# Hyper-Frequency Wave Guides—Mathematical Theory

By JOHN R. CARSON, SALLIE P. MEAD and S. A. SCHELKUNOFF

Following a brief historical sketch, this paper deals with the mathematical theory of wave transmission in two novel kinds of cylindrical wave guides of circular cross section; namely, the hollow conductor and the dielectric wire. These transmission systems behave as high pass filters with exceedingly high critical frequencies.

The attenuation and impedance characteristics of the hollow conductor, heretofore ignored as far as the writers are aware, are given especial attention. This investigation discloses the remarkable fact that there exists in this system *one* and *only one* type of wave, the attenuation of which *decreases* with increasing frequency, a characteristic which attaches to no other type of guided wave known to the writers.

## I. Introduction

THE object of this paper is to derive and discuss the characteristics of two novel guided wave *transmission* systems. Structurally one consists simply of a straight hollow [1] conducting cylinder of circular cross-section. The electromagnetic wave is confined inside the cylindrical sheath and is propagated along the axis of the cylinder. The other consists simply of a dielectric wire, within which the major part of the electric field is confined. The mathematical theory developed below does not deal with the question as to how such waves are established nor with the reflection phenomena which must occur at the terminals and other points of discontinuity. The analysis is limited to finding the types of waves which are possible in such systems, and to investigating and describing their characteristics.

The historical background of the problem is interesting. In 1897 Rayleigh published a paper entitled "On the Passage of Electric Waves through Tubes, or the Vibrations of Dielectric Cylinders." [2] Dealing solely with ideal cylinders of perfect conductivity he showed that for all types of waves that can exist inside the cylinders there are critical frequencies below which the waves are attenuated and above which they are freely transmitted. The first paper on transmission along dielectric wires was that published in 1910 by Hondros and Debye entitled "Elektromagnetische Wellen an dielektrischen Drähten." [3] This deals theoretically with transmission along cylinders of ideally non-conducting material, somewhat along the lines followed in Section IV

[1] The term *hollow* means that the interior of the cylinder is electrically non-conducting.

[2] *Phil. Mag.*, Vol. 43, 1897, pp. 125–132.

[3] *Ann. der Phys.*, Vol 32, 1910, pp. 465–476.

of this paper. Another paper, entitled "Über den Nachweis elektromagnetischer Wellen an dielektrischen Drähten," [4] published in 1916 by Zahn, is of interest because of the historical note attached, which indicates that experimental work was begun in 1914 by Rüter and Schriever, two students of Zahn, and continued with such diligence as the exigencies of war permitted until the date of Zahn's paper, at least. In 1920 Southworth, then working at Yale University, accidentally observed such waves in a trough of water which he was using in connection with some high-frequency studies, measured their wave-lengths and recognized their identity with those discussed by Schriever [5] in a paper which appeared at about that time. In 1924 Carson rediscovered the transmission characteristics of the hollow conducting cylinder, and disclosed it in an unpublished memorandum entitled "Hyper-Frequency Wave Filters." Finally, in 1931, Southworth, then a research engineer with the American Telephone and Telegraph Company, returned to the subject and initiated the comprehensive investigation which he is reporting in a companion paper.[6] Independently, and almost simultaneously, Hartley, at the Bell Telephone Laboratories, suggested the possibility of guided transmission along a *hollow* cylindrical dielectric wire; and these two (Southworth and Hartley) enlisted our cooperation in a mathematical investigation.

In the theoretical parts of these papers dissipation was always neglected, though obviously the attenuation would be a controlling factor in practical applications. The writers, on the other hand, have given especial attention to this factor. Out of this research there emerged the remarkable fact that with hollow conducting guides there exists *one* and *only one* type of wave the attenuation of which *decreases* with increasing frequency; a unique characteristic which does not attach to dielectric wires, nor so far as the writers are aware, to any other type of guided wave.

## IA. Transmission Through Hollow Conducting Cylinders

Throughout this paper it will be assumed that the cylindrical sheath possesses high conductivity and that the losses in the internal dielectric medium are either small or negligible. Subject to these assumptions the effect of dissipation on the attenuation of the wave is formulated in

[4] *Ann. der Phys.*, Vol. 49, 1916, pp. 907–933. This paper contains several collateral references.

[5] "Elektromagnetischen Wellen an Dielektrischen Drähten," *Ann. der Phys.*, Vol. 63, 1920, pp. 645–673.

[6] "Hyper-Frequency Wave Guides—General Considerations and Experimental Results," G. C. Southworth, this issue of the *Bell System Technical Journal*.

Section III. First, however, in the general discussion which immediately follows and, in particular, in the comparison with the usual guided wave transmission systems, attention will be confined to the ideal non-dissipative structure. This simplification brings out, in a simpler and more striking way, the peculiar transmission characteristics of the system, while, at the very high frequencies involved, it introduces negligible error except as regards the attenuation due to dissipation.

In the ordinary type of guided-wave systems, such for example, as that composed of two concentric conductors, or two parallel wires, the guiding conductors form two sides of a *circuit* in which equal and opposite currents flow, and the transverse lines of electric intensity terminate on the two sides of the circuit. In the system under consideration there is only *one* conductor and consequently there is no *circuit* in the usual sense. Corresponding to this difference in physical structure there are striking differences in the character of the waves propagated.

In the first place, in the ordinary type of guided wave system, the wave employed for the transmission of power and intelligence is the *Principal Plane Wave.* For the ideal non-dissipative case, the field of this wave is entirely transverse to the axis of the system; that is, the axial components of the electric and magnetic intensities are everywhere zero. Furthermore all frequencies are transmitted without attenuation with the same phase velocity; that of light in the medium. (Of course dissipation modifies the phenomena somewhat but in actual systems designed for efficient transmission the Principal Wave approximates to that just described.)

In the hollow conducting cylinder, on the other hand, *no principal transverse wave can exist;* that is, there must exist inside the cylinder either an axial component of the electric or the magnetic intensity, or both. Physically this is answerable to the absence of a circuit on which the transverse lines of force might terminate. Thus in the hollow conducting cylinder all the possible waves must be *complementary* waves;[7] a type which is ignored in the ordinary transmission system.

A second outstanding distinction is that in the hollow conducting cylinder, all frequencies below a critical frequency are attenuated while frequencies above the critical frequency are freely transmitted without attenuation.[8] In this respect the system behaves like a Campbell high-

[7] See "Guided and Radiated Energy in Wire Transmission," John R. Carson, *Jour. A.I.E.E.*, October 1924.

[8] It will be understood, of course, that this is strictly true only in the ideal case of no dissipation.

pass wave-filter. The exact value of the critical frequency depends, as shown later, on the type of wave transmitted; roughly speaking, however, the internal diameter must be approximately equal to one-half a wave-length in the internal dielectric medium at the lowest critical frequency. (The exact formula is diameter $> \frac{3.68}{2\pi}$ times the wave-length.) Since we are interested in freely transmitted waves it is evident at once that for a cylinder of practicable dimensions the frequencies employed must be relatively enormous. For this reason it may be appropriately said that the hollow conducting cylinder is applicable to the transmission of *hyper-frequency* waves alone.

The types of waves which can exist inside the cylinder are broadly classifiable as $E$-waves and $H$-waves.[9] By the term $E$-wave is to be understood a wave in which the axial component of the magnetic force is everywhere absent; correspondingly in the $H$-wave the axial component of the electric force is everywhere absent. In the $E$-waves the surface currents in the cylinder are entirely parallel to the axis thereof. On the other hand, in the $H$-waves the currents may have both transverse and axial components; that is, circulatory components around the periphery of the cylinder in planes normal to its axis as well as components parallel thereto.

In each class of wave there may exist a fundamental wave and in addition *geometrically* harmonic[10] waves. In the fundamental wave the phenomena do not vary around the periphery of the cylinder. In the $n$th harmonic wave ($E_n$- or $H_n$-wave) the phenomena vary around the periphery as $\cos n(\theta - \theta_n)$.

Each component $E$- or $H$-wave has its own individual critical frequency. Curiously enough the lowest critical frequency is possessed by the first harmonic $H$-wave; that is the $H_1$-wave. For this wave the critical frequency is given by the formula $d > \frac{3.68}{2\pi}\lambda$ where $d$ is the internal diameter of the cylinder and $\lambda$ the wave-length. In general, however, the critical frequency increases with the order of the harmonic.

In the usual transmission system, the transmission phenomena are determined and described in terms of the characteristic impedance and the propagation constant. By characteristic impedance the engineer understands the impedance actually presented by an infinitely long line to an electromotive force connected across the terminals of the circuit. Now since in the hollow conducting cylinder there is only one

[9] This terminology has been adopted as a matter of convenience. It is suggested by equations (1) where the field is expressed in terms of $E_z$ and $H_z$. Another terminology is *transverse magnetic* and *transverse electric* waves.

[10] This term must not be confused with *frequency* harmonics.

conductor and hence no circuit, this concept breaks down. There is another way, however, in which the characteristic impedance may be defined, and by aid of which it remains a useful concept in hollow cylinder transmission. Writing $K = K_R + iK_I$ as the complex expression for the characteristic impedance, then it may be shown that

$$K = \overline{W} + i2\omega(\overline{T} - \overline{U}),$$

where $\overline{W}$ is the mean power transmitted, $\overline{T}$ is the mean stored magnetic energy, and $\overline{U}$ the mean stored electric energy, corresponding to an unit current. Now in the hollow conducting cylinder, for, say the $E_0$-wave, we can calculate

$$\overline{W} + i2\omega(\overline{T} - \overline{U})$$

for an *unit axial* current, and call this the characteristic impedance. Again for the $H_0$-wave we can calculate this quantity for an *unit circulating current per unit length* and designate it as the characteristic impedance. In addition, somewhat similar conventions apply to the harmonic waves.

One of the chief uses of the foregoing concept of characteristic impedance is in the calculation of the attenuation in the dissipative system. For, if corresponding to $\overline{W}$ we calculate the mean dissipation $\overline{Q}$ per unit length, then the attenuation $\alpha$ is given by

$$\alpha = \overline{Q}/2\overline{W}.$$

All actual systems are of course dissipative and consequently the wave is attenuated. If the hollow conducting cylinder were to be employed in practice for hyper-frequency wave transmission the securing of low and desirable attenuation characteristics would probably be the controlling consideration.

The attenuation in the free transmission range is due to (1) dissipation in the cylinder or sheath and (2) dissipation in the internal dielectric medium. The former is inherent and can be reduced only by employing a sheath of high conductivity and by properly designing the dimensions of the system. As regards the dielectric loss, this may be substantially eliminated by employing air as the dielectric medium. The use of a dielectric medium of high specific inductive capacity has the advantage of substantially reducing the critical frequency; on the other hand it inevitably introduces heavy losses and thus sharply increases the attenuation. The analysis of Section III brings out the remarkable fact that for the fundamental $H$-wave the attenuation decreases with increasing frequency; for all the other types it increases.

For the very high frequencies with which we shall be concerned in the following analysis, a physically very thin cylindrical metallic sheath behaves electrically as though it were infinitely thick. This fact greatly simplifies the mathematical treatment; its real importance, however, is that external interference is entirely eliminated.

As stated at the outset, this paper will not attempt to deal with the problem of the reflection phenomena which occur at the terminals of the system and at points of discontinuity. For a discussion of the general character of the boundary problem the reader is referred to "Guided and Radiated Energy in Wire Transmission."[7] It may be remarked here, however, that the simple engineering boundary conditions (continuity of current and potential) are entirely inadequate.

## IB. Transmission Through Dielectric Guides

The greater part of this paper deals with transmission in thin hollow conducting cylinders; the last section, however, discusses briefly transmission along the dielectric wire.[3] Theoretically this type of transmission is extremely interesting and the mathematical theory resembles to a considerable extent that of hollow cylinder transmission. Unfortunately, however, dielectric losses are usually high. Hence our discussion of dielectric waves will be limited to a development of the fundamental equation from which the critical frequencies and the phase velocities can be determined.

## II. Non-Dissipative Hollow Conducting Guides

In dealing with the propagation of hyper-frequency electromagnetic waves inside a long hollow conducting cylinder parallel to the $z$-axis, it is convenient to write the field equations in the appropriate cylindrical coordinates $(\rho, \theta, z)$ in the form,[11]

$$
\begin{aligned}
\lambda^2 H_\rho &= \frac{h^2}{\mu i\omega}\frac{1}{\rho}\frac{\partial}{\partial\theta}E_z - \gamma\frac{\partial}{\partial\rho}H_z,\\
\lambda^2 H_\theta &= -\frac{h^2}{\mu i\omega}\frac{\partial}{\partial\rho}E_z - \frac{\gamma}{\rho}\frac{\partial}{\partial\theta}H_z,\\
\lambda^2 E_\rho &= -\gamma\frac{\partial}{\partial\rho}E_z - \frac{\mu i\omega}{\rho}\frac{\partial}{\partial\theta}H_z, \qquad (1)\\
\lambda^2 E_\theta &= -\frac{\gamma}{\rho}\frac{\partial}{\partial\theta}E_z + \mu i\omega\frac{\partial}{\partial\rho}H_z,\\
\operatorname{div} E &= 0, \quad \operatorname{div} H = 0.
\end{aligned}
$$

[11] In this form the field is expressed explicitly in terms of the axial electric and magnetic intensities and their spatial derivatives. This is highly advantageous for the purposes of this paper.

In these equations the symbols have the following significance:

$E_\rho, E_\theta, E_z$ = components of electric force,
$H_\rho, H_\theta, H_z$ = components of magnetic force,
$\lambda^2 = \gamma^2 - h^2$,
$\gamma$ = propagation constant,
$h^2 = \mu i\omega(4\pi\sigma + \epsilon i\omega/c^2) = 4\pi\sigma\mu i\omega - (\omega^2/v^2)$,
$v = c/\sqrt{\epsilon\mu}$ = velocity of light in the medium,
$c$ = velocity of light in air,
$\mu$ = permeability of the medium in electromagnetic units,
$\sigma$ = conductivity of the medium in electromagnetic units,
$\epsilon$ = dielectric constant of the medium in electrostatic units,
$\omega/2\pi$ = frequency,
$i = \sqrt{-1}$.

The solutions of these equations for the axial components of electric and magnetic force, $E_z$ and $H_z$ respectively, in the region, $0 \leqq \rho \leqq a$, $a$ being the internal radius of the conductor, are of the form

$$E_z = \sum_{n=0}^{\infty} J_n(\rho\lambda)(A_n \cos n\theta + B_n \sin n\theta) \exp. (i\omega t \pm \gamma z),$$
$$H_z = \sum_{n=0}^{\infty} J_n(\rho\lambda)(C_n \cos n\theta + D_n \sin n\theta) \exp. (i\omega t \pm \gamma z), \tag{2}$$

where $A_n$, $B_n$, $C_n$ and $D_n$ are arbitrary constants to be determined by boundary conditions and $J_n$ is the Bessel function of the first kind or the internal Bessel function. The components of the transverse electromagnetic field may then be expressed by introducing (2) in (1).

We shall first discuss the simplest case, that in which there is no dissipation. The current will then be in a sheet on the surface, $\rho = a$, of the perfectly conducting cylinder. But the axial current density $u_z$ and the circulating current density $u_\theta$ are given by

$$u_z = \frac{1}{4\pi} H_\theta, \quad \rho = a \tag{3}$$

and

$$u_\theta = \frac{1}{4\pi} H_z, \quad \rho = a. \tag{4}$$

Thus it follows that $H_z$ and $H_\theta$ are discontinuous at the surface $\rho = a$ and the boundary conditions are simply $E_z = E_\theta = 0$. These conditions can be fulfilled by two types of waves: (1) a wave for which $H_z$

is zero everywhere, which will be called generically the $E$-wave and (2) a wave for which $E_z$ is zero everywhere, which will be designated generically as the $H$-wave. (If the cylinder is dissipative, however, the $E$- and $H$-waves can exist alone only for the case of circular symmetry. In other words, unless $\partial/\partial\theta = 0$, neither the $E_z$ nor the $H_z$ component of the field can be identically zero. This will be discussed further in Section III.)

Assuming first a non-dissipative system, it will be seen that when $H_z$ is zero everywhere,

$$E_z \text{ and } E_\theta \sim J_n(\lambda\rho) \begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix}.$$

Thus the possible $E$-waves are determined by the boundary equation

$$J_n(\lambda a) = 0, \tag{5}$$

where

$$\lambda^2 = \gamma^2 + \omega^2/v^2.$$

This has an infinite number of real roots in $\lambda$ determining an infinite number of possible waves. Only a finite number, $m$, of these waves will be unattenuated, however, for, if $\lambda$ is to be real and $\gamma$ pure imaginary, the frequency must be so high that

$$\omega/v > \lambda_{nm}, \tag{6}$$

where $\lambda_{nm}a$ is the $m$th root of $J_n(\lambda a) = 0$. It is therefore convenient to designate as the $E_{nm}$-wave that component of the $E$-wave for which

$$E_z \sim J_n(\lambda_{nm}\rho) \begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix}.$$

Thus if

$$\lambda_{n,\,m+1} > \omega/v > \lambda_{nm},$$

the components $E_{n,\,m+1}$, $E_{n,\,m+2}$, $\cdots$ of the $E$-wave will all be attenuated but $E_{n1}$, $E_{n2}$, $\cdots$, $E_{nm}$ will be unattenuated. There will also be only a finite number $n + 1$ of the components $E_{01}$, $E_{11}, \cdots E_{n1}$, for the frequency must be at least sufficiently high so that

$$\omega/v > \lambda_{n1},$$

where $\lambda_{n1}a$ is the lowest root (excluding zero) of $J_n(\lambda a) = 0$, in order to transmit the component $E_{n1}$ of the $E$-wave without attenuation. Hence the $E$-wave consists of a doubly terminating series of possible components; for each of the finite number $k + 1$ possible values of $n$ there will be $m_n$ possible values of $\lambda a$ or a total of

$$m_0 + m_1 + m_2 + \cdots + m_k$$

possible modes of propagation.

For the $H$-wave, $E_z$ is zero everywhere,

$$E_\theta \sim J_n'(\lambda\rho) \begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix}$$

and the possible waves are determined by the transcendental equation

$$J_n'(\lambda a) = 0, \tag{7}$$

where

$$\lambda^2 = \gamma^2 + \omega^2/v^2$$

and $J_n'(z) = (d/dz)J_n(z)$. These values of $\lambda$ and consequently of $\gamma$ will, of course, differ from those characterizing the $E$-waves. Similarly, however, there will be a doubly terminating series of possible components, $H_{nm}$.

Hence for both types of wave the hollow conducting cylinder constitutes a high-pass wave-filter. The critical frequency $f_{nm}$ of the $E_{nm}$-wave is given by

$$f_{nm} = r_{nm}(c/2\pi a\sqrt{\epsilon\mu}), \tag{8}$$

where $r_{nm}$ is the $m$th root of $J_n(\lambda a) = 0$ or $r_{nm} = \lambda_{nm}a$. Similarly for the $H_{nm}$-wave, the critical frequency is

$$f_{nm}' = r_{nm}'(c/2\pi a\sqrt{\epsilon\mu}), \tag{8$'$}$$

where

$$r_{nm}' \text{ is the } m\text{th root of } J_{nm}'(\lambda a) = 0.$$

The propagation constant $\gamma_{nm}$ is then

$$\gamma_{nm} = \frac{i\omega}{c} \cdot \frac{c}{v_{nm}'} = \frac{i\omega}{v_{nm}'}, \tag{9}$$

where the ratio $c/v_{nm}'$ of the velocity of light in air to the phase velocity of the wave in response to any frequency $f$ is given by

$$\begin{aligned} c/v_{nm}' &= \sqrt{\epsilon\mu}\sqrt{1 - (f_{nm}/f)^2} \\ &\to 0 \quad \text{when} \quad f \to f_{nm}, \\ &\to \sqrt{\epsilon\mu} \quad \text{when} \quad f \to \infty \end{aligned} \tag{10}$$

for the $E$-wave and

$$c/v_{nm}' = \sqrt{\epsilon\mu}\sqrt{1 - (f_{nm}'/f)^2}$$

for the $H$-wave.

For the $E$-wave we have

$$\begin{aligned} r_{01}, r_{02}, \cdots &= 2.405, 5.52, \cdots \\ r_{11}, r_{12}, \cdots &= 3.83, 7.02, \cdots \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

and for the $H$-wave

$$r_{01}', r_{02}', \cdots = 3.83, 7.02, \cdots$$
$$r_{11}', r_{12}', \cdots = 1.84, 5.33, \cdots.$$

Hence, it is possible to transmit a fundamental $E$-wave if the radius, dielectric constant, permeability and frequency are so related that,

$$fa\sqrt{\epsilon\mu} \geqq 2.405(c/2\pi), \tag{11}$$

a fundamental $H$-wave provided,

$$fa\sqrt{\epsilon\mu} \geqq 3.83(c/2\pi), \tag{12}$$

the component $E_{11}$ of the $E$-wave provided

$$fa\sqrt{\epsilon\mu} \geqq 3.83(c/2\pi) \tag{13}$$

and the component $H_{11}$ of the $H$-wave provided

$$fa\sqrt{\epsilon\mu} \geqq 1.84(c/2\pi). \tag{14}$$

Thus from the standpoint of minimum physical constants and dimensions the component $H_{11}$ of the $H$-wave is most advantageous. The consideration of the attenuation characteristics below will show, however, that this advantage is outweighed, since in practice the attenuation will be the controlling factor.

We shall now consider the characteristic impedance of the system.[12] While the derivation of the characteristic impedance is interesting and valuable on its own merits, it also provides the basis for a quasi-synthetic and approximate method of deriving the attenuation which will be developed below. The results obtained here on the assumption of a perfect conductor will be valid in the dissipative case of the next section provided the conductivity is sufficiently high so that the relation, $4\pi\sigma \gg \epsilon\omega/c^2$, obtains among the constants of the sheath.

The characteristic impedance, $K$, is here defined as the transverse Complex Poynting Vector, $P$, integrated over the cross section of the system divided by the mean square current. Thus we have, in general,

$$\begin{aligned} P &= \frac{1}{8\pi}\int dS[E\cdot H^*]_z \\ &= \overline{W} + i2\omega(\overline{T} - \overline{U}), \end{aligned} \tag{15}$$

[12] See the discussion of the characteristic impedance in Section I of this paper. Equation (15) below is in agreement with the definition there given.

where $\overline{W}$ is the mean energy transmitted through the cylinder, $\overline{T}$ is the mean stored magnetic energy and $\overline{U}$ the mean stored electric energy, $H^*$ denoting the conjugate imaginary of $H$. Then

$$K = K_R + iK_I \tag{16}$$

and

$$\tfrac{1}{2}K_R|I|^2 = W, \tag{16a}$$

while

$$\tfrac{1}{2}K_I|I|^2 = 2\omega(\overline{T} - \overline{U}), \tag{16b}$$

$I$ being the total current. (In a non-dissipative system $\overline{T} = \overline{U}$ and $K = K_R$.) Rewriting the integral in (15) we therefore have

$$W = \tfrac{1}{2}K_R|I|^2 = \frac{1}{8\pi}\left[\int_0^{2\pi}\int_0^a \rho(E_\rho H_\theta^* - E_\theta H_\rho^*)d\rho d\theta\right]_{\text{Real Part}}. \tag{17}$$

(From equations (1) it readily follows, that for any $E$- or $H$-wave, $K$ may be made to depend upon either the transverse electric or transverse magnetic force alone by substituting in formula (17)

$$E_\rho H_\theta^* - E_\theta H_\rho^* = \frac{1}{c}\sqrt{\frac{\epsilon}{\mu}\frac{v'}{v}}\,[E]^2 = c\sqrt{\frac{\mu}{\epsilon}\frac{v}{v'}}\,[H]^2 \tag{18}$$

for the $E$-wave, and

$$E_\rho H_\theta^* - E_\theta H_\rho^* = \frac{1}{c}\sqrt{\frac{\epsilon}{\mu}\frac{v}{v'}}\,[E]^2 = c\sqrt{\frac{\mu}{\epsilon}\frac{v'}{v}}\,[H]^2 \tag{19}$$

for the $H$-wave, where $[E]^2$ and $[H]^2$ are defined as

$$[E]^2 = |E_\rho|^2 + |E_\theta|^2 \quad \text{and} \quad [H]^2 = |H_\rho|^2 + |H_\theta|^2.)$$

Consider first the fundamental $E$-wave. $H_z$, $H_\rho$ and $E_\theta$ are zero and

$$\begin{aligned} E_z &= AJ_0(\rho\lambda), \\ E_\rho &= \frac{\gamma}{\lambda}AJ_1(\rho\lambda), \\ H_\theta &= \frac{\epsilon i\omega}{c^2}\frac{1}{\lambda}AJ_1(\rho\lambda), \end{aligned} \tag{20}$$

where

$$\lambda^2 = \gamma^2 + \omega^2/v^2$$

and

$$\lambda = r_{0m}/a. \quad (J_0(r_{0m}) = 0.)$$

From (3) the total axial current $I_z$ in the sheath is given by

$$I_z = \frac{a}{2}H_\theta, \quad \rho = a. \tag{21}$$

Putting $I_z = 1$ then gives

$$A = \frac{c^2}{\epsilon i\omega} \frac{2\lambda}{aJ_1(\lambda a)} \cdot$$

Thus

$$\begin{aligned} K &= \frac{c}{\epsilon} \frac{c}{v'} \frac{2}{a^2} \frac{1}{(J_1(\lambda a))^2} \int_0^a \rho(J_1(\lambda\rho))^2 d\rho \\ &= \frac{c}{\epsilon} \cdot \frac{c}{v'} \left[ 1 + \left( \frac{J_0(\lambda a)}{J_1(\lambda a)} \right)^2 - \frac{2}{\lambda a} \frac{J_0(\lambda a)}{J_1(\lambda a)} \right] \\ &= c\sqrt{\mu/\epsilon}\,\sqrt{1 - (f_{0m}/f)^2}. \end{aligned} \tag{22}$$

Now, for the fundamental component $H_0$ of the $H$-wave, $E_z$, $E_\rho$ and $H_\theta$ are zero and

$$\begin{aligned} H_z &= CJ_0(\lambda\rho), \\ H_\rho &= \frac{\gamma}{\lambda} CJ_1(\lambda\rho), \\ E_\theta &= -\frac{\mu i\omega}{\lambda} CJ_1(\lambda\rho), \end{aligned} \tag{23}$$

where

$$\lambda^2 = \gamma^2 + \omega^2/v^2$$

and

$$\lambda = r_{0m}'/a. \qquad (J_0'(r_{0m}) = 0.)$$

There is no axial current transmitted by this wave but there is a circulating current in the sheath. From (4) this circulating current, $I_\theta$, per unit length is given by

$$I_\theta = \frac{a}{2} H_z \quad \text{when} \quad \rho = a. \tag{24}$$

Thus, for the $H_0$-wave, we calculate the characteristic impedance with respect to unit circulating current per unit length of conductor. This gives

$$C = \frac{4\pi}{J_0(\lambda a)}$$

and

$$\begin{aligned} K &= \frac{1}{2v'} \left( \frac{4\pi\omega}{\lambda} \right)^2 \frac{1}{(J_0(\lambda a))^2} \int_0^a \rho(J_1(\lambda))^2 d\rho \\ &= (2\pi a)^2 \frac{\mu}{v'} \left( \frac{f}{r_{0m}'} \right)^2, \end{aligned} \tag{25}$$

where, as given above, $r_{0m}'$ is the $m$th root of $J_0'(\lambda a)$, and, by (10),

$$v' = \frac{c}{\sqrt{\epsilon\mu}} \frac{1}{\sqrt{1 - (f_{0m}'/f)^2}} \cdot$$

So we see that, while the characteristic impedance of the $E_0$-wave approaches a constant at very high frequencies, for the $H_0$-wave we have

$$K \sim \omega^2.$$

In other words, while the energy transmitted by the $E_0$-wave is independent of the frequency at sufficiently high values of frequency, that transmitted by the $H_0$-wave increases as the square of the frequency.

For the harmonic $E$- and $H$-waves, the currents vary as $\cos n\theta$ around the periphery of the sheath. Hence the total harmonic current is zero over any axial or normal cross-section. For these waves, however, it is possible and convenient to calculate the Complex Poynting Vector on the basis of the average mean square current intensities,

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}\left|\frac{H_\theta}{4\pi}\right|^2 d\theta \quad \text{and} \quad \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}\left|\frac{H_z}{4\pi}\right|^2 d\theta, \qquad \rho = a,$$

which we may assume for convenience to be of the same value, 1/2, as the mean square currents associated with the fundamental components.

On this basis we shall obtain first the characteristic impedance of any harmonic component $E_n$ of the $E$-wave, ignoring dissipation. Putting

$$J_n(\lambda a) = 0 \quad \text{and} \quad \lambda a = r_{nm},$$

the Complex Poynting Vector becomes

$$\overline{W} = \frac{a^4}{16c}\sqrt{\frac{\epsilon}{\mu}\frac{v'}{v}}\left(\frac{\omega}{v'}\right)^2 \frac{|A_n|^2 + |B_n|^2}{r_{nm}^2}(J_{n-1}(r_{nm}))^2. \tag{26}$$

On the basis of the current value which we are assuming

$$\frac{|A_n|^2 + |B_n|^2}{\lambda^2}(J_{n-1}(r_{nm}))^2 = 32\pi^2\left(\frac{c^2}{\epsilon\omega}\right)^2. \tag{27}$$

Thus

$$K_R = (2\pi a)^2 c\sqrt{\mu/\epsilon}\sqrt{1 - (f_{nm}/f)^2}. \tag{28}$$

Similarly, for the component $H_n$ of the $H$-wave, we put

$$J_n'(\lambda a) = 0 \quad \text{and} \quad \lambda a = r_{nm}',$$

getting

$$\overline{W} = \frac{a^4 c}{16}\sqrt{\frac{\mu}{\epsilon}\frac{v'}{v}}\left(\frac{\omega}{v'}\right)^2 (|C_n|^2 + |D_n|^2)\frac{(J_n(r_{nm}'))^2}{(r_{nm}')^2}\left(1 - \frac{n^2}{(r_{nm}')^2}\right), \tag{29}$$

where

$$(|C_n|^2 + |D_n|^2)(J_n(r_{nm}'))^2 = 32\pi^2. \tag{30}$$

Thus

$$K_R = (2\pi a)^4\left(1 - \frac{n^2}{(r_{nm}')^2}\right)\frac{\mu}{v'}\left(\frac{f}{r_{nm}'}\right)^2, \tag{31}$$

where, as given above, $r_{nm}'$ is the $m$th root of $J_n'(\lambda a)$ and, by (10)

$$v' = \frac{c}{\sqrt{\epsilon\mu}}\frac{1}{\sqrt{1 - (f_{nm}'/f)^2}}.$$

Thus the mean transmitted energy and the characteristic impedance of all components of the $H$-wave increase as the square of the frequency whereas these characteristics of the $E$-wave are constant with respect to frequency. To appreciate the bearing of this difference upon the comparative attenuations consider the following argument.

Since the wave varies along the $z$-axis of the transmission system as exp. $((-\alpha - i\beta)z)$, $\alpha$ and $\beta$ denoting the attenuation and phase constants per unit length, respectively,

$$\frac{\partial \overline{W}}{\partial z} = -2\alpha\overline{W}. \tag{32}$$

But, denoting by $Q$ the dissipation loss per unit length of the transmission system, we have also

$$\frac{\partial \overline{W}}{\partial z} = -\overline{Q}. \tag{33}$$

Hence,

$$\alpha = \overline{Q}/2\overline{W} \tag{34}$$

$$= (4\pi\overline{Q}\int dS[E\cdot H^*]_z)_{\text{Real Part}}. \tag{35}$$

Thus, we see that, if the mean dissipation loss, $\overline{Q}$, is known or readily obtainable, the Complex Poynting Vector, $\overline{W}$, leads immediately to the attenuation.

To obtain $\overline{Q}$ we have the formula

$$\overline{Q} = -\left(\frac{1}{8\pi}\int dS[E\cdot H^*]_r\right)_{\text{Real Part}}. \tag{36}$$

Thus $\alpha$ may also be written

$$\alpha = \left(\frac{-\int dS[E\cdot H^*]_r}{2\int dS[E\cdot H^*]_z}\right)_{\text{Real Part}} \tag{37}$$

in which it is evident the current is not explicitly involved. If we write

$$\overline{Q} = R(I^2)_m$$

and

$$\overline{W} = K_R(I^2)_m,$$

$R$ being the resistance per unit length and $K_R$ the characteristic impedance with respect to the mean square current $(I^2)_m$ we have in addition

$$\alpha = R/2K_R. \tag{38}$$

Before continuing our discussion of attenuation, we shall, therefore, have to calculate the losses in the sheath and the internal dielectric medium.

## III. Dissipative Hollow Conducting Guides

In the ideal case of the preceding section, where the conductivities $\sigma_1$ and $\sigma_2$ of the dielectric and conductor are, respectively, zero and infinity, the boundary conditions are simply that $E_z = E_\theta = 0$ at the surface, $\rho = a$. When we take into account the dissipation which is actually present in the conductor (and the dielectric as well) the boundary conditions are the continuity of both the tangential electric and tangential magnetic forces. This double set of boundary conditions makes the problem inherently more difficult, of course. As we are assuming a good conductor and dielectric, we shall treat the dissipative case as a departure of the first order from the ideal case. Thus, since the dissipation has a negligible first order effect upon the phase velocity, the propagation constant $\gamma$ will now be

$$\gamma = i\omega/v' + \alpha,$$

where $\alpha$ denotes the attenuation.

We must now consider the field in the sheath as well as the field in the inner dielectric medium. When necessary we distinguish between the electrical constants of the two media by the subscripts 2 and 1, respectively. We suppose that the sheath is electrically very thick, a legitimate assumption at the very high frequencies in which we are interested, and write for $\rho > a$,

$$\begin{aligned} E_z &= \sum_{n=0}^{\infty} K_n(\rho\lambda_2)(A_n' \cos n\theta + B_n' \sin n\theta) \exp. (i\omega t \pm \gamma z), \\ H_z &= \sum_{n=0}^{\infty} K_n(\rho\lambda_2)(C_n' \cos n\theta + D_n' \sin n\theta) \exp. (i\omega t \pm \gamma z), \end{aligned} \tag{39}$$

where

$$\lambda_2^2 = \gamma^2 - h_2^2$$

and $K_n$ is the Bessel function of the second kind[13] (or the external Bessel function) and obtain $H_\rho$, $H_\theta$, $E_\rho$ and $E_\theta$ from (39) and (1). Putting

$$\lambda_1 a = y \quad \text{and} \quad \lambda_2 a = x,$$

and equating the tangential electric and magnetic forces $E_z$, $E_\theta$ and $H_z$, $H_\theta$ at the boundary surface $\rho = a$, we obtain eight homogeneous equations in the eight arbitrary constants. A non-trivial solution requires the vanishing of the determinant; this condition leads to the transcendental equation:

$$\left(\frac{h_1^2}{\mu_1}\frac{J_n'(y)}{yJ_n(y)} - \frac{h_2^2}{\mu_2}\frac{K_n'(x)}{xK_n(x)}\right)\left(\mu_1\frac{J_n'(y)}{yJ_n(y)} - \mu_2\frac{K_n'(x)}{xK_n(x)}\right) - n^2\gamma^2\left(\frac{1}{y^2} - \frac{1}{x^2}\right)^2 = 0, \quad (40)$$

where

$$y^2 = a^2(\gamma^2 - h_1^2) \quad (40a)$$

and

$$x^2 = a^2(\gamma^2 - h_2^2). \quad (40b)$$

The propagation constant $\gamma$ is then determined by equation (40).

We mentioned in Section II that the $E$- or $H$-waves cannot exist alone in the dissipative case unless they are circularly symmetrical and it may be noticed that both $E_z$ and $H_z$ were required in the analysis of the preceding paragraph. To show that $E_z$ and $H_z$ must coexist when the conductor is dissipative, assume for the moment that $E_z = 0$. The boundary equations when $n \neq 0$ are then

$$\begin{aligned} C_nJ_n(y) &= C_n'K_n(x), & D_nJ_n(y) &= D_n'K_n(x),\\ \frac{C_n}{y^2}J_n(y) &= \frac{C_n'}{x^2}K_n(x), & \frac{D_n}{y^2}J_n(y) &= \frac{D_n'}{x^2}K_n(x),\\ \frac{\mu_1C_n}{y}J_n'(y) &= \frac{\mu_2C_n'}{x}K_n'(x), & \frac{\mu_1D_n}{y}J_n'(y) &= \frac{\mu_2D_n'}{x}K_n'(x), \end{aligned} \quad (41)$$

six equations which cannot be satisfied by four arbitrary constants. When $n = 0$, however, $H_\theta$ is everywhere zero and the boundary equations are simply

$$\begin{aligned} C_0J_0(y) &= C_0'K_0(x),\\ \frac{\mu_1C_0}{y}J_0'(y) &= \frac{\mu_2C_0'}{x}K_0'(x). \end{aligned} \quad (42)$$

[13] This is the Hankel function given in Jahnke und Emde, "Funktionentafeln," p. 94, 1st ed., and denoted by $H_n^{(1)}(z)$ when $\arg z < \pi$. To avoid confusion with the $n$th harmonic of the $H$-wave, we shall use $K_n$ as a generic symbol to denote the external Bessel function.

Similarly the boundary equations can be satisfied when $H_z = 0$ provided $n = 0$ but not when $n \neq 0$.

Although $E_z$ and $H_z$ must co-exist in the dissipative case, one or the other will predominate in the actual wave provided the conductivity is so high that $4\pi\sigma_2 \gg \epsilon_2\omega/c^2$, a condition which is true of a good conductor unless $f \to \infty$. That this is so or, in other words, that the actual wave approximates either an $E$- or an $H$-wave will now be shown from equation (40). Since it is assumed that the conductivity is high or that

$$4\pi\sigma_2 \gg \epsilon_2\omega/c^2 \quad \text{and} \quad h_2^2 \gg \gamma^2, \tag{43}$$

$x = a\sqrt{-4\pi\sigma_2\mu_2 i\omega}$ and the asymptotic values of $K_n(x)$ and $K_n'(x)$ are valid. Equation (40) may then be written

$$\left(\frac{h_1^2}{\mu_1}\frac{J_n'(y)}{yJ_n(y)} - \frac{h_2}{\mu_2 a}\right)\left(\mu_1\frac{J_n'(y)}{yJ_n(y)} - \frac{\mu_2}{ah_2}\right) - n^2\gamma^2\left(\frac{1}{y^2} + \frac{1}{a^2h_2^2}\right) = 0. \tag{44}$$

When $h_2 = \infty$, (44) reduces to

$$J_n'(y) = 0 \text{ provided } J_n(y) \neq 0 \tag{45}$$

and to

$$J_n(y) = 0 \text{ provided } J_n'(y) \neq 0. \tag{46}$$

Thus there are two possible solutions of (44). These are in the neighborhood of $y = r$ and of $y = r'$, where $r$ and $r'$, respectively, are roots of $J_n(y) = 0$ and of $J_n'(y) = 0$, the equations characterizing the $E$- and the $H$-wave, respectively. We shall, therefore, refer to $E$- and $H$-waves in the dissipative case with the understanding that the actual wave approximates either one or the other type in a cylinder of sufficiently high conductivity.

As stated above, the propagation constant $\gamma$ may be determined by solving equation (40). The procedure is straightforward but is complicated by the necessity for approximations and does not easily admit of physical interpretation. We may obtain the same attenuation formulas by means of the quasi-synthetic method developed at the end of Section II.

The high-frequency attenuation of the symmetric $E$- and $H$-waves is easily derived from equation (38). Here $R$, the resistance per unit length of the cylinder for the $E$-wave at sufficiently high frequencies, is given by

$$R = \frac{\sqrt{\mu_2 f/\sigma_2}}{a}. \tag{47}$$

Introducing $K$ of (22) for $K_R$, and understanding that $\epsilon = \epsilon_1$, while it is assumed that $\epsilon_2 = 1$, we have

$$\alpha = \frac{1}{2ac}\sqrt{\frac{\epsilon\mu_2 f}{\mu_1\sigma_2}}\frac{1}{\sqrt{1-(f_{0m}/f)^2}} \tag{48}$$

to a high precision at high frequencies ($f > f_{0m}$). This is, of course, the contribution of the conductor and ignores the effect of the conductivity of the dielectric.

Similarly, for the fundamental $H$-wave, the resistance per unit length of the cylinder at sufficiently high frequencies, from equations (1) and (39) and the relations

$$\overline{Q} = \left[\frac{a}{8\pi}\int_0^{2\pi} E_\theta H_z^* d\theta\right]_{\text{Real Part}}$$

and

$$(I^2)_m = \frac{1}{2}|I_\theta|^2 = \frac{1}{2},$$

is given by

$$R = \frac{\sqrt{\mu_2 f/\sigma_2}}{a}. \tag{49}$$

Putting $K$ of (25) for $K_R$, gives, to the same precision as (48), when $f > f_{0m}'$,

$$\alpha = \frac{\sqrt{\epsilon\mu_2/\mu_1\sigma_2}(f_{0m}')^2}{2ac}\frac{f^{-3/2}}{\sqrt{1-(f_{0m}'/f)^2}}. \tag{50}$$

Formulas (48) and (50), respectively, may be written in the form

$$\alpha = \frac{\alpha_0}{\sqrt{1-(f_c/f)^2}}, \qquad f > f_c \tag*{(48)$'$}$$

and

$$\alpha = \alpha_0\frac{(f_c'/f)^2}{\sqrt{1-(f_c'/f)^2}}, \qquad f > f_c' \tag*{(50)$'$}$$

where

$$\alpha_0 = \frac{1}{2ac}\sqrt{\frac{\epsilon\mu_2 f}{\mu_1\sigma_2}}$$

and $f_c$ and $f_c'$ are the critical frequencies of the fundamental $E$- and $H$-waves, respectively, as given by (8) and (8)$'$.

Thus, in the neighborhood of their respective critical frequencies, the attenuations of the two waves are functionally the same; ultimately, however, while the attenuation of the fundamental $E$-wave *increases* as $f^{1/2}$, the attenuation of the fundamental $H$-wave *decreases* as $f^{-3/2}$; a remarkable property peculiar to this type of wave alone.

By extending the preceding treatment to the harmonic waves, it is found after some rather laborious analysis that for all the component $E$-waves,

$$\alpha = \frac{\alpha_0}{\sqrt{1 - (f_n/f)^2}}, \qquad f > f_n. \tag{51}$$

Care must be taken, of course, to choose the correct critical frequency ($f_n = f_{nm}$) for the particular component wave under consideration.

For all the $H$-waves (including the fundamental $H$-wave) it is found that

$$\alpha = \frac{\alpha_0}{\sqrt{1 - (f_n'/f)^2}} \left( (f_n'/f)^2 + \frac{(n/r')^2}{1 - (n/r')^2} \right). \tag{52}$$

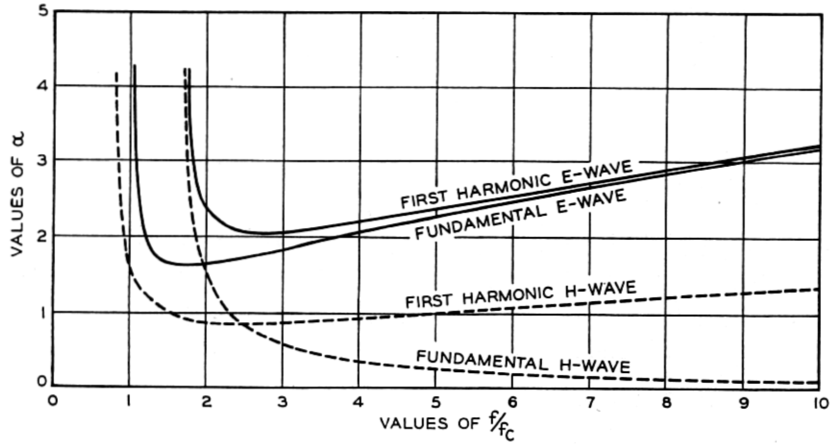
Here $n$ is the order of the geometric harmonic wave ($H_n$-wave) and $r'$ is the root of $J_n'(y)$ corresponding to the particular component wave under consideration.

The foregoing formulates the attenuation due to dissipation in the sheath alone. If we suppose that the dielectric has a very small but finite conductivity $\sigma_1$, then there must be added to the attenuation, for all types of waves, a term

$$\frac{2\pi\sigma_1 c\sqrt{\mu_1/\epsilon}}{\sqrt{1 - (f_n/f)^2}}. \tag{53}$$

To a first order approximation the dissipation has no effect on the phase velocity, which is simply $v'$.

Comparative values of attenuation are shown on the accompanying drawing for the fundamental and for the first harmonic $E$- and $H$-waves. This is the attenuation due to the loss in the conductor only. That due to the dielectric loss, the term given by (53), must be added. In many instances, we cannot say how large this term will be, for the losses in many dielectrics at the high frequencies involved herein are not known with any certainty at present. Such approximate calculations as we have made, however, have shown them to be very large except in the case of air.




Attenuation, $\alpha$, in Hollow Conducting Cylinder.

Multiply ordinates by $A_0 = \dfrac{4.66}{d}\sqrt{\dfrac{f_c}{\sigma \times 10^4}}$ to read db per mile.

For copper, $A_0 = \dfrac{1.89\sqrt{f_c}}{d}$.

Multiply abscissae by $f_c = (2.30/d)10^4$ to read frequency in megacycles. Here

$f_c$ = critical frequency of fundamental $E$-wave in megacycles,
$d$ = inner diameter of cylinder in centimeters,
$\sigma$ = conductivity of cylinder in emu
= $6.06 \times 10^{-4}$ for copper.

## IV. Dielectric Cylindrical Guides

We shall now pass to the mathematical theory of waves in dielectric "wires" of circular cross-section, immersed in air. We assume that the dielectric is perfect. The field in such a dielectric guide, and in the air outside, can be represented by the same general expressions as in hollow tubes. Thus for the $n$th harmonic wave, we have

$$\begin{aligned} E_z &= A_n J_n(\lambda_1\rho)\cos n\theta, \qquad H_z = B_n J_n(\lambda_1\rho)\sin n\theta, \text{ in the guide,} \\ E_z &= C_n K_n(\lambda_2\rho)\cos n\theta, \qquad H_z = D_n K_n(\lambda_2\rho)\sin n\theta, \text{ in the air.} \end{aligned} \tag{54}$$

The exponential factor $e^{-\gamma z+i\omega t}$ is implied in these as well as in the subsequent expressions for the field intensities. Another fundamental solution is obtained by changing $\theta$ into $\theta + \pi/2n$.

The transverse components of $E$ and $H$ are obtainable from $E_z$ and $H_z$ by differentiation. For our present purposes we need only $E_\theta$ and $H_\theta$; these are

$$
\begin{aligned}
E_\theta &= \left[ A_n \frac{n\gamma}{\lambda_1^2 \rho} J_n(\lambda_1 \rho) + B_n \frac{i\omega\mu_1}{\lambda_1} J_n'(\lambda_1 \rho) \right] \sin n\theta, \text{ in the guide,} \\
H_\theta &= -\left[ A_n \frac{i\omega\epsilon_1}{\lambda_1 c^2} J_n'(\lambda_1 \rho) + B_n \frac{n\gamma}{\lambda_1^2 \rho} J_n(\lambda_1 \rho) \right] \cos n\theta, \text{ in the guide,} \\
E_\theta &= \left[ C_n \frac{n\gamma}{\lambda_2^2 \rho} K_n(\lambda_2 \rho) + D_n \frac{i\omega\mu_2}{\lambda_2} K_n'(\lambda_2 \rho) \right] \sin n\theta, \text{ in the air,} \\
H_\theta &= -\left[ C_n \frac{i\omega\epsilon_2}{\lambda_2 c^2} K_n'(\lambda_2 \rho) + D_n \frac{n\gamma}{\lambda_2^2 \rho} K_n(\lambda_2 \rho) \right] \cos n\theta, \text{ in the air.}
\end{aligned} \tag{55}
$$

The boundary conditions require the continuity of the tangential components of $E$ and $H$. Hence if $a$ is the radius of the guide, we have

$$
A_n J_n(\lambda_1 a) = C_n K_n(\lambda_2 a), \quad B_n J_n(\lambda_1 a) = D_n K_n(\lambda_2 a), \tag{56}
$$

$$
A_n \frac{n\gamma}{\lambda_1^2 a} J_n(\lambda_1 a) + B_n \frac{i\omega\mu_1}{\lambda_1} J_n'(\lambda_1 a) = C_n \frac{n\gamma}{\lambda_2^2 a} K_n(\lambda_2 a) + D_n \frac{i\omega\mu_2}{\lambda_2} K_n'(\lambda_2 a),
$$

$$
A_n \frac{i\omega\epsilon_1}{\lambda_1 c^2} J_n'(\lambda_1 a) + B_n \frac{n\gamma}{\lambda_1^2 a} J_n(\lambda_1 a) = C_n \frac{i\omega\epsilon_2}{\lambda_2 c^2} K_n'(\lambda_2 a) + D_n \frac{n\gamma}{\lambda_2^2 a} K_n(\lambda_2 a).
$$

This is a homogeneous set of linear equations in the coefficients $A$, $B$, $C$ and $D$ from which only the ratios of these coefficients can be determined. But there are only *three* independent ratios and *four* equations; eliminating these ratios we shall obtain the *characteristic equation* of our boundary value problem from which the propagation constant $\gamma$ can be calculated in terms of the frequency, the radius of the guide and the electromagnetic constants of the guide.

If $n = 0$, the above set of equations breaks up into two independent sets connecting the pairs $A$, $C$ and $B$, $D$. Hence non-trivial solutions are possible by letting $A = C = 0$ or $B = D = 0$. In one case $E_z$ is zero everywhere and in the other $H_z$ vanishes. Thus in the circularly symmetric case we have waves of either the $E$-type or $H$-type in the sense previously defined. But if $n \neq 0$, then $E_z$ and $H_z$ must be present simultaneously.

The case $n = 0$ is so much simpler than the others that we shall examine it separately. Thus the characteristic equation for an $E_0$-wave is

$$
\frac{\epsilon_1 J_1(\lambda_1 a)}{\lambda_1 a J_0(\lambda_1 a)} = \frac{\epsilon_2 K_1(\lambda_2 a)}{\lambda_2 a K_0(\lambda_2 a)}, \tag{57}
$$

and that for an $H_0$-wave is

$$
\frac{\mu_1 J_1(\lambda_1 a)}{\lambda_1 a J_0(\lambda_1 a)} = \frac{\mu_2 K_1(\lambda_2 a)}{\lambda_2 a K_0(\lambda_2 a)}. \tag{58}
$$

In addition to either of these equations, we have

$$\gamma = \sqrt{\lambda_1^2 - \mu_1\epsilon_1\omega^2/c^2} = \sqrt{\lambda_2^2 - \mu_2\epsilon_2\omega^2/c^2}, \quad (59)$$

and the condition that for truly guided waves $\gamma$ and $\lambda_2$ must be pure imaginary while $\lambda_1$ is real. When $\lambda_2$ is pure imaginary, the Hankel function of the second kind will decrease almost exponentially with increasing distance from the guide if this distance is sufficiently large.

If $\lambda_1$ and $\lambda_2$ are taken from (59) and substituted in (57) and (58) we shall have equations determining $\gamma$ in terms of $\omega$. Unfortunately these equations do not admit of an explicit solution for $\gamma$. It is possible, however, to carry out the numerical calculations in the following manner. We plot the left and the right terms of (57), let us say, against their arguments; then we select a pair of values of these arguments corresponding to equal ordinates. Let us suppose that we obtain

$$(\lambda_1 a)^2 = p^2, \quad (\lambda_2 a)^2 = -q^2, \quad (60)$$

where $p$ and $q$ are real. Referring to section III, we have $p = y$ and $iq = x$. Substituting these in (59) and solving, we have

$$\omega = \frac{c\sqrt{p^2 + q^2}}{a\sqrt{\mu_1\epsilon_1 - \mu_2\epsilon_2}}, \quad \gamma = i\sqrt{\frac{\mu_2\epsilon_2\omega^2}{c^2} + \frac{q^2}{a^2}}. \quad (61)$$

Since $\mu_1$ usually equals $\mu_2$, the guided waves are possible only if the dielectric constant of the guide is higher than that of the surrounding medium.

The lowest value of $q$ is zero; the right member of (57) is then infinite and the corresponding value of $p$ must then be a root of

$$J_0(p_m) = 0. \quad (62)$$

Corresponding to each root we have a different mode of propagation. The lowest frequency which can be transmitted in any particular mode and the corresponding propagation constant are given by

$$\omega_m = \frac{cp_m}{a\sqrt{\mu_1\epsilon_1 - \mu_2\epsilon_2}}, \quad \gamma = \frac{i\omega\sqrt{\mu_2\epsilon_2}}{c}. \quad (63)$$

At this frequency the phase velocity of propagation is equal to that of light in air. Since $\lambda_2$ is small, the field extends to great distances outside the guide. As $q$ increases indefinitely, the right part of (57) approaches zero and $p$ must approach the root of $J_1(x)$ near the particular root of $J_0$ that we happen to be considering. Thus for large values of $q$, we have approximately

$$\omega = \frac{cq}{a\sqrt{\mu_1\epsilon_1 - \mu_2\epsilon_2}}, \quad \gamma = \frac{i\omega\sqrt{\mu_1\epsilon_1}}{c}. \quad (64)$$

Hence at high frequencies the propagation takes place substantially with the velocity of light appropriate to the substance of the guide. The constant $\lambda_2$ being large, the field is concentrated largely in the guide.

Returning to the general $n$-th harmonic wave, we set

$$\begin{aligned} A_n &= SK_n(\lambda_2 a), \quad C_n = SJ_n(\lambda_1 a), \\ B_n &= TK_n(\lambda_2 a), \quad D_n = TJ_n(\lambda_1 a). \end{aligned} \tag{65}$$

Substitute in the last two equations of (56) and eliminate $S$ and $T$. Thus we obtain

$$\begin{aligned} \frac{n\gamma}{a} J_n K_n \left(\frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2}\right) S &= i\omega \left(\frac{\mu_2 J_n K_n'}{\lambda_2} - \frac{\mu_1 K_n J_n'}{\lambda_1}\right) T, \\ \frac{i\omega}{c^2}\left(\frac{\epsilon_2 J_n K_n'}{\lambda_2} - \frac{\epsilon_1 K_n J_n'}{\lambda_1}\right) S &= \frac{n\gamma}{a} J_n K_n \left(\frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2}\right) T. \end{aligned} \tag{66}$$

Subsequently

$$\begin{aligned} \frac{\epsilon_1\mu_1 J_n'^2}{p^2 J_n^2} - \frac{i(\epsilon_1\mu_2 + \mu_1\epsilon_2)J_n'K_n'}{pqJ_nK_n} - \frac{\epsilon_2\mu_2 K_n'^2}{q^2K_n^2} \\ = n^2\left(\frac{1}{p^2} + \frac{1}{q^2}\right)\left(\frac{\epsilon_1\mu_1}{p^2} + \frac{\epsilon_2\mu_2}{q^2}\right), \end{aligned} \tag{67}$$

and finally

$$\begin{aligned} \epsilon_2\mu_2 \frac{K_{n-1}K_{n+1}}{q^2K_n^2} - \epsilon_1\mu_1 \frac{J_{n-1}J_{n+1}}{p^2J_n^2} - \frac{i(\epsilon_1\mu_2 + \mu_1\epsilon_2)J_n'K_n'}{pqJ_nK_n} \\ = n^2 \frac{\epsilon_1\mu_1 + \epsilon_2\mu_2}{p^2q^2}. \end{aligned} \tag{68}$$

Allowing $q$ to approach zero, we shall obtain in the limit an equation whose roots in conjunction with (61) determine the critical frequencies.

Thus if $n > 1$, we obtain

$$(\epsilon_1\mu_2 + \mu_1\epsilon_2)\frac{pJ_{n-1}(p)}{J_n(p)} = n(\epsilon_1 - \epsilon_2)(\mu_2 - \mu_1) + \frac{\epsilon_2\mu_2}{n-1}p^2. \tag{69}$$

Since ordinarily $\mu_1 = \mu_2$, (69) becomes

$$\frac{J_{n-1}(p)}{pJ_n(p)} = \frac{\epsilon_2}{(n-1)(\epsilon_1 + \epsilon_2)}. \tag{70}$$

If the dielectric constant of the guide is very much higher than that of the surrounding air, the first few roots of (70) are very close to those of $J_{n-1}(p) = 0$. As $q$ increases indefinitely (68) degenerates into

$$\frac{J_{n-1}(p)J_{n+1}(p)}{J_n^2(p)} = 0. \tag{71}$$

Thus in the limit the roots of (68) will be *exactly* those of $J_{n-1}(p) = 0$. In other words as $q$ varies from 0 to $\infty$ the corresponding value of $p$ as given by (68) will not change much. It might appear that the limiting values of $p$ could be roots of $J_{n+1}(p) = 0$; this is not possible, however, because in the process of transition $p$ would have to pass through the intermediate zero of $J_n(p)$ and no real value of $q$ is consistent with such zeros.

The case $n = 1$ requires a special examination. After multiplying (68) by $q^2$ and permitting $q$ to approach zero, we find that the first term tends to infinity while the last term becomes a constant. Since the limit of $\frac{qK_1'}{K_1}$ is finite, $J_1(p)$ must approach zero. Thus for $n = 1$, the critical frequencies are determined by the zeros of $J_1(p)$.

One interesting point may be mentioned in conclusion. If the guide were surrounded by a hypothetical medium of zero dielectric constant, equation (57) for the $E_0$-waves would become

$$\frac{J_1(\lambda_1 a)}{\lambda_1 a J_0(\lambda_1 a)} = 0, \quad J_1(\lambda_1 a) = 0. \tag{72}$$

Thus the critical frequencies would be given by the roots of $J_1(p) = 0$ and not by those of $J_0(p) = 0$ as is the case for *any* ratio $\frac{\epsilon_2}{\epsilon_1}$ *different from zero* no matter how small it may be. Our first impression is that this result does violence to our physical common sense which demands that the hypothetical idealized case should be an approximation to the real one when one dielectric constant is large in comparison with the other. And indeed common sense is justified if one does not adhere too closely to the exact mathematical definition of the expression "critical frequency." In the region between any particular zero of $J_0(p)$, giving the true critical frequency, and the corresponding zero [14] of $J_1(p)$, giving the "approximate" critical frequency, most of the energy travels *outside* the guide, with a velocity substantially equal to that of light in the surrounding medium. The "approximate" critical frequency marks the region of the most rapid transition from wave propagation outside the guide to that inside the guide.

[14] This zero is always larger than that of $J_0(p)$.