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A Laplacian Expansion for Hermitian-Laplace Functions of High Order*

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Among the wide variety of practical and theoretical problems confronting the telephone engineer, there is a surprisingly large number to whose solution mathematics has made notable contribution. In his kit of mathematical tools the theory of probability is a frequently used and most effective instrument. This theory of probability contains a large number of theorems, a large number of functions, which permit of application to telephony. Among these is a particular tool, a particular group of mathematical functions known as the "Hermitian Functions," each of which is identified by a number called its "order." These mathematical functions or relations have no practical utility until the variables in the equation can be assigned numerical values and the resultant numerical value of the function calculated. Tables of the numerical values of Hermitian functions of low order exist; for example, Glover's Tables of Applied Mathematics cover the ground for those of the first eight orders. But tables for the functions of higher order are still a desideratum. This paper presents an expansion by means of which the evaluation of a high order function can be readily accomplished with a considerable degree of accuracy.

The development of the expansion is prefaced by some remarks on the early history of the Hermitian functions and the relation of this history to modern theoretical physics.

I

Among contributions made by Laplace to the domain of pure and applied mathematics, two of great practical value are:

(*a*) His method of evaluating definite integrals [1] whose integrands involve factors raised to high powers;

(*b*) The pair of orthogonal polynomial functions [2] which he defined by the following Equations (1) and (2)

$$(1) \qquad [(2n)!\sqrt{\pi}/2^{2n}n!]U_n(u) = \int_{-\infty}^{\infty} e^{-x^2}(x - iu)^{2n}dx = 2e^{u^2}\int_0^\infty e^{-x^2}x^{2n}\cos(2ux)dx;$$

$$(2) \qquad [(2n+1)!\sqrt{\pi}/2^{2n}n!]U_n'(u) = i\int_{-\infty}^{\infty} e^{-x^2}(x - iu)^{2n+1}dx = 2e^{u^2}\int_0^\infty e^{-x^2}x^{2n+1}\sin(2ux)dx.$$

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These polynomial functions formed the coefficients in a series satisfying the partial differential equation

$$\frac{\partial U}{\partial r'} = 2U + 2u\frac{\partial U}{\partial u} + \frac{\partial^2 U}{\partial u^2}$$

to which Laplace reduced the solution of the following ball problem: [3]

Consider two urns A and B each containing n balls and suppose that of the total number of balls, $2n$, as many are white as black. Conceive that we draw simultaneously a ball from each of the urns, and that then we place in each urn the ball drawn from the other. Suppose that we repeat this operation any number, r, of times, each time shaking well the urns in order that the balls be thoroughly mixed; and let us find the probability that after the r operations the number of white balls in urn A be x.

Under the caption "The Statistical Meaning of Irreversibility" Lotka [4] has pointed out the significance of Laplace's ball problem in the modern kinetic theory of matter. Moreover, Hostinsky [5] has shown the bearing of the same problem on the theory of Brownian movements and said "In effect, the partial differential equation obtained by Laplace has been refound by Smoluchowski."

To avoid confusion with the Laplace functions which one encounters in spherical harmonic analysis, the functions defined by Equations (1) and (2) are herein designated as Hermitian-Laplace functions. Such a designation is justified by the Equations (3) and (4) derived in the next paragraph.

II

We also find in Laplace [6]

$$I_n(u)^* = \int_0^\infty e^{-x^2}x^{2n}\cos(2ux)dx = \frac{(-1)^n\sqrt{\pi}}{2^{2n+1}}\left(\frac{d^{2n}e^{-u^2}}{du^{2n}}\right),$$

$$I_n'(u) = \int_0^\infty e^{-x^2}x^{2n+1}\sin(2ux)dx = \frac{(-1)^{n+1}\sqrt{\pi}}{2^{2n+2}}\frac{(d^{2n+1}e^{-u^2})}{du^{2n+1}}.$$

Comparing these Laplacian expressions for the definite integrals $I_n(u)$ and $I_n'(u)$ with the Equations (1) and (2) we see immediately that

$$(3) \qquad U_n(u) = (-1)^n[n!/(2n)!]H_{2n}(u),$$

$$(4) \qquad U_n'(u) = (-1)^{n+1}[n!/2(2n+1)!]H_{2n+1}(u),$$

where H_{2n} and H_{2n+1} are the original Hermite polynomials [7] of order $2n$ and $2n+1$, respectively. These equations connecting the Her-

* The symbols $I_n(u)$ and $I_n'(u)$ are introduced here as convenient abbreviations for the integrals to which they are equated; these symbols do not appear in Laplace.

mite with the Laplace polynomials have been presented in an earlier paper.[8]

Appell and Feriet, Arne Fisher, T. C. Fry, H. L. Rietz and others base their definitions of the Hermite polynomials on $e^{-x^2/2}$ instead of e^{-x^2}. We shall write $A_n(u)$ for the nth polynomial as defined by these authors, reserving $H_n(u)$ to symbolize the Hermitian polynomial as defined in his paper of 1864. Thus, in what follows,

$$A_n(x) = (-1)^n e^{x^2/2}(d^n e^{-x^2/2}/dx^n), \qquad H_n(u) = (-1\sqrt{2})^n A_n(u\sqrt{\ }$$

III

Laplacian expansions * for the U, H, and A polynomials follow immediately from those obtainable by applying to the integrals $I_n(u)$ and $I_n'(u)$ his method of evaluating definite integrals whose integrands embrace factors raised to high powers. As will be shown in Part IV of this paper, we have

$$I_n(u)/[\sqrt{\pi}(Y\sqrt{N})^N] = [S\cos(u\sqrt{2N})] + [S'\sin(u\sqrt{2N}], \qquad N = 2n,$$

$$I_n'(u)/[\sqrt{\pi}(Y\sqrt{N})^N] = [S\sin(u\sqrt{2N})] - [S'\cos(u\sqrt{2N}], N = 2n+1,$$

where $Y = (xe^{-x^2})$ for $x = X = 1/\sqrt{2}$ and

$$S = \sum_{s=0}^{\infty}\left(\frac{-1}{4N}\right)^s [u^{-(2s+1)}K_{2s}],$$

$$S' = \left(\frac{1}{2\sqrt{N}}\right)\sum_{s=1}^{\infty}\left(\frac{-1}{4N}\right)^{(s-1)}[u^{-2s}K_{2s-1}].$$

The explicit expressions for K_0, K_2, K_4 and K_1, K_3, K_5 are given in Section V of this paper. The desired expansions are then given by the equations:

$$\begin{aligned} e^{u^2}I_n(u)/\sqrt{\pi} &= U_n(u)[(2n)!/2^{2n+1}n!] \\ &= H_{2n}(u)[(-1)^n/2^{2n+1}] \\ &= A_{2n}(u\sqrt{2})[(-1)^n/2^{n+1}], \\ e^{u^2}I_n'(u)/\sqrt{\pi} &= U_n'(u)\,[(2n+1)!/2^{2n+1}n!] \\ &= H_{2n+1}(u)[(-1)^{n+1}/2^{2n+2}] \\ &= A_{2n+1}(u\sqrt{2})[(-1)^n/2^{n+1}\sqrt{2}]. \end{aligned}$$

The numerical results shown below in Table I indicate the efficacy

* It may be of interest to compare the expansions presented in this paper with the asymptotic forms of the Hermite functions given by N. Schwid[9] and by M. Plancherel and M. Rotach.[10]

TABLE I

N	$u\sqrt{2} = x$	True Value of $A_N(x)$	$[{}_sA_N(x) - A_N(x)]/A_N(x)$		
			$s = 1$	$s = 2$	$s = 3$
9	0.1	9.32438×10^1	0.0089	0.0000	0.0000
	0.5	3.26533×10^2	0.0084	0.0001	0.0000
	1	2.80000×10^1	0.0263	− 0.0010	− 0.0002
	2	-1.90000×10^2	0.0002	0.0018	0.0001
	3	1.62000×10^3	− 0.0474	− 0.0027	0.0002
	4	-1.74680×10^4	− 0.0283	− 0.0082	− 0.0027
10	0.1	-8.98064×10^2	0.0084	− 0.0001	0.0000
	0.5	4.90439×10^1	− 0.0027	0.0013	0.0000
	1	1.21600×10^3	0.0046	0.0003	0.0000
	2	-2.62100×10^3	− 0.0147	0.0002	0.0001
	3	9.50400×10^3	− 0.0436	− 0.0044	− 0.0004
	4	-5.18090×10^4	0.0445	0.0013	− 0.0018
15	0.1	-1.98001×10^5	0.0054	0.0000	0.0000
	0.5	-5.05845×10^5	0.0052	0.0000	0.0000
	1	4.69456×10^5	0.0022	0.0002	0.0000
	2	-1.41980×10^6	− 0.0102	0.0000	0.0000
	3	4.38955×10^6	− 0.0284	− 0.0017	− 0.0001
	4	-1.85644×10^7	− 0.0338	− 0.0041	− 0.0006
20	0.1	5.90233×10^8	0.0042	0.0000	0.0000
	0.5	-4.45178×10^8	0.0035	0.0000	0.0000
	1	-1.61935×10^8	0.0046	0.0000	0.0000
	2	-1.62882×10^9	− 0.0081	0.0000	0.0000
	3	4.60718×10^9	− 0.0212	− 0.0009	0.0000
	4	8.53219×10^9	0.1241	0.0068	0.0000

of the Laplacian expansion as applied to the evaluation of $A_n(x)$, $x = u\sqrt{2}$, for values of x ranging from 0.1 to 4 and for $N = 9$, 15, 10 and 20, respectively.

Designating by ${}_sA_N(x)$ the approximate value obtained for $A_N(x)$ when one takes into account the first s terms in each of the two series S, S', the last three columns of the table show the proportional errors incurred when $s = 1$, 2 and 3, respectively. It will be noted that for $N = 10$ and $x = 4$ the second approximation is closer than the third; this situation will occasion no surprise if it is recalled that to obtain the best results the natural order of the terms may have to be altered when, for example, one expresses a term of the binomial expansion in a series of Hermite functions.

For the convenience of one who wishes to calculate $A_N(x)$ for values of N other than 9, 10, 15 and 20, there are given in Table II the values of $u^{-(m+1)}K_m$ for $m = 0, 1, 2, 3, 4, 5$ and those values of u covered by Table I.

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tories, for the computations involved in the preparations of Tables I and II.

TABLE II

$u\sqrt{2}$	$u^{-1}K_0$	$u^{-3}K_2$	$u^{-5}K_4$
0.1	0.7053412	0.234229	0.034490
0.5	0.6642654	0.198970	− 0.069716
1	0.5506953	0.0936947	− 0.273541
2	0.2601300	− 0.158968	− 0.018172
3	0.07452849	− 0.121885	0.514828
4	0.01295111	0.0244632	− 0.078479
	$u^{-2}K_1$	$u^{-4}K_3$	$u^{-6}K_5$
0.1	− 0.03520828	0.00602953	0.091905
0.5	− 0.1591469	0.0450683	0.40870
1	− 0.2294564	0.150921	0.48658
2	− 0.08671002	0.255634	− 0.78050
3	0.05589638	− 0.104689	− 0.35023
4	0.04317037	− 0.162097	0.98512

IV

A simple change of variable gives

$$(5) \quad I_n(u) = (\sqrt{N})^{N+1} \int_0^\infty (e^{-x^2}x)^N \cos(x2u\sqrt{N})dx, \qquad N = 2n,$$

$$(6) \quad I_n'(u) = (\sqrt{N})^{N+1} \int_0^\infty (e^{-x^2}x)^N \sin(x2u\sqrt{N})dx, \qquad N = 2n + 1.$$

Set $y(x) = e^{-x^2}x$, and note that $dy/dx = 0$ for $x = X = 1/\sqrt{2}$. Now set $Y = y(X)$,

$$[g(x)]^2 = (\log Y - \log y)/(x - X)^2$$
$$= \frac{1}{X^2}\left[1 - \frac{1}{3}\left(\frac{x-X}{X}\right) + \frac{1}{4}\left(\frac{x-X}{X}\right)^2 - \frac{1}{5}\left(\frac{x-X}{X}\right)^3 + \cdots\right],$$

(7), $t = (x - X)g(x)$.

These transformations give

$$I_n(u) = (\sqrt{N})^{N+1} Y^N \int_{-\infty}^\infty e^{-Nt^2}(dx/dt) \cos(x2u\sqrt{N})dt.$$

By (7) and the Lagrange-Laplace expansion for a function of x in powers of t we obtain

$$I_n(u) = (\sqrt{N})^{N+1} Y^N \sum_{m=0}^{\infty} \int_{-\infty}^\infty e^{-Nt^2}[t^{2m}A_{2m}/(2m)!]dt$$

or

$$I_n(u)/\sqrt{\pi}(Y\sqrt{N})^N = \sum_{m=0}^{\infty} (1/2\sqrt{N})^{2m}(A_{2m}/m!),$$

where, writing D for the differential operator d/dx,

$$A_{2m} = [D_x^{2m} g^{-(2m+1)} \cos(x2u\sqrt{N})]_{x=X}$$

or, by the Leibnitz theorem for the product of two functions,

$$A_{2m} = \sum_{r=0}^{2m} \binom{2m}{r} (2u\sqrt{N})^r \cos(u\sqrt{2N} + r\pi/2)[D_x^{2m-r} g^{-(2m+1)}]_{x=X}$$

and, therefore,

$$\frac{A_{2m}}{m!} = \cos(u\sqrt{2N}) \sum_{r=0}^{m} \binom{2m}{2r} \frac{(-1)^r(u2\sqrt{N})^{2r}}{m!} [D_x^{2m-2r} g^{-(2m+1)}]_{x=X}$$
$$- \sin(u\sqrt{2N}) \sum_{r=0}^{m-1} \binom{2m}{2r+1} \frac{(-1)^r(u2\sqrt{N})^{2r+1}}{m!} [D_x^{2m-2r-1} g^{-(2m+1)}]_{x=X},$$

on separating the even and odd terms in r. Now setting $m - r = s$ and summing with reference to s and r, instead of m and r, gives

$$\sum_{m=0}^{\infty} (1/2\sqrt{N})^{2m} A_{2m}/m!$$
$$= \cos(u\sqrt{2N}) \sum_{s=0}^{\infty} \left(\frac{1}{2\sqrt{N}}\right)^{2s} \frac{1}{(2s)!} \sum_{r=0}^{\infty} \frac{(2r+2s)!}{(2r)!} \frac{(-u^2)^r}{(r+s)!}$$
$$\times [D_x^{2s} g^{-(2r+2s+1)}]_{x=X} - \sin(u\sqrt{2N}) \sum_{s=1}^{\infty} \left(\frac{1}{2\sqrt{N}}\right)^{2s-1} \frac{1}{(2s-1)!}$$
$$\times \sum_{r=0}^{\infty} \frac{(2r+2s)!}{(2r+1)!} \frac{(-1)^r u^{2r+1}}{(r+s)!} [D_x^{2s-1} g^{-(2r+2s+1)}]_{x=X}.$$

But, writing $u/g = v$, we have

$$\sum_{r=0}^{\infty} \left(\frac{(2r+2s)!}{(2r)!}\right) \frac{(-1)^r u^{2r}}{(r+s)!} [D_x^{2s} g^{-(2r+2s+1)}]$$
$$= (-1)^s u^{-(2s+1)} \left[D_x^{2s} v^{2s+1} \sum_{r=0}^{\infty} \frac{(-1)^{r+s} v^{2r}}{(r+s)!} \frac{(2r+2s)!}{(2r)!} \right]$$
$$= (-1)^s u^{-(2s+1)} \left[D_x^{2s} v^{2s+1} D_v^{2s} \sum_{r=0}^{\infty} \frac{(-1)^{r+s} v^{2r+2s}}{(r+s)!} \right]$$
$$= (-1)^s u^{-(2s+1)} [D_x^{2s} v^{2s+1} D_v^{2s} e^{-v^2}],$$

since $D_v^{2s} v^{2m} = 0$ for m less than s.

Likewise

$$\sum_{r=0}^{\infty} \frac{(2r+2s)!}{(2r+1)!} \frac{(-1)^r u^{2r+1}}{(r+s)!} [D_x^{2s-1} g^{-(2r+2s+1)}]$$
$$= (-1)^s u^{-2s} [D_x^{2s-1} v^{2s} D_v^{2s-1} e^{-v^2}].$$

Therefore, finally,

$$\frac{I_n(u)}{(Y\sqrt{N})^N \sqrt{\pi}} = \cos(u\sqrt{2N}) \sum_{s=0}^{\infty} \left(\frac{1}{2\sqrt{N}}\right)^{2s} [u^{-(2s+1)} K_{2s}](-1)^s$$
$$+ \sin(u\sqrt{2N}) \left(\frac{1}{2\sqrt{N}}\right) \sum_{s=1}^{\infty} \left(\frac{1}{2\sqrt{N}}\right)^{2(s-1)} [u^{-2s} K_{2s-1}](-1)^{s-1},$$

where

$$(2s)!K_{2s} = [D_x^{2s} v^{2s+1} e^{-v^2} H_{2s}(v)]_{x=X},$$
$$(2s-1)!K_{2s-1} = [D_x^{2s-1} v^{2s} e^{-v^2} H_{2s-1}(v)]_{x=X}.$$

Substituting $\sin(x2u\sqrt{N})$ for $\cos(x2u\sqrt{N})$ in the equations defining A_{2m} and then proceeding exactly as above we derive the corresponding expansion for $I_n'(u)$.

V

To obtain the values of K_{2s} and K_{2s-1} note that

$$Xg^2 = 1 - (x-X)/3X + (x-X)^2/4X^2 - (x-X)^3/5X^3 + \cdots$$

gives, for $x = X = 1/\sqrt{2}$,

$$g = \sqrt{2}, \qquad v = (1/\sqrt{2})u,$$
$$dg/dx = -1/3, \qquad dv/dx = (1/6)u,$$
$$d^2g/dx^2 = (4\sqrt{2})/9, \qquad d^2v/dx^2 = -(1/3\sqrt{2})u,$$
$$d^3g/dx^3 = -88/45, \qquad d^3v/dx^3 = (53/90)u,$$
$$d^4g/dx^4 = (824\sqrt{2})/135, \qquad d^4v/dx^4 = -(211\sqrt{2}/135)u,$$
$$d^5g/dx^5 = -28184/567, \qquad d^5v/dx^5 = (79/7)u,$$

etc.

Therefore

$$\sqrt{2}(u^{-1}K_0) = e^{-\frac{1}{2}u^2},$$
$$36\sqrt{2}(u^{-3}K_2) = e^{-\frac{1}{2}u^2}(u^6 - 6u^4 - 9u^2 + 12),$$
$$7776\sqrt{2}(u^{-5}K_4) = e^{-\frac{1}{2}u^2}(u^{12} - 12u^{10} - 183.6u^8 + 1432.8u^6$$
$$+ 2889u^4 - 10368u^2 + 432), \text{ etc.}$$

$$6(u^{-2}K_1) = ue^{-\frac{1}{2}u^2}(u^2 - 3),$$

$$648(u^{-4}K_3) = ue^{-\frac{1}{2}u^2}(u^8 - 9u^6 - 59.4u^4 + 279u^2 + 54),$$

$$233280(u^{-6}K_5) = ue^{-\frac{1}{2}u^2}(u^{14} - 15u^{12} - 414u^{10} + 4494u^8 + 25152.4u^6 - 168723u^4 - 119340u^2 + 304560).$$

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