The Number of Impedances of an n Terminal Network

By JOHN RIORDAN

This paper gives the enumeration of impedances measurable at the n terminals of a linear passive network. The enumeration supplies background for the study of network representations and the numerical results which are given up to ten terminals are perhaps surprising in the rapidity of the rise of the number of impedances with the number of terminals; almost 126,000,000 impedances, e.g., are measurable for ten terminals.

A LINEAR passive network having n accessible terminals may be completely represented by an equivalent direct impedance network,[1] consisting of branches, devoid of mutual impedance, connecting the terminals in pairs. The number of elements (branches) in this representation is equal to the number of combinations of n things taken two at a time, i.e., $\frac{1}{2}n(n-1)$. Each of the elements is defined by an impedance measured by energizing between one of the terminals it connects and the remaining terminals connected together and taking the ratio of the driving voltage to the current into the other terminal it connects. The network then is represented by a particular set, of $\frac{1}{2}n(n-1)$ members, of impedances measurable at its terminals; as will appear later, the set is of short-circuit transfer impedances.

The direct impedance network is one among many network representations; it is taken as illustrative of two aspects, (i) the necessity of a certain number of elements $\frac{1}{2}n(n-1)$ and (ii) the expression of these elements in terms of measurable impedances. It is well known that any linearly independent set, of $\frac{1}{2}n(n-1)$ members, of the measurable impedances of an n-terminal network will serve as a network representation; hence the enumeration of representations may be taken in two steps, the first of which, the enumeration of measurable impedances, is dealt with in the present paper.

The number of measurable impedances for two to ten terminal linear passive networks is given in Table I, which lists the driving-point impedances, D_n, transfer impedances (open or short circuit), T_n, certain additional transfer impedances to be described later, U_n, and the total N_n. As mentioned below, this total counts once only

[1] Item (*b*) in the list of equivalent networks given by G. A. Campbell "Cisoidal Oscillations," *Trans. A.I.E.E. 30*, pp. 873–909 (1911), p. 889; or p. 81, "Collected Papers of George Ashley Campbell," Amer. Tel. & Tel. Co., New York, 1937.

impedances which are equal by the reciprocity theorem; the doubling of T_n in forming the total is due to the equality in number of open-circuit and short-circuit transfer impedances. The numbers increase rapidly with n, reaching almost 126,000,000 for ten terminals. The number of representations, which is the number of combinations of the measurable impedances $\frac{1}{2}n(n - 1)$ at a time less the number of non-independent sets, at a guess increases even more rapidly, indicating a variety of equivalents, few of which seem to have been investigated.

TABLE I

MEASURABLE IMPEDANCES OF AN n-TERMINAL NETWORK

n	D_n	T_n	U_n	$N_n = D_n + 2T_n + U_n$
2	1	0	0	1
3	6	3	0	12
4	31	33	60	157
5	160	270	1,050	1,750
6	856	2,025	12,540	17,446
7	4,802	14,868	129,570	164,108
8	28,337	109,851	1,257,060	1,505,099
9	175,896	827,508	11,889,990	13,720,902
10	1,146,931	6,397,665	111,840,180	125,782,441

Because the field of the work is somewhat unusual, considerable space is given to details in the formulation of the problem before proceeding to the enumeration proper. The enumerating expressions obtained are found susceptible of some mathematical development which, though subsidiary to the main object of the paper, seems of sufficient interest to justify the relatively brief exposition given. The arrangement is such that readers not interested in this mathematical half may obtain the substance of the paper without it.

FORMULATION OF THE PROBLEM

The enumerating problem is essentially one of combinations, as indicated schematically in Fig. 1, which shows the n terminals of a linear passive network together with the apparatus required for impedance measurement, that is, a source, a voltmeter and an ammeter, each supplied with two terminals (shown solid to distinguish them from the network terminals). Each of these latter may be connected across any pair of the n terminals except that the ammeter, which constitutes a short circuit, may not be connected to terminals to which either the source or voltmeter is connected; in the former case no current will be supplied to the network and in the latter the voltmeter will read zero. The ammeter may be connected in series with the source to read the source current, of course.

Although but one source, voltmeter, and ammeter are shown, as many of each as will produce distinct impedances should of course be included. Multiple sources are not required because if the source voltages are in defined proportions, as is necessary to determine impedances independent of source voltage, the corresponding measurable admittances are linear combinations of single-source admittances, by the principle of superposition; a similar requirement on source currents produces impedances which are linear combinations of single-source impedances. A single voltmeter is sufficient because it has no effect on network currents or voltages and it is immaterial whether

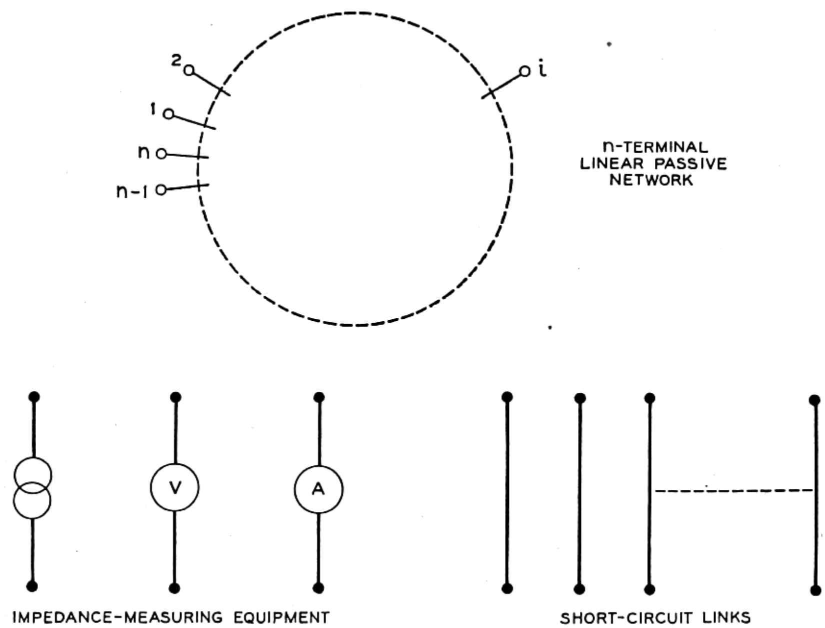

Fig. 1.—Elements involved in impedance enumeration.

impedances are supposed measured by successive positions of a single voltmeter or by many voltmeters. The connection of an ammeter is equivalent to a short circuit (except of course when in series with a source) across the terminals the ammeter connects; this alters network voltages and currents and the impedances measured without the ammeter differ from those with it. Hence a plurality of ammeters or its equivalent is required; for convenience, all ammeters except that one determining a specific impedance under consideration are supposed replaced by the short-circuiting links on the right of Fig. 1, thus focussing attention on the single items of the enumeration.

The classification under which the enumeration is conducted is illustrated by Fig. 2, which shows typical positions of source, voltmeter and ammeter for measuring impedances of three classes. In the first of these, the ammeter reads the source current, the voltmeter source voltage (across some pair of the network terminals) and the class is that of driving-point impedances, D_n. In the second class, that of transfer impedances T_n, there are two types of connection: in the first the ammeter reads the source current, the voltmeter a non-source voltage, the voltage-current ratios being open-circuit transfer impedances; in the second the voltmeter reads the source voltage and

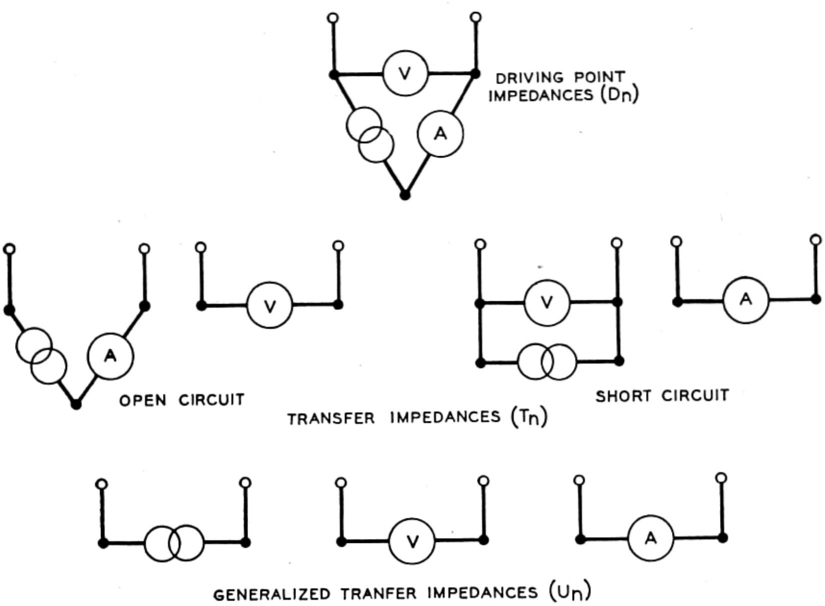

FIG. 2—Arrangement of apparatus for measuring impedances of three classes.

the ammeter a non-source current, the voltage-current ratios being short-circuit transfer impedances. It will be noted that the two connections differ only in that the ammeter and voltmeter are interchanged. The third class is that of generalized transfer impedances U_n, in which both voltmeter and ammeter are across non-source terminals.

The last class, of course, might be supposed to include the two preceding ones but the separation proves convenient not only for numerical work, as will appear, but also for keeping distinct both well-recognized and formally different classes.

The most important of these differences in classes is that arising from the reciprocity theorem. Of the three classes, only the T (open-circuit and short-circuit transfer impedances) includes members which are equal by the reciprocity theorem; this follows because the reciprocity theorem requires interchange of voltmeter (or ammeter) and source with associated ammeter (or voltmeter) and the T class alone permits this. It is a matter of taste whether such duplicates should be counted separately or as one; in the interest of keeping large figures as low as possible they are here counted as one, since the classification is such that the other alternative may be taken merely by doubling the T_n.

Another reason for keeping the T class distinct is that total open-circuit and short-circuit transfer impedances for a given number of terminals are equal in number. This is proved immediately by observing that the two connections shown in Fig. 2 for this class are in one-one correspondence: each may be obtained from the other by interchanging voltmeter and ammeter. Moreover, if $T^o{}_{x,\,n}$ and $T^s{}_{x,\,n}$ are the numbers of open-circuit and short-circuit transfer impedances measurable when short circuits have been placed across the n terminals in all possible ways so as to realize x terminals, each merged group of terminals counting as a single terminal, the correspondence leads to the relation

$$T^o{}_{x+1,\,n} = T^s{}_{x,\,n}, \tag{1}$$

since the interchange of voltmeter and ammeter in the measuring arrangement for open-circuit transfer impedances results in one less available terminal, two terminals being merged by the ammeter short circuit. Note that $T^o{}_{2,\,n} = T^s{}_{n,\,n} = 0$, since with just two terminals, no non-source voltages and with n terminals no non-source currents are measurable.

Equation (1) is important in determining enumerating expressions in the section following.

Enumerating Expressions

The laws of enumeration appear most simply exposed by examining the simplest cases first.

For two terminals, there is but one measurable impedance, the driving-point impedance between the terminals.

For three terminals, with the terminals distinct, there are three driving-point and three open-circuit transfer impedances, for there are three ways of selecting driving pairs of the three terminals and for each selection two ways of selecting pairs for open-circuit voltage measurement, the total of six transfer impedances being halved to eliminate reciprocity theorem duplicates. With two of the terminals connected by an ammeter, there are again three driving-point and three transfer impedances, the latter being short-circuit transfer impedances, for there are three ways of connecting pairs of terminals and one driving-point and two transfer impedances for each, the total of transfer impedances again being halved to eliminate duplicates.

There are no generalized transfer impedances because with an ammeter connected, there is only one measurable voltage, the driving-circuit voltage.

With terminals designated by t_1, t_2 and t_3, the conditions arising from connection of terminals may be exhibited as follows:

Terminals distinct	$t_1 \vert t_2 \vert t_3$		
Pairs connected	$t_1t_2 \vert t_3$	$t_1t_3 \vert t_2$	$t_1 \vert t_2t_3$

the lines of separation dividing the terminals into groups such that the terminals in any group are merged into a single terminal. Paying attention only to the number of terminals in each group, the groups illustrated may be designated by the partition notation (111) or (1^3) and (21), the numbers in the designation being partitions[2] of the number 3.

The enumeration for three terminals may then be exhibited as follows:

MEASURABLE IMPEDANCES

Group	Driving Point	Open-Circuit Transfer	Short-Circuit Transfer	Total
(1^3)	3	3	0	6
(21)	3	0	3	6
	6	3	3	12

It will be noted that the open-circuit and short-circuit transfer impedances satisfy equation (1), that is, $T^o{}_{3,\,3} = T^s{}_{2,\,3}$.

This table and its correspondents for larger values of n show that the impedances may be expressed as sums with respect to x, where x is the number of terminals defined as in equation (1), from 2 to n; thus e.g., $D_n = \sum D_{x,\,n}$ where $D_{x,\,n}$ is the number of driving-point impedances measurable for all conditions of merging of n terminals such that the resulting number of terminals is x. Moreover, con-

[2] A partition of a number n is any collection of positive integers whose sum is equal to n. It may be noted that the number of parts of a partition is the number x of equation (1); the partition (1^3) has three parts corresponding to the three distinct terminals; (21) has two parts corresponding to two terminals, each merged pair of terminals counting singly.

sidering for the moment only the driving-point and open-circuit transfer impedances, the numbers $D_{x,\,n}$ and $T^o_{x,\,n}$ are the products of two factors: (i) the number of such impedances measurable for x terminals, which is independent of n and (ii) the number of ways the n terminals may be merged so as to result in x terminals, which is independent of the impedance classes. By equation (1) this result applies also to $T^s_{x,\,n}$ and, as $U_{x,\,n}$ is related to $T^s_{x,\,n}$ by a factor independent of n, as will be shown, it applies generally.

This leads to the following equation:

$$\begin{Bmatrix} D_n \\ T_n \\ U_n \end{Bmatrix} = \sum_{x=2}^{n} \begin{Bmatrix} d_x \\ t_x \\ u_x \end{Bmatrix} S_{x,\,n}. \tag{2}$$

The small letters are the several factors of the first kind and $S_{x,\,n}$ is the common second factor.

The small letters are determined as follows: A driving point impedance may be measured between every pair of terminals; hence d_x is the number of combinations of x things taken two at a time, that is:

$$d_x = \binom{x}{2} = \tfrac{1}{2}x(x-1) = \tfrac{1}{2}(x)_2, \tag{3}$$

where $(x)_i$ is the factorial symbol $x(x-1)\cdots(x-i+1)$.

For a given pair of driving terminals, there are $\binom{x}{2} - 1$ measurable open-circuit transfer impedances since a voltmeter can be connected to every pair of the x terminals except the driving pair; hence, multiplying by the number of driving terminals and by the factor one-half to eliminate reciprocity theorem duplicates:

$$\begin{aligned} t_x &= \tfrac{1}{2}\binom{x}{2}\left[\binom{x}{2} - 1\right], \\ &= \tfrac{1}{8}[4(x)_3 + (x)_4]. \end{aligned} \tag{4}$$

The second, factorial, form is given for convenience of later development.

By equation (1) this serves for enumeration of both open-circuit and short-circuit transfer impedances; the direct enumeration of the latter appears more difficult.

Considering, for the generalized transfer impedances, a fixed source and an ammeter in a fixed (non-source) position, the voltmeter may be connected across $\binom{x}{2}$ pairs of terminals when x terminals are available; one of these pairs is the source pair measuring a short-circuit transfer impedance which must be excluded; hence, remembering that reciprocity theorem duplicates are eliminated in the latter:

$$U_{x,\,n} = 2\left[\binom{x}{2} - 1\right] T^{s}{}_{x,\,n}$$
$$= 2\left[\binom{x}{2} - 1\right] T^{o}{}_{x+1,\,n} = 2\left[\binom{x}{2} - 1\right] t_{x+1} S_{x+1,\,n}.$$

Degrading x by unity to obtain the form of equations (2), the third of the lower case factors is reached as follows:

$$u_x = \binom{x}{2}\left[\binom{x}{2} - 1\right]\left[\binom{x-1}{2} - 1\right] \tag{5}$$
$$= \tfrac{1}{8}[20(x)_4 + 10(x)_5 + (x)_6].$$

The common factor $S_{x,\,n}$ remains for determination.

Returning to the connection conditions illustrated for three terminals, this number is the number of ways separators may be placed between letters of the collection $t_1, t_2 \cdots t_n$ symbolizing the terminals so as to produce x compartments, symbolizing merged terminals. The terminal symbols $t_1 \cdots t_n$ may be thought of as the prime distinct factors (excluding unity) of some number and the number $S_{x,\,n}$ is then identically the number of ways a number having n distinct prime factors may be expressed as a product of x factors. The enumeration for this latter problem is given by Netto,[3] who gives the recurrence relation

$$S_{x,\,n+1} = xS_{x,\,n} + S_{x-1,\,n}$$

with

$$S_{n,\,n} = 1, \qquad S_{x,\,n} = 0,\ x > n, \qquad S_{0,\,n} = 0,\ n \neq 0.$$

This is the recurrence relation for the Stirling numbers of the second kind,[4] the notation for which has been adopted in anticipation of the result. These numbers are perhaps better known as the "divided differences of nothing," that is, as defined by the equation:

$$S_{x,\,n} = \lim_{z=0} \frac{1}{x!} \Delta^x z^n = \frac{1}{x!} \Delta^x 0^n,$$

where Δ^x denotes x iterations of the difference operator with unit

[3] "Lehrbuch der Combinatorik," Leipzig, 1901, pp. 169–170; Whitworth, "Choice and Chance," Cambridge, 1901, Prop. XXIII, p. 88, gives a generating function for the solution of this problem which, it is not difficult to show, leads to the same answer.

[4] Ch. Jordan, "Statistique Mathematique," Paris, 1927, p. 14.

interval, that is, of the operator defined by

$$\Delta f(z) = f(z + 1) - f(z).$$

For convenience of reference, a short table of the numbers follows:

$n \backslash x$	$S_{x,n}$ 0	1	2	3	4	5
0	1					
1	0	1				
2	0	1	1			
3	0	1	3	1		
4	0	1	7	6	1	
5	0	1	15	25	10	1

The table may be verified and extended readily by the recurrence relation.

With this table (extended to $n = 10$) and corresponding tables of d_x, t_x and u_x running to $x = 10$, the values given in Table I may be calculated by equations (2) and in this sense this paper is completed at this point. The sections below contain an algebraic and arithmetical examination of the numbers.

Generating Identities

The generating identity for the function

$$\sum_{x=0}^{n} a^x S_{x,\,n}$$

is [5]

$$\exp\left[a(e^t - 1)\right] = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{x=0}^{n} a^x S_{x,\,n}.$$

This leads, by differentiating s times with respect to a and setting a equal to unity, to the generating identity:

$$(e^t - 1)^s \exp(e^t - 1) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{x=0}^{n} (x)_s S_{x,\,n}.$$

This relation may be rendered more summarily by introducing the notation of the symbolic or umbral calculus [6] of Blissard; the expression on the right is written $\exp t\delta$ where δ is an umbral symbol standing for the sequence $(\delta_0, \delta_1, \cdots \delta_n \cdots)$ in this case infinite, through the relation $\delta^n = \delta_n$ and:

[5] E. T. Bell, "Exponential Polynomials," *Annals of Math. 35*, 2 (April, 1934) p. 265; or J. Riordan, "Moment Recurrence Relations . . . ," *Annals of Math. Statistics 8*, 2, pp. 103–111 (June, 1937), eq. 3.4.

[6] Cf. Bell, l.c. p. 260 where further references are given.

$$\delta_n = \sum_{x=0}^{n} (x)_s S_{x,\, n}.$$

All algebraic operations on umbral symbols are carried out as in ordinary algebra except that the degrading of subscripts must not be performed until operations are completed. It must be noted that $\delta^0 = \delta_0$, hence is unity only when $\delta_0 = 1$, as in the present case and not always as in ordinary algebra.

The umbrae for the impedance numbers are written D, T and U, and by use of the generating identity above have the following generating identities:

$$\begin{aligned}
\exp tD &= \tfrac{1}{2}(e^t - 1)^2 \exp (e^t - 1),\\
\exp tT &= \tfrac{1}{8}[4(e^t - 1)^3 + (e^t - 1)^4] \exp (e^t - 1), \qquad (6)\\
\exp tU &= \tfrac{1}{8}[20(e^t - 1)^4 + 10(e^t - 1)^5 + (e^t - 1)^6] \exp (e^t - 1).
\end{aligned}$$

These follow immediately from the base generating identity and the factorial expressions for d_x, t_x and u_x.

Expanding these expressions in powers of e^t gives alternate expressions as follows:

$$\begin{aligned}
\exp tD &= \tfrac{1}{2}(e^{2t} - 2e^t + 1) \exp (e^t - 1),\\
\exp tT &= \tfrac{1}{8}(e^{4t} - 6e^{2t} + 8e^t - 3) \exp (e^t - 1), \qquad (6.1)\\
\exp tU &= \tfrac{1}{8}(e^{6t} + 4e^{5t} - 15e^{4t} + 35e^{2t} - 36e^t + 11) \exp (e^t - 1).
\end{aligned}$$

To recapitulate, these expressions mean that D_n, T_n and U_n are the coefficients of $t^n/n!$ in the expansions of the right-hand sides; taking D_n, for example, the first equation of (6) is equivalent to the equation:

$$D_n = \lim_{t=0} \frac{d^n}{dt^n} [\tfrac{1}{2}(e^t - 1)^2 \exp (e^t - 1)],$$

which may be shown to be equivalent to the first of equations (2).

The generating identities lead immediately to recurrence relations, as will now appear.

Recurrence Relations

Recurrence relations to be derived are all obtained by differentiation with respect to t. Under this operation umbrae behave like ordinary variables; thus

$$\begin{aligned}
\frac{d}{dt} \exp tD &= D \exp tD\\
&= D_1 + D_2 t + D_3 \frac{t^2}{2!} + \cdots D_{n+1} \frac{t^n}{n!} + \cdots,
\end{aligned}$$

as may be verified readily.

In the first type of recurrence only successive values of the numbers themselves appear. The derivation is illustrated for the D_n, the simplest case. Differentiating the first of equations (6) leads to the relation:

$$D \exp tD = \tfrac{1}{2}(e^{3t} - e^t) \exp (e^t - 1),$$

or

$$\begin{aligned}(e^t - 1)D \exp tD &= e^t(e^t + 1)\tfrac{1}{2}(e^t - 1)^2 \exp (e^t - 1)\\ &= (e^{2t} + e^t) \exp tD.\end{aligned}$$

Equating coefficients of $t^n/n!$ in this relation gives the umbral recurrence:

$$D(D + 1)^n - D_{n+1} = (D + 2)^n + (D + 1)^n,$$

which in ordinary form is:

$$(n - 2)D_n = \sum_{i=1}^{n} \left[\binom{n}{i} (2^i + 1) - \binom{n}{i + 1} \right] D_{n-i}.$$

The process is common to the three classes of numbers and produces similar results which may be put in general form as follows:

$$a_n A_n = \sum_{i=1}^{n} \left[\binom{n}{i} b_i - \binom{n}{i + 1} c_{i+1} \right] A_{n-i}, \qquad (7)$$

where A_n, a_n, b_i and c_i are defined for the three cases by the following table:

A_n	a_n	b_i	c_i
D_n	$n-2$	2^i+1	1
T_n	$4n-12$	$3^i+6\cdot 2^i+5$	2^i+2
U_{n-3}	$480\left[\binom{n}{4}-\binom{n}{3}\right]$	$7^i+10\cdot 6^i+5\cdot 5^i-60\cdot 4^i$ $+35\cdot 3^i+34\cdot 2^i-25$	$6^i+4\cdot 5^i-15\cdot 4^i$ $+35\cdot 2^i-36$

Somewhat more convenient recurrences may be obtained by allowing the presence of numbers other than those for which the recurrence is sought. For this purpose it is expedient to introduce the exponential numbers ϵ_n of E. T. Bell.

These are defined by the generating identity:

$$\exp t\epsilon = \exp (e^t - 1)$$

or by the equivalent formula:

$$\epsilon_n = \lim_{t=0} \frac{d^n}{dt^n} \exp (e^t - 1) = \sum_{x=1}^{n} S_{x,\,n},$$

which shows their close relation with the impedance numbers. They have the recurrence relation:

$$\epsilon_{n+1} = (\epsilon + 1)^n$$

and

$$\epsilon_0 = \epsilon_1 = 1.$$

Now, returning to the first of equations (6) and again differentiating:

$$\begin{aligned} D \exp tD &= \tfrac{1}{2}[2(e^t - 1)e^t + (e^t - 1)^2 e^t] \exp (e^t - 1), \\ &= (e^t - 1)e^t \exp (e^t - 1) + e^t \exp tD, \\ &= 2 \exp tD + (e^t - 1) \exp (e^t - 1) + \exp t(D + 1), \\ &= 2 \exp tD + \exp t(\epsilon + 1) - \exp t\epsilon + \exp t(D + 1), \end{aligned}$$

from which, passing to the coefficient relation, comes the umbral recurrence:

$$D_{n+1} = 2D_n + (D + 1)^n + \epsilon_{n+1} - \epsilon_n.$$

Similar recurrences for the T and U numbers are derived in the same way; writing $\Delta\epsilon_n = \epsilon_{n+1} - \epsilon_n$, the results may be summarized as follows:

$$\begin{aligned} D_{n+1} &= 2D_n + (D + 1)^n + \Delta\epsilon_n, \\ T_{n+1} &= 4T_n + (T + 1)^n + 3D_n, \\ U_{n+1} &= 6U_n + (U + 1)^n + 46T_n - 4T_{n+1} \\ &\qquad + 30D_n - 6D_{n+1} + 6\Delta\epsilon_n. \end{aligned} \tag{8}$$

The expressions in parentheses, it will be remembered, are shorthand binomial expansions; thus:

$$(D + 1)^n = \sum_{i=0}^{n} \binom{n}{i} D_i.$$

Relations with the Exponential Integers

The generating identities in equations 6.1 furnish immediate relations with the exponential integers, ϵ_n. Writing $\exp (e^t - 1)$ as $\exp t\epsilon$, as above, and passing from generating relations to coefficient relations, these results are as follows:

$$\begin{aligned} D_n &= \tfrac{1}{2}[(\epsilon + 2)^n - 2(\epsilon + 1)^n + \epsilon_n], \\ T_n &= \tfrac{1}{8}[(\epsilon + 4)^n - 6(\epsilon + 2)^n + 8(\epsilon + 1)^n - 3\epsilon_n], \\ U_n &= \tfrac{1}{8}[(\epsilon + 6)^n + 4(\epsilon + 5)^n - 15(\epsilon + 4)^n \\ &\qquad + 35(\epsilon + 2)^n - 36(\epsilon + 1)^n + 11\epsilon_n]. \end{aligned} \tag{9}$$

Expanding internal parentheses by the binomial theorem, the general result is as follows:

$$A_n = \sum_{i=1}^{n} \binom{n}{i} \alpha_i \epsilon_{n-i},$$

where the coefficients α_i for the three cases are as follows:

A_n	α_i
D_n	$2^{i-1} - 1$
T_n	$(2^{i-1} - 1)(2^{i-2} - 1)$
U_n	$\frac{1}{8}[6^i + 4 \cdot 5^i - 15 \cdot 4^i + 35 \cdot 2^i - 36]$

Note that in the first case $(D_n)\alpha_1 = 0$, in the second $(T_n)\alpha_1 = \alpha_2 = 0$, in the third $(U_n)\alpha_1 = \alpha_2 = \alpha_3 = 0$. Thus a given table of values of ϵ_n up to $n = k$ determines D_n up to $k + 2$, T_n up to $k + 3$, and U_n up to $k + 4$.

Somewhat simpler relations may be derived as follows. Repeated differentiation of the generating identity of the ϵ_n with respect to t, and passage from the generating relations to coefficient relations leads to the following:

$$\epsilon_{n+1} = (\epsilon + 1)^n,$$
$$\epsilon_{n+2} = (\epsilon + 1)^n + (\epsilon + 2)^n,$$
$$\epsilon_{n+3} = (\epsilon + 1)^n + 3(\epsilon + 2)^n + (\epsilon + 3)^n,$$

or, in general:

$$\epsilon_{n+m} = \sum_{x=1}^{m} (\epsilon + x)^n S_{x,\, m}.$$

This formula may be inverted by the reciprocal relations for the Stirling numbers of the first and second kinds [7] which run as follows: If

$$a_m = \sum_{x=1}^{m} b_x S_{x,\, m}$$

then

$$b_m = \sum_{x=1}^{m} a_x s_{x,\, m}$$

where $s_{x,\, m}$ is the Stirling number of the first kind defined by the recurrence relation

$$s_{x,\, m+1} = s_{x-1,\, m} - m s_{x,\, m}$$

and the boundary conditions $s_{m,\, m} = 1$, $s_{x,\, m} = 0$ $x > m$, $s_{0,\, m} = 0$, $m > 0$.

[7] Nielsen: "Handbuch der Gamma Funktion," Leipzig, 1906, p. 69.

The inverted formula[8] is:

$$(\epsilon + m)^n = \sum_{x=1}^{m} \epsilon_{n+x} s_{x,\,m}.$$

A short table of the Stirling numbers of the first kind follows:

$m \backslash x$	$s_{x,\,m}$ 0	1	2	3	4	5	6
0	1						
1	0	1					
2	0	−1	1				
3	0	2	−3	1			
4	0	−6	11	−6	1		
5	0	24	−50	35	−10	1	
6	0	−120	274	−225	85	−15	1

The three equations resulting from applying this transformation to equations (9) are as follows:

$$\begin{aligned} D_n &= \tfrac{1}{2}[\epsilon_{n+2} - 3\epsilon_{n+1} + \epsilon_n], \\ T_n &= \tfrac{1}{8}[\epsilon_{n+4} - 6\epsilon_{n+3} + 5\epsilon_{n+2} + 8\epsilon_{n+1} - 3\epsilon_n], \\ U_n &= \tfrac{1}{8}[\epsilon_{n+6} - 11\epsilon_{n+5} + 30\epsilon_{n+4} + 5\epsilon_{n+3} \\ &\qquad - 56\epsilon_{n+2} - 5\epsilon_{n+1} + 11\epsilon_n]. \end{aligned} \tag{10}$$

For computing purposes, values of ϵ_n and $\Delta\epsilon_n$ up to $n = 10$ are given in Table II.

TABLE II

Exponential Numbers

n	ϵ_n	$\Delta\epsilon_n$
0	1	0
1	1	1
2	2	3
3	5	10
4	15	37
5	52	151
6	203	674
7	877	3,263
8	4,140	17,007
9	21,147	94,828
10	115,975	562,595

[8] Noting that $\sum_{x=1}^{m} a^x s_{x,\,m} = (a)_m$, where $(a)_m$ is the factorial symbol used throughout, the inverse relation may also be written:

$$(\epsilon + m)^n = \epsilon^n(\epsilon)_m.$$

In this notation, the inverses to equations (2) for the impedance numbers have the following simple forms which are worth noting:

$$\begin{aligned} (D)_n &= d_n \\ (T)_n &= t_n \\ (U)_n &= u_n \end{aligned}$$

Congruences

For numerical checks, it is convenient to note the simplest congruences[9] for the three numbers. These follow from the Touchard congruence for the ϵ numbers[10] which runs as follows:

$$\epsilon_{p+n} \equiv \epsilon_{n+1} + \epsilon_n \mod p,$$

where p is a rational prime greater than 2.

Since by equations (10) each of the impedance numbers is a linear function of the ϵ numbers, each has a similar congruence as follows:

$$\begin{aligned} D_{p+n} &\equiv D_{n+1} + D_n \mod p, \\ T_{p+n} &\equiv T_{n+1} + T_n \mod p, \\ U_{p+n} &\equiv U_{n+1} + U_n \mod p. \end{aligned} \tag{11}$$

Special values for the first few congruences are as follows:

n	Remainder, mod p		
	D_{p+n}	T_{p+n}	U_{p+n}
0	0	0	0
1	1	0	0
2	7	3	0
3	37	36	60

These are sufficient for checking every value in Table I at least once and the values for $n = 5, 6, 7, 8$ are checked twice.

Acknowledgment

This paper arose as a result of a suggestion made by R. M. Foster on a former paper[11] and thanks are also due him for continuous counsel and critical scrutiny which have enlarged the boundary and sharpened the outline of the problem.

[9] The congruence $D_n = r \mod p$ is equivalent to the equation $D_n = mp + r$, where m is an integer; that is, r is the remainder after division by p (or the remainder plus some multiple of p).

[10] See E. T. Bell, "Iterated Exponential Integers," *Annals of Math.*, *39*, 3 (July, 1938), eq. 1.101, p. 541.

[11] "A Ladder Network Theorem," *Bell System Technical Journal 16*, pp. 303–318 (July, 1937); see especially footnote 3; I take this opportunity to draw attention to an error in that footnote: for four terminals (see Table I) there are 157, not 64, measurable impedances; hence the upper bound to the number of representations is 18,883,356,492, not 74,974,368.