{-1} + G'L' &= 0 \end{aligned} \tag{A5.5}$$

where Λ is the diagonal matrix whose elements are $\lambda_1, \lambda_2, \cdots \lambda_m$. These relations are consequences of the properties of k_r and l_r. Two more relations may be obtained by transposition. From the first of (A5.5) and the transposed of the second it follows that

$$\begin{aligned} GK\Lambda K^{-1} + H + G'K\Lambda^{-1}K^{-1} &= 0 \\ L^{-1}\Lambda LG + H + L^{-1}\Lambda^{-1}LG' &= 0 \end{aligned} \tag{A5.6}$$

where it is assumed that the reciprocal matrices K^{-1} and L^{-1} exist. Combinations similar to $K\Lambda K^{-1}$, $K\Lambda^{-1}K^{-1}$, etc. enter the expressions (2.59) for Z_o and Y_o.

By differentiating the equation

$$f(\lambda)F(\lambda) = \Delta(\lambda)I,$$

where $\Delta(\lambda)$ is the determinant

$$\Delta(\lambda) = |f(\lambda)| = |G\lambda^2 + H\lambda + G'|,$$

it may be proved that

$$\begin{aligned} GK\Lambda K^{-1} + H + L^{-1}\Lambda LG &= L^{-1}EK^{-1} \\ G'K\Lambda^{-1}K^{-1} + H + L^{-1}\Lambda^{-1}LG' &= -L^{-1}EK^{-1} \end{aligned} \tag{A5.7}$$

in which E is the diagonal matrix

$$E = \begin{bmatrix} \overset{(1)}{\Delta}(\lambda_1) & 0 & \cdots & 0 \\ 0 & \overset{(1)}{\Delta}(\lambda_2) & & \\ \vdots & & & \\ 0 & & & \overset{(1)}{\Delta}(\lambda_m) \end{bmatrix}$$

and

$$\overset{(1)}{\Delta}(\lambda_r) = \left[\frac{d}{d\lambda}\Delta(\lambda)\right]_{\lambda=\lambda_r}$$

Since the roots λ_r are assumed to be distinct, $\overset{(1)}{\Delta}(\lambda_r) \neq 0$.

We also have the equations

$$\begin{aligned} KE^{-1}L &= L'E^{-1}K' \\ GK\Lambda E^{-1}L - GL'\Lambda^{-1}E^{-1}K' &= I \end{aligned} \tag{A5.8}$$

The first equation of (A5.8) shows that $KE^{-1}L$ is a symmetrical matrix. From this and the second equation it follows that

$$GK\Lambda K^{-1} - GL'\Lambda^{-1}L'^{-1} = L^{-1}EK^{-1} \tag{A5.9}$$

From the first of equations (A5.7) and the second of (A5.6)

$$GK\Lambda K^{-1} - L^{-1}\Lambda^{-1}LG' = L^{-1}EK^{-1} \tag{A5.10}$$

and the comparison with (A5.9) shows that the matrix $GL'\Lambda^{-1}L'^{-1}$ is symmetrical. The other matrices of this type are also symmetrical as may now be seen from equations (A5.6) and (A5.7). Results of this sort may be obtained from physical principles by noting that Z_o and Y_o must be symmetrical matrices.

As an example of the application of these formulas we assume that G and H are expressed in terms of the Y's. Then we may take K and L' to be particular choices of P and $\overline{P}$ and equation (A5.9) becomes

$$Y_{12}P\Lambda P^{-1} - Y_{12}\overline{P}\Lambda^{-1}\overline{P}^{-1} = (\overline{P}')^{-1}EP^{-1}.$$

From equations (2.59) and (2.62)

$$Y_o + \overline{Y}_o = Y_{12}P\Lambda P^{-1} - Y_{12}\overline{P}\Lambda^{-1}\overline{P}^{-1}$$

and hence

$$\overline{P}'(Y_o + \overline{Y}_o)P = E.$$

For the more general choice of P and $\overline{P}$ allowed in §2.8 the diagonal matrix E is replaced by a general diagonal matrix. Similarly it follows that

$$\overline{Q}'(Z_o + \overline{Z}_o)Q$$

is a diagonal matrix.