Equivalent Networks of Negative-Grid Vacuum Tubes at Ultra-High Frequencies

By F. B. LLEWELLYN

It is shown that the equivalent network of negative-grid vacuum tubes both at low and at very high frequencies may be expressed in many different forms. Several are suggested and the advantages of two are described in some detail. One of these is closely analogous to that which is in general use at low frequencies and requires only the addition of resistive components in series both with the cathode-grid and the grid-plate capacitances to make it applicable to frequencies where transit time effects are appreciable though moderately small. The resistance in series with the grid-plate capacitance is negative in sign. In this form of the equivalent network, electron transit times do not introduce a phase angle into the amplification factor.

The paper is divided into two parts. The first gives a descriptive interpretation of the results while the second contains the mathematical manipulations.

Part I

When the equivalent network of a vacuum tube is mentioned, it brings to the mind of practically every radio engineer a certain combination of resistances and capacitances together with an internal μ-generator which has become familiar through years of use. Historically, this equivalent network did not spring into being full grown like Athena from the forehead of Zeus, but was the result of a slow and painful development. The beginnings of the equivalent network of negative-grid vacuum tubes are to be found in the work of Nichols where it was pointed out that a non-linear resistance is the equivalent of a fixed resistance in series with a generator. As a second step, Van der Bijl's relation states that the plate current in a vacuum tube is a function of the plate voltage plus a constant times the grid voltage. This constant was identified with our well-known amplification factor μ and it was an easy step thereafter to combine the Van der Bijl and Nichols relations and represent the vacuum tube by the equivalent network shown in Fig. 1.

Here the cathode is located at C and the plate at P. Between them the vacuum tube is represented by the internal plate resistance r_p acting in series with the fictitious generator $\mu_0 V_g$. This equivalent naturally represents conditions between the cathode and plate at very low frequencies only, because the low-frequency impedance between the grid element and the other electrodes is so high that it can safely be disregarded. Such an equivalent network was satisfactory only so

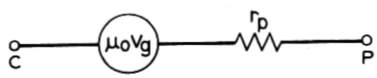

FIGURE 1

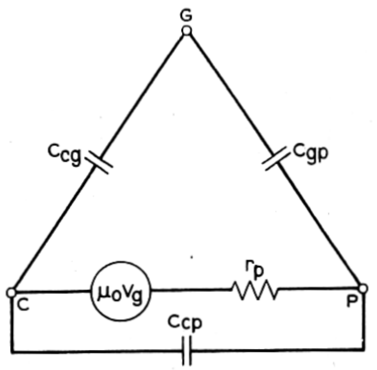

FIGURE 2

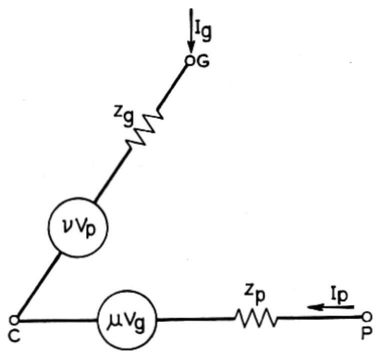

FIGURE 3

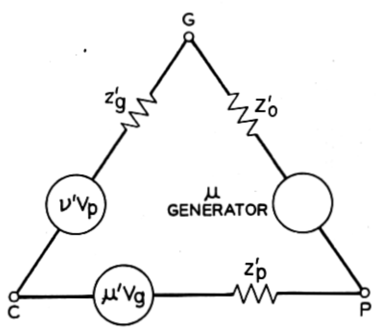

FIGURE 4

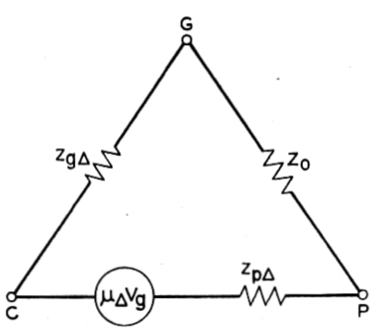

FIGURE 5

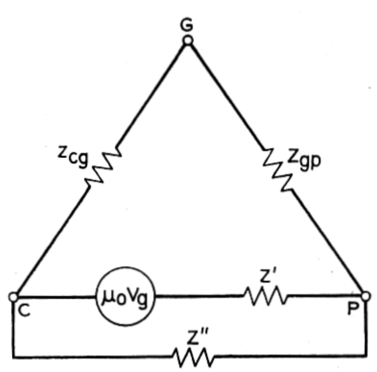

FIGURE 6

Figs. 1–6—Equivalent vacuum tube networks.

long as frequencies were low enough to allow this last approximation to remain valid. With the advent of higher frequencies it became evident that the internal tube capacitances played an important role in the operation of the device. The lengths to which early workers went to include the capacitance effects are illustrated by the complicated formulas on page 207 of Van der Bijl's well-known book on Thermionic Vacuum Tubes. Further study, however, showed that the complication could be overcome largely by a modification of the simple network of Fig. 1 so that capacitances are introduced between all three elements of the vacuum tube. The result is Fig. 2 which has been adequate in the past for all purposes. In comparatively recent years, however, increasing frequencies demand that a further revision be made.

The necessity for revision first became evident with the discovery that the impedance measured between grid and cathode when a very large condenser was placed between plate and cathode, showed an important resistive component at very high frequencies so that the simple combination of Fig. 2 involving only capacitances for the grid-cathode and grid-plate impedance was no longer valid. The tools for effecting the modification of Fig. 2 are available [1] and already have been employed to a certain extent. These tools are the result of a theoretical analysis of the motions of electrons within vacuum tubes and started from fundamentals. With the reservation that they apply strictly to planar rather than cylindrical tube structures, the results should therefore require little further modification for some time to come.

The first result of theoretical analysis was to produce an equivalent network which, on the face of it, resembles Fig. 2 only remotely, but which can be shown [1] to be exactly equivalent at low frequencies. This generalized theoretical network is shown in Fig. 3. It may be seen to consist of two branches only, which exist respectively between cathode and grid and between cathode and plate. Both branches contain internal generators and, in general, the impedance in neither branch is a pure resistance but depends upon a number of factors including the time required by electrons in traversing the vacuum tube. The immediate query which results from inspection of Fig. 3 is "What has become of the grid-plate path?" The answer to this lies in the definition of current in Fig. 3 so that the cathode-plate path is included in the network as shown. This definition of current is merely the generalized one adopted years ago by Maxwell when he

[1] F. B. Llewellyn, "Operation of Ultra-High-Frequency Vacuum Tubes," *Bell Sys. Tech. Jour.*, Vol. XIV, pp. 632–665, October 1935.

realized that a change in electric intensity produces precisely the same effect in a circuit as does an actual motion of charge. In Fig. 3 this means that the current entering the branch between cathode and grid for example consists not only of ordinary conduction current but also of displacement current so that the current in the cathode-grid mesh is the whole current flowing into the grid element. Likewise, the current in the cathode-plate mesh is the whole current flowing into the plate element of the tube.

In Part II straightforward transformation of the equations representing Fig. 3 shows that it may be represented just as well by an infinite number of other equivalent networks. Naturally our aim is to choose the form of network which is easily adaptable to the greatest number of practical applications, and the one that suggests itself primarily for this purpose contains the fewest number of internal generators. A second consideration in the choice of the best equivalent network is that the network should resemble the familiar delta equivalent of Fig. 2 as closely as may be, so that results based on that figure may be interpreted readily in terms of the more general network.

Fig. 2 is actually a modified form of a delta network. The most general delta would be the one shown in Fig. 4 which consists of three series branches, each containing an internal generator in series with an impedance. When the mathematical transformations from Fig. 3 to Fig. 4 are carried through, it is found that a proper choice of definitions for the various impedances reduces Fig. 4 to the network shown in Fig. 5. Here only one internal generator remains, but that generator acts in series with the internal plate impedance of the tube so that Fig. 5 does not quite conform to the popular network where a capacitance is assumed to shunt the internal generator by acting directly between plate and cathode. However, again it can be shown that Fig. 5 may be transformed to Fig. 6 and by a proper choice of the two impedances Z' and Z'', the internal generator reduces merely to our familiar low-frequency amplification factor multiplied by the grid potential variation.

Thus Fig. 6 with the associated definitions of impedance represents the generalized form of the equivalent network of negative-grid vacuum tubes and is valid until the velocity of the electrons approaches that of light or until the distance between elements of the vacuum tube becomes comparable to the free-space wave-length of any ultra-high frequency considered. The expressions for the various impedances in Fig. 6 are naturally long and complicated. However, at frequencies where the effects of transit time of the electrons are only moderately important, the complication reduces enormously and we have Fig. 7.

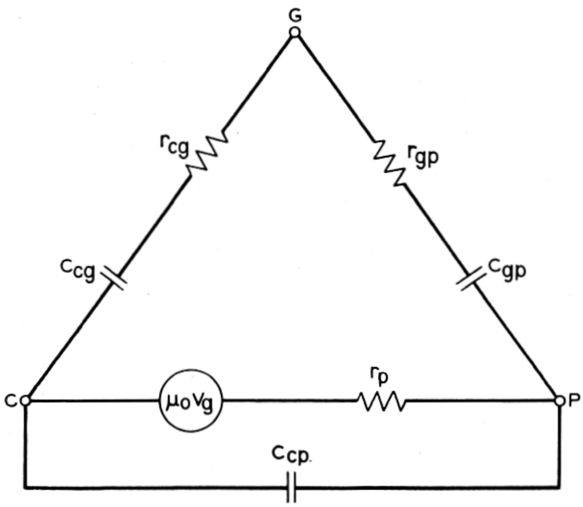

Fig. 7—General equivalent of vacuum tubes valid for moderately high frequencies.

This is nearly identical with the well-known equivalent which was shown in Fig. 2, and the modification consists only of the addition of series resistors in the internal cathode-grid and cathode-plate paths. A phase angle in the amplification factor is avoided by the resistances in series with the capacitances.[2] The impedance in series with the μ-generator is the well-known internal plate resistance as given by the slope of the static characteristic of the tube. The capacitances are likewise those we have used all along at lower frequencies, but the mathematics now enables their dielectric constants to be computed. Fig. 7 will be found to be valid at any of the frequencies for which negative-grid tubes are now contemplated for commercial application, including those where the transit angle is about half a radian.

Much has been said and written of late years about the active grid loss.[1, 3, 4, 5] In Fig. 7 this would be determined by placing a large

[2] It can be shown that measurements published in a paper "Phase Angle of Vacuum Tube Transconductance," F. B. Llewellyn, *Proc. I. R. E.*, Vol. 22, August 1934, may be interpreted as well in terms of the phase angle of the grid-plate impedance. If the phase angle of the latter is α, and the angle measured for the paper is θ, then

$$\sin^2 \alpha = 1 - \frac{C'}{C} + \frac{\sin \phi \cos \phi \tan \theta}{\omega^2 LC},$$

where C' is the grid-plate capacitance of the cold tube, C of the hot tube, and ϕ is the phase angle of the inductive branch of the tuned circuit used in the experiments.

[1] Loc. cit.

[3] J. G. Chaffee, "The Determination of Dielectric Properties at Very High Frequencies," *Proc. I. R. E.*, Vol. 22, August 1934.

[4] W. R. Ferris, "Input Resistance of Vacuum Tubes as Ultra-High Frequency Amplifiers," *Proc. I. R. E.*, Vol. 24, January 1936.

[5] D. O. North, "Analysis of the Effects of Space Charge on Grid Impedance," *Proc. I. R. E.*, Vol. 24, January 1936.

condenser between cathode and plate and measuring the input impedance. The result of computing the impedance from Fig. 7 agrees with that formerly presented[1] as, of course, it should, since both results were derived from the same fundamental analysis. This agreement, however, is mentioned by way of giving an example which checks the algebraic manipulations which were employed in arriving at Fig. 7.

Finally the values of the various elements in Fig. 7 are summarized in Table I. The formulas naturally are very long and their greatest

TABLE I

REFERRING TO FIG. 7

Let C_{cg}', C_{gp}', C_{cp}' be capacitances of cold tube,

$$y = \frac{x_p}{x_c} \text{ be ratio of } g - p \text{ to } c - g \text{ spacing,}$$

$$h = \frac{T_p}{T_c} \text{ be ratio of } g - p \text{ to } c - g \text{ transit time,}$$

$$M = \frac{4}{3}(y - h^3)(1 + h) - 2h^2 + h^4,$$

$$N = (y - h^3)(9 + 44h + 45h^2) - 51h^2 - 105h^3 - 27h^4 + 27h^5,$$

$$C_{cp} = \frac{4}{3} C_{cp}' \left[\frac{1 + y + \dfrac{\mu_0 y}{y - h^3}}{1 + M + \mu_0} \right] \left[1 + h - \frac{3}{2} \frac{h^2}{y - h^3} \right],$$

$$r_p = \text{same as at low frequencies,}$$

$$C_{cg} = C_{cg}' \left[\frac{1 + y + \dfrac{\mu_0 y}{y - h^3}}{1 + M + \mu_0} \right] \frac{M}{y},$$

$$r_{cg} = \left[\frac{r_p(y - h^3)}{45\mu_0(1 + M + \mu_0)M^2} \right] \left[(\mu_0 + 1)N + \frac{45\mu_0 h^4}{(y - h^3)} M \right],$$

$$C_{gp} = C_{gp}' \left[\frac{1 + y + \dfrac{\mu_0 y}{y - h^3}}{1 + M + \mu_0} \right],$$

$$r_{gp} = - \left[\frac{r_p(y - h^3)}{45\mu_0(1 + M + \mu_0)} \right] \left[N - \frac{45\mu_0 h^4}{y - h^3} \right].$$

use probably is in describing the simple circuit of Fig. 7 where the values of the various elements can actually be measured or computed as convenient.

The easiest way to visualize the equations is to apply them to a special case which can be approached experimentally; namely the condition that the time required by electrons in moving from grid to plate is much shorter than the cathode-grid time. When this is the case, the formulas reduce to those shown in Table II. These show

TABLE II

REFERRING TO FIG. 7

Let C_{cg}', C_{gp}', C_{cp}' be capacitances of cold tube,

$$y = \frac{x_p}{x_c} = \text{ratio of } g - p \text{ to } c - g \text{ spacing},$$

$$h = \frac{T_p}{T_c} = \text{ratio of } g - p \text{ to } c - g \text{ transit time}.$$

Then when $h \to 0$:

$$C_{cp} = \frac{4}{3} C_{cp}' \left[\frac{1 + y + \mu_0}{1 + \frac{4}{3} y + \mu_0} \right],$$

$$r_p = \text{same as at low frequencies},$$

$$C_{cg} = \frac{4}{3} C_{cg}' \left[\frac{1 + y + \mu_0}{1 + \frac{4}{3} y + \mu_0} \right],$$

$$r_{cg} = \frac{9}{80} \frac{r_p}{\mu_0} \left[\frac{1 + \mu_0}{1 + \frac{4}{3} y + \mu_0} \right],$$

$$C_{gp} = C_{gp}' \left[\frac{1 + y + \mu_0}{1 + \frac{4}{3} y + \mu_0} \right],$$

$$r_{gp} = -\frac{1}{5} \frac{r_p}{\mu_0} \left[\frac{y^2}{1 + \frac{4}{3} y + \mu_0} \right].$$

that the cathode-plate and cathode-grid capacitances have dielectric constants greater than unity, but that the grid-plate capacitance has a dielectric constant less than unity. The cathode-grid resistance is positive, and the grid-plate resistance is negative.

The outstanding result of this investigation of the network representing the negative-grid tube is the demonstration of the slight modification required in our conventional network to make it accurate even in the ultra-high-frequency range. The amplification factor is the familiar low-frequency one, and at moderately high frequencies, the only alteration needed in the conventional diagram is the addition of two small but very important resistances, one in the cathode-grid path and one in the grid-plate path, where the resistance in the latter path is negative in sign.

PART II

In a recent paper,[1] general equations have been derived which describe the behavior of vacuum tubes at ultra-high frequencies. In

[1] Loc. cit.

the case of negative-grid triodes and referred to Fig. 8, these equations take the general form:

$$V_p + \mu V_g = I_p z_p, \tag{1}$$

$$V_g - \nu V_p = I_g z_g, \tag{2}$$

where

$$\left.\begin{aligned} \mu &= \frac{(Z_1 + Z_2) - (Z_2 + Z_3 + Z_c)}{Z_c + Z_g}, \\ z_p &= \frac{(Z_2 + Z_3 + Z_c)Z_g + (Z_1 + Z_2)Z_c}{Z_c + Z_g}, \\ \nu &= \frac{Z_c}{Z_2 + Z_3 + Z_c}, \\ z_g &= \frac{(Z_2 + Z_3 + Z_c)Z_g + (Z_1 + Z_2)Z_c}{Z_2 + Z_3 + Z_c}, \end{aligned}\right\} \tag{3}$$

and the Z's may be expressed in terms of the tube geometry and d-c. current or voltage by means of equations (80)–(84) in the reference. In these relations the currents, I_p and I_g, denote the total current reaching plate or grid, respectively, and hence include both the conduction current carried by the electrons themselves and the displacement current arising from the change of electric force. With this meaning of current, (1) and (2) contain the complete description of the performance of the tube, and separate consideration of the grid-plate current, usual in low-frequency methods, is unnecessary because that current is already included in I_p in (1).

The equivalent network represented by (1) and (2) is shown in Fig. 8. Only two currents are involved, I_p and I_g, but, also two internal generators, μV_g and νV_p, are required. For some purposes, an equivalent network which corresponds more nearly with the usual low-frequency delta arrangement is desirable. Such an equivalent may be obtained from (1) and (2) in conjunction with Fig. 9 which shows the relation between currents in a delta network and those of Fig. 8. In Fig. 9 no restriction is yet placed upon the three currents, I_1, I_2 and I_3, so that in general they all may be allowed to include both conduction and displacement components. From Fig. 9

$$I_p = I_1 + I_2, \tag{4}$$

$$I_g = I_3 - I_2. \tag{5}$$

Here are two equations expressing the three unknowns, I_1, I_2 and I_3, in terms of the currents I_p and I_g, which are assumed to be known.

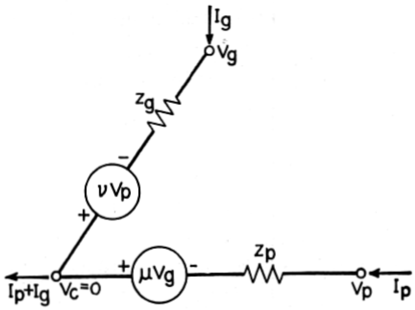

FIGURE 8

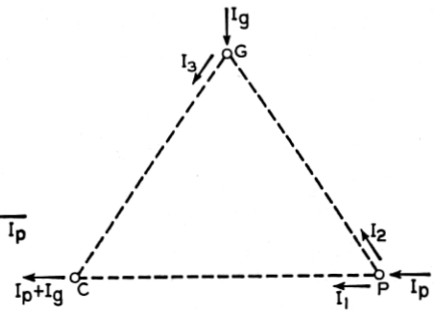

FIGURE 9

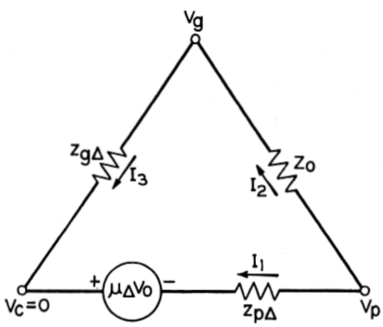

FIGURE 10

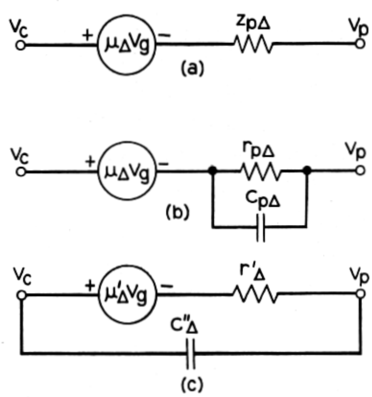

FIGURE 11

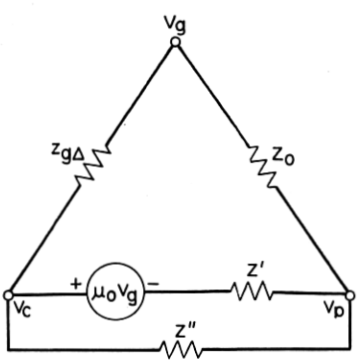

FIGURE 12

Fig. 8—Equivalent network of vacuum tube from equations (1) and (2).
Fig. 9—Relation between currents of delta network and those of Fig. 8.
Fig. 10—Equivalent delta network.
Fig. 11—Equivalent cathode-plate paths.
Fig. 12—Modified delta network equivalent to Fig. 10.

It is obvious that a third equation is needed before the unknowns can be found. But (4) and (5) express all the relationships that are necessary for the equivalence of Figs. 8 and 9. It follows that we are at liberty to impose arbitrarily a third restriction upon the currents in Fig. 9.

The choice of this restriction may be made in many ways, each resulting in a different network, all equivalent however to Fig. 8 and to the vacuum tube. For example, I_1 might be defined as consisting of conduction current only. Such a choice might seem at first sight to be a desirable one because it corresponds rather well with the conception of the cathode-plate path as being determined by electron movement at low frequencies. If it were adopted, however, the generalized network resulting would be found to be quite awkward, involving two or more internal generators and complex amplification factors.

The simplest network would be the one involving the fewest number of internal generators, and the restriction adopted in the following analysis for the currents in Fig. 9 is made with that object in view. The result, as will be shown, corresponds at low frequencies with the usual concept of the tube, and gives a high-frequency network where neither the cathode-grid nor the grid-plate paths contain internal generators.

The restriction which accomplishes this result is obtained by placing

$$V_p - V_g = I_2 Z_0, \tag{6}$$

so that (4), (5) and (6) determine the internal currents, I_1, I_2 and I_3, in terms of the external currents I_p and I_g, and the, as yet, arbitrary impedance Z_0.

The solution of (1) to (6) yields

$$V_p + V_g\left[\frac{\mu + \dfrac{z_p}{Z_0}}{1 - \dfrac{z_p}{Z_0}}\right] = I_1\left[\frac{z_p}{1 - \dfrac{z_p}{Z_0}}\right], \tag{7}$$

$$V_g - V_p\left[\frac{\nu - \dfrac{z_g}{Z_0}}{1 - \dfrac{z_g}{Z_0}}\right] = I_3\left[\frac{z_g}{1 - \dfrac{z_g}{Z_0}}\right]. \tag{8}$$

The choice of (6) eliminates the internal generator from the grid-plate path, but still leaves the impedance Z_0 to be defined at will. From (8) it is evident that the internal generator is eliminated from the cathode-grid path if Z_0 is chosen so that

$$\nu = z_g/Z_0. \tag{9}$$

The impedance Z_0 will accordingly be taken to be defined by (9). The result is that the fundamental equations for Fig. 9 become:

$$V_p + \mu_\Delta V_g = I_1 z_{p\Delta}, \tag{10}$$

$$V_g = I_3 z_{g\Delta}, \tag{11}$$

$$V_p - V_g = I_2 Z_0, \tag{12}$$

where

$$\left.\begin{aligned} \mu_\Delta &= \frac{Z_1 - Z_3}{Z_g}, \\ z_{p\Delta} &= Z_2 + Z_3 + Z_c + \frac{Z_c}{Z_g}(Z_1 + Z_2), \\ z_{g\Delta} &= Z_g + \frac{Z_c(Z_1 + Z_2 + Z_g)}{Z_2 + Z_3}, \\ Z_0 &= Z_1 + Z_2 + \frac{Z_g}{Z_c}(Z_2 + Z_3 + Z_c). \end{aligned}\right\} \tag{13}$$

The various definitions in (13) are seen to be slightly simpler than those in (3) but it must be remembered that the delta-network involves one more current-path than the original ultra-high-frequency network, Fig. 1. The delta corresponding to (10)–(13) is shown in Fig. 10.

At frequencies only moderately high, all of the impedances in Fig. 10 are composed of combinations of ordinary resistances and capacitances. Both $z_{g\Delta}$ and Z_0 consist of a condenser and resistor in series.

In the case of the cathode-plate path in Fig. 10, the equivalent combination of resistance and capacitance may be represented by either a parallel combination of resistance and capacitance which is in series with the μ-generator, as shown at (*b*) in Fig. 11, or a μ-generator of different value acting in series with a resistance, and the whole being shunted by a capacitance connected between cathode and plate, as shown at (*c*) in Fig. 11. The latter picture is in more strict accord with conventional practice but the mathematical relationships involve a choice of definitions for μ. It is found by trial that this choice may be made so that μ in Fig. 12 is defined merely as μ_0, its low-frequency value, and is independent of frequency. When this definition is adopted, we have in general the equivalent network of Fig. 12 which holds at high as well as low frequencies. This differs from Fig. 10 in the cathode-plate path only, and Z', Z'' are defined as follows, where μ_0

is the low-frequency amplification factor:

$$Z' = \mu_0 \left[\frac{(Z_2 + Z_3 + Z_c)Z_g + (Z_1 + Z_2)Z_c}{Z_1 - Z_3} \right], \tag{14}$$

$$Z'' = \mu_0 \left[\frac{(Z_2 + Z_3 + Z_c)Z_g + (Z_1 + Z_2)Z_c}{\mu_0 Z_g - Z_1 + Z_3} \right]. \tag{15}$$

Figure 7 is the form taken by Fig. 12 for moderately high frequencies where transit angle effects are just beginning to become noticeable.

In general, the formulas for the impedances in the delta network are just as long as in the original network of Fig. 8. The delta may, however, have an advantage in view of its wide use in low-frequency work, and of the fact that the amplification factor for the delta can be expressed without involving the transit angle, and hence does not contain a phase shift.