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Electrostatic Electron-Optics

By FRANK GRAY

Certain types of electrostatic fields may be used as lenses to focus electron beams. The theory of these lenses is developed for electric fields that are symmetrical about a central axis. The introduction of two velocity functions exactly reduces the partial differential equations of electron motion to a series of ordinary differential equations. The first equation describes the action of a lens for electron paths near the axis; the remaining equations determine the higher order aberration terms. Sections on the following subjects are included: the general equations of electron-optics, thin lenses, thick lenses, aberration, the reduction of aberration, apertured plates, and concentric tubes. A list of symbols and lens equations is also included at the end of the article.

IN certain types of modern vacuum tubes, a beam of electrons is brought to a focus by an electrostatic field whose action on the beam is analogous to that of an optical lens on a beam of light. An electrostatic field which acts in this manner is called an electron lens. Such lenses are rapidly finding applications in amplifier tubes, television and oscillograph tubes, electron microscopes, and various types of experimental apparatus. As the extent of their application widens, the theory of these lenses naturally assumes a corresponding importance.

The first articles on the new science of electron-optics were published by Bush [1] in 1926–1927, and the next important step in its development was taken by Davisson and Calbick [2] and by Brüche and Johannson [3] working independently in 1931–1932. The following years marked an increased interest in the subject, with comprehensive articles by various authors, and its literature expanded rapidly. An

[1] H. Bush, *Ann. d. Physik, 81*, 974, 1926 and *Arch. f. Elecktrotech., 18*, 583, 1927.
[2] C. J. Davisson and C. J. Calbick, *Phys. Rev., 38*, 585, 1931 and *Phys. Rev., 42*, 580, 1932.
[3] E. Brüche and N. Johannson, *Ann. d. Physik, 15*, 145, 1932.

excellent review of this literature and the history of electron-optics are given in a symposium[4] of papers published in 1936, and the various practical applications of electron lenses are well described in the books on that subject.[5]

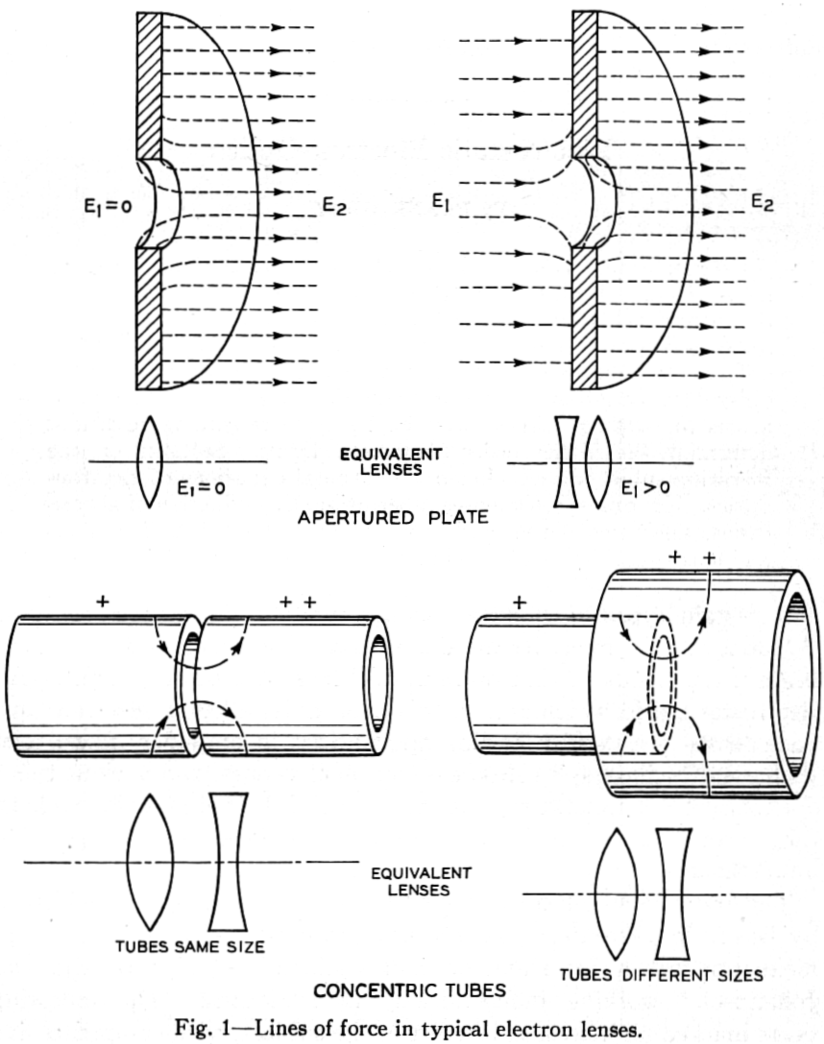


Fig. 1—Lines of force in typical electron lenses.

[4] *Zeit. f. techn. Physik*, *17*, 584–645, 1936.

[5] E. Brüche and O. Scherzer, *Geometrische Electronenoptik* (Springer, 1934). J. T. MacGregor-Morris and J. A. Henley, "Cathode Ray Oscillography" (Monographs on Electrical Engineering), 1936. Maloff and Epstein, "Electron Optics in Television" (McGraw-Hill, 1938).

The theory of electron-optics is thus well established and any further attempts at the subject must lead to substantially the same results. There is, however, a need for a precise development of the theory in a simpler manner. With this need in mind, the present article approaches the subject in a manner that appeals to the reader who is more familiar with electrical theory than he is with the concepts of geometrical optics, and this approach leads clearly to the various approximations that are needed in the development of the theory. With the aid of two velocity functions, the partial differential equations of electron motion are briefly and exactly reduced to a series of ordinary differential equations; the theory is then developed in terms of their approximate solutions.

Attention is confined to systems in which the electric fields are symmetrical about a central axis. In such systems any field having a radial component of electric intensity changes the radial velocity of an electron passing through it, and thus behaves—to some extent at least—as an electron lens. A uniform field parallel to the axis and field-free space are the only regions in which there is no lens action. Typical electron lenses are shown in the figures on the second page. As illustrated by these examples, a practical electron lens is characterized by a short region in which there is an abrupt change in the electric intensity parallel to the axis. Lines of force are continuous, and the field parallel to the axis can change only by lines of force coming into it, or going out from it, in a radial direction. In the region of the abrupt change, there are consequently strong radial fields which can deflect an electron in a radial direction. The region changes the focus of an electron beam passing through it, and its action is analogous to that of an optical lens.

Section I—The General Equations

In the present paper it is assumed that the initial electron source has perfect symmetry of form about the central axis, and that the electrons have no appreciable velocities of emission from the source. An electron thus has no angular velocity about the axis, and its motion may be described in terms of a coordinate z taken along the axis and a radial coordinate r measured from the axis.

If an electron's velocity vector is projected at any point along its path, it intersects the axis at some point p, as illustrated in Fig. 2, and the electron may be regarded as instantaneously moving either away from, or else toward, this point of intersection. The distance d along the axis from the electron to the point of intersection is called the instantaneous focal distance.[6] Defined in this manner, the focal distance conforms with the optics of light; it is positive when an electron is moving toward a focal point; and is negative when the

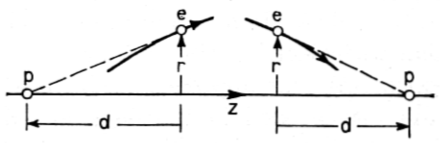


Fig. 2—Focal distance.

electron is moving away from such a point. From the geometry of the figure, it is seen that

$$d = -\frac{r\dot{z}}{\dot{r}}, \tag{1}$$

where $\dot{r}$ and $\dot{z}$ are the instantaneous components of electron velocity.

The focal distance of an electron varies continuously as the electron moves along. The simplest variation occurs in field-free space, where the electron travels in a straight line and the focal point remains stationary; but even then the focal distance varies as the electron moves; for the focal distance is measured from the moving electron to the stationary focal point, as illustrated in Fig. 3. In an electron

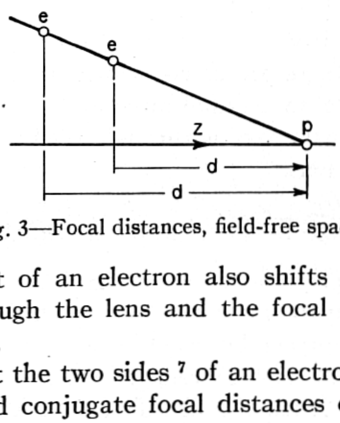


Fig. 3—Focal distances, field-free space.

lens, the focal point of an electron also shifts continuously as the electron moves through the lens and the focal distance varies in a complicated manner.

The values of d at the two sides [7] of an electron lens, for any electron path, are called conjugate focal distances of the lens, and are usually designated as d_1 and d_2. The theory of electron-optics is largely concerned with the derivation of an equation relating these conjugate focal distances.

[6] The term is here used in a broad sense to include the distance to any intersection point on the z-axis, even though the latter is not the point of convergence of an electron beam.

[7] The value of d as an electron enters the non-uniform field of the lens, and the value of d as it leaves the non-uniform field.

Before passing on to such a derivation, it is well to introduce another quantity, which is analogous to focal distance and very useful in making approximations. Suppose that, from any point along its path, an electron were to continue on with its instantaneous velocity in a straight line. Its velocity along the axis would continue to have the instantaneous value $\dot{z}$, and the electron would travel over the distance d and arrive at the focal point in a period of time T given by

$$T = d/\dot{z} \tag{2}$$

or from equation 1

$$T = -r/\dot{r}. \tag{3}$$

This period of time is analogous to focal distance, and we therefore call it focal time. The values of T at the two sides of an electron lens, for any electron path, are in a corresponding manner called conjugate focal times of the lens.

To obtain an equation relating the conjugate focal distances of a lens, we must consider the path of an electron through the lens. The path is determined by the initial velocity and coordinates of the electron as it enters the lens and by its acceleration in the electric field of the lens. By defining electrical units in the proper manner the ratio e/m is eliminated from the equations of acceleration and they assume the simple form

$$\ddot{r} = \frac{\partial \Phi}{\partial r}, \tag{4}$$

$$\ddot{z} = \frac{\partial \Phi}{\partial z}, \tag{5}$$

where Φ is the potential at points in space.[8]

The first solution of these equations gives the well known energy relation

$$\dot{r}^2 + \dot{z}^2 = 2\Phi, \tag{6}$$

where the electron source is taken as zero potential.

With the exception of special cases, the equations are not further soluble in the usual sense, and one resorts to solution in series.

As they stand, the two equations for acceleration are inconvenient; they involve partial derivatives of potential with respect to space and ordinary derivatives of velocity with respect to time, and the latter cannot be transformed to partial derivatives with respect to space, for the simple reason that the velocity of an electron does not exist

[8] The final equations of electron-optics involve the potentials only in the form of ratios which are independent of the electrical units.

at points off its path. The equations may, however, be reduced to a more convenient form by the introduction of two velocity functions [9] defined as follows.

Let u and w be any two functions of r and z that satisfy the equations

$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} = 0, \tag{7}$$

$$u^2 + w^2 = 2\Phi. \tag{8}$$

Consider now an imaginary point moving with velocity components

$$\dot{r} = u, \qquad \dot{z} = w. \tag{9}$$

The derivative of $\dot{r}$ with respect to time is

$$\ddot{r} = \frac{\partial u}{\partial r}\dot{r} + \frac{\partial u}{\partial z}\dot{z} = \frac{\partial u}{\partial r}u + \frac{\partial u}{\partial z}w \tag{10}$$

and from equations 7 and 8

$$\ddot{r} = \frac{\partial u}{\partial r}u + \frac{\partial w}{\partial r}w = \frac{\partial \Phi}{\partial r} \tag{11}$$

the component $\dot{r}$ thus satisfies differential equation (4) for electron motion. In a similar manner it may be shown that the velocity component $\dot{z}$ satisfies equation (5). The motion of the imaginary point is thus the same as the motion of an electron, and the velocity functions u and w are therefore the velocity components of electron motion.

The velocity functions are solutions of equations 7 and 8, one of which is a simple algebraic equation and the other a partial differential equation with respect to space alone. The inconvenient time derivatives have been eliminated in these new equations for electron velocity.

The existence of a velocity function is not confined to a single electron path; it exists over the electric field in general. Any pair of particular solutions for u and w thus corresponds to an infinite number of possible electron paths. In the converse manner, there are an infinite number of particular solutions for any electric field, and there is a pair of particular solutions corresponding to any given electron path through the field.[10]

[9] These functions are the components of the generalized vector function described in Appendix 4.

[10] The existence of such solutions is proved by the existence of the series solutions, which are derived in the following pages.

Solutions for the velocity functions are obtained by expressing them as power series in r.

$$u = Ar + Br^3 + Cr^5 + \cdots, \tag{12}$$

$$w = a + br^2 + cr^4 + \cdots, \tag{13}$$

where the coefficients are functions of z alone. The above powers of r are the ones required in a system symmetrical about the z-axis. In such a system $\dot{r}$ reverses in sign with r and the u-series is odd; $\dot{z}$ does not reverse sign with r and the w-series is even. Aside from such reasoning, the choice of the two series is justified provided they lead to solutions of the differential equation in a form suitable for the purposes of electron-optics.

The potential Φ obeys the equation

$$\Delta\Phi = \frac{\partial^2\Phi}{\partial z^2} + \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} = 0 \tag{14}$$

and it may likewise be expressed as a power series in r. This well known series is

$$\Phi = v - \frac{v''}{2^2}r^2 + \frac{v''''}{(2\cdot 4)^2}r^4 \cdots, \tag{15}$$

where v is the potential on the axis of the system, and the primes indicate differentiation with respect to z.

On substituting the three series in equations 7 and 8 and equating the coefficients of the various powers of r in each equation we obtain a series of ordinary differential equations for the coefficients of the u-series.

$$\sqrt{2v}A' + A^2 = -\frac{v''}{2}, \tag{16}$$

$$\sqrt{2v}B' + 4AB = \frac{v''''}{16} - \frac{(A')^2}{2}, \tag{17}$$

$$\sqrt{2v}C' + 6AC = -\frac{v''''''}{384} - 3B^2 - 3/4A'B', \tag{18}$$

$$\cdots\cdots\cdots\cdots$$

and the coefficients of the w-series are

$$\begin{aligned} a &= \sqrt{2v}, \\ b &= A'/2, \\ c &= B'/4. \\ &\cdots \end{aligned} \tag{19}$$

The solution of the partial differential equations for electron velocity is thus reduced to the solution of a series of ordinary differential equations, which in themselves contain no approximations.

From equation 1, the inverse focal distance is now obtained by dividing u by r and w, which gives

$$\frac{1}{d} = -\frac{A + Br^2 + Cr^4 + \cdots}{\sqrt{2v} + \frac{A'}{2}r^2 + \frac{B'}{4}r^4 + \cdots}. \tag{20}$$

This is the general equation for focal distance as it is affected by aberration. In using this equation, we are at liberty to set the higher coefficients equal to zero at the incident side of the lens. This determines the initial value of A in terms of the first conjugate focal distance. The second conjugate focal distance is then determined by solving for the coefficients at the exit side of the lens. Due to the presence of the terms in r, this second focal distance varies slightly with the radial distance at which an electron passes through the lens, and the focus is therefore diffused along the axis. This diffusion of the focus is called aberration.

The coefficient A is of particular importance in the theory of electron-optics. For paraxial rays, that is, rays near the axis, the higher terms in the two series are negligibly small compared to their first terms, and for such rays

$$\frac{1}{d} = -\frac{A}{\sqrt{2v}}. \tag{21}$$

With the exception of aberration, the single coefficient A thus determines the complete performance of a lens, and the principal constants of a lens are determined by its differential equation alone. In lenses where the rays are confined to a region near the axis with proper diaphragms, the aberration terms are small and the coefficient A describes the performance of a lens sufficiently well.

The next section is devoted to the derivation of the principal lens equations from this coefficient. The aberration terms are considered only in the last section of the paper.

Section II—Rays Near the Axis

For rays near the axis the optical characteristics of an electric field are determined by the differential equation for A alone,

$$\sqrt{2v}A' + A^2 = -\frac{v''}{2}. \tag{16}$$

For such rays the higher terms in r may be neglected in the general equations and we obtain the following useful relations

$$A = \dot{r}/r = -\frac{\sqrt{2v}}{d}, \tag{22}$$

$$\dot{z} = \sqrt{2v}, \tag{23}$$

$$T = d/\sqrt{2v} = -\frac{1}{A}, \tag{24}$$

$$dt = dz/\dot{z} = dz/\sqrt{2v}. \tag{25}$$

A Uniform Electric Field

A uniform electric field parallel to the axis is not usually regarded as an electron lens,[11] but it does shift the focal point of a beam of electrons passing through it. In a uniform field, v'' is zero and the differential equation for A may be written in the form

$$\frac{dA}{A^2} = -\frac{dz}{\sqrt{2v}}. \tag{26}$$

An integration of this equation from any point z_1 to any other point z_2, in the uniform field, gives

$$\frac{1}{A_2} - \frac{1}{A_1} = \frac{2(z_2 - z_1)}{\sqrt{2v_2} + \sqrt{2v_1}}, \tag{27}$$

where A_1 and A_2 are the values of A at z_1 and z_2. On substituting $-\sqrt{2v}/d$ for A in this equation, it may be transformed to

$$\left(1 + \sqrt{\frac{v_2}{v_1}}\right) d_1 - \left(1 + \sqrt{\frac{v_1}{v_2}}\right) d_2 = 2(z_2 - z_1), \tag{28}$$

which is the equation relating the conjugate focal distances at any two planes—located at z_1 and z_2—in the uniform field.

The shift in the focal point of an electron beam as it passes through a uniform field is illustrated in Fig. 4.

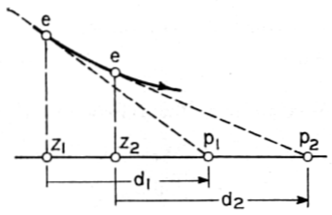


Fig. 4—Focal distances in a uniform field.

[11] Electron rays parallel to the axis are not bent by the field, and it does not magnify an electron image.

Thin Lenses

Approximate solutions of the differential equation 16 for A are obtained more clearly by first changing the space variables to time variables. This is done by using relations 24 and 25, which transform the equation to

$$\frac{1}{T^2}\left(\frac{dT}{dt} + 1\right) = -\frac{v''}{2} \tag{29}$$

or

$$\frac{1}{T^2}\frac{d}{dt}(T + t) = -\frac{v''}{2}. \tag{30}$$

The new equation tells how the focal time T varies with time as an electron moves along.[12]

A thin lens is defined as a region of non-uniform field extending over such a short distance along the axis that an electron traverses it in a period of time small compared to the focal times involved: the thickness of the lens is small compared to the conjugate focal distances. By taking the origin of time t at the middle of an electron's period of transit through a lens, t in the lens is not greater than half the period of transit, and t may therefore be neglected in comparison to T in a thin lens. With this approximation in equation 31, it reduces to

$$\frac{d}{dt}\left(\frac{1}{T}\right) = \frac{v''}{2}. \tag{31}$$

In integrating this equation through a lens we choose two points z_1 and z_2 at the approximate boundaries of the non-uniform field, that is, the points where v'' substantially drops to zero as illustrated in Fig. 5. Then, remembering that dt is $dz/\sqrt{2v}$, an integration from z_1

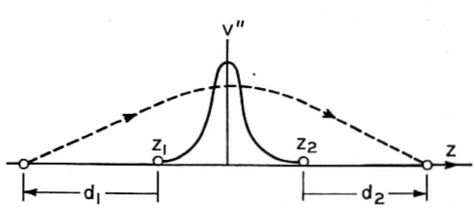


Fig. 5—Conjugate focal distances, thin lens.

to z_2 gives

$$\frac{1}{T_2} - \frac{1}{T_1} = \frac{1}{F}, \tag{32}$$

[12] The period of time that a train requires to reach its terminal point also varies with time as the train moves along.

where the inverse focal term is

$$\frac{1}{F} = \int_{z_1}^{z_2} \frac{v''}{2\sqrt{2v}} dz \tag{33}$$

or on integration by parts

$$\frac{1}{F} = \frac{1}{2}\left[\left(\frac{v'}{\sqrt{2v}}\right)_2 - \left(\frac{v'}{\sqrt{2v}}\right)_1\right] + \frac{1}{2}\int_{z}^{z_2} (v')^2(2v)^{-3/2}dz. \tag{34}$$

The substitution in equation 32 of the values for T_1 and T_2 as given by equation 24 now gives the lens equation

$$\frac{\sqrt{2v_2}}{d_2} - \frac{\sqrt{2v_1}}{d_1} = \frac{1}{F}. \tag{35}$$

This equation is analogous to the equation for a thin optical lens

$$\frac{\mu_2}{d_2} - \frac{\mu_1}{d_1} = \frac{1}{F}, \tag{36}$$

bounded on its two sides by media with different refractive indices μ_1 and μ_2, the $\sqrt{2v}$ corresponding to refractive index.

Electron rays parallel to the axis do not come to a focus at a distance F from an electron lens; in other words, F is not a principal focal distance. There are, in general, two principal focal points on opposite sides of an electron lens. Their principal focal distances f_1 and f_2 are found by setting first d_1, and then d_2, equal to infinity in equation 35. This gives

$$f_1 = -\sqrt{2v_1}F, \qquad f_2 = \sqrt{2v_2}F \tag{37}$$

as the two principal focal distances. It may be shown from equation 33 that these principal focal distances really involve the voltages only in the form of the ratio v_2/v_1. By substituting them in the lens equation 35, it may be written in the convenient form

$$\frac{f_2}{d_2} + \frac{f_1}{d_1} = 1, \tag{38}$$

which likewise involves the voltages only in the form of a ratio.

There are two types of electron lenses that deserve special consideration. The first is a small aperture in a thin plate separating two uniform fields of different intensities—as a special case one of the fields may be zero. An example of such a lens is illustrated in Fig. 6.

In this type of lens, the non-uniform field at the aperture covers a distance along the axis about equal to its diameter.[13] If the diameter is small compared to v/v', there is little change in potential throughout

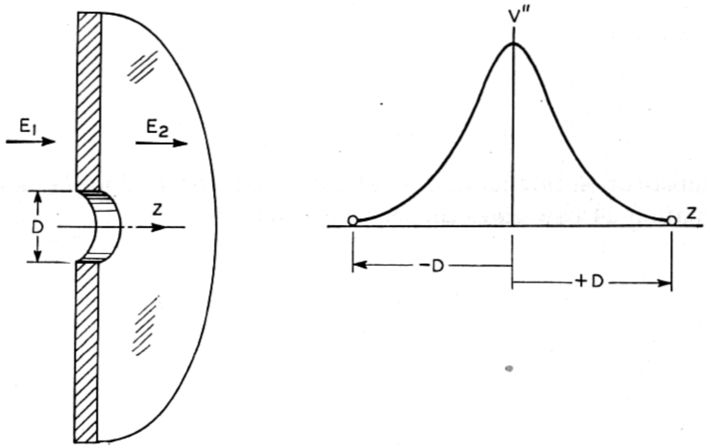


Fig. 6—An apertured plate.

the lens and the $\sqrt{2v}$ may be considered as a constant in the integration 33 for the inverse focal term. With this approximation,

$$\frac{1}{F} = \frac{1}{2}\left[\frac{v_2' - v_1'}{\sqrt{2v}}\right], \tag{39}$$

when v is the potential of the plate and v_1' and v_2' are the electric intensities of the two uniform fields. The lens equation 35 is then

$$\frac{\sqrt{2v}}{d_2} - \frac{\sqrt{2v}}{d_1} = \frac{1}{2}\left[\frac{v_2' - v_1'}{\sqrt{2v}}\right], \tag{40}$$

which may be written in the simpler form

$$\frac{1}{d_2} - \frac{1}{d_1} = \frac{v_2' - v_1'}{4v}. \tag{41}$$

In this type of lens, the electrical refractive index $\sqrt{2v}$ is the same on both sides of the lens and the two principal focal distances are equal, just as they are for a thin optical lens when it is bounded by air on both sides.[14]

[13] "Two Problems in Potential Theory," T. C. Fry, *Bell Telephone System Monograph B-671.*

[14] A complete electron-optical system usually involves a combination of lenses. The calculations for a combination are illustrated by the example in Appendix 1.

The apertured plate between two uniform fields is the only lens that permits such a simple calculation of focal distances. In all other lens structures the potential varies appreciably throughout the lens and the integration for the focal term is complicated. The actual numerical calculations have been carried out for only a few of these cases.

The second type of lens deserving special consideration is a lens bounded on both sides by field-free space. For such boundaries the first term in the last member of equation 34 vanishes, and $1/F$ is determined by the integral term alone. This integral in inherently positive, and a lens bounded on both sides by field-free space is thus always a convergent lens. The two concentric tubes of Fig. 7 give

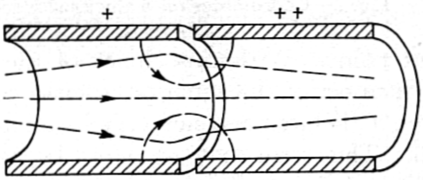


Fig. 7—Concentric tubes—lines of force and electron paths.

a lens of this type, the electric field in each tube dropping to zero at a short distance from its end. It is true that there is a divergent field of the same intensity as the convergent field; but an electron is at a higher potential in the divergent field and traveling faster, so it receives a smaller radial deflection in that field and the lens is convergent. It is interesting to note that the lens is still positive even when the potentials on the electrodes are reversed; in other words, a lens of this type is positive irrespective of the direction of the electric field.[15]

An Approximation for Certain Thick Lenses

In certain electron lenses there is a short region of strong lens action accompanied by more extended regions of weaker action; the large values of the derivative v'' are confined to a short distance along the axis, but the derivative does have appreciable values over a more extended region. A lens of this type can be treated in the following approximate manner, provided that there is but one maximum of $|v''|$ in the lens.

For this purpose, the differential equation 31 is rewritten in the form

$$d\left(\frac{1}{T+t}\right) = \frac{v''}{2}\left(1 + \frac{t}{T}\right)^{-2} dt, \qquad dt = dz/\sqrt{2v} \tag{42}$$

[15] The principal focal distances of concentric tubes are calculated in Appendix 2.

and the lens equation is derived by integrating it from a point z_1 to a point z_2, where the two points are taken at the substantial boundaries of the non-uniform field. In carrying out this integration, the origin of time is taken at the instant that the electron is at the maximum of $|v''|$, as illustrated in Fig. 8, and for convenience the origin of z is

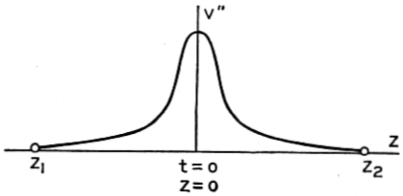


Fig. 8—Coordinates for a thick lens.

also taken at that point. With this choice of the origin, the term t/T in the second member of the equation is small compared to unity in the region where v'' is large and not very important in the regions where v'' is small. This term may therefore be neglected in lenses when the time of transit is not too great a fraction of the focal times involved. The integration of the equation then gives

$$\frac{1}{T_2 + t_2} - \frac{1}{T_1 + t_1} = \frac{1}{F} \tag{43}$$

when the inverse focal term is again

$$\frac{1}{F} = \int_{z_1}^{z_2} \frac{v''}{2\sqrt{2v}}\, dz \tag{44}$$

and

$$t_2 = \int_0^{z_2} \frac{dz}{\sqrt{2v}}, \qquad t_1 = \int_0^{z_1} \frac{dz}{\sqrt{2v}}. \tag{45}$$

A transformation to space variables by means of equations 24 and 25 gives the lens equation in a form analogous to that for a thick optical lens,

$$\frac{\sqrt{2v_2}}{d_2 - \alpha_2} - \frac{\sqrt{2v_1}}{d_1 - \alpha_1} = \frac{1}{F}, \tag{46}$$

where

$$\begin{aligned} \alpha_2 &= -\sqrt{2v_2}t_2 = -\int_0^{z_2} \sqrt{\frac{v_2}{v}}\, dz, \\ \alpha_1 &= -\sqrt{2v_1}t_1 = -\int_0^{z_1} \sqrt{\frac{v_1}{v}}\, dz. \end{aligned} \tag{47}$$

A plane located at a distance α_1 from the point z_1 is the approximate first principal plane of the lens; and a plane located at a distance α_2 from the point z_2 is the approximate second principal plane of the lens. In the lens equation, $d_1 - \alpha_1$ and $d_2 - \alpha_2$ are the conjugate focal distances measured from the principal planes. If the focal distances measured in this manner are designated as D_1 and D_2 respectively, the lens equation assumes the simpler form

$$\frac{\sqrt{2v_2}}{D_2} - \frac{\sqrt{2v_1}}{D_1} = \frac{1}{F}. \tag{48}$$

An electron lens frequently has both a positive and a negative maximum of v'', and the preceding approximation cannot be applied to the lens as a whole. There is, however, necessarily a point between the two maxima where v'' is zero and by taking this as a division point, the lens can be separated into two components. The approximation can then be separately applied to each component, and the whole lens treated as a combination of two lenses.

The General Theory of Thick Lenses

The equation for the coefficient A,

$$\frac{dA}{dz} + \frac{A^2}{\sqrt{2v}} = -\frac{v''}{2\sqrt{2v}},$$

is a Racciti equation and, with the exception of special cases, it has no exact solution in the usual sense. Particular solutions can be obtained only by integration in series. It is, however, possible to express the general solution of a Racciti equation in terms of any two particular solutions, and this property enables us to develop the general theory of a thick lens in terms of its principal focal distances.

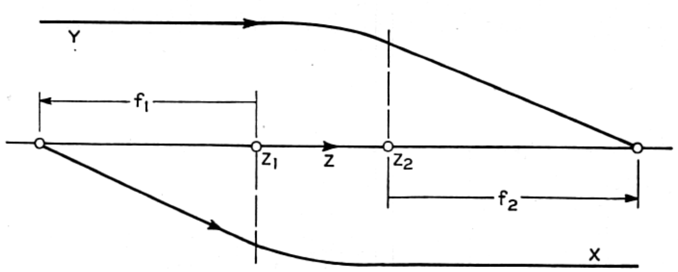


Fig. 9—Paths corresponding to X and Y.

In considering a thick lens, two points z_1 and z_2 are again taken at the substantial boundaries of the non-uniform field constituting the lens. The differential equation for A necessarily has a particular solution equal to zero at z_1. This solution is designated as Y, and it corresponds to an electron ray entering the lens parallel to the axis. At z_2 this solution is equal to $-\sqrt{2v_2}/f_2$, where f_2 is the second principal focal distance measured from z_2. The path of such a ray is illustrated in Fig. 9. This particular solution obeys the same differential equation as A. By subtracting the differential equation of A from that of Y and making a slight transformation, we obtain

$$\frac{d}{dz}\log(A - Y) = -\frac{A + Y}{\sqrt{2v}}, \tag{49}$$

and it should be noted that

$$A/\sqrt{2v} = \frac{\dot{r}}{r\sqrt{2v}} = \frac{d}{dz}\log r. \tag{50}$$

An integration from z_1 to z_2 and a transformation to focal distances then gives the relation

$$\frac{(f_2 - d_2)d_1}{f_2 d_2} = k_2\sqrt{\frac{v_1}{v_2}\frac{r_1}{r_2}}, \tag{51}$$

where k_2 is a constant of the lens, given by

$$1/k_2 = \exp.\int_{z_1}^{z_2}\frac{Y dz}{\sqrt{2v}}. \tag{52}$$

By proceeding in the same manner with a particular solution X for a ray leaving the lens parallel to the axis, we obtain a second relation

$$\frac{(f_1 - d_1)d_2}{f_1 d_1} = k_1\sqrt{\frac{v_2}{v_1}\frac{r_2}{r_1}}, \tag{53}$$

where

$$k_1 = \exp.\int_{z_1}^{z_2}\frac{X dz}{\sqrt{2v}}. \tag{54}$$

The differential equations of X and Y may also be subtracted and integrated, and this gives a third relation

$$f_1/f_2 = -\frac{k_2}{k_1}\sqrt{\frac{v_1}{v_2}}. \tag{55}$$

A multiplication of the first two relations 51 and 53 gives

$$(f_2 - d_2)(f_1 - d_1) = k_1 k_2 f_1 f_2, \tag{56}$$

which is one form of the equation relating the conjugate focal distances of a lens. This equation may be converted into a more useful form by the following considerations.

A combination of the three preceding relations gives

$$\frac{r_2}{d_2}(d_2 - \alpha_2) = \frac{r_1}{d_1}(d_1 - \alpha_1), \tag{57}$$

where

$$\alpha_1 = f_1(1 - k_1), \qquad \alpha_2 = f_2(1 - k_2). \tag{58}$$

To interpret this equation, we erect two imaginary planes as shown in Fig. 10. The first plane is located at a distance α_1 from z_1. If the

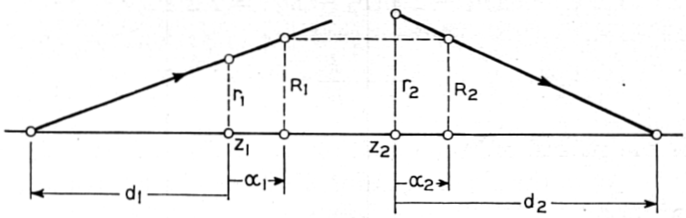


Fig. 10—The principal planes.

path of the incident ray is projected it intersects this plane at some radial distance R_1. The second plane is erected at a distance α_2 from z_2. The path of the exit ray intersects it at a radial distance R_2. The equation says—from simple geometry—that the two radial distances R_1 and R_2 are equal. The path of an electron through the lens is therefore the same as if the electron proceeded in a straight line to the first plane, passed parallel to the axis to the second plane, and then proceeded again in a straight line to the second conjugate focal point. These two planes are called the first and second principal planes of the lens. The action of a thick lens is the same as if the space between the principal planes were non-existent, leaving them in coincidence, and a thin lens were located at the plane of coincidence.

The principal planes of a lens may lie either inside or outside of the lens. In most convergent lenses, α_1 is positive and α_2 negative, and the two planes both lie inside the lens.

The first conjugate focal distance measured from the first principal plane is designated as D_1, and the second conjugate focal distance measured from the second principal plane is designated as D_2. When they are measured in this manner, the two conjugate distances are

$$D_1 = d_1 - \alpha_1, \qquad D_2 = d_2 - \alpha_2. \tag{59}$$

The two principal focal distances measured from the principal planes are, in a similar manner, designated as F_1 and F_2; then, from equation 58,

$$\begin{aligned} F_1 &= f_1 - \alpha_1 = k_1 f_1, \\ F_2 &= f_2 - \alpha_2 = k_2 f_2, \end{aligned} \tag{60}$$

and, from equation 55,

$$F_1/F_2 = -\sqrt{\frac{2v_1}{2v_2}}. \tag{61}$$

Substitution of the new focal distances in the lens equation 56 now gives

$$(F_2 - D_2)(F_1 - D_1) = F_1 F_2 \tag{62}$$

or

$$\frac{F_2}{D_2} + \frac{F_1}{D_1} = 1. \tag{63}$$

This is the general equation relating the conjugate focal distances in any lens. With the aid of equation 61, it may be written in the more familiar form

$$\frac{\sqrt{2v_2}}{D_2} - \frac{\sqrt{2v_1}}{D_1} = \frac{1}{F}, \tag{64}$$

where

$$\frac{1}{F} = \frac{\sqrt{2v_2}}{F_2} = -\frac{\sqrt{2v_1}}{F_1}. \tag{65}$$

The Principal Points of a Lens

The points locating the two principal planes on the axis of a lens and its two principal focal points are called the cardinal points of the lens. The preceding theory of a thick lens shows that its performance is completely determined by the locations of these four points.[16] The theory does not furnish a general method for calculating their locations, but it does show that they can be determined from a knowledge of two so-called principal rays. The first is a ray leaving the lens parallel to the axis. If its entrance and exit paths are projected, they intersect as shown in Fig. 11, and the intersection locates the first principal plane. The projected incident ray also intersects the axis, and this intersection locates the first principal focal point. The second principal plane and the second principal focal point may be located in a similar manner from the entrance and exit paths of a ray entering the lens parallel to the axis.

[16] We are here speaking only of rays near the axis.

The required paths of the two rays must in general be determined either by a series or step-by-step integration of the differential equation for A, or else by actual measurements on the physical lens.[17]

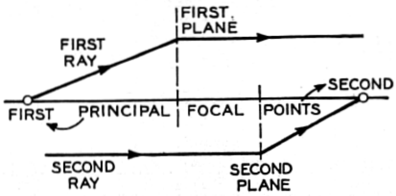


Fig. 11—The cardinal points of a lens.

Magnification

Electron object and electron image are defined as their optical analogies. The electron object may be an actual source of electrons, or the real image of such a source, or it may be a virtual image from which the electrons are apparently coming as they enter a lens.

The magnification by an electron lens may be treated in the following manner. Let S_1 be the size of an electron object located at a distance D_1 from the first principal plane of a lens. Two electron rays from the edge of the object are considered—as shown in Fig. 12. The ray

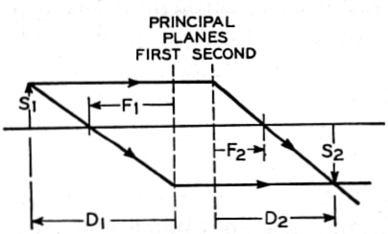


Fig. 12—Magnification.

entering the lens parallel to the axis may be regarded as passing on to the second principal plane and then bending sharply to pass through the second principal focal point; the ray through the first principal focal point may be considered as passing on to the first principal plane and then proceeding parallel to the axis. The intersection of the two rays locates the electron image and determines its size S_2. The magnification M is defined as S_2/S_1, and it follows from simple geometry that

$$M = \frac{D_2 - F_2}{F_2} = \frac{F_1}{D_1 - F_1}. \tag{65}$$

[17] Other step-by-step methods can be used when a map of the equipotential surface is available.

A more convenient expression for magnification is now obtained by combining the two preceding expressions to give

$$M = \frac{F_1 D_2}{F_2 D_1} \tag{66}$$

and from equation 61

$$M = -\sqrt{\frac{2v_1}{2v_2}}\frac{D_2}{D_1}. \tag{67}$$

The magnification is not in general equal to the ratio of the image distance to the object distance, as it is for an optical lens in air. It is only equal to that ratio when the voltage is the same on both sides of the lens.

Section III—Aberration in a Lens

Returning to the first section, we see that the general expression for focal distance is

$$\frac{1}{d} = -\frac{A + Br^2 + Cr^4 \cdots}{\sqrt{2v} + \frac{A'}{2}r^2 + \frac{B'}{4}r^4 \cdots}. \tag{20}$$

The exact focal distance of an electron thus depends on its radial coordinate r, and a ray passing through a lens at a distance from the axis does not come to the same focus as a ray near the axis. A precise, general theory for rays at a distance from the axis could—in theory at least—be derived by solving the differential equations for as many of the higher coefficients as desired and substituting them in the above equation. Such a general solution would, however, be very difficult indeed, and one is content—as he usually is in optics—to treat the performance of a lens in a much more restricted manner.

The equation for focal distance can be simplified to some extent by noting that its denominator is the velocity component $\dot{z}$. With the aid of the energy equation 6, this component can be written in the form

$$\dot{z} = \sqrt{2v}\left[1 + \left(\frac{r}{d}\right)^2\right]^{-1/2}. \tag{68}$$

In most lenses r is small compared to d, and the last factor in the above equation may be approximately set equal to unity. This approximation is accurate to one per cent even for a lens with an angular aperture corresponding to F3.5—an F2 lens is a very fast camera lens. With this approximation the inverse focal distance is

$$\frac{1}{d} = -\frac{A + Br^2 + Cr^4 + \cdots}{\sqrt{2v}}. \tag{69}$$

The presence of the terms in r causes a diffusion of the focus in a lens, and a clearer picture of this diffusion is obtained by expressing it as lateral aberration. So we now proceed to derive an expression for this aberration, and the meaning of the term becomes apparent from the derivation. For this purpose we consider electrons entering a lens as if they all came from a point source at a distance d_1 from the first side of the lens. We are at liberty to set the higher coefficients equal to zero at that side of the lens, and this gives

$$A_1 = -\frac{\sqrt{2v_1}}{d_1}, \tag{70}$$

$$B_1 = C_1 = \cdots = 0.$$

At the exit side of the lens, the focal distance is

$$\frac{\sqrt{2v_2}}{d_2} = -(A_2 + B_2r^2 + C_2r^4 + \cdots), \tag{71}$$

where the coefficients are solutions of their differential equations subject to the initial conditions 70. The focal distance d_0 for rays near the axis is given by

$$\frac{\sqrt{2v_2}}{d_0} = -A_2. \tag{72}$$

The difference between the focal distance d_2 of a ray leaving the lens at a distance r from the axis and the focal distance d_0 of a ray near the axis is

$$d_2 - d_0 = \frac{d_0^2}{\sqrt{2v_2}}(B_2r^2 + C_2r^4 + \cdots). \tag{73}$$

This difference is called the longitudinal aberration of the lens. It is the distance that the focal point is diffused along the axis, when the lens is limited by an exit diaphragm or radius r.

If a screen is placed at a distance d_0 from the lens, rays near the axis will come to a point focus on the screen; but rays leaving the lens at a distance r from the axis will strike the screen along a circular line. The radius s of this circle is called the lateral aberration of the lens. It follows rather simply, from the value of the longitudinal aberration, that the lateral aberration is

$$s = \frac{d_0}{\sqrt{2v_2}}(B_2r^3 + C_2r^5 \cdots). \tag{74}$$

This is the radius of the diffuse image of a point source, when the lens is limited by a diaphragm of radius r.

The differential equations 17, 18 $\cdots$ for the aberration coefficients are linear and subject to solution in the usual manner when A and v are known functions of z. The solutions for the higher coefficients would of course be quite involved. The higher terms are, however, small compared to the second term, which causes most of the aberration, and the approximate distortion is given by the second term alone. This term is called the second order aberration term.

The Reduction of Aberration

The coefficient of any aberration term vanishes when conditions are arranged so that the last member of its differential equation is zero, for the coefficient may be arbitrarily set equal to zero at the first side of the lens, and the solution of its linear equation is then zero throughout the lens.

The important second order aberration term can thus be made to vanish by arranging conditions so that

$$\frac{V''''}{16} - \frac{(A')^2}{2} = 0. \tag{75}$$

In a lens that is not too thick compared to the focal distances involved, we have seen that the term A^2 may be neglected in the differential equation for A, and

$$A' = -\frac{v''}{2\sqrt{2v}}. \tag{76}$$

The substitution of this value for A' in the above equation gives

$$v'''' - \frac{(v'')^2}{v} = 0 \tag{77}$$

as the differential equation for electric fields that are approximately free from second order aberration, when the focal distances are reasonably large compared to the length of the field along the axis.

The general solution of this equation is a series solution, but several particular solutions have been obtained in terms of known functions. The potentials corresponding to these particular solutions are given by the following equations:

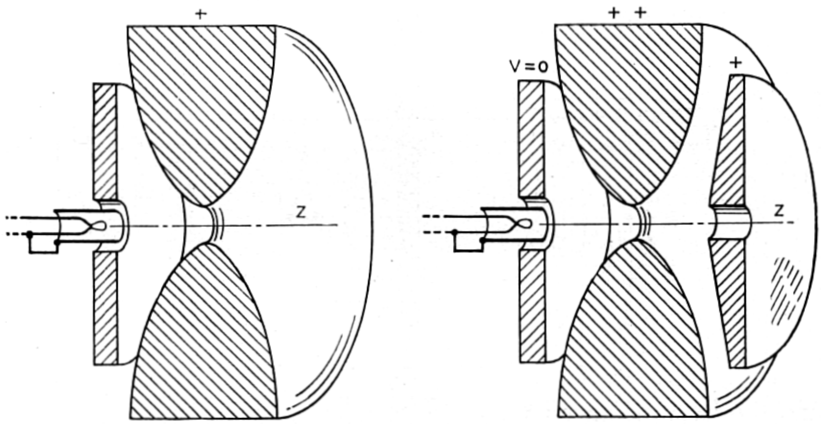
$$\Phi = ae^{\pm\omega z}J_0(\omega r), \tag{78}$$

$$\Phi = (a \sin \omega z + b \cos \omega z)J_0(i\omega r), \tag{79}$$

$$\Phi = (a \sinh \omega z + b \cosh \omega z)J_0(\omega r), \tag{80}$$

$$\Phi = 3az^{3/2}\left[\frac{1}{3} - \frac{1}{4}\left(\frac{r}{2z}\right)^2 + \frac{3}{64}\left(\frac{r}{2z}\right)^4 \cdots\right]. \tag{81}$$

Any one of these electric fields can be produced by shaping and positioning electrodes to correspond with two of its equipotential surfaces. These fields are, however, in general not well adapted to production with practical electrode structures. The one exception is the field defined by equation 79, and electrodes for producing it in a practical form are shown in Fig. 13. They are suitable for giving an



Figs. 13, 14—Lenses with reduced aberration.

electron stream its initial acceleration. The electric field constitutes a divergent lens, as do practically all initial accelerating fields.

As expressed by equation 79, this field is followed by a symmetrically reversed field, and for some purposes it may be desirable to include the reversed field. This is done by locating a low potential electrode along its corresponding equipotential surface as shown in Fig. 14. A small aperture may be cut in this electrode for the passage of electrons. The aperture then acts as a lens to bring the beam to a focus, but this lens has its own aberration, and the whole system is then only partially free from first order aberration.

Appendix I—Calculations for a Complete System

The electrode arrangement of Fig. 15 is chosen for giving a simple example of the calculations for a complete optical system. The final focal distance is found by calculating the focal distances at the points m, n, o, p in succession. Electrons leave the cathode and travel parallel to the axis in the uniform field between the first and second plates, so their focal distance is $-\infty$ when they arrive at the point m. The electrons then pass through the aperture in the second plate, and their focal distance at n is calculated from the lens equation 41, which gives

$$\frac{1}{d_n} + \frac{1}{\infty} = \frac{1}{4v_1}\left[\frac{v_1 - v_2}{l} - \frac{v_1}{l}\right] \tag{1}$$

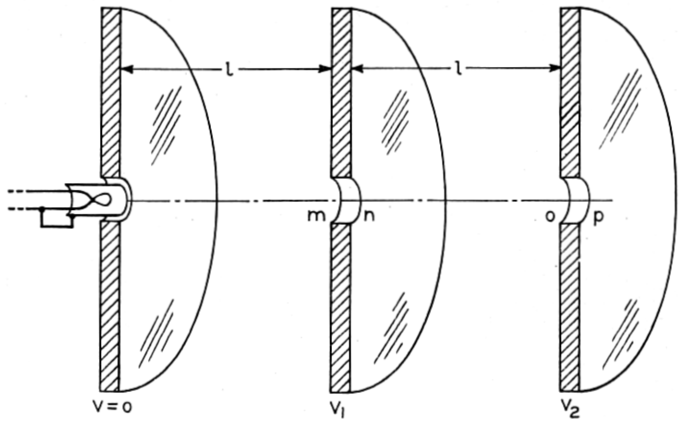


Fig. 15—Example of a complete system.

and the focal distance at n is

$$d_n = \frac{4l}{\beta^2 - 2}, \tag{2}$$

where β is $\sqrt{v_2/v_1}$. The beam then passes through the uniform field between the second and third plates, and the focal distance at 0 is calculated from equation 28 for a uniform field, which becomes

$$(1 + \beta)d_n - (1 + 1/\beta)d_0 = 2l \tag{3}$$

and gives

$$d_0 = 2\beta l \frac{4 + 2\beta - \beta^2}{(1 + \beta)(\beta^2 - 2)}. \tag{4}$$

The beam then passes through the aperture in the third plate into field-free space, and the lens equation for this aperture is

$$\frac{1}{d_p} - \frac{1}{d_0} = \frac{1}{4v_2}\left[0 - \frac{v_2 - v_1}{l}\right]. \tag{5}$$

Substitution for d_0 now gives

$$\frac{1}{d_p} = \frac{1 + \beta}{2\beta l}\left[\frac{\beta^2 - 2}{4 + 2\beta - \beta^2} + \frac{1 - \beta}{2\beta}\right], \tag{6}$$

which is the reciprocal of the final focal distance measured from the last plate.

In complete lens systems, where the symbolic calculations are complicated, it is frequently simpler to introduce specific numerical values and carry the successive steps of the calculation through in a numerical manner. By doing this for a few suitably chosen numerical values one can obtain the particular information that is desired.

Appendix II—Concentric Tubes

Two concentric tubes at different potentials form an electron lens that is well adapted to practical tube construction. When the two tubes are of the same diameter, the approximate constants of the lens may be determined as follows.[18]

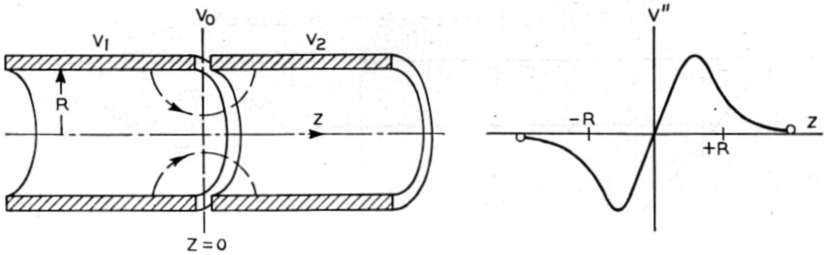


Fig. 16—Concentric tubes.

In this type of lens, the electric intensity is symmetrical with respect to an imaginary plane drawn between the two tubes—as illustrated in Fig. 16—and the plane is therefore an equipotential surface. Its potential v_0 is the mean potential of the two tubes. This plane is regarded as a division plane separating the lens into two component electric fields.

We first consider the component to the right of the plane. The solution for the potential inside of the tube may be obtained in the form of a Bessel Function series, and it follows from this series that the potential on the axis is

$$v = v_2 - (v_2 - v_0)\sum_{\mu} \frac{2}{\mu J_1(\mu)} \exp.\left(-\frac{\mu z}{R}\right), \tag{1}$$

where R is the radius of the tubes, and μ takes on discrete values equal to the successive roots of

$$J_0(\mu) = 0. \tag{2}$$

We find that an approximation to the exponential series is given by

$$\sum_{\mu} \frac{2}{\mu J_1(\mu)} \exp.\left(-\frac{\mu z}{R}\right) = 1 - \tanh \omega z, \tag{3}$$

[18] We assume that the separation between their ends is negligibly small compared to their diameter.

where ω is equal to $1,32/R$. The closeness of this approximation is shown in Fig. 17. Its introduction gives

$$v = v_0 + (v_2 - v_0) \tanh \omega z. \tag{4}$$

A similar approximation is found for the potential on the axis to the left of the division plane,

$$v = v_0 - (v_1 - v_0) \tanh \omega z, \tag{5}$$

and it turns out that the potential on the axis of both tubes can be expressed by the single equation

$$v = \tfrac{1}{2}[(v_2 + v_1) + (v_2 - v_1) \tanh \omega z]. \tag{6}$$

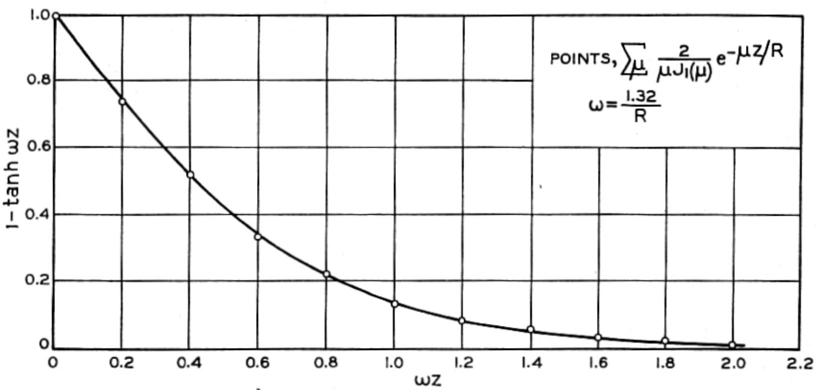


Fig. 17—An approximation for the exponential series.

With the potential on the axis expressed in terms of a known function of z, various series methods may be used for locating the principal planes and calculating the principal focal distances. They are, however, complicated and it may be preferable to use the approximate lens equation obtained by treating the structure as a thin lens.

When treated in this manner, the expression 33 for the inverse focal term can be exactly integrated, and the lens equation is

$$\frac{\sqrt{2v_2}}{d_2} - \frac{\sqrt{2v_1}}{d_1} = \frac{\omega\sqrt{2}}{3(\sqrt{v_2} + \sqrt{v_1})}(\sqrt{v_2} - \sqrt{v_1})^2, \qquad \omega = 1,32/R. \tag{7}$$

Division by either $\sqrt{2v_2}$ or $\sqrt{2v_1}$—as desired—reduces this equation to one that involves the voltages only in the form of a ratio. The error in a focal distance d calculated from this equation is of the order of R, when the focal distance is measured from the division plane of the tubes.

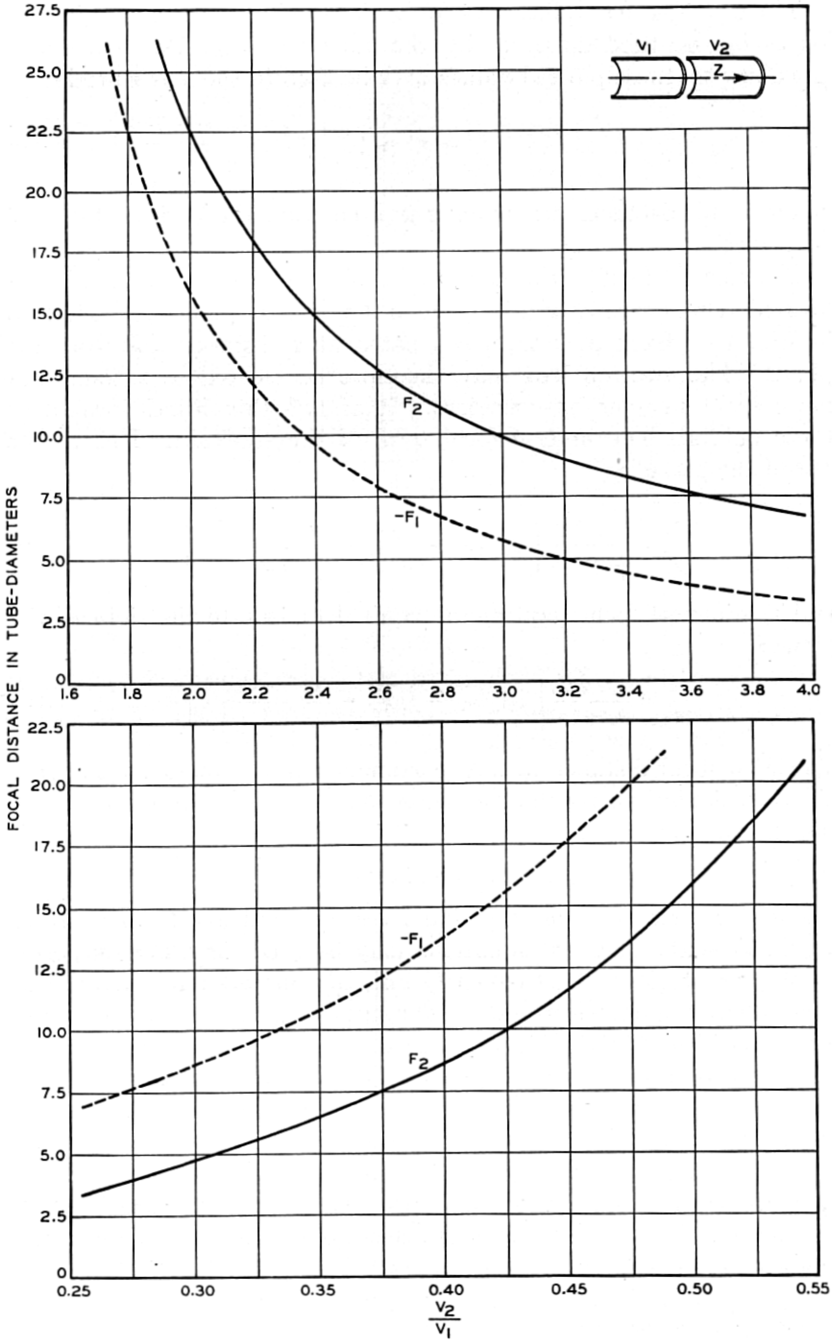


Fig. 18—Principal focal distances—concentric tubes.

The principal focal distances for various voltage ratios are given in terms of the tube diameter by the curves of Fig. 18. For rough calculations, these plotted values may be used in the lens equation

$$\frac{f_2}{d_2} + \frac{f_1}{d_1} = 1, \tag{8}$$

where focal distances are again measured from the division plane of the tubes.

The electric field of the concentric tubes has two maxima of $|v''|$ located symmetrically with respect to the division plane, as illustrated in Fig. 16. Each maximum is located at a distance $.5R$ from the plane. The electron lens may therefore be treated in a somewhat more exact manner by considering it as two thin lenses located at these points. The inverse focal term of the equivalent lens to the left of the plane is

$$\frac{1}{F} = \frac{\omega\sqrt{2}}{v_0 - v_1}\left[v_0(\sqrt{v_0} - \sqrt{v_1}) - \frac{v_0^{3/2} - v_1^{3/2}}{3}\right], \tag{9}$$

and the inverse focal term of the equivalent lens to the right of the plane is

$$\frac{1}{F} = \frac{\omega\sqrt{2}}{v_2 - v_0}\left[v_0(\sqrt{v_2} - \sqrt{v_0}) - \frac{v_2^{3/2} - v_0^{3/2}}{3}\right]. \tag{10}$$

The final focal distance in any particular case is found by carrying out the calculations for the two lenses in succession, with their separation taken equal to R.

Appendix III

A Plane Electrode at the End of a Tube.—In addition to their above application, the last two equations may be used for other purposes. In electron devices, one frequently puts a plane electrode at the end of another, tubular electrode.[19] The approximate lens action of the electric field between the plate and tube is then described by one or the other of these equations. Equation 9 applies when the plane follows the tube in the direction of electron motion; and equation 10 applies when the plane precedes the tube.

In structures of this type, the plate is usually pierced with an aperture for the passage of electrons. When the aperture is small compared to the tube diameter, the lens system can be treated in the following manner.

[19] We assume the separation between the plate and the end of a tube to be negligible compared to the tube diameter.

We first consider the case of the plane preceding the tube. The electric intensity at the plate is found by differentiating equation 4 with respect to z and then setting z equal to zero. A substitution of this intensity in equation 41 of the text gives the lens equation of the aperture. In addition to this lens there is an equivalent thin lens located inside the tube at a distance $.5R$ from the plate, and having the inverse focal distance of equation 10. The system is considered as a combination of the two lenses and the calculations are carried through in the usual manner. When the plane follows the tube, the constants of the two lenses are determined from equations 5 and 9, and the combination is treated in a similar manner.

Appendix IV—The Velocity Function

The auxiliary functions u and w are a special case of the components of a generalized vector function that is useful in developing series solutions for electron motion. The equations of this function are equivalent to the Hamilton-Jacobi equation; they are briefly outlined in the present system of units as follows.

In a field that may comprise both an electric intensity E and a magnetic intensity H, let v be any vector function of x, y, z that satisfies the equations

$$\text{curl } v = H/c, \tag{1}$$

$$1/2|v|^2 = \phi + W, \tag{2}$$

where W is a constant equal to the energy of electron emission from the source. Then v is a possible vector velocity for electron motion in the field.

If the magnetic intensity is zero, the vector function v has a potential ψ, which may be any solution of the equation

$$1/2|\text{grad } \psi|^2 = \phi + W \tag{3}$$

and grad ψ is then a possible vector velocity for electron motion in the field.

The validity of these equations is established by transforming them to the usual equations for electron acceleration.

A List of the More Important Symbols and Equations

In the present theory of electron-optics, *all distances* along the axis are measured in the direction of motion, as they are in the optics of light.

r, z —cylindrical coordinates
t —time

Φ —potential at point in space, the electron source taken as zero potential
v —potential on the axis
v' —derivative of v with respect to z
v —is also used for the voltage of electrodes
d —focal distance in general
T —focal time in general
A —the important coefficient for rays near the axis, a function of z alone
u, w —velocity functions corresponding to $\dot{r}$ and $\dot{z}$
d_1, d_2—conjugate focal distances measured from the two sides of a lens
f_1, f_2 —principal focal distances measured from the two sides of a lens

As an approximation in thin lenses, the focal distances are measured either from the mid-point of the lens, or from the point where $|v''|$ is a maximum, provided that there is but one maximum in the lens.

α_1, α_2 —location of the principal planes with respect to the sides of a lens
D_1, D_2—conjugate focal distances measured from the principal planes
F_1, F_2—principal focal distances measured from the principal planes
F —the focal term of a lens, not a focal distance

Equations for Rays Near the Axis

$$\dot{z} = \sqrt{2v},$$

$$A = -\frac{\sqrt{2v}}{d} = -\frac{1}{T} \doteq \dot{r}/r,$$

$$\sqrt{2v}A' + A^2 = -\frac{v''}{2},$$

$$\frac{1}{T^2}\frac{d}{dt}(T + t) = -\frac{v''}{2}.$$

The important equations for a thin lens are:

$$\frac{\sqrt{2v_1}}{d_2} - \frac{\sqrt{2v_1}}{d_1} = \frac{1}{F},$$

$$\frac{f_2}{f_1} = -\sqrt{\frac{2v_2}{2v_1}},$$

$$\frac{f_2}{d_2} + \frac{f_1}{d_1} = 1,$$

$$\frac{1}{F} = \int_{z_1}^{z_2} \frac{v''}{2\sqrt{2v}}\, dz.$$

The following equations hold for any lens:

$$\frac{\sqrt{2v_2}}{d_2 - \alpha_2} - \frac{\sqrt{2v_1}}{d_1 - \alpha_1} = \frac{1}{F},$$

$$\frac{\sqrt{2v_2}}{D_2} - \frac{\sqrt{2v_1}}{D_1} = \frac{1}{F},$$

$$\frac{F_2}{F_1} = -\sqrt{\frac{2v_2}{2v_1}},$$

$$\frac{F_2}{D_2} + \frac{F_1}{D_1} = 1,$$

$$M = \frac{F_1 D_2}{F_2 D_1} = -\sqrt{\frac{2v_1}{2v_2}}\frac{D_2}{D_1},$$

where M is magnification.