Operation of Ultra-High-Frequency Vacuum Tubes

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Previous electronics analyses are extended by the introduction of more general boundary conditions. The results are applied to the calculation of the rectifying properties of diodes at very high frequencies and to the amplifying properties of negative grid triodes at both low and high frequencies. The effect of space charge on the various capacitances in triodes is discussed, and formulas for the amplification factor and plate impedance are presented in terms of the tube geometry. Finally, a discussion of the input impedance of negative grid triodes is given together with a comparison of the theoretical value with the results of measurements made by several well-known experimenters.

IN the study of the functioning of vacuum tubes at ultra-high frequencies it has been necessary to retrace the steps followed in the early history of vacuum tube performance but with the difference that a microscope for viewing the path at close range must be substituted for the telescope with which the original trail was mapped from afar off. As a result of this microscopic survey, formulas have been developed which are applicable to frequencies so high that the time of flight of the electron across the tube may occupy several cycles of the high-frequency oscillation. In addition to this result, several by-products of the study are found to have a useful application in the low-frequency field and to throw additional light on the multitudinous activities of the electrons and their effect upon the external circuit.

In this respect it is particularly interesting to see the way in which the geometry of the vacuum tube enters into the determination of the amplification factor of negative grid triodes and to compare the results now obtained with the earlier results of such workers as Abraham, King, Schottky, Lane and Van der Bijl. The effect of the negative grid on the transit time of the electrons also yields low-frequency relations in which certain new facts concerning the plate resistance are brought out.

Various papers [1, 2, 3, 4, 5] published within the last few years have dealt with the general problem of vacuum tubes in which the electron transit time is of importance and have derived results which are useful in several practical applications. However, in none of these have the initial relations been general enough to allow more than a very rough application to be made to the most widely used tube of all—the negative grid triode. A paper [2] by the present writer contains certain general conclusions concerning the negative grid tube at very high frequencies and derives expressions for the complex values taken by the amplification factor and plate impedance. While these expressions represent the general trend of the variations, they are based on certain a priori assumptions, as was emphasized in the paper, and hence partake of the telescopic viewpoint of the low-frequency vacuum tube analysis rather than the microscopic viewpoint which is now necessary.

It was with the aim of overcoming this limitation that the work described in the present paper was undertaken. Its successful outcome was made possible by a very simple generalization of the methods described in the references, but one which has such far-reaching consequences that it appears worthwhile to start the analysis at the very beginning, and abbreviate only to the extent that intermediate algebraic steps are omitted because of their unwieldy length.

For convenience, the paper is divided into four parts. In Part I the mathematical analysis is outlined in its fundamental form and general working formulas are developed. In a way similar to low-frequency analysis these formulas may be divided into constant or d.-c. relations, first-order relations, second-order relations, and so on; where the first-order relations apply to a.-c. effects for small amplitudes only, the second-order relations contain rectification and distortion terms; and so on, in exact correspondence with the well known low-frequency relations. Part II contains the solution of the first-order relations expressed in appropriate form for later computations, while Part III does the same thing for second-order relations. Also in Part III, by way of illustration, the effect of frequency on the rectifying properties of parallel plane diodes is discussed. Part IV applies the general first-order and d.-c. solutions to the negative grid triode and shows how its various properties depend on frequency. A discussion of the important effect of active grid loss is included.

In certain cases the same formulas are expressed in several different ways. This is done because of the difficulty of determining the most useful method of expression before a large number of applications shall have been made. In most cases the general equations have been arranged to conform as far as possible with the most widely used modes of expression of the corresponding low-frequency equations. Where a choice of modes of expression is available, both modes have usually been given, it being left to future experience to determine the more useful one. While this procedure results in some repetition, the two modes of expression of the same equation are found in many instances to have their individual advantages, the one being more suitable for one type of application while the other is more particularly adapted to a different application.

Part I—General Analysis

In line with most previous electronic papers consideration is here directed to the behavior of electrons between two parallel planes of practically infinite extent. Differing, however, from earlier works, neither plane is to be regarded as constituting a thermionic emitter or cathode in the general sense. For certain special applications conditions may be chosen so that one of the planes coincides with, and assumes the properties of a zero potential cathode, but in developing the general relations this idea is strictly avoided. It will therefore be premised as a starting point that the velocity and the acceleration of the electrons at one of these two planes are given as initial conditions. It will be found that this generalization completely avoids certain ambiguities which were discussed in the February 1935 issue of the *Proceedings of the Institute of Radio Engineers.*[4] It also allows application of the results to be made to a much wider range of devices, including a fairly rigorous treatment of the negative grid triode.

In the following analysis, and differing from previous references, a change in the units employed has been made so that all quantities are expressed immediately in the practical system of engineering units (amperes, volts, ohms, coulombs, etc.) instead of in the electrostatic and electromagnetic systems. This change has been found to be of great advantage in the use of the equations, since it obviates all necessity for the continuous and irritating transformation of units that accompanies the electrostatic and electromagnetic systems.

The analysis starts with the expression for the total current density

$$I = \rho u + \epsilon \frac{\partial E}{\partial t}, \tag{1}$$

where I is current density, amperes per cm.^{2},
ρ is charge density, coulombs per cm.^{3},
u is electron, or charge velocity, cm. per sec.,
ϵ is permittivity of a vacuum, which is $1/36\pi 10^{11} = 8.85 \times 10^{-14}$ farads/cm.,
E is the electric intensity, volts per cm.

The equation of motion of an electron is

$$eE = kma, \tag{2}$$

where e is electronic charge, coulombs,
m is electronic mass, grams,

$e/m = -1.77 \times 10^8$ coulombs per gm.,*
$k = 10^{-7}$ is the ratio of dyne-cm. to joules,
a is electron acceleration, cm. per sec.^{2}

In this equation the effect of a magnetic field is disregarded. This is thoroughly justified until electron velocities approach that of light or the spacing between the two parallel planes becomes comparable with the wave-length of any alternating field considered. A more detailed discussion is given by Benham.[1]

A third fundamental equation is

$$\text{div } \epsilon E = \rho,$$

which, for the parallel planes now considered, becomes

$$\epsilon \frac{\partial E}{\partial x} = \rho. \tag{3}$$

From (1), (2) and (3) is obtained

$$\frac{eI}{km\epsilon} = \frac{da}{dt}. \tag{4}$$

The total current, I, may be considered to be composed of two parts, the first being a constant component and the second being a function of time only. On this basis we can write

$$\frac{eI}{km\epsilon} = K + \varphi'''(t), \tag{5}$$

where K is the constant part, and $\varphi'''(t)$ is the variable part, the primes denoting derivatives with respect to the argument in parentheses. Inserting (5) into (4) and integrating once with respect to time, we find:

$$a = K(t - t_a) + \varphi''(t) - \varphi''(t_a) + a_a + \alpha(t_a), \tag{6}$$

where $a_0 + \alpha(t_a)$ is the acceleration when $t = t_a$ and a_a is independent of t_a.

Another integration gives

$$u = K\frac{(t - t_a)^2}{2} + \varphi'(t) - \varphi'(t_a) - (t - t_a)\varphi''(t_a) + (t - t_a)a_a + (t - t_a)\alpha(t_a) + u_a + \mu(t_a), \tag{7}$$

where $u_a + \mu(t_a)$ is the velocity when $t = t_a$ and u_a is independent of t_a.

* This value is based on deflection measurements (which are applicable to vacuum tube analysis) rather than on spectroscopic measurements which give 1.76 $\times 10^8$. Compton and Langmuir[6] use the spectroscopic figure. For a comprehensive discussion of values of physical constants, see Birge, *Phys. Rev. Supp.*, Vol. 1, July, 1929.

A third integration gives

$$x = K\frac{(t-t_a)^3}{6} + \varphi(t) - \varphi(t_a) - (t-t_a)\varphi'(t_a) - \frac{(t-t_a)^2}{2}\varphi''(t_a)$$
$$+ \frac{(t-t_a)^2}{2}a_a + \frac{(t-t_a)^2}{2}\alpha(t_a) + (t-t_a)u_a + (t-t_a)\mu(t_a), \quad (8)$$

where x is zero when $t = t_a$.

This choice of initial conditions allows one of the two parallel planes to be located at the position where x is zero and where the electron velocities and accelerations are given as above in terms of the time instant t_a when the electron crosses the plane, which may be referred to as the " a " plane. When these initial conditions are specified, then (6), (7), and (8) allow the acceleration, velocity and position, respectively, to be determined at any time t, thereafter.

These quantities are expressed in terms of t and t_a whereas it is more convenient in vacuum tube work to have the acceleration and velocity expressed in terms of t and x. Ideally, this could be done by solving (8) for t_a and thence eliminating t_a from (6) and (7). Practically, (8) cannot be solved directly for t_a because it is a higher order equation and involves φ, α and μ which may be (and usually are) transcendental functions of t_a. However, an indirect method can be employed.

If φ, α and μ were zero, then x would be given by the relatively simple equation

$$x = K\frac{(t-t_a)^3}{6} + a_a\frac{(t-t_a)^2}{2} + u_a(t-t_a). \quad (9)$$

Although this is a cubic, nevertheless $(t - t_a)$ may be obtained with relative ease in any particular case. The use of a new variable T to replace x is suggested by (9) and accordingly the defining equation of T under all conditions is taken to be:

$$x = K\frac{T^3}{6} + a_a\frac{T^2}{2} + u_aT, \quad (10)$$

which holds even when φ, α and μ are not zero. It is evident when φ, α and μ are small that T does not differ very much from $t - t_a$ as (10) must then become nearly equivalent to (9). It thus seems expedient to write in general

$$t - t_a = T + \delta, \quad (11)$$

and note that δ becomes very small when φ, α and μ are small.

On the basis of (11), functions of $(t - t_a)$ may be expanded into series in powers of δ as follows:

$$f(t-t_a) = f(T+\delta) = f(T) + f'(T)\delta + \frac{1}{2!}f''(T)\delta^2 + \cdots, \quad (12)$$

and similarly functions of t_a may be written

$$f(t_a) = f(t - T - \delta) = f(t - T) - f'(t - T)\delta + \frac{1}{2!}f''(t - T)\delta^2 - \cdots. \quad (13)$$

When (10), (11), (12) and (13) are used in conjunction with (8) the result is a relation between t, T and δ as follows:

$$\begin{aligned}
0 = &\frac{K}{6}[3T^2\delta+3T\delta^2+\delta^3] \\
&+\frac{a_a}{2}[2T\delta+\delta^2]+u_a\delta \\
&+\varphi(t)-\left[\varphi(t-T)-\varphi'(t-T)\delta+\frac{1}{2!}\varphi''(t-T)\delta^2-\cdots\right] \\
&-(T+\delta)\left[\varphi'(t-T)-\varphi''(t-T)\delta+\frac{1}{2!}\varphi'''(t-T)\delta^2-\cdots\right] \\
&-\frac{1}{2}(T^2+2T\delta+\delta^2)\left[\varphi''(t-T)-\varphi'''(t-T)\delta+\frac{1}{2!}\varphi''''(t-T)\delta^2-\cdots\right] \\
&+\frac{1}{2}(T^2+2T\delta+\delta^2)\left[\alpha(t-T)-\alpha'(t-T)\delta+\frac{1}{2!}\alpha''(t-T)\delta^2-\cdots\right] \\
&+(T+\delta)\left[\mu(t-T)-\mu'(t-T)\delta+\frac{1}{2!}\mu''(t-T)\delta^2-\cdots\right]. \quad (14)
\end{aligned}$$

This equation may be written in the form of a power series in δ. It cannot yet be solved directly for δ. It has the advantage, however, that δ is not involved in the transcendental functions φ, α and μ, so that an indirect method may be used. Let δ, φ, α and μ each be split up into series as follows:

$$\left.\begin{aligned}
\delta &= \delta_1 + \delta_2 + \delta_3 + \text{etc.} \\
\varphi &= \varphi_1 + \varphi_2 + \varphi_3 + \text{etc.} \\
\alpha &= \alpha_1 + \alpha_2 + \alpha_3 + \text{etc.} \\
\mu &= \mu_1 + \mu_2 + \mu_3 + \text{etc.}
\end{aligned}\right\} \quad (15)$$

These are to be substituted into (14) and the resulting expression may then be expressed as an infinite series of separate equations such that the first equation includes all linear terms which have the subscript 1, but no other terms; The second equation includes all linear terms having the subscript 2 and also all quadratic terms having the subscript 1; the third equation includes all linear terms having the subscript 3, cubic terms with subscript 1, and also products of quadratic terms with subscript 1 and linear terms with subscript 2. The rules for succeeding equations are analogous, so that in general, the sum of the subscripts of each term of the nth equation is equal to n.

The first of these equations may be solved for δ_1, the second for δ_2, the third for δ_3, and so on, giving the following:

$$\delta_1 = -\frac{\left[\varphi_1(t) - \varphi_1(t-T) - T\varphi_1'(t-T) - \frac{1}{2}T^2\varphi_1''(t-T) + \frac{1}{2}T^2\alpha_1(t-T) + T\mu_1(t-T)\right]}{K\frac{T^2}{2} + a_aT + u_a}, \tag{16}$$

$$\delta_2 = -\frac{\left[\begin{aligned}&\varphi_2(t) - \varphi_2(t-T) - T\varphi_2'(t-T) - \frac{1}{2}T^2\varphi_2''(t-T)\\ &+ \frac{1}{2}T^2\alpha_2(t-T) + T\mu_2(t-T)\\ &+ \delta_1\Big(\frac{1}{2}T^2\varphi_1'''(t-T) - \frac{1}{2}T^2\alpha_1'(t-T)\\ &\qquad + T\alpha_1(t-T) - T\mu_1'(t-T) + \mu_1(t-T)\Big)\\ &+ \delta_1^2\left(K\frac{T}{2} + \frac{1}{2}a_a\right)\end{aligned}\right]}{K\frac{T^2}{2} + a_aT + u_a}, \tag{17}$$

$$\delta_3 = -\frac{\left[\begin{aligned}&\varphi_3(t) - \varphi_3(t-T) - T\varphi_3'(t-T) - \frac{1}{2}T^2\varphi_3''(t-T)\\ &+ \frac{1}{2}T^2\alpha_3(t-T) + T\mu_3(t-T)\\ &+ \delta_2\Big(\frac{1}{2}T^2\varphi_1'''(t-T) - \frac{1}{2}T^2\alpha_1'(t-T)\\ &\qquad + T\alpha_1(t-T) - T\mu_1'(t-T) + \mu_1(t-T)\Big)\\ &+ \delta_1\Big(\frac{1}{2}T^2\varphi_2'''(t-T) - \frac{1}{2}T^2\alpha_2'(t-T)\\ &\qquad + T\alpha_2(t-T) - T\mu_2'(t-T) + \mu_2(t-T)\Big)\\ &+ \delta_1^2\Big(-\frac{1}{4}T^2\varphi_1''''(t-T) + \frac{1}{2}T\varphi_1'''(t-T)\\ &+ \frac{1}{4}T^2\alpha_1''(t-T) - T\alpha_1'(t-T) + \frac{1}{2}\alpha_1(t-T)\\ &\qquad + \frac{1}{2}T\mu_1''(t-T) - \mu_1'(t-T)\Big)\\ &+ \delta_1\delta_2\left(KT + a_a\right) + \delta_1^3\frac{K}{6}\end{aligned}\right]}{K\frac{T^2}{2} + a_aT + u_a}, \tag{18}$$

$$\delta_4 = \cdots. \tag{19}$$

These formulas together with (11) and (15) allow the acceleration (6) and velocity (7) to be written in terms of t and T, and hence effectively in terms of t and x, for T is a function of x only, as given by (10). The resulting equations for acceleration and velocity are conveniently broken up into a series of equations in accord with the same rules formulated for obtaining the δ's from (14). Thus, we write

$$a = a_0 + a_1 + a_2 + a_3 + \text{etc.}, \tag{20}$$

where a_0 comprises those terms having zero or no subscript, and obtain from (6):

$$a_0 = KT + a_a, \tag{21}$$

$$a_1 = K\delta_1 + \varphi_1''(t) - \varphi_1''(t - T) + \alpha_1(t - T), \tag{22}$$

$$\begin{aligned} a_2 = K\delta_2 &+ \varphi_2''(t) - \varphi_2''(t - T) + \varphi_1'''(t - T)\delta_1 \\ &+ \alpha_2(t - T) - \alpha_1'(t - T)\delta_1, \end{aligned} \tag{23}$$

$$\begin{aligned} a_3 = K\delta_3 &+ \varphi_3''(t) - \varphi_3''(t - T) + \varphi_1'''(t - T)\delta_2 \\ &+ \varphi_2'''(t - T)\delta_1 - \frac{1}{2}\varphi_1''''(t - T)\delta_1^2 + \alpha_3(t - T) \\ &- \alpha_2'(t - T)\delta_1 - \alpha_1'(t - T)\delta_2 + \frac{1}{2}\alpha_1''(t - T)\delta_1^2, \end{aligned} \tag{24}$$

$$a_4 = \cdots. \tag{25}$$

Treating the velocity in a similar way we write

$$u = u_0 + u_1 + u_2 + u_3 + \text{etc.} \tag{26}$$

and obtain from (7)

$$u_0 = K\frac{T^2}{2} + a_aT + u_a, \tag{27}$$

$$\begin{aligned} u_1 = KT\delta_1 &+ a_a\delta_1 + \varphi_1'(t) - \varphi_1'(t - T) \\ &- T\varphi_1''(t - T) + T\alpha_1(t - T) + \mu_1(t - T), \end{aligned} \tag{28}$$

$$\begin{aligned} u_2 = KT\delta_2 &+ \frac{K}{2}\delta_1^2 + a_a\delta_2 + \varphi_2'(t) - \varphi_2'(t - T) \\ &- T\varphi_2''(t - T) + T\varphi_1'''(t - T)\delta_1 + T\alpha_2(t - T) \\ &+ \alpha_1(t - T)\delta_1 - T\alpha_1'(t - T)\delta_1 + \mu_2(t - T) - \mu_1'(t - T)\delta_1, \end{aligned} \tag{29}$$

$$\begin{aligned} u_3 = KT\delta_3 &+ K\delta_1\delta_2 + a_a\delta_3 + \varphi_3'(t) - \varphi_3'(t - T) \\ &+ \frac{1}{2}\varphi_1'''(t - T)\delta_1^2 - T\varphi_3''(t - T) + T\varphi_1'''(t - T)\delta_2 \end{aligned}$$

$$+ T\varphi_2'''(t - T)\delta_1 - \frac{1}{2}T\varphi_1''''(t - T)\delta_1^2 + T\alpha_3(t - T)$$
$$- T\alpha_1'(t - T)\delta_2 - T\alpha_2'(t - T)\delta_1 + \frac{1}{2}T\alpha_1''(t - T)\delta_1^2$$
$$+ \alpha_1(t - T)\delta_2 + \alpha_2(t - T)\delta_1 - \alpha_1'(t - T)\delta_1^2 + \mu_3(t - T)$$
$$- \mu_1'(t - T)\delta_2 - \mu_2'(t - T)\delta_1 + \frac{1}{2}\mu_1''(t - T)\delta_1^2. \tag{30}$$

Aside from their length these equations are not complicated and in applications to special cases many terms are apt to vanish, leaving relatively compact expressions.

In circuit work the potential difference between the two parallel planes, " a " and " b," say, is more often required than the electron acceleration. This may be found from the definition of the potential difference, namely

$$V_a - V_b = \int_a^b E\partial x, \tag{31}$$

in which t remains constant during the integration. From (10)

$$\partial x = \left(K\frac{T^2}{2} + a_a T + u_a \right) dT = u_0 dT, \tag{32}$$

so that with the aid of this equation and (2) the potential difference is given by:

$$W_a - W_b = \int_a^b a dx = \int_0^T a \left(K\frac{T^2}{2} + a_a T + u_a \right) dT, \tag{33}$$

where the symbol, W, is used as an abbreviation for eV/km.

In the same way as the acceleration and velocity are divided into components, the potential difference may be split up into $(W_a - W_b)_0$, $(W_a - W_b)_1$, $(W_a - W_b)_2$, etc. which are defined by:

$$(W_a - W_b)_0 = \int_0^T a_0 u_0 dT, \tag{34}$$

$$(W_a - W_b)_1 = \int_0^T a_1 u_0 dT, \tag{35}$$

$$(W_a - W_b)_2 = \int_0^T a_2 u_0 dT, \tag{36}$$

$$(W_a - W_b)_3 = \int_0^T a_3 u_0 dT, \tag{37}$$

and where the δ's given by (16), (17), (18), etc., are to be inserted in the a's given by (21), (22), (23), etc., before the integration can be carried out.

The formal solution of the problem has been reached with the attainment of (34), (35), (36), etc. As soon as specific functions are chosen for the current, $[K + \varphi'''(t)]km\epsilon/e$, the initial acceleration, $a_a + \alpha(t)$, and the initial velocity, $u_a + \mu(t)$, the integrations can be performed to give the potential difference between two planes located respectively at $x = 0$ and $x = x$.

While the general relation between current and potential is non-linear, the first order current $\varphi'''(t)e/km\epsilon$ is linearly related to the first order potential difference $(W_a - W_b)e/km$, and the results for this case can be expressed conveniently by using complex functions in the manner usual with electrical engineers. This will be done in some of the following applications.

In the treatment of second and higher-order components, it is convenient to select the current components to correspond with powers of the potential rather than vice versa. For example, $(V_a - V_b)_1$ is taken to be the complete expression for the fluctuating component of potential, thus causing $(V_a - V_b)_2$, $(V_a - V_b)_3$, etc. to vanish. Equation (36) then yields an expression for the second-order current in terms of the first-order current, and (37) does likewise for the third-order current.

The general equations are, however, applicable to any converging method of selecting the components, and the proper one for any particular case is to be determined by considerations of simplicity and convenience.

Part II—First-Order Solution

This is the linear case, so that to each component of current, $A \sin \omega t$, say, there corresponds a potential component of the form $B \sin (\omega t + \eta)$. It follows that a current of the form Ae^{pt} will produce a corresponding potential difference Pe^{pt}, and that p may be taken to be a generalized exponent having the value $i\omega$ when sinusoidal currents are considered. In the latter case, P will usually be complex. The generalized exponent p results in a more compact symbolism than would be possible with the imaginary exponent, $i\omega$.

Thus, we write for the current

$$\frac{eI}{km\epsilon} = K + \varphi_1'''(t) = K + Je^{pt}. \qquad (38)$$

In a similar way the initial fluctuating components of acceleration and velocity are taken to be

$$\alpha_1(t - T) = Ge^{p(t-T)} = (Ge^{-pT})e^{pt}, \tag{39}$$

$$\mu_1(t - T) = He^{p(t-T)} = (He^{-pT})e^{pt}. \tag{40}$$

With this nomenclature, all terms appearing in the first-order equations will contain the factor e^{pt}, which may accordingly be omitted throughout. Thus, instead of writing Ge^{pt} for the acceleration at the " a " plane, the single symbol α_1 will be used, where the omission of the functional notation $\alpha_1(t - T)$ indicates that the acceleration is taken at the " a " plane where T is zero, and that the multiplying factor e^{pt} is understood. In a similar way the symbol μ_1 will indicate the first-order fluctuating component of velocity at the " a " plane with the factor e^{pt} understood.

A still further symbolism will be of assistance. The quantity pT will be denoted by β. When p is the imaginary $i\omega$ then ωT is the transit angle, θ, which has been defined in previous papers,[1, 2] and in the sinusoidal case $\beta = i\theta$.

With this nomenclature, the first-order potential difference, (35), acceleration, (22), and velocity (28) may be written:

$$\begin{aligned}(W_a - W_b)_1 = \frac{J}{p^4}\Bigg[K\left(\frac{\beta^3}{6} - \beta - 2e^{-\beta} - \beta e^{-\beta} + 2\right) \\ + pa_a\left(\frac{\beta^2}{2} + \beta e^{-\beta} + e^{-\beta} - 1\right) + p^2u_a(\beta + e^{-\beta} - 1)\Bigg] \\ - \frac{\alpha_1}{p^2}\left[a_a(\beta e^{-\beta} + e^{-\beta} - 1) + u_a p(e^{-\beta} - 1)\right] \\ + \frac{\mu_1}{p^2}\left[K(\beta e^{-\beta} + e^{-\beta} - 1)\right],\end{aligned} \tag{41}$$

$$\begin{aligned}a_1 = \frac{J}{p}\left[1 - e^{-\beta} - \frac{K}{p^2u_0}\left(1 - e^{-\beta} - \beta e^{-\beta} - \frac{\beta^2}{2}e^{-\beta}\right)\right] \\ + \alpha_1\left(1 - \frac{KT^2}{2u_0}\right)e^{-\beta} - \mu_1\frac{KT}{u_0}e^{-\beta},\end{aligned} \tag{42}$$

$$\begin{aligned}u_1 = \frac{J}{p^2}\left[1 - e^{-\beta} - \beta e^{-\beta} - \frac{a_0}{pu_0}\left(1 - e^{-\beta} - \beta e^{-\beta} - \frac{\beta^2}{2}e^{-\beta}\right)\right] \\ + \frac{\alpha_1}{p}\left(1 - \frac{a_0T}{2u_0}\right)\beta e^{-\beta} + \mu_1\left(1 - \frac{a_0T}{u_0}\right)e^{-\beta}.\end{aligned} \tag{43}$$

An alternate method of procedure, which is particularly useful at moderately low frequencies where the transit angle θ is only a few radians, is to expand functions of $(t - T)$ into power series in T. The same result may be obtained from (41), (42) and (43) above by writing $e^{-\beta}$ in series form. The result is:

$$(W_a - W_b)_1 = \sum_{n=0}^{\infty} (-\beta)^n \left[J\left(\frac{KT^4(n+2)}{(n+4)!} + \frac{a_aT^3(n+2)}{(n+3)!} + \frac{u_aT^2}{(n+2)!}\right) + \alpha_1\left(\frac{a_aT^2(n+1)}{(n+2)!} + \frac{u_aT}{(n+1)!}\right) - \mu_1\left(\frac{KT^2(n+1)}{(n+2)!}\right)\right], \quad (41a)$$

$$a_1 = \sum_{n=0}^{\infty} \frac{(-\beta)^n}{n!}\left[JT\left(\frac{1}{n+1} - \frac{KT^2}{2u_0}\frac{1}{(n+3)}\right) + \alpha_1\left(1 - \frac{KT^2}{2u_0}\right) - \mu_1\left(\frac{KT}{u_0}\right)\right], \quad (42a)$$

$$u_1 = \sum_{n=0}^{\infty} \frac{(-\beta)^n}{n!}\left[JT^2\left(\frac{1}{n+2} - \frac{a_0T}{2u_0}\frac{1}{(n+3)}\right) + \alpha_1T\left(1 - \frac{a_0T}{2u_0}\right) + \mu_1\left(1 - \frac{a_0T}{u_0}\right)\right]. \quad (43a)$$

These latter forms of expression clearly show what happens at low frequencies and they prove that the equations do not give infinite values for any of the components so long as a_0T/u_0 and KT^2/u_0 remain finite. Now in general

$$\frac{a_0T}{u_0} = \frac{(KT + a_a)T}{K\frac{T^2}{2} + a_aT + u_a}$$

and

$$\frac{KT^2}{u_0} = \frac{KT^2}{K\frac{T^2}{2} + a_aT + u_a},$$

and these remain finite for all finite positive values of T, K, a_a and u_a. Thus the difficulty at the origin which was discussed [4] in the *Proceedings of the Institute of Radio Engineers*, February, 1935 is overcome by the generalized definition of T in (10). It is true that δ_1 still tends toward infinity when T approaches zero if a_a and u_a are zero and μ_1 is different from zero. This means that the ratio of the variation in transit time, δ_1, to the transit time T tends toward infinity when T approaches zero, and when variations in initial velocity are still present with no constant initial acceleration. This is a logical and expected result, but it leads also to the conclusion that the electrons actually halt their forward motion and reverse their direction. When this happens, (1) is no longer applicable because the velocity at a given point has become multi-valued. This restriction was pointed out by Müller [3] as limiting any analysis which starts with (1).

It may be concluded then that the inherent limitation on (41), (42) and (43) is a singly valued velocity rather than the behavior of the δ's in the neighborhood of a point where the acceleration is zero.

Part III—Second-Order Solution

The second-order solution has practical utility in the computation of distortion and of the modulating and detecting properties of ultra-high-frequency thermionic systems. Even in low-frequency applications such computations are long and tedious. The introduction of the added complication of appreciable transit angles causes further difficulty because of the unwieldy length of the equations. Accordingly, instead of a general exposition of second-order effects, a greatly simplified special case will be treated at the present time, leaving the details of a more general solution until the need for it has become more acute.

The rectifying properties of a parallel plane diode operating with complete space charge will be calculated. The complete space charge condition is defined by placing initial velocities and accelerations equal to zero. When this has been done in (16), (17) and (23) and the resulting values of the δ's have been substituted in (23), functions of $(t - T)$ may be expanded into power series to give the following:

$$a_2 = -2\sum_{n=0}^{\infty}\frac{(-T)^{n+1}\varphi_2^{(n+3)}(t)}{(n+3)(n+1)!} + \frac{1}{K}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-T)^{n+m+1}\varphi_1^{(n+3)}(t)\,\varphi_1^{(m+3)}(t)}{(n+3)(m+3)n!m!}. \tag{44}$$

The second-order potential difference is given by (36) where $u_0 = KT^2/2$ for complete space charge. Thus, from (44)

$$(W_a - W_b)_2 = K\sum_{n=0}^{\infty}\frac{(n+2)}{(n+4)!}(-T)^{n+5}\varphi_2^{(n+3)}(t) - \frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-T)^{n+m+4}\varphi_1^{(n+3)}(t)\,\varphi_1^{(m+3)}(t)}{(n+m+4)(n+3)(m+3)n!m!}. \tag{45}$$

The second-order potential difference $(W_a - W_b)_2$ will now be taken as zero, which implies that the first-order potential difference only is impressed on the diode. This gives from (45)

$$K\left[\frac{1}{12}T^4\varphi_2'''(t) - \frac{1}{40}T^5\varphi_2^{\text{iv}}(t) + \frac{1}{180}T^6\varphi_2^{\text{v}}(t) - \cdots\right]$$
$$= \frac{1}{2}\left[\frac{T^4\varphi_1'''(t)\varphi_1'''(t)}{36} + \frac{T^6\varphi_1^{\text{iv}}(t)\varphi_1^{\text{iv}}(t)}{96} + \frac{T^8\varphi_1^{\text{v}}(t)\varphi_1^{\text{v}}(t)}{800} + \cdots\right]$$
$$+ \left[-\frac{T^5\varphi_1'''(t)\varphi_1^{\text{iv}}(t)}{60} + \frac{T^6\varphi_1'''(t)\varphi_1^{\text{v}}(t)}{180} - \frac{T^7\varphi_1'''(t)\varphi_1^{\text{vi}}(t)}{756}\right.$$
$$\left. + \frac{T^8\varphi_1'''(t)\varphi_1^{\text{vii}}(t)}{4032} - \frac{T^7\varphi_1^{\text{iv}}(t)\varphi_1^{\text{v}}(t)}{224} + \frac{T^8\varphi_1^{\text{iv}}(t)\varphi_1^{\text{vi}}(t)}{1152} - \cdots\right]. \quad (46)$$

If the first-order voltage which is impressed on the diode is taken to be a single sine wave, it follows from the linearity of the first-order relations that the first-order current is likewise a single sine wave of the same angular frequency, ω. Thus, let

$$\varphi_1'''(t) = A \sin \omega t. \quad (47)$$

Then

$$\begin{aligned}\varphi_1^{\text{iv}}(t) &= A\omega \cos \omega t,\\ \varphi_1^{\text{v}}(t) &= -A\omega^2 \sin \omega t,\\ \varphi_1^{\text{vi}}(t) &= -A\omega^3 \cos \omega t.\end{aligned}$$

When these are substituted into (46) it is seen that the right hand side of the equation contains a d.-c. term and a double-frequency term. The left hand side must accordingly contain terms of the same frequencies. Hence the most general form which can be assumed for the second-order current $\varphi_2'''(t)e/km\epsilon$ is:

$$\begin{aligned}\varphi_2'''(t) = (a_0 + a_1\theta + a_2\theta^2 + \cdots)&\\ + (b_0 + b_1\theta + b_2\theta^2 + \cdots) \sin 2\omega t&\\ + (c_0 + c_1\theta + c_2\theta^2 + \cdots) \cos 2\omega t,& \quad (48)\end{aligned}$$

where $\theta = \omega T$ is the transit angle, and the a's have no reference to the symbols previously used for acceleration.

When (47) and (48) are substituted into (46) it will be seen that the coefficients of the d.-c. term, the sin $2\omega t$ term and the cos $2\omega t$ term respectively may be equated on the two sides of the equation. This gives three equations and in each of these, coefficients of corresponding powers of the transit angle may be equated, thus providing the values of all of the coefficients in (48).

Without carrying out this procedure in detail it is possible to find the d.-c. component directly from (46) and (47) by noting that time derivatives of the d.-c. component of $\varphi_2'''(t)$ are zero. Hence the left hand side of (46) reduces to its first term. Substitution of (47) into the right hand side and selection of d.-c. components then gives:

$$\varphi_2'''(t)_{\text{d.-c.}} = \frac{1}{12}\frac{A^2}{K}\left[1 - \frac{\theta^2}{40} + \frac{\theta^4}{2800} - \cdots\right]. \tag{49}$$

This indicates that the rectified current decreases when the transit angle θ becomes appreciable. In the second part of his first electronics paper,[1] Benham reached the conclusion that the rectified current increases with frequency. To reconcile his result with (49) it is only necessary to note that (49) indicates a decrease in rectified current with frequency provided that the amplitude A of the first-order current remains constant, whereas Benham's result was based on a constant amplitude of the first-order voltage. A direct comparison therefore necessitates the computation of the first-order voltage. From (41a) with zero initial acceleration and velocity we have:

$$(W_a - W_b)_1 = \frac{JKT^4}{12}\left[1 - \frac{3}{10}i\theta - \frac{1}{15}\theta^2 + \frac{1}{84}i\theta^3 + \frac{1}{560}\theta^4 \cdots\right]. \tag{50}$$

Taking the amplitude of the current factor $\varphi_1'''(t)$ to be A as in (47) and the amplitude of the voltage factor $(W_a - W_b)_1$ to be B, we find from (50) that

$$B^2 = A^2\frac{K^2T^8}{144}\left[\left(1 - \frac{1}{15}\theta^2 + \frac{1}{560}\theta^4 - \cdots\right)^2 + \left(-\frac{3}{10}\theta + \frac{1}{84}\theta^3 - \cdots\right)^2\right]$$

or

$$B^2 = A^2\frac{K^2T^4}{144}\left[1 - \frac{13}{300}\theta^2 + \frac{11}{12600}\theta^4 - \cdots\right]. \tag{51}$$

The A^2 may thus be eliminated between (51) and (49) giving

$$\varphi_2'''(t)_{\text{d.-c.}} = \frac{12B^2}{K^3T^8}\left[1 + \frac{11}{600}\theta^2 + \frac{39}{140000}\theta^4 + \cdots\right], \tag{52}$$

which is in agreement with the result obtained by Benham.

The physical explanation of the increase in rectified current which occurs when the transit angle becomes appreciable follows directly when it is realized that as the frequency is made higher, the fundamental component of the alternating current increases when the a.-c. voltage is maintained at constant amplitude. This is because of the capacitance of the diode. It is this increase in fundamental current which causes the increase in rectified current, and not the effect of transit angle directly.

PART IV—NEGATIVE GRID TRIODES

The general aspect of the negative grid triode is shown in Fig. 1. The upper diagram shows a plan view of the electrode arrangement and

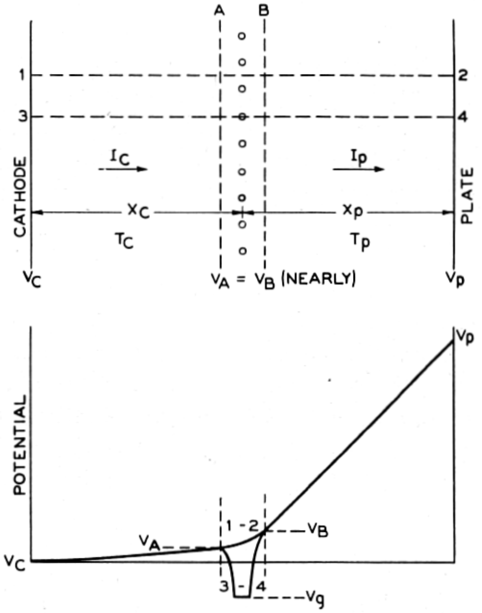

Fig. 1.—Nomenclature and potential distribution in negative grid triodes.

indicates the nomenclature which will be adopted. The planes at V_c and V_p constitute the cathode and plate, respectively, and the grid wires are indicated by the small circles. The planes *A* and *B* are imaginary planes located on opposite sides of the grid and only far enough away so that the irregularities in the potential distribution caused by the individual negative grid wires have practically disappeared. An analysis of the potential distribution in the immediate neighborhood of a shielding screen has been made by Maxwell (Treatise on Electricity and Magnetism Vol. I) and this applies to the grid of the triode in the absence of electron flow. The presence of electrons between the negative grid wires will tend to decrease the irregularities shown in Maxwell's analysis and therefore allow the planes *A* and *B* to be placed somewhat closer together than his figures would indicate. As far as the writer is aware, exact relations for the potential distribution near the grid wires in the presence of electron flow have not been worked out so that Maxwell's analysis is our only basis at the present time for determining how near to the grid we can imagine A and B to be fixed.

The distance from cathode to grid will be called x_c and from grid to plate will be called x_p. Corresponding to x_c there will be an electron transit time T_c, and to x_p there will be the transit time T_p. The density of the total current leaving the cathode is I_c so that the current factor $eI/km\epsilon$ will be J_c. The factor for the density of the total current reaching the plate will be J_p.

The general scheme for analysis of the triode is to regard the structure as composed of two parallel plane diodes. The first of these comprises the cathode and the imaginary plane A, while the second comprises the imaginary plane B and the plate or anode. In accord with normal operation, complete space charge is postulated at the cathode, so that initial velocities and accelerations are zero, and the general parallel plane equations, (41), (42) and (43) may be applied directly to conditions between the cathode and the plane A.

The plane B and the plate also constitute a structure to which the general equations can be applied, but first the initial velocities and accelerations at B will have to be found, and finally a relation between J_c and J_p will be needed. Moreover, the relation of the potentials at A and B to the potential V_g of the grid wires themselves must be found, because it is only the latter that is available for external use or measurement.

As an aid to finding these various relations the lower diagram in Fig. 1 will be of assistance. This is a graph showing the general form of the d.-c. potential as a function of the distance from the cathode. Between cathode and plane A the potential curve is the same regardless of whether the line 1–2 in the upper diagram which passes between two grid wires is followed, or whether the line 3–4 which passes through a grid wire is followed. The same thing applies between plane B and the plate.

Between planes A and B, however, conditions become vastly different according to whether the potential curve is drawn for the line 1–2 or the line 3–4, and the general shape of the potential curve for the two conditions is marked 1–2 and 3–4 respectively in the lower diagram. Along 1–2 the potential is everywhere positive, as otherwise electrons would not be able to penetrate the grid mesh and reach the plate. On the other hand, the grid wires themselves are at a negative potential, and the curve 3–4 shows the way in which the potential surface forms into pockets surrounding the grid wires. The size of these pockets, and hence the location of the planes A and B is determined by the relative potentials of grid and plate, the size of the grid wires, and the spacing between them.

In applying the general analysis for parallel planes to this type of structure it is possible to find all of the desired initial conditions at the B-plane in terms of conditions at the A-plane as well as to find the actual a.-c. potential of the grid wires themselves and the relation between plate current and cathode current provided that a very simple condition is fulfilled. This condition is merely that the size of the potential pockets surrounding the grid wires is so small that the planes A and B may be taken close enough together to cause the electron transit time between the two to be negligibly small compared with the transit time from cathode to A or from B to the anode. Without this condition, the problem appears almost hopeless. With it, the procedure is straightforward and simple.

For the factors affecting the fulfillment of this condition, Maxwell's analysis fortunately provides a guide that is at least safe, because if the planes can be located near together compared with the distances x_c and x_p on the basis of his analysis, then the smoothing effect of the electrons between grid wires will make the actual operating conditions still better. The adaptation of his equations gives the following expression for the potential distribution and for small diameter grid wires:

$$V = \left[\frac{V_g - \dfrac{x_c}{x_c + x_p} V_p}{\left(\dfrac{x_c x_p}{x_c + x_p} \right) \dfrac{4\pi}{a} - 2 \log_e \left(2 \sin \dfrac{\pi c}{a} \right)} \right] \left[\frac{4\pi x_p}{a} \left(\frac{x_c + d}{x_c + x_p} \right) - \log_e (1 - 2e^{2\pi d/a} \cos^{2\pi z/a} + e^{4\pi d/a}) \right] + V_p \left(\frac{x_c + d}{x_c + x_p} \right). \quad (53)$$

Here d is the distance from the grid to the place where the potential is to be computed, and is considered positive when directed from the grid toward the plate, z is distance measured parallel to the grid, a is the distance between centers of grid wires, c is the wire radius, and the cathode potential, V_c is taken to be zero. The only term here contributing to the potential pockets surrounding grid wires is the logarithmic one which alone involves z, the distance along plane A or B. The condition for the variations to be smoothed out with either positive or negative values of d is

$$e^{2\pi D/a} \gg 2,$$

where D represents the magnitude of d. Below are given some of the values of $e^{2\pi D/a}$

D/a	$e^{2\pi D/a}$
0.1	1.875
0.2	3.514
0.3	6.587
0.4	12.35
0.5	23.17
0.6	43.39
0.7	81.34
0.8	152.5
0.9	285.5
1.0	536.0

This shows that if the magnitude of the cosine term which involves z is to be limited to one per cent of the largest of the three terms constituting the logarithm, then D/a should be greater than 0.84. On the other hand if a ten per cent variation is tolerated, then D/a may be as small as 0.48.

There is a further consideration that tends to smooth out the corrugations in the potential caused by the grid wires. The relation (53) was derived on the assumption that the cathode and plate were quite distant from the grid. When this is not the case, the fact that both are equipotential surfaces tends to crowd the irregularities in toward the grid, and hence to decrease the area of the pockets.

The general conclusion to be drawn from this investigation into the potential distribution is that the electronics analysis can be applied with greatest accuracy to tubes where the grid wires are very close together relative to the distance between grid and either cathode or anode. When this is not the case, the analysis may be expected to show the general trend of the performance in most cases but to be unreliable for quantitative computation. In extreme cases, where the grid wires are very far apart, the grid action approximates more nearly to a change in the effective cathode area than a uniform action over the entire surface. The electronics analysis can be adapted to this extreme case when the change in effective cathode area occurs instantaneously with a change in grid potential, for then the diode analysis applies directly between the effective cathode area and the anode. Means for doing this will become evident when the theory for grids with fine mesh is understood so that the details will not be described.

The analysis for fine-mesh grids now follows, based on the assumption applicable to them that the planes A and B in Fig. 1 are very near together compared with x_c and x_p. As a first consequence of this assumption, the electron velocities at B must be the same as those at A, and hence one of the initial conditions at the B-plane is provided for. A second consequence is that the potential at B is the same as that at A, and therefore the potential between plate and cathode is the sum of the potentials between cathode and A and between B and the plate.

The accelerations at the two planes are not the same. This can be seen from the lower diagram in Fig. 1 when it is remembered that the acceleration is proportional to the slope of the potential curve. The accelerations can be found, however, by a relatively simple calculation and this will be done in the course of the following analysis.

D.-C. Relations

As a preliminary to the treatment of first order effects in negative grid triodes, certain d.-c. relations will be determined.

The distance is related to the transit time by (10). At the cathode and for complete space charge the acceleration and velocity are zero, so that the cathode-grid transit time T_c is given by

$$x_c = K\frac{T_c^3}{6}. \tag{54}$$

At B the acceleration is yet to be found, but may be taken to be g times that at A. Hence, from (21)

$$a_B = gKT_c. \tag{55}$$

The velocity at B is the same as that at A, so that from (27)

$$u_B = K\frac{T_c^2}{2}. \tag{56}$$

These values, (55) and (56), may now be inserted as initial conditions in (10) to give a relation between x_p and T_p. Calling the ratio T_p/T_c by the symbol h and x_p/x_c by the symbol y we thus obtain:

$$gh^2 = y/3 - h - h^3/3. \tag{57}$$

In this way the acceleration is related to the transit times. These have now to be expressed in terms of quantities that can be measured directly, namely the current and the plate voltage. To do this, the general expression for potential difference is obtained by integration of (34) which gives:

$$(W_a - W_b)_0 = \frac{1}{8}K^2T^4 + \frac{1}{2}Ka_aT^3 + \frac{1}{2}Ku_aT^2 + \frac{1}{2}a_a^2T^2 + a_au_aT. \quad (58)$$

Putting in the initial conditions we have

$$(W_c - W_A)_0 = \frac{1}{8}K^2T_c^4 \quad (59)$$

and

$$(W_B - W_p)_0 = \frac{1}{8}K^2T_c^4[4gh(1 + h^2) + 4g^2h^2 + 2h^2 + h^4]. \quad (60)$$

From these, and placing the cathode potential equal to zero, we get

$$\sqrt{\frac{V_p}{V_A}} = 2gh + 1 + h^2.$$

With (57) the ratio g may be eliminated from this, giving

$$\sqrt{\frac{V_p}{V_A}} = \frac{2y}{3h} - 1 + \frac{1}{3}h^2. \quad (61)$$

When h is small, as it normally is, the last term may be disregarded, giving

$$h = \frac{2/3\ y}{1 + \sqrt{V_p/V_A}} \text{ (approximately)}. \quad (61a)$$

Equations (57) and (61) allow g and h to be found in terms of V_p, which can be measured directly, and V_A which can be found in terms of the current and the distance x_c by combining (54) and (59) to give:

$$(W_c - W_A)_0 = \frac{1}{8}K^{2/3}(6x_c)^{4/3}. \quad (62)$$

This is the well-known Child's equation, and when W and K are expressed in terms of voltage and current appears in the usual form

$$I_0 = -2.34 \times 10^{-6}V_A^{3/2}/x_c^2 \text{ amperes/cm.}^2 \quad (63)*$$

Several other expressions which are useful for computation purposes may be found from these d.-c. relations:

* The negative sign occurs because of the assumed current direction, from the cathode. The numerical factor is 2.34 instead of 2.33 given by Compton and Langmuir [6] because of the value of e/m which was used, q.v.

The transit angle $\theta_c = \omega T_c$ may be written in terms either of the voltage V_A or the current I_0, thus

$$\theta_c = \frac{9500 x_c}{\lambda\sqrt{V_A}} = \frac{-126}{\lambda}\left(\frac{x_c}{I_0}\right)^{1/3} \text{radians,} \tag{64}$$

where λ is the free-space wave-length in centimeters of an alternating current of angular frequency ω.

The slope of the static characteristic of a diode coinciding with the cathode and the plane A may be expressed

$$-\left.\frac{\partial V_A}{\partial I_0}\right|_{x_c} = r_c = -\frac{2}{3}\frac{V_A}{I_0} = \frac{285{,}000 x_c^2}{\sqrt{V_A}} = \frac{-3780 x_c^{4/3}}{I_0^{1/3}}, \tag{65}$$

where r_c is the low-frequency resistance in ohms of a square centimeter of area.

From (64) and (65) it can be seen that

$$r_c = 30\lambda x_c \theta_c. \tag{66}$$

A further expression that occurs frequently in following equations is:

$$\frac{KT_c^4}{\epsilon} = 12 r_c. \tag{67}$$

Later we shall be able to show how the low-frequency plate resistance of the triode is related to r_c and the amplification factor, as well as how the inter-electrode capacitances of the "cold" tube are involved in these quantities. A simple approximation for the transit time ratio h will also be derived.

First-Order Relations

In the picture shown by Fig. 1 an alternating current is assumed to flow from the cathode to the plane A. This current I_c is related to the quantity J in the general equations (41) or (41a) by the expression $J_c = eI_c/km\epsilon$, and the current includes both conduction and displacement components. Complete space charge at the cathode allows initial velocities and accelerations to be placed equal to zero, so that the first order potential difference between cathode and plane A is given by (41a) as follows:

$$(W_c - W_A)_1 = J_c K T_c^4 \sum_{n=0}^{\infty} (-\beta_c)^n \frac{(n+2)}{(n+4)!}. \tag{68}$$

In a similar way, the potential between plane B and the plate can be written when the initial first-order acceleration at B has been found, the d.-c. initial conditions being given in the previous section, and the first order initial velocity being the same as that at A. From (43a) we have

$$\mu_B = \mu_A = J_c T_c^2 \sum_{m=0}^{\infty} \frac{(-\beta_c)^m (m+1)}{(m+3)!}. \tag{69}$$

The computation of the a.-c. component of acceleration at B is based on the particular property possessed by negative grid tubes that no electrons reach the negative grid. Then, because the velocity at B is the same as that at A it follows that the conduction component of the current is the same at both planes. The total first-order current may be written from (1)

$$I_1 = (\rho u)_1 + \epsilon \frac{\partial E_1}{\partial t}.$$

Multiplying through by $e/km\epsilon$ we can write

$$J = Q + \frac{\partial a_1}{dt}, \tag{70}$$

where $Q = (\rho u)_1 e/km\epsilon$ measures the conduction current, and is the same at planes A and B. Hence at plane A, and since $\partial/\partial t = p$ for exponential currents and voltages:

$$J_c = Q + pa_{1A}. \tag{71}$$

At plane B we have similarly

$$J_p = Q + p\alpha_B. \tag{72}$$

From (71) and (72) there results

$$p\alpha_B = J_p - J_c + pa_{1A}. \tag{73}$$

The value of a_{1A} may be obtained from (42a) so that the acceleration at B is given by

$$p\alpha_B = J_p - J_c - 2J_c \sum_{m=0}^{\infty} \frac{(-\beta_c)^{m+1}(m+2)}{(m+3)!}. \tag{74}$$

All of the initial values at the plane B have been obtained and are expressed in (55), (56), (69) and (74). The potential between B and the plate is now obtained from (41a) and follows, where $h = T_p/T_c$:

$$\frac{(W_B - W_p)_1}{KT_c^4} = (J_p - J_c)\frac{h(gh+1)}{2\beta_c} + J_p h^4 \sum_{n=0}^{\infty} \frac{(-h\beta_c)^n (n+2)}{(n+4)!}$$

$$+ J_c \left[h^2 \sum_{n=0}^{\infty} (-h\beta_c)^n \left(\frac{2gh(n+2)+(n+3)}{2(n+3)!} \right) \right.$$
$$+ h \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h^n (-\beta_c)^{n+m} \quad (75)$$
$$\left. \times \left(\frac{2gh(n+1)(m+2)+(n+2)(m+2)-h(n+1)(m+1)}{(n+2)!(m+3)!} \right) \right].$$

The attainment of (68) and (75) does not completely solve the problem because it is the potential of the grid wires that can be measured rather than the potential at A and B. The transformation may be readily accomplished, however, by writing

$$(V_A - V_g)_1 = I_g Z_g, \quad (76)$$

where Z_g is the effective impedance between the plane A (or B) and the grid wires. In the negative grid tube when A and B are close together, Z_g is a pure capacitance, C_g.

Writing (68) and (75) respectively in terms of V and I instead of W and J we now have in symbolic form

$$(V_c - V_A)_1 = I_c Z_c, \quad (77)$$
$$(V_A - V_p)_1 = I_c(Z_3 - Z_1) + I_p(Z_1 + Z_2), \quad (78)$$

where

$$I_c = I_g + I_p. \quad (79)$$

The four equations, (76), (77), (78) and (79) are the basis of negative grid triode analysis. The impedances involved refer to a square centimeter of area and are as follows:

$Z_g = \frac{1}{pC_g}$ = impedance between plane A (or B) and the grid wires, (80)

$$Z_c = 12r_c \sum_{n=0}^{\infty} (-\beta_c)^n (n+2)/(n+4)!, \quad (81)$$

$$Z_1 = 12r_c h(gh+1)/2\beta_c, \quad (82)$$

$$Z_2 = 12r_c h^4 \sum_{n=0}^{\infty} (-h\beta_c)^n (n+2)/(n+4)!, \quad (83)$$

$$Z_3 = 12r_c h^2 \left[\sum_{n=0}^{\infty} (-h\beta_c)^n \left(\frac{2gh(n+2)+(n+3)}{2(n+3)!} \right) \right]$$
$$+ 12r_c h \left[\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-\beta_c)^{n+m} h^n \right.$$
$$\left. \times \left(\frac{2gh(n+1)(m+2)+(n+2)(m+2)-h(n+1)(m+1)}{(n+2)!(m+3)!} \right) \right]. \quad (84)$$

The value of r_c is expressed by (65).

For large values of transit angle it is convenient to have these impedances expressed in closed instead of in series form. By using (41) instead of (41a) throughout the analysis the result may be obtained. The expressions for Z_g and Z_1 will not differ from those given above, but Z_c, Z_2 and Z_3 may be written in the following forms:

$$Z_c = \frac{12r_c}{\beta_c^4}\left[\frac{\beta_c^3}{6} - \beta_c(e^{-\beta_c} + 1) - 2(e^{-\beta_c} - 1)\right], \tag{81a}$$

$$Z_2 = \frac{12r_c}{\beta_c^4}\left[\frac{h^3\beta_c^3}{6} - h\beta_c(e^{-h\beta_c} + 1) - 2(e^{-h\beta_c} - 1)\right], \tag{83a}$$

$$\begin{aligned} Z_3 = \frac{12r_c}{\beta_c^4}\Bigg[& \beta_c^3\frac{1}{2}(h + gh^2) \\ & + \beta_c[(h - 2gh - 1)e^{-(h+1)\beta_c} + he^{-h\beta_c} + e^{-\beta_c}] \\ & + [2(h - gh - g)e^{-(h+1)\beta_c} + 2(1 - h + gh)e^{-h\beta_c} + 2ge^{-\beta_c} - 2] \\ & + \frac{2}{\beta_c}(1 - g)(e^{-(h+1)\beta_c} - e^{-h\beta_c} - e^{-\beta_c} + 1)\Bigg]. \end{aligned} \tag{84a}$$

Of the various impedances Z_3 is the only one that is really troublesome. The values of Z_c and Z_2 may be obtained from data given in published papers [1, 2] when it is noticed that Z_2/h^4 may be calculated from Z_c if β_c is replaced by $\beta_p = h\beta_c$. In the treatment of Z_3 there seems to be no easy road, although the series form (84) may be expanded with comparative ease.

Further steps consist in the transposition of these equations to obtain convenient forms and to show how they harmonize with low-frequency theory. A useful expression may be obtained from the fundamental relations (76), (77), (78), (79) by eliminating V_A, I_c and I_g. The result is

$$\begin{aligned} (V_c - V_p)_1 + \frac{(Z_1 - Z_c - Z_3)}{Z_c + Z_g}(V_c - V_g)_1 \\ = I_p\left[\frac{[(Z_c + Z_2 + Z_3)Z_g + (Z_1 + Z_2)Z_c]}{Z_c + Z_g}\right]. \end{aligned} \tag{85}$$

This begins to look like the familiar low-frequency equation

$$(V_c - V_p)_1 + \mu(V_c - V_g)_1 = I_pZ_p, \tag{86}$$

where μ is the amplification factor and Z_p is the internal plate impedance. The two are equivalent at all frequencies if I_p is interpreted as the density of the total plate current, and not the conduction component, only. Then we have:

$$\mu = \frac{Z_1 - Z_c - Z_3}{Z_c + Z_g} = \frac{(Z_1 + Z_2) - (Z_c + Z_2 + Z_3)}{Z_c + Z_g}, \tag{87}$$

$$Z_p = \frac{(Z_c + Z_2 + Z_3)Z_g + (Z_1 + Z_2)Z_c}{Z_c + Z_g}. \tag{88}$$

The relations involved in these two equations may be made somewhat clearer by finding what they resolve into at low frequencies. This may be done by going to (80)–(84). The first thing to notice is that Z_1 and Z_g become very large because the frequency term $p = i\omega$ appears in their denominators. The other terms are relatively small at low frequencies and we have:

$$\mu_0 \rightarrow Z_1/Z_g, \tag{87a}$$

$$r_p = Z_p \rightarrow Z_c(1 + \mu_0) + Z_2 + Z_3. \tag{88a}$$

It will be shown below that this formulation for the amplification factor is in accord with that derived by Maxwell in his "Treatise on Electricity and Magnetism" for the shielding effect of a grid mesh.

Low-Frequency Relations in Negative Grid Triodes

In (87a) and (88a) the general form of the low-frequency triode relations is given. It is instructive to compute these in some detail so that the role played by the capacitance C_g between the planes A or B and the grid wires is demonstrated.

To do this, (82) which gives the impedance Z_1 may first be transformed by aid of (57) and (67) to give

$$Z_1 = \frac{x_p}{\epsilon p}(1 - h^3/y). \tag{89}$$

The impedance Z_g may be written as in (80) so that the low-frequency amplification factor is

$$\mu_0 = \frac{x_p}{\epsilon} C_g(1 - h^3/y). \tag{90}$$

It is of interest to note that ϵ/x_p is the capacitance per unit area between the plate and a solid plane at the grid. As expressed by Compton and Langmuir [6] from Maxwell's analysis the low-frequency amplification factor is

$$\mu_0' = \frac{x_p}{\dfrac{a}{2\pi}\log_e \dfrac{a}{2\pi c}}, \tag{91}$$

where a is the distance between centers of grid wires and c is the wire radius.

Comparing (90) and (91) we see that both are proportional to x_p, but that (90) contains a correction term, h^3/y which is normally very small as h is usually of the order of one-third to one-tenth. The presence of the correction term may be explained by remembering that (90) was derived for conditions holding when electrons are flowing while (91) applies strictly only to a cold tube in which no free electrons are present. This being the case, it is to be expected that the two equations would be equivalent if the correction term were omitted from (90). The term being small, this is very nearly true in any case and gives

$$C_g = \frac{2\pi\epsilon}{a \log_e \dfrac{a}{2\pi c}} \text{ farads in cm.}^2 \tag{92}$$

The capacitance C_g which was introduced as existing between the electron stream and the grid wires is thus shown to have a value which can be calculated with fair exactness under the conditions when Maxwell's equation (91) holds, namely when the grid wires are small compared with their separation.

Attention is now directed to the low-frequency value of the plate impedance, Z_p. From (88a) together with (81), (83) and (84) it may be shown that

$$r_p = r_c \left[\mu_0 + \frac{4}{3}(1 + y)(1 + h) - \frac{1}{3}(1 + h)^4 \right]. \tag{93}$$

This is thought to be the first instance of an expression for the plate resistance derived on strictly theoretical grounds. The formula as it stands contains r_c which is given by (65) and h which is given by (61), and both involve V_A. This latter may be found from (63) in terms of the direct current. A convenient approximation for V_A is obtained by making the assumption that the presence of electrons changes the d.-c. potential V_A by a small amount only, so that its value may be calculated from the static capacitances of a cold tube. The appropriate diagram is shown on Fig. 2.

From the figure, putting $V_c = 0$ we obtain

$$V_{A0} = \frac{V_{p0} + \dfrac{C_g}{C_p} V_{g0}}{1 + \dfrac{C_g}{C_p} + \dfrac{C_c}{C_p}}.$$

It was shown in (90) that C_g/C_p is the static, or " cut-off " amplification factor, μ_0' of the tube. Again, $y = C_c/C_p$ so that

$$V_{A0} = \frac{V_{p0} + \mu_0' V_{g0}}{1 + \mu_0' + y} \text{ (approximately).}$$

When the cathode is heated so that electrons flow, the potential V_A becomes depressed below this value which should be used only in forming rough estimates.

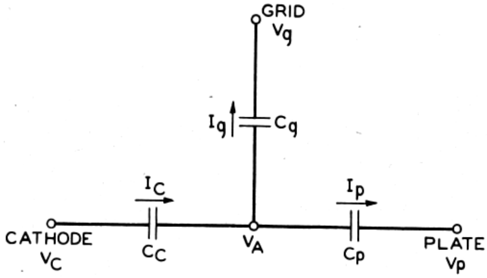

Fig. 2.—Equivalent network of negative grid triode in the absence of electrons.

A more accurate formula for V_A involves the transit time ratio h, but may be obtained as follows: The plate current of an ideal negative grid triode is proportional to $(V_{p0} + \mu V_{g0})^{3/2}$ so that

$$r_p = \frac{2}{3} \frac{(V_{p0} + \mu_0 V_{g0})}{I_0} .$$

Similarly

$$r_c = \frac{2}{3} \frac{V_{A0}}{I_0} .$$

From these two equations together with (93) is obtained

$$V_{A0} = \frac{V_{p0} + \mu_0 V_{g0}}{\mu_0 + \frac{4}{3}(1 + y)(1 + h) - \frac{1}{3}(1 + h)^4},$$

which may also be written

$$V_{A0} = \frac{V_{p0} + \mu_0 V_{g0}}{1 + \mu_0 + \frac{4}{3} y(1 + h) - \frac{1}{3} h^2(6 + 4h + h^2)} . \qquad (94)$$

This is in a form for comparison with the approximate equation above and shows the modification produced by the presence of electrons.

General Relations in Negative Grid Triodes

The high-frequency values of the amplification factor and the plate impedance could be computed in detail from (87) and (88) together with the various expressions given for the impedances involved. To do this would require an enormous amount of computation, so that the details are deferred until such time as it becomes evident that (87) and (88) express the high-frequency properties of tubes in the most useful way. The fact that they are analogous to the ordinary low-frequency conventions does not at all assure their general utility in the high-frequency field, and the fundamental equations (76) to (79) may be arranged in a wide variety of forms. For example, a companion equation to (85) may be obtained from the fundamental equations by eliminating V_A, I_p and I_c and thus obtaining:

$$(V_c - V_g)_1 - \left(\frac{Z_c}{Z_2 + Z_3 + Z_c}\right)(V_c - V_p)_1 = I_g\left[\frac{(Z_2 + Z_3 + Z_c)Z_g + (Z_1 + Z_2)Z_c}{Z_2 + Z_3 + Z_c}\right]. \quad (95)$$

Just as (85) gives an equivalent circuit between cathode and plate involving the whole current reaching the plate, so does (95) give an equivalent circuit between cathode and grid involving the whole current reaching the grid. The two equations completely describe the tube performance when the external connections are known. Because of the way in which I_g and I_p are defined they include the so-called grid-plate path, which is treated as a separate circuit at low frequencies.

The impedance presented by a tube to an e.m.f. applied between cathode and grid may now be calculated. To carry out the computation in general, it would be necessary to know the impedance attached between plate and cathode in the external circuit so that $(V_c - V_p)_1$ could be obtained from (85). However, the high-frequency properties of negative grid tubes may be illustrated more directly by choosing for consideration a special case that avoids having to take this additional step involving (85). This special case is the one where such a large capacitance is connected between the plate and cathode that $(V_c - V_p)_1$ is zero for any of the frequencies to be considered. The result will therefore be particularly applicable to finding the input impedance of screen tubes where the requirement for the special case is fulfilled.

For this special case where $(V_c - V_p)_1$ is zero, (95) may be solved directly for the impedance Z_a presented to an input applied between grid and cathode and gives

$$Z_a = Z_g + \frac{(Z_1 + Z_2)Z_c}{Z_2 + Z_3 + Z_c}. \tag{96}$$

The impedance Z_g is a pure capacitance, but the second term on the right of (96) contains both reactive and resistive components which latter account for the active grid loss which has been the subject of several investigations both of a theoretical and experimental nature.[2, 7, 8] At very low frequencies the capacitance represented by Z_1 predominates and the resistive component vanishes leaving for the input impedance Z_a merely the following

$$Z_a \to Z_g + Z_1\left(\frac{Z_c}{Z_2 + Z_3 + Z_c}\right)$$
$$= \frac{(y - h^3)}{i\omega\mu_0 C_c}\left[\frac{r_p/r_c}{\frac{4}{3}(1 + y)(1 + h) - \frac{1}{3}(1 + h)^4}\right]. \tag{97}$$

This expression may be written in several different forms but in none of them does a simplification occur in the way in which the transit time ratio h enters the equation. Perhaps the best mode of expression is a comparison of the "hot" capacitance C_a given by (97) where $Z_a = 1/i\omega C_a$ with the capacitance C_0 of a cold tube with plate and cathode tied together. This latter may be written

$$C_0 = \frac{\mu_0' C_p(1 + y)}{1 + y + \mu_0'}, \tag{98}$$

where μ_0' is the "cold" amplification factor, (91), and is related to μ_0 as shown by (90) so that $\mu_0 = \mu_0'(1 - h^3/y)$. The ratio C_a/C_0 is the "dielectric constant" of the hot tube and is

$$\frac{C_a}{C_0} = \left(\frac{1 + y + \mu_0'}{1 + y}\right)\left[\frac{\frac{4}{3}(1 + y)(1 + h) - \frac{1}{3}(1 + h)^4}{\mu_0 + \frac{4}{3}(1 + y)(1 + h) - \frac{1}{3}(1 + h^4)}\right]. \tag{99}$$

For illustration suppose that a certain tube has the following values: $y = 1$, $\mu_0 = 10$, $h = 1/5$. Then from (99) the dielectric constant of the hot tube would be 1.19. This is somewhat less than the value of

4/3 obtained by Benham [1] in an analysis which considered effects between cathode and grid only. Benham's value applies strictly to the capacitance between cathode and the plane A in Fig. 1 while (99) applies to the parallel combination of cathode-grid and grid-plate capacitances. The constants of the individual capacitances may be calculated from the fundamental equations (76)–(79) but are omitted here because of space limitations. A series of experiments performed several years ago by Mr. A. J. Rack and the writer and covering a wide range of operating conditions with several vacuum tubes showed values greater than unity for the dielectric constant of the cathode-grid capacitance and less than unity for the grid-plate capacitance. The parallel combination had a dielectric constant which was greater than unity in the range investigated, being thus in accord with (99) which always gives constants greater than unity for normal values of h.

The input capacitance of detector- and voltmeter-tubes being thus a somewhat complicated function, it is to be expected that the calculation of the impedance at higher frequencies where transit times are appreciable will be similarly complicated. To avoid undue length only the first term contributing to the resistive component of the active grid loss will be computed in detail. It must be pointed out, however, that the series in powers of transit angle which represents the input impedance converges slowly so that the first term is useful only when the transit time is small.

Keeping this in mind, we go to (96) and write the impedance in series form, obtaining finally the following expression for the equivalent shunting resistance between cathode and grid:

$$\frac{1}{R_a} = \frac{\theta_c^2}{180}\frac{\mu_0}{r_p}\left[\frac{y^2A - yB + C}{(y - h^3)^2}\right] \times \left[\frac{\mu_0}{\mu_0 + \frac{4}{3}(1 + y)(1 + h) - \frac{1}{3}(1 + h)^4}\right], \tag{100}$$

where

$$A = 9 + 44h + 45h^2,$$
$$B = 51h^2 + 123h^3 + 55h^4 + 3h^5,$$
$$C = 45h^4 + 51h^5 + 24h^6 + 11h^7 + 3h^8.$$

As in previous cases, this may be written in several ways, depending on the mode of expression of θ_c and r_p. For example

$$\theta_c = 2r_c\omega C_c,$$

so that

$$\frac{1}{R_a} = \frac{(\omega C_c)^2}{45} \frac{r_p}{\mu_0} \left[\frac{y^2 A - yB + C}{(y - h^3)^2} \right] \times \left[\frac{\mu_0}{\mu_0 + \frac{4}{3}(1 + y)(1 + h) - \frac{1}{3}(1 + h)^4} \right]^3. \quad (100a)$$

Comparison of (100) and (100a) shows that the transconductance μ_0/r_p appears in the numerator of the former, but in the denominator of the latter, and illustrates the care that must be taken in deriving sweeping conclusions concerning the effect of various tube parameters without taking all of the contributing factors into consideration. In the case of (100) the conclusion is that the loss may be reduced by decreasing the transconductance, but only if the transit angle θ_c is unchanged. On the other hand, (100a) says that the loss may be reduced by increasing the transconductance, but only if this is accomplished without change in the cathode-grid spacing, and without altering h by an amount large enough to affect materially the factors in square brackets.

Experimentally, it is found in many tubes that the loss increases when the transconductance is increased by changing the voltages applied to a given tube. This would seem to be at variance with (100a), for the cathode-grid spacing, and hence C_c, has not been altered by the voltage change. The explanation of the difficulty apparently lies in the departure of the static characteristics of many tubes from the 3/2 power law, which again may be explained in part by the presence of initial velocities and the large size of the potential pockets surrounding the grid wires. In a rough way the action of the latter is to vary the effective cathode area when the voltages are changed, producing an increased area with increase of current, and hence producing a current variation greater than the 3/2 power law.

In a recent paper, D. O. North [11] derives a formula for the active grid loss by neglecting space charge between grid and plate. His result is similar in many respects to (100) and both contain the factors $\theta_c^2\mu_0/r_p$. In an experimental check, W. R. Ferris [10] secures excellent results by obtaining the transconductance from the static characteristics of the tubes used, but computing θ_c by a formula which differs only slightly from (94). It can be shown that such a procedure would give a computed loss which increases with transconductance when the static characteristic is of the form $I = K(V_{p0} + \mu_0 V_{g0})^n$ and when n is greater than 2. The static characteristics are not given in Mr.

Ferris' paper so that it is not evident whether the exponent is greater than 2. The loss did, however, increase with transconductance and was checked by the computations in a satisfactory manner, which would imply either that the exponents were actually greater than 2 in Mr. Ferris' tubes, or that their cylindrical shape caused a decrease in the effect of the transconductance on the transit angle.

The equations in general indicate that the shunting resistance between cathode and grid is proportional to the square of the wave-length. This is in accord with the theory and experiments of Thompson and Ferris,[7, 10] and with the experiments of J. G. Chaffee [8] on tubes biased as class *A* amplifiers. However, when the tubes were biased as detectors, Chaffee [8, 9] found that the resistance varied more nearly as the first power of the wave-length. There are several factors which may contribute to this difference. With detector bias near cut-off the transit angles are large so that more terms of the fundamental equations may be needed. In Chaffee's work these were computed to be of the order of two or three radians which was scarcely enough to cause the entire effect observed by him. Another cause is thought to be an actual reversal in the direction of motion of electrons caused by the alternating potential operating in the vicinity of cut-off. Further study both of experimental and theoretical nature is required, however, before the point can be considered to be satisfactorily explained.

Conclusion

In general, the analysis presented in the foregoing pages is capable of serving as a guide to indicate the kind of results to be expected in the operation of vacuum tubes at ultra-high frequencies. In those cases where the physical structure of the tube complies with the conditions laid down for the theoretical treatment, a quantitative agreement can be anticipated. The importance of departures of the physical structure from this ideal can be evaluated in many instances by a careful comparison of the actual with the ideal structure.

Much yet remains to be done in the way of computing and tabulating the various factors involved in the equations and of investigating the effects of such things as initial electron velocities from a hot cathode, large size grid wires, coarse mesh grids, and cylindrical structures.

References

1. W. E. Benham, "Theory of the Internal Action of Thermionic Systems at Moderately High Frequencies." *Phil. Mag.*, p. 641, March 1928. Part II, *Phil. Mag. Supp.*, Vol. 11, p. 457, February 1931.
2. F. B. Llewellyn, "Vacuum Tube Electronics at Ultra-High Frequencies." *Proc. I.R.E.*, Vol. 21, November 1933; also *B.S.T.J.*, Vol. XIII, January 1934.

3. Johannes Müller, "Elektronenschwingungen im Hochvakuum." *Hochfrequenztechnik und Elektroakustik*, Vol. 41, May 1933.
4. F. B. Llewellyn, "Note on Ultra-High-Frequency Electronics." *Proc. I.R.E.*, Vol. 23, February 1935.
5. W. E. Benham, "Electronic Theory and the Magnetron Oscillator." *Proc. Phys. Soc.*, Vol. 47, January 1935.
6. Compton and Langmuir, "Electrical Discharges in Gases," Part II. *Rev. of Mod. Physics*, Vol. 3, April 1931.
7. B. J. Thompson and W. R. Ferris, Oral paper at Rochester I.R.E., 1934.
8. J. G. Chaffee—Forthcoming discussion on (7).
9. J. G. Chaffee, "The Determination of Dielectric Properties at Very High Frequencies." *Proc. I.R.E.*, Vol. 22, August 1934.
10. W. R. Ferris, "Input Resistance of Vacuum Tubes as Ultra-High Frequency Amplifiers." *
11. D. O. North, "Analysis of the Effects of Space Charge on Grid Impedance." *

* To be published shortly. Much of the material was presented orally by B. J. Thompson and W. R. Ferris at various technical meetings including:

U. R. S. I. and *I. R. E.*, Washington, April 27, 1934.
Philadelphia section of *I. R. E.*, January 3, 1935.
Reference (7) above.