Some Equivalence Theorems of Electromagnetics and Their Application to Radiation Problems

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After a review of the general aspects of the classical electromagnetic theory several "equivalence" theorems are established and illustrated with a number of examples from the diffraction theory. Then follows a discussion of possible applications of these theorems to radiation problems. The latter part of the paper is devoted to the calculation of the power radiated from an open end of a coaxial pair.

THE usual methods of calculating the power radiated by an electric circuit depend upon a determination of the electromagnetic field from the electric current distribution in the circuit. The best known of these methods consists in integrating the Poynting vector over the surface of an infinite sphere surrounding the circuit. This method has been used exclusively until recent years; to facilitate its application, John R. Carson obtained a compact general formula for the radiated power.[1] Another method [2] consists in calculating the work done against the forces of the field in supporting a given current distribution in the circuit. Theoretically either of the two methods is sufficient for solving any radiation problem. Practically, aside from inherent difficulties involved in the calculation of the electric current distribution in the first place, the preliminary integration for determining the field components E and H may be rather complex. Thus in obtaining the power radiated by a semi-infinite pair of perfectly conducting coaxial cylinders this preliminary integration has to be extended over the infinite surfaces of the two conductors. And yet by the Maxwell-Poynting theory, no energy can flow through the walls of the outer cylinders since the electric intensity E and hence the Poynting vector vanish there. Any energy which is radiated away must pass through the open end and it is natural to expect that there must be a method for calculating this energy from the conditions at the open end. The integration involved in this method would extend only over a comparatively small area of the open end. It is in search of a method of this type for calculating the radiated power that I was led some time ago to certain "equivalence theorems." Subsequently I learned that

[1] John R. Carson, "Electromagnetic Theory and the Foundations of Electric Circuit Theory," *The Bell System Technical Journal*, pp. 1–17, January 1927.

[2] A. A. Pistolkors, "The Radiation Resistance of Beam Antennas," *Proc. I. R. E.*, Vol. 17, No. 3 (1929). R. E. Bechmann, "On the Calculation of Radiation Resistance of Antennas and Antenna Combinations," *Proc. I. R. E.*, Vol. 19, p. 1471 (1931).

one of these theorems was discovered long ago, first by A. E. H. Love [3] and then by H. M. MacDonald [4] and proved by the latter [5] for the case of non-dissipative media in 1911. Another proof of this theorem, believed to be helpful from the physical point of view and extended so as to include the dissipative media, is given in this paper. After a brief review of some fundamental concepts we shall prove these equivalence theorems, discuss their significance, and solve one or two simple examples for illustrative purposes.

The physical sources of electromagnetic fields are electric and magnetic charges in motion, that is electric and magnetic currents. The radio engineer has never been interested in shaking magnets for the purpose of radiating energy and has settled into a habit of ignoring magnetic currents altogether as if they were non-existent. It is true that there are no magnetic conductors and no magnetic conduction currents in the same sense as there are electric conductors and electric conduction currents but magnetic convection currents are just as real as electric convection currents, although the former exist only in doublets of oppositely directed currents since magnetic charges themselves are observable only in doublets. And, of course, the magnetic displacement current, defined as the time-rate of change of the magnetic flux, is exactly on the same footing as the electric displacement current defined by Maxwell as the time-rate of change of the electric displacement. We shall find it convenient, at least for analytical purposes, to employ the concept of magnetic current on a par with the concept of electric current.

The two fundamental electromagnetic laws can now be stated in a symmetric form. Ampère's law as amended by Maxwell is: *An electric current is surrounded by a magnetic field of force; the induced magnetomotive force in a closed curve is equal to the electric current passing through any surface bounded by the curve.* In its original form, the "electric current" meant only the conduction current so that the law was applicable only to closed conduction currents. Maxwell's amendment consisted in including the displacement currents, thereby making the law applicable to open conduction currents. The second law is due to Faraday: *A magnetic current is surrounded by an electric field of force; the induced electromotive force in a closed curve is equal to the negative of the magnetic current passing through any surface bounded by the curve.* The rule for algebraic signs is as follows: choose some direction of the closed curve as positive and have an observer placed in such a way that

[3] A. E. H. Love, "The Integration of the Equations of Propagation of Electric Waves," *Phil. Trans.* A, Vol. 197, pp. 1–45 (1901).

[4] H. M. MacDonald, "Electric Waves," p. 16 (1902).

[5] H. M. MacDonald, "The Integration of the Equations of Propagation of Electric Waves," *Proc. London Mathematical Society*, Series E, Vol. 10, pp. 91–95 (1911).

this direction appears to him counterclockwise; then the positive direction of either the electric or the magnetic current is chosen *toward* the observer. If the currents are flowing toward the reader, the directions of the E.M.F. and the M.M.F. are as indicated in Fig. 1.

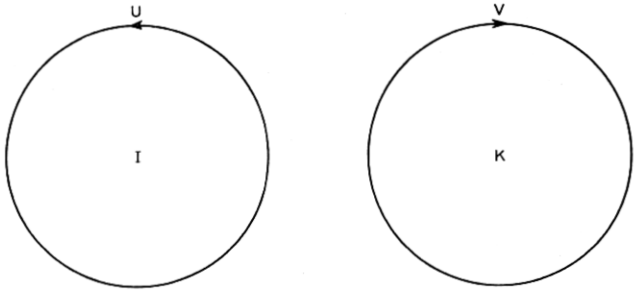

Fig. 1—The relative directions of the E.M.F. and the M.M.F. induced respectively by the electric current I and the magnetic current K are indicated by the arrows. Both I and K are directed toward the reader.

In the well-known way these two physical laws lead to a pair of partial differential equations

$$\text{curl } E = -M, \qquad \text{curl } H = J, \tag{1}$$

where J and M are respectively the total electric current density and the total magnetic current density. The electric density is composed of several parts; namely: the conduction current density, the displacement current density and the applied current density. The first of these components is, in many substances, proportional to the electric intensity E; the second is proportional to $\partial E/\partial t$; and the third is due to forces other than those of the field, mechanical or chemical, for instance. Similarly the magnetic current density is the sum of the magnetic displacement density proportional to $\partial H/\partial t$ and the impressed magnetic current density. Thus, we write

$$\text{curl } E = -M_0 - \mu \frac{\partial H}{\partial t}, \qquad \text{curl } H = J_0 + gE + \epsilon \frac{\partial E}{\partial t}, \tag{2}$$

where J_0 and M_0 are the densities of the impressed currents and the constants of proportionality g, ϵ and μ are respectively the conductivity, the dielectric constant and the permeability.[6]

The functions J_0 and M_0 are supposed to be known functions of coordinates and of time, representing the distribution of the physical

[6] A consistent practical system of units is used in this paper. Thus the E.M.F. is measured in volts, the electric current in amperes, E in volts per centimeter, H in amperes per centimeter, etc. The permeability of vacuum is then $4\pi 10^{-9}$ henries per centimeter and the dielectric constant $(1/36\pi)10^{-11}$ farads per centimeter.

sources in the space-time. If they are zero everywhere and at all times, the only physically significant solution of (2) must be $E = H = 0$ throughout the entire space and at all times.[7] If there are other solutions of (2), they are extraneous and some rule must be found for excluding them. Such extraneous solutions often find their way into mathematical equations because it is usually impossible to express *all* physical conditions by an equation or a system of equations. Naturally these remarks do not apply to a limited region of space or a finite interval of time. In fact, in many physical problems these "extraneous in the large" solutions of (2) can be advantageously used for expressing the general character of electromagnetic phenomena in a limited region and then obtaining, with the aid of the boundary and the initial conditions, the complete answer. But the philosophy of causality demands the dictum "no sources, no field" when considering the whole space-time. It may seem unnecessary to dwell at length on such obvious matters but they happen to be essential in the subsequent discussion if the arguments are to be taken as positive proofs rather than as plausible justifications.

Equations (2) are linear and the principle of superposition is applicable. This is in accordance with physical intuition which tells us that we can subdivide the impressed currents into elementary cells of volume dv, calculate the field due to a typical element, and obtain the total field by integration. For the typical element (2) becomes

$$\operatorname{curl} E = -\mu \frac{\partial H}{\partial t}, \qquad \operatorname{curl} H = gE + \epsilon \frac{\partial E}{\partial t}, \tag{3}$$

everywhere except in the infinitely small volume occupied by the element. The product of the current density and the volume of the element is called the *moment* of the element.

At times the impressed currents are confined to sheets so thin that their thickness can be disregarded without introducing a serious error in the result. This leads to a hypothetical infinitely thin *current sheet.* We pass from a real current sheet to an ideal one by assuming that the thickness of the former decreases and the current density increases in such a way that their product remains constant. This product is called the linear density of the sheet and it represents the current per unit length perpendicular to the lines of flow. The moment of a current element is now the product of the density of the sheet and the area of the element. Finally if the impressed current is confined to a

[7] We assume that all the electric and magnetic charges were originally in the neutral state, in which case their separation could be effected only through their motion. The argument could be extended so as to include purely static fields that may constitute an integral part of the universe but it is of no particular interest to us.

very thin filament, the moment is the product of the current and the length of the element.

It is the moment of the current element that determines its electromagnetic field. If the medium is non-dissipative, the actual expressions for the field components are obtained in terms of an auxiliary function called by Lorentz the *retarded magnetic vector potential.* For an electric current element of moment $p(t)$ this vector potential at any point P is parallel to the current density and is a function of the distance r from the element to P

$$A = \frac{p\left(t - \frac{r}{c}\right)}{4\pi r}. \tag{4}$$

The quantity c has the dimensions of a velocity and it appears that the action of the source travels outward with this velocity. But there is another solution of (3)

$$A_1 = \frac{p\left(t + \frac{r}{c}\right)}{4\pi r}. \tag{5}$$

One might wonder if this solution appertains in any way to the source; that is not the case, however. If the moment $p(t)$ is identically zero prior to some instant $t = t_0$, the field which can legitimately be attributed to the action of this source is also identically zero for any instant $t < t_0$. But (5) implies a non-vanishing field at distant points; it is as if the effect appeared before the cause. Any other solution is a combination of (4) and (5) and has to be rejected on the same grounds.

In terms of this auxiliary vector potential the field components can be expressed as follows

$$H = \text{curl}\, A, \qquad \frac{\partial E}{\partial t} = \frac{1}{\epsilon}\,\text{curl}\, H, \qquad E = \frac{1}{\epsilon}\int_{-\infty}^{t} \text{curl}\, H\, dt. \tag{6}$$

If the moment is harmonic of frequency f, we regard it as the real part of $pe^{i\omega t}$. Then the vector potential and the field components are the real parts of the following expressions

$$A = \frac{pe^{-i\beta r}}{4\pi r}, \qquad H = \text{curl}\, A, \qquad E = -\, i\omega\mu A + \frac{\text{grad div}\, A}{i\omega\epsilon}, \tag{7}$$

where the *phase constant*

$$\beta = \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda},$$

λ is the wave-length, and the time factor $e^{i\omega t}$ is implied.

If the medium is dissipative, we have

$$A = \frac{pe^{-\sigma r}}{4\pi r}, \qquad H = \text{curl } A, \qquad E = -i\omega\mu A + \frac{\text{grad div } A}{g + i\omega\epsilon}. \tag{8}$$

The quantity σ is the *intrinsic propagation constant* of the medium and is defined by

$$\sigma = \sqrt{i\omega\mu(g + i\omega\epsilon)}. \tag{9}$$

In this case the action of the source at some point is not only delayed by the time needed for the disturbance to travel the intervening distance but also exponentially attenuated.

If instead of an electric current element, we are dealing with a magnetic element, the field components can be expressed in terms of an *auxiliary electric vector potential.* This vector F is given by

$$F = \frac{Pe^{-\sigma r}}{4\pi r}, \tag{10}$$

where the moment P of the element is the product of the magnetic current density and the volume of the element. The field components are then given by

$$E = -\text{ curl } F, \qquad H = -(g + i\omega\epsilon)F + \frac{\text{grad div } F}{i\omega\mu}. \tag{11}$$

In the periodic case the general mathematical solution for the vector potential of an element is found to be a linear combination of any two of the following functions

$$\frac{e^{-\sigma r}}{r}, \qquad \frac{e^{\sigma r}}{r}, \qquad \frac{\cosh \sigma r}{r}, \qquad \frac{\sinh \sigma r}{r}. \tag{12}$$

All of these except the first become exponentially infinite at an infinite distance from the source and cannot be taken to represent the vector potential of a physical source. The last function is finite in any finite region; conceivably it can represent an electromagnetic field in the finite region free from physical sources. If the medium is non-dissipative it is impossible to exclude any of the solutions given by (12) on the grounds of their behavior at infinity—they all vanish there. But we may regard the non-dissipative case as the limit of the dissipative one and in this way establish a rule for finding the proper unique solution.

In the presence of a current sheet, equations (2) are valid on either side of it but not on it. Let us consider a cross-section of an electric current sheet, perpendicular to the lines of flow, and a curvilinear rectangle $A'B'B''A''$ with two of its sides parallel to the sheet (Fig. 2). We assume that the current flows toward the reader and that $A'A''$ and $B'B''$ are vanishingly small. Since the M.M.F. around this rectangle is equal to the electric current passing through it and since this M.M.F. is merely the difference between the M.M.F.'s along the sides $A'B'$ and $A''B''$, we obtain

$$H_t' - H_t'' = J_l \tag{13}$$

by simply calculating these quantities per unit length of the rectangle. The tangential components of the magnetic intensity are regarded as

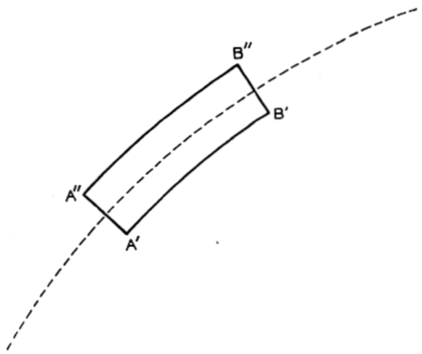

Fig. 2—A cross-section of a current sheet perpendicular to the lines of flow. The positive direction of the current is toward the reader.

positive when directed from A to B. Thus the tangential component of the magnetic intensity is discontinuous across an electric current sheet and the amount of the discontinuity is equal to the density of the sheet.

Similarly across a magnetic current sheet the tangential component of the electric intensity is discontinuous and the amount of this discontinuity is equal to the negative of the magnetic current density of the sheet; thus

$$E_t' - E_t'' = - M_l. \tag{14}$$

In deriving equations (2) it is also necessary to assume that g, μ and ϵ are continuous throughout the region under consideration. They have no meaning on the boundary between two different media. Since the boundary is a geometric surface, it cannot constitute either an electric or a magnetic current sheet. Hence the components of E and H tangential to such a boundary are continuous across it. These *boundary conditions* provide a link between the fields in the two media.

Let us suppose that we have a continuous distribution of sources on a closed surface C (Fig. 3) and that there are no other sources. We assume that the sources are harmonic of frequency f. The electromagnetic field $\mathfrak{F}$ produced by these sources can be calculated directly from this distribution with the aid of the above mentioned vector potentials. On the other hand, we can reason as follows.

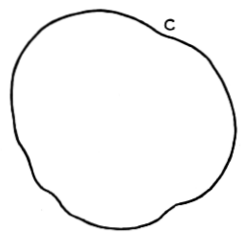

Fig. 3—A cross-section of a closed surface C.

There are no sources either inside or outside of C; hence everywhere except on C, we have

$$\text{curl } E = -i\omega\mu H, \qquad \text{curl } H = (g + i\omega\epsilon)E. \tag{15}$$

In the region inside of C we take that solution of (15) which is finite throughout this region and outside of C we select the solution vanishing at infinity. Both solutions will contain constants which can be determined from conditions (13) and (14) across the surface. The field $\mathfrak{F}'$ obtained in this manner is identical with $\mathfrak{F}$ because the difference $\mathfrak{F} - \mathfrak{F}'$ is everywhere source-free and thus must vanish.

Let us now reverse the process and, instead of starting with the known distribution of sources on C, suppose that we know the field and wish to find its sources. Let the known field $\mathfrak{F}$ be source free everywhere except on C. In order to determine these sources S we merely calculate the discontinuities in the tangential components of E and H across C. We can utilize this result to establish the major *Equivalence Principle.* For the outside portion of $\mathfrak{F}$ we can choose the outside portion of the field $\mathfrak{F}'$ produced by a given system of sources S' situated inside C and for the inside part of $\mathfrak{F}$ we take any field which is source-free there. The latter may be, for instance, the inside portion of the field $\mathfrak{F}''$ produced by some sources S'' situated outside C. Thus we arrive at the following *Equivalence Principle* discovered by Love and Macdonald [8]: a distribution of electric and magnetic currents on a given surface C can be found such that *outside* C it produces the

[8] See references 3 and 4 and also H. M. Macdonald, "Electromagnetism" (1934).

same field as that produced by given sources *inside* C; and also the field *inside* C is the same as that produced by given sources *outside* C. One of these systems of sources can be identically equal to zero.

The actual calculations are made as follows. From the discontinuities in the tangential components of E and H, we obtain J and M by (13) and (14). From these currents we find the two vector potentials

$$\begin{aligned} A &= \frac{1}{4\pi} \iint\limits_{(C)} \frac{J(x', y', z')}{r} e^{-i\beta r}\, dS, \\ F &= \frac{1}{4\pi} \iint\limits_{(C)} \frac{M(x', y', z')}{r} e^{-i\beta r}\, dS, \end{aligned} \tag{16}$$

where $r = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$ is the distance between a point $P(x, y, z)$ somewhere in space and a point $P'(x', y', z')$ on C. From these potentials we calculate the electric intensity and the magnetic intensity by

$$\begin{aligned} E &= -\, i\omega\mu\, A + \frac{\text{grad div } A}{i\omega\epsilon} - \text{curl } F, \\ H &= \text{curl } A + \frac{\text{grad div } F}{i\omega\mu} - i\omega\epsilon\, F. \end{aligned} \tag{17}$$

The proof of the Equivalence Principle can be modified so as to throw some additional light on it. Let us suppose that given sources S' are inside the closed surface C and let us make our new synthetic field by obliterating the old field outside C and leaving everything as it was inside C. The new field has the same sources S' and besides it is discontinuous across C. These discontinuities are the additional sources S whose densities are calculable from (13) and (14). Since the new field is identically zero outside C, the field produced by S is such as to cancel the field produced by S' outside C. Thus the system of sources S acts as a perfect absorber for the electromagnetic wave produced by S'. Reversing the directions of the current distributions on C, we conclude that the system of sources $-S$ produces outside C exactly the same field as S'.

The Equivalence Principle is closely related to another theorem which we may call the Induction Theorem. Let us suppose that a closed surface C subdivides the entire space into two homogeneous media and that a system of sources S is given in one of those regions (Fig. 4). Let E, H be the field due to these sources on the assumption that the medium inside C is the same as that outside. The true field outside C must vanish at infinity but it need not be the same as E, H; let it be $E + E'$, $H + H'$. The field E', H' must be source-free outside C. Inside C the field must be source-free; we shall designate it by E'', H''. The field E', H' is called the *reflected field* and E'', H'' the *refracted field*. The boundary conditions are such that the components of the electric and the magnetic intensities tangential to C must be continuous. Thus over the surface C, we have

$$\bar{E}_t + \bar{E}_t' = \bar{E}_t'', \qquad \bar{H}_t + \bar{H}_t' = \bar{H}_t''. \tag{18}$$

The bar over the letters is used to designate the values of the corresponding quantities on C. From (18) we obtain

$$\bar{E}_t'' - \bar{E}_t' = \bar{E}_t, \qquad \bar{H}_t'' - \bar{H}_t' = \bar{H}_t. \tag{19}$$

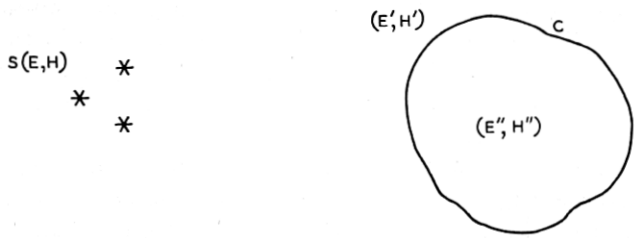

Fig. 4—The closed surface C is the boundary between two homogeneous regions in space. (E, H) designates the field produced by some system of sources S; (E', H') is the field reflected by the body C; and (E'', H'') is the field in the body.

Hence *the reflected and the refracted fields together constitute an electromagnetic field in the entire space; this field is source-free everywhere except on C and the distribution of sources on C is calculable from the given sources S.* This Induction Theorem is a generalization of the well-known theorem used in calculating the response of an electric circuit to an impressed field. Since the wires constituting the circuit are very thin, only the tangential components of E in the direction of the wires need be considered.

It may be noted that if the medium inside C is identical with that outside C, the "reflected field" must be absent and the "refracted field" must be identical with the field E, H due to the sources S. Thus the Induction Theorem leads to the Equivalence Principle.

The Equivalence Principle is evidently an extension of Kirchhoff's theorem. The latter deals with a single wave function instead of two vectors. Kirchhoff derived a formula for computing the wave function in the source-free region from its values and the values of its normal derivative over a closed surface separating the source-free region from the region containing the sources of the wave functions. In the Theory of Sound the wave function represents the excess pressure or the velocity potential and Kirchhoff's theorem is valuable in the analysis of diffraction phenomena. Kirchhoff's theorem is also used in dealing with optical diffraction. We may also remark that Kirchhoff's formula is a mathematical expression of a principle governing compressional wave motion. This principle was first formulated by Huygens in the following form: each particle in any wave front acts as a new source of disturbance, sending out secondary waves, and these secondary waves combine to form the new wave front.[9]

Let us now examine one of the familiar diffraction problems in the light of the Equivalence Principle. Consider a source S and a perfectly absorbing screen (Fig. 5*a*). Such a screen will be defined in the usual manner: the impressed wave enters it without reflection but does not pass through it. If the screen is infinitely thin, this definition implies the existence of electric and magnetic currents in the screen whose densities are given by the postulated discontinuity in the field. In reality the "black bodies" absorb not by virtue of the coexistence of electric and magnetic currents but by virtue of electric currents alone with the aid of reflections taking place between atomic layers. The true mechanism of absorption is complex and requires more than a mere surface. In diffraction studies it has become a habit with us to ignore the precise nature of absorption and confine outselves to its implications; but it is just as well to know the nature of the ideal mechanism which we are substituting for the true mechanism.

We can apply the Equivalence Principle to the present problem in two ways. We can choose as our surface C a surface (1234) just on the other side of the screen. The part (23) contributes nothing; the equivalent distribution of sources S' over the parts (12) and (34) gives us a complete field to the right of the screen. On the other hand if S'' is the field due to the electric and magnetic currents in the screen induced by S, the total field is $S + S''$. The choice of the "surface C" that would yield this result is shown in Fig. 5*b* although the conclusion is obvious without recourse to the Equivalence Principle. Since to the right of C in Fig. 5*a* the two alternative fields must be the same, we have

$$S' = S + S'' \qquad \text{and} \qquad S' - S'' = S. \tag{20}$$

Incidentally the last equation is the expression for the Equivalence Principle as applied to S in the absence of the screen since $-S''$ is

[9] A. E. Caswell, "An Outline of Physics," p. 544 (1929).

the contribution of that portion of the equivalent layer which was removed by the screen.

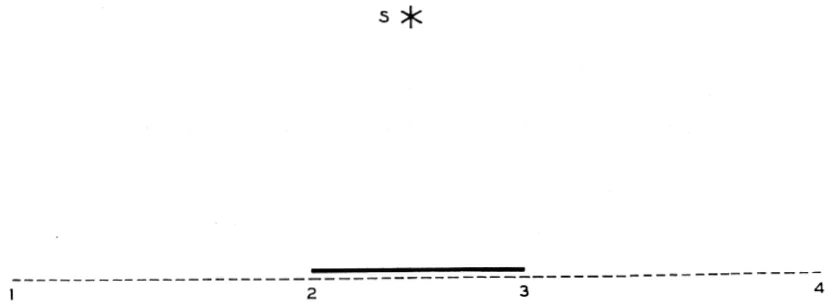

Fig. 5*a*—A source S in front of a screen the cross-section of which is shown in heavy lines.

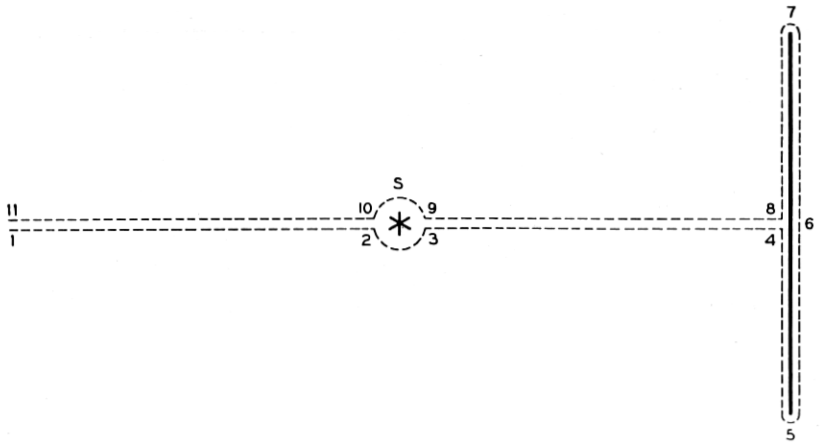

Fig. 5*b*—A source S in front of a screen the cross-section of which is shown in heavy lines.

The case of a hole in a perfectly absorbing screen (Fig. 6*a* and Fig. 6*b*) can be treated in the same manner and the reciprocity existing between this and the preceding case is quite evident. In terms of the sources previously defined the field to the right of the screen is $-S''$; by (20) this is the same as $S - S'$.

If the screen is a perfect conductor, the problem is much more complex. The screen will support electric currents but not magnetic currents. The densities of the electric currents are not calculable directly from the field S but from the condition that the component of the electric intensity tangential to the screen vanishes. The problem is very difficult and its solution has been found in only a few special cases. It is true that once we know the electric currents in the screen, we can determine the field on both sides of the screen; but there is no simple way of calculating these currents exactly. Frequently it is assumed that, in so far as the side opposite to the source is concerned, a perfectly conducting screen is equivalent to a perfectly absorbing screen of the same geometric character. This is equivalent to a

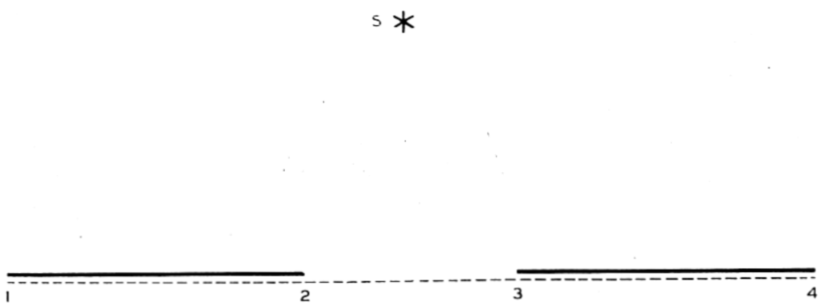

Fig. 6*a*—A source *S* in front of a screen the cross-section of which is shown in heavy lines.

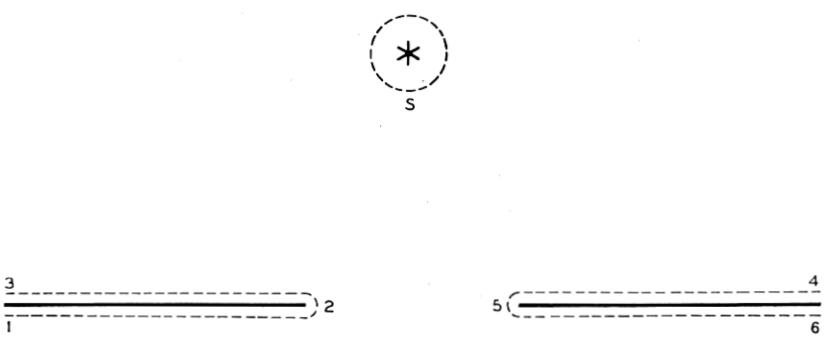

Fig. 6*b*—A source *S* in front of a screen the cross-section of which is shown in heavy lines.

hypothesis that the electric current density of the screen is determined by the magnetic intensity impressed directly by the source *S*. We could take the results obtained from this hypothesis as a first approximation to the true results. The tangential component of the electric intensity calculated on the basis of this hypothesis does not vanish on the screen which means that we have violated the original hypothesis that the screen is a perfect conductor. If the discrepancy is not too great we might look for an additional electric current distribution to reduce this discrepancy.

There are times, however, when the current distribution in the "screen" can be determined with a fair accuracy without elaborate mathematics. It is so, for instance, in the case of a pair of perfectly conducting coaxial cylinders (Fig. 7*a* and Fig. 7*b*) in which the radii

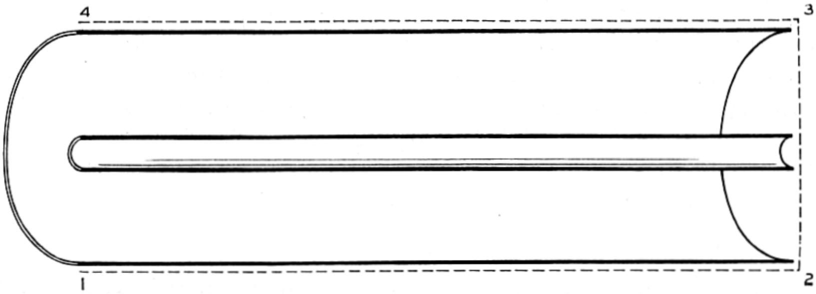

Fig. 7*a*—An axial cross-section of a coaxial pair.

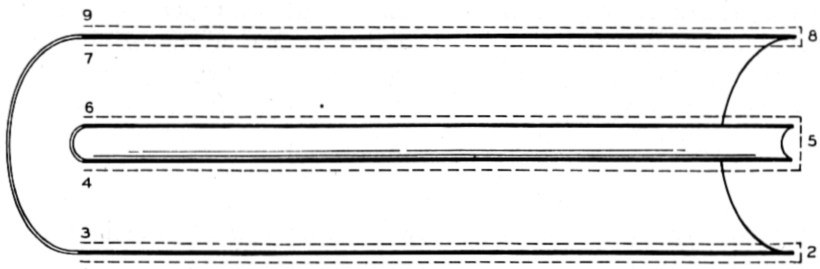

Fig. 7*b*—An axial cross-section of a coaxial pair.

are small by comparison with the wave-length. We shall assume that the coaxial pair is semi-infinite. Trusting his common sense, the engineer assumes that inside this structure the magnetic lines are circles coaxial with the cylinders. The electric lines are the radii and the electric current in the cylinders as well as the transverse voltage between the cylinders vary along the length in the same way as in a transmission line with uniformly distributed series inductance and shunt capacity. A careful analysis by John R. Carson indicates that this simple picture is justifiable if the cross-section of the coaxial pair is small by comparison with the wave-length.[10] While a whole series of electric waves can exist in such a structure, all of these waves except the one recognized by the engineer, the *principal wave*, are attenuated very rapidly and are significant only very close to the generator and

[10] John R. Carson, "The Guided and Radiated Energy in Wire Transmission," *A. I. E. E. Journal*, pp. 908–913, October, 1924.

very near the open end. The complementary waves are needed only for logical consistency and to satisfy the boundary conditions.

Thus let us suppose that the field distribution in the coaxial pair is known to a high degree of accuracy. In order to calculate the field outside the coaxial pair and hence obtain the radiated power we can use the Equivalence Principle in two ways. We can fit our surface C smoothly over the outer cylinder and the open end (Fig. 7*a*) or, regarding this surface as a perfectly elastic rubber sheet, we can press it through the open end and fit it smoothly over the inner surface of the outer conductor and the outer surface of the inner conductor (Fig. 7*b*). Since by hypothesis the conductors are perfect, the components of E tangential to the cylinders vanish; hence in the second choice of C the equivalent layer consists of only an electric current sheet. Naturally this current distribution is precisely that which actually exists in the conductors so that this choice of C leads to something that we knew beforehand, namely: if the actual sources, that is, if the electric currents in the structure are known exactly or approximately, the entire field can be calculated exactly or approximately.

The first choice of C is more important. Over the lateral portion (12, 34) of C the equivalent magnetic current sheet vanishes as in the preceding case on account of the perfect conductivity of the cylinders. The magnetic intensity just outside the coaxial pair is also zero except near the open end where it must be exceedingly small. To see this, we need only recall that the electric currents in the two cylinders are equal and opposite and that except in the neighborhood of the open end the displacement currents are transverse. Thus the equivalent electric current sheet can be ignored altogether. What is left is the magnetic current sheet over the surface of the open end; the density of this sheet is determined by the radial component of the electric intensity and in the final analysis by the voltage existing between the ends of the inner and outer conductors. Presently we shall carry out the actual calculations but just now we shall examine the question of the accuracy of the results. Of course, the results would be exact if we knew the equivalent electric and magnetic sheets accurately; and the above approximations appear to be reasonable. We shall not be able to find out how good these approximations are but we can prove that they are just as good as the approximations usually made in calculating the radiated power from the distribution of electric currents. The only virtue of the Equivalence Principle is to save a certain amount of mathematical work and furnish a further insight into the phenomena of radiation.

If a progressive wave is advancing from left to right in a semi-infinite coaxial pair (Fig. 7) and if the generator is at infinity, we can assume it to be the principal wave. At the open end this wave is reflected. It is usually assumed that the reflected wave is also the principal wave but moving in the opposite direction. In other words, it is assumed that the total field is such that the electric lines are radial and the magnetic lines are circular. Since the electric lines are radial, there is no longitudinal displacement current; and since the conduction current at the open end must be zero, the magnetic intensity is zero over the entire open end. This is what follows if we neglect the complementary waves.

These approximate results correspond to the *exact* results in the following hypothetical situation. If a hypothetical perfect magnetic conductor is fitted over the open end of the coaxial pair so that it closes it entirely, then the reflection is complete and there are no complementary waves. Perfect magnetic conductors are defined by analogy with perfect electric conductors—the tangential component of the magnetic intensity vanishes at the surface of the former just as the tangential component of the electric intensity vanishes at the surface of the latter. Magnetic conductors support magnetic current sheets just as electric conductors support electric current sheets. The densities of the sheets are given by the discontinuities of the tangential components of E in the former case and H in the latter.

In the hypothetical case in which the open end is closed with a perfect magnetic conductor, no energy can flow outside the coaxial pair. This is because the flow of energy is given by $\frac{1}{2}E \times H$ and either one or the other factor vanishes over the outer boundary of the structure. The field outside the coaxial pair must now be identically zero. Our sources are the electric current in the walls of the coaxial pair and the magnetic currents in the cap. If one field is designated by S and the other by S', then $S + S' = 0$ and $S' = - S$. Thus the field produced by the electric currents in the conductors on a supposition that principal waves alone exist, is the same as the field produced by a hypothetical distribution of magnetic currents over the surface of the open end.

Let us examine another case. It is usual to assume that the electric current in a thin wire (Fig. 8) in free space is distributed sinusoidally. Experimental evidence shows that the radiated power calculated on this assumption is very nearly correct. On the other hand the sinusoidal distribution of the electric current in the wire corresponds to a hypothetical case in which a perfect magnetic conductor is introduced in the shape of a sphere concentric with the center of the wire and passing through its ends. Thus we could calculate the radiated power from an appropriate distribution of magnetic currents over this sphere but in this case such a procedure would involve more difficult integrations than the usual method.

Before considering the more general case of radiation from a semi-infinite coaxial pair let us assume that the radii of the two conductors are nearly equal. We have seen that in applying the Equivalence Principle we need take into account only the magnetic current sheet over the open end of the pair. In the present instance this sheet is merely a circular loop of magnetic current equal to the voltage V between the ends of the conductors. If we were to treat in the same

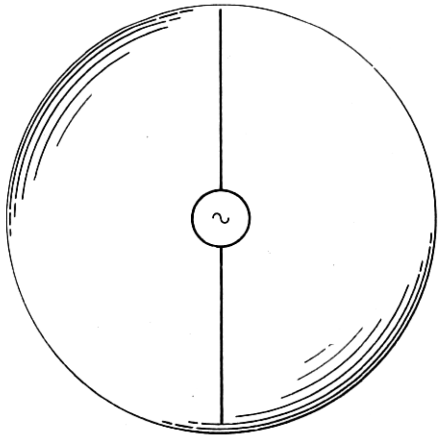

Fig. 8—A vertical antenna and a cross-section of an imaginary sphere passing through the ends of the antenna.

manner a condenser made of two parallel circular plates, we should come to the conclusion that it is also equivalent to a magnetic loop around its periphery. Thus in both cases the radiated power is the same. But the power radiated by an electric doublet is known to be $(40\pi^2 I^2 l^2)/\lambda^2$ watts where I is the amplitude of the electric current, l the length of the doublet and λ the wave-length. In applying this formula to a condenser it is better to express it in terms of the voltage V across the condenser and its area S. The capacity of the condenser is $C = S/(36\pi 10^{11} l)$ farads and $I = \omega CV = SV/60\lambda l$ amperes. Hence the power radiated by the condenser is $(\pi^2 S^2 V^2)/90\lambda^4$ watts. This is also the approximate power radiated by the coaxial pair if we interpret S as the cross-sectional area of either conductor.

Let us now calculate the more general expression for the power radiated from an open end of a coaxial pair. The cylindrical conductors whose cross-sections are shown in Fig. 9 are supposed to extend

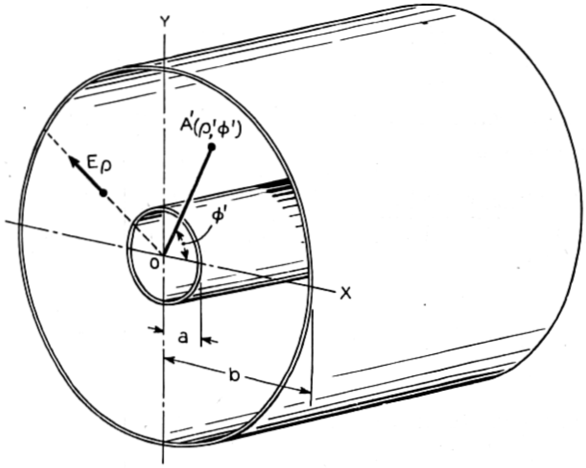

Fig. 9—The end view of a coaxial pair of cylindrical conductors.

below the surface of the paper and the z-axis of the coordinate system is directed toward the reader. The primed letters will refer to points situated in the opening, the unprimed letters being reserved for typical points in space.

The electric intensity in the coaxial pair varies inversely as the distance from the axis

$$E_{\rho'} = \frac{P}{\rho'}, \qquad E_{\varphi'} = 0. \tag{21}$$

In accordance with the Equivalence Principle we assume that the field below the xy-plane is wiped out and the discontinuity in E arising as the result of the separation is replaced by a magnetic current sheet. This magnetic current is perpendicular to E; by (14) its density is

$$M_{\varphi'} = -E_{\rho'} = -\frac{P}{\rho'}, \qquad M_{\rho'} = 0. \tag{22}$$

The constant E is related to the voltage V between the ends of the coaxial conductors; in fact, we have

$$V = \int_a^b E_{\rho'}d\rho' = P\log\frac{b}{a}, \qquad P = \frac{V}{\log\dfrac{b}{a}}\cdot \tag{23}$$

In order to calculate the field at some point A due to the distribution (20), we must determine the retarded electric vector potential. Since the integration is vectorial, it is convenient to deal with the cartesian components of the magnetic current density

$$M_{x'} = \frac{P \sin \varphi'}{\rho'}, \qquad M_{y'} = -\frac{P \cos \varphi'}{\rho'}. \tag{24}$$

The area of the element is $\rho' d\rho' d\varphi'$ so that the components of the retarded potential are

$$\begin{aligned} F_x &= \frac{1}{4\pi} \int_a^b \int_0^{2\pi} \frac{M_{x'} e^{-i\beta AA'} \rho' d\rho' d\varphi'}{AA'} \\ &= \frac{P}{4\pi} \int_a^b \int_0^{2\pi} \frac{e^{-i\beta AA'} \sin \varphi'}{AA'} d\rho' d\varphi', \\ F_y &= -\frac{P}{4\pi} \int_a^b \int_0^{2\pi} \frac{e^{-i\beta AA'} \cos \varphi'}{AA'} d\rho' d\varphi', \quad \beta = \omega\sqrt{\mu\epsilon} = \frac{\omega}{c} = \frac{2\pi}{\lambda}. \end{aligned} \tag{25}$$

Hence the components in the polar coordinates are

$$\begin{aligned} F_\varphi &= -F_x \sin \varphi + F_y \cos \varphi \\ &= -\frac{P}{4\pi} \int_a^b \int_0^{2\pi} \frac{e^{-i\beta AA'} \cos (\varphi - \varphi')}{AA'} d\rho' d\varphi', \\ F_\rho &= 0. \end{aligned} \tag{26}$$

The distance AA' is

$$AA' = \sqrt{r^2 - 2r\rho' \cos \vartheta + \rho'^2}, \tag{27}$$

where r is the distance OA, and ϑ is the angle between OA and OA'. Since we are interested in the radiation field alone, we need retain only those terms in (24) which vary inversely as the distance; the other terms contribute nothing to the radiated power. Thus we let r increase indefinitely, obtaining

$$AA' = r - \rho' \cos \vartheta, \tag{28}$$

and then

$$F_\varphi = -\frac{Pe^{-i\beta r}}{4\pi r} \int_a^b \int_0^{2\pi} e^{i\beta\rho' \cos \vartheta} \cos (\varphi - \varphi') d\rho' d\varphi'. \tag{29}$$

If θ and θ' are the angles made by OA and OA' with OZ, we have

$$\begin{aligned} \cos \vartheta &= \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\varphi - \varphi') \\ &= \sin \theta \cos (\varphi - \varphi'). \end{aligned} \tag{30}$$

Since ρ' is small by comparison with the wave-length λ, we can expand the exponential term in the integrand into a power series and retain only the first two terms

$$e^{i\beta\rho'\cos\vartheta} \doteq 1 + i\beta\rho'\cos\vartheta = 1 + i\beta\rho'\sin\theta\cos(\varphi - \varphi'). \qquad (31)$$

We need the second term because the integral of the first vanishes. Integrating the second term, we obtain

$$\begin{aligned} F_\varphi &= -\frac{i\beta P e^{-i\beta r}\sin\theta}{4\pi r}\int_a^b \rho' d\rho' \int_0^{2\pi} \cos^2(\varphi - \varphi')d\varphi' \\ &= -\frac{1}{8} i\beta(b^2 - a^2)P\frac{e^{-i\beta r}}{r}\sin\theta. \end{aligned} \qquad (32)$$

The magnetic current is uniform around the axis and there is no accumulation of magnetic charge anywhere; hence the second term in the expression for H as given by (11) vanishes. Therefore

$$H_\varphi = -\frac{P}{8}\omega\epsilon\beta(b^2 - a^2)\frac{e^{-i\beta r}}{r}\sin\theta. \qquad (33)$$

At a great distance from the source the wave tends to become plane so that in the radiation field the electric intensity is perpendicular to OA and to H and is given by

$$E_\theta = \sqrt{\frac{\mu}{\epsilon}}H_\varphi = 120\pi H_\varphi. \qquad (34)$$

According to Poynting the radiated power is the real part of the following integral

$$W = \frac{1}{2}\int_0^\pi \int_0^{2\pi} E_\theta H_\varphi{}^* r^2 \sin\theta \, d\theta \, d\varphi \text{ watts}, \qquad (35)$$

where $H_\varphi{}^*$ is the complex number conjugate to H_φ. Substituting from (31) and (32), we obtain

$$\begin{aligned} W &= \frac{\pi^3}{980}\left(\frac{b^2 - a^2}{\lambda^2}\right)^2 P^2 \int_0^\pi \sin^3\theta \, d\theta \int_0^{2\pi} d\varphi \\ &= \frac{\pi^4}{360}\left(\frac{b^2 - a^2}{\lambda^2}\right)^2 P^2 \text{ watts}. \end{aligned} \qquad (36)$$

Introducing from (21) the expression for P and designating by S the area of the opening, we have

$$W = \frac{\pi^2}{360}\left(\frac{S}{\lambda^2 \log\frac{b}{a}}\right)^2 V^2 \text{ watts}. \qquad (37)$$

The effect of radiation on the transmission line can be simulated by a resistance R,

$$R = \frac{180}{\pi^2}\left(\frac{\lambda^2 \log \frac{b}{a}}{S}\right)^2 \text{ohms} \tag{38}$$

shunted across the open end. This is not the resistance seen by the generator. If V and I are the amplitudes of the voltage and the electric current at their antinodes and Z_0 the characteristic impedance of the coaxial pair, then

$$V = Z_0 I = \left(60 \log \frac{b}{a}\right) I. \tag{39}$$

Since the end of the coaxial pair is a voltage antinode, the radiated power may be expressed as

$$W = 10\pi^2 \left(\frac{S}{\lambda^2}\right)^2 I^2 \text{ watts.} \tag{40}$$

Hence the radiation resistance seen by a generator placed at a current antinode is

$$R_G = \frac{20\pi^2 S^2}{\lambda^4} \text{ohms.} \tag{41}$$

With this simple illustration, we conclude the present paper.