Mutual Impedances of Parallel Wires *

By RAY S. HOYT and SALLIE PERO MEAD

This is a theoretical paper relating to circuits of straight parallel wires traversed by alternating currents under such conditions (frequency of the alternating currents, diameter and spacing of the wires) that the resulting non-uniformity of the current distribution is sufficient to play an important part in determining the mutual and self impedances. The paper deals primarily with the mutual impedances; but incidentally the self impedances are dealt with almost as fully, except that no numerical calculations are made for them.

Part I is mainly a discussion of the physical nature of the mutual and self impedances in the generalized manner necessitated by the non-uniformity of the current distribution. It deals with wires which are short enough compared with the wave-length so that the complicating effects of propagation are negligible and so that the current in each wire can be regarded as an aggregate of filamentary currents.

Part II establishes, by recourse to electromagnetic wave theory, calculation formulas for the mutual and self impedances per unit length of a pair of long straight parallel transmission circuits forming a square array. Values of the mutual impedance are calculated over a frequency-range of 1 to 1000 kilocycles per second, for three cases of the circuits, and are compared with measured values.

INTRODUCTION

THE concept of the mutual impedance per unit length between two straight parallel filamentary conductors is well understood by engineers, and its calculation formula is simple. This mutual impedance is a pure reactance (directly proportional to the frequency), the induced electromotive force being in phase quadrature with the inducing current.

In the case of open-wire circuits, even when operating with carrier currents of very high frequency, the mutual impedance can be calculated with high accuracy by regarding the wires as filamentary.

For cable circuits, however, the foregoing statement is not true, because of the close juxtaposition of the wires. In such circuits the wires may be termed "thick," meaning that their diameter is appreciable compared with their interaxial separation. Depending in a complicated manner on the conductivity, permeability, diameter, and interaxial separation of such wires, the frequency may easily be so high as to render the filamentary formulas for the mutual impedance of even straight wires quite inaccurate and unreliable. In such cases it is necessary to consider the current distribution over the cross-section

* The two parts of this paper are distinct, though complementary. Part I was written by Ray S. Hoyt, Part II by Sallie Pero Mead.

of the wires.[1] When this is done it is found that the mutual impedance comprises not only a reactance component, which is no longer proportional to the frequency, but also a resistance component, which does not vary in any simple way with the frequency. Both of these component departures of the mutual impedance from its simple filamentary value increase the difficulties of balancing out crosstalk, and the resistance component has also an important effect on the attenuation at carrier frequencies. These matters have recently assumed considerable importance on account of the rapidly increasing interest in the possibilities of communication transmission over non-loaded cable circuits with the aid of carrier currents having frequencies high compared with those of speech. As an approximate guide to the behavior of twisted circuits in cables the theory and formulas for straight wires, as developed in this paper, have proved to be of considerable service.

The present paper deals with the mutual impedances of two or more straight parallel wires from two aspects: In Part I the physical theory is developed and expounded. The current in a wire is there regarded as made up of an indefinitely large number of parallel filamentary current elements. On this basis it is shown (among other things) that the current distribution over the cross-section of each conductor is necessarily non-uniform, and that this non-uniformity gives rise to a mutual resistance term in the mutual impedance, besides a change in the mutual reactance term. In Part II electromagnetic wave theory is applied to develop formulas for the mutual and self impedances of a pair of long straight parallel transmission circuits in close juxtaposition. Calculations of the mutual impedance made with these formulas over a very wide range of frequencies (1 to 1000 kilocycles per second) are found to be in very satisfactory agreement with available experimental results.[2] In both parts of the paper an endeavor has been made to bring engineering concepts and formulas into closer relationship with electromagnetic theory.

[1] The convenient term "proximity effect" when applied to the distribution of the current over the cross-section of a given conductor means the deviation of this distribution from the "intrinsic distribution," the latter meaning the distribution when the given conductor is far enough from all other conductors so that the distribution in it is sensibly unaffected by them.

When the given conductor is a straight uniform wire of circular cross-section, its "intrinsic distribution" is of course axially symmetrical.

Not every axially symmetrical distribution is the same as the corresponding intrinsic distribution, as is evidenced by the case of two coaxial conductors, where the proximity effect in the outer conductor may be large although the current is axially symmetrical in each conductor.

[2] See the paper by R. N. Hunter and R. P. Booth, in the April issue of this *Journal*, entitled "Cable Crosstalk—Effect of Non-Uniform Current Distribution in the Wires," which includes the results of some rather extensive sets of measurements of the mutual impedance of straight wire circuits, and also of twisted circuits in cables, and a brief physical discussion with particular regard to the effect of non-uniform current distribution.

PART I

Physical Theory

The Physical System; Analysis of the Wire Currents into Filaments

Since the mutual impedance between any two parallel circuits can be expressed wholly in terms of the mutual impedances between the various wires composing the circuits,[3] it will suffice in Part I to discuss the mutual impedance between the two wires A and B in Fig. 1. These are each of uniform cross-section, but they need not be alike in cross-sectional shape and area nor in material.

The wires in Fig. 1 will be assumed very long compared to the dis-

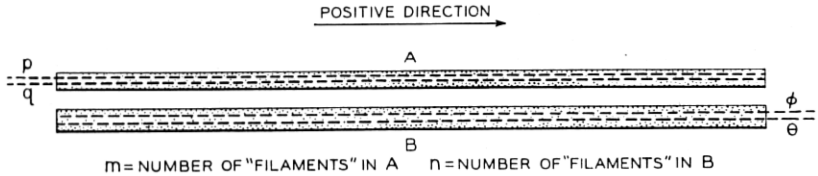

Fig. 1—Two "thick" straight parallel wires. p, q designate any two "filaments" of wire A; ϕ, θ any two of B.

tance between them, so that the end-effects[4] in the current distribution will be negligible, yet short enough compared with the wave-length so that the charging current will be a negligible fraction of the total current and therefore the current in each wire of sensibly the same value throughout its length.[5] These circumstances enable the current in each wire to be treated as an aggregate of filamentary currents which are purely longitudinal, and correspondingly enable the mutual and self impedances of the wires to be described and formulated in terms of the mutual and self impedances of such filaments, thus correlating well with the familiar treatment of a system of fine parallel wires. This treatment by analysis into filaments has been chosen

[3] For example, the mutual impedance Z_{ab} between two circuits a and b, of which a comprises wires 1 and 2 and b comprises 3 and 4, is given by $Z_{ab} = Z_{13} - Z_{14} - Z_{23} + Z_{24}$. However, since the wires are in general "thick," the value of each mutual impedance (also each self impedance) must depend on the presence of all four of the wires.

[4] These consist in the currents not being purely longitudinal near the ends of the wires.

[5] Negligibility of the charging current does not by any means imply that the distributed charges on the surfaces of the wires are negligible as regards the voltages which they produce, for extremely small charging currents suffice to establish charges which can produce relatively large voltages.

For a discussion of this very important fact and other underlying concepts of circuit theory, the reader is referred to a paper by John R. Carson, "Electromagnetic Theory and the Foundations of Electric Circuit Theory," published in this *Journal* for January, 1927.

because it lends itself well to a physical exposition and to the derivation of the simple formulas needed in that exposition.

Since in general the various filamentary currents in a wire are not in phase the total, or resultant, current in the wire, which is the complex algebraic sum of the filamentary currents, must be less than the arithmetic sum of the filamentary currents. An extreme instance of this fact is presented by a wire, short compared with the wave-length, which is on open circuit and is situated in the field due to other currents; for although the total, or resultant, current traversing any cross-section of this open wire must be zero, the individual filamentary currents are not zero.

The Two Parts of a Voltage, and Their Resultant [6]

For clearness in describing and formulating the mutual and self impedances of the wires, even when these are filamentary, it is necessary to recognize that the voltage along any specified path (which may, in particular, be a filament in a conductor) is in general the sum, or resultant, of two voltages which are simultaneously present along the path, namely the voltage due to all charges, and the voltage due to all currents; for brevity, these two parts of the total voltage will be called merely the "charge voltage" and the "current voltage" respectively—or, somewhat more fully, the "charge-produced voltage" and the "current-produced voltage." They will be denoted by V and U respectively, and their resultant by W, so that $W = V + U$.

The two parts of a voltage have the sharply contrasting properties constituting principles "1" and "2" in the following set of four principles, all of which are of much importance for the understanding of electric circuit theory and transmission theory.

1. A "charge voltage" (V) has exactly the same value along every path between any two fixed points, and hence is zero around every closed path.

2. A "current voltage" (U) has in general unequal values along any two different paths between any two fixed points, the difference in these values being accounted for by the time rate of change of the magnetic flux in the space between the two paths; thus a "current voltage" is in general not zero around a closed path.

3. The total, or resultant, voltage (W) must evidently have the same properties as the "current voltage" (U) in "2."

4. For any current filament f in a conductor the product of the resistance R_f of the filament and its current I_f is, by Ohm's law, equal

[6] This section is based on certain fundamentals of electromagnetic theory summarized in an appendix placed at the end of the whole paper.

to the total, or resultant, voltage along the filament; that is $R_f I_f = W_f = V_f + U_f$. Hence $V_f = R_f I_f - U_f$, which is the most convenient form in many applications, particularly those involving inductances.

Before taking up (in the next section) the more complicated subject of the mutual and self impedances of "thick" wires, some of the foregoing principles will be illustrated by applying them to the simple system represented by Fig. 2, which comprises two filamentary wires

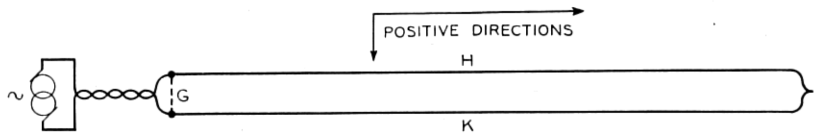

Fig. 2—An illustrative circuit of two "filamentary" wires, H and K.

H and K forming a loop. The dotted line G is merely a geometrical path traced directly between the initial terminals of the two wires.

The first form of principle "1," when applied to the two separate paths G and HK between the initial terminals, gives:

$$V_G = V_H + (-V_K). \tag{1}$$

The following two equations result from the last form of principle "4," when supplemented by the definitions of the self and mutual inductances of filamentary wires, which enable the U's to be expressed in terms of the I's:

$$V_H = R_H I_H - U_H = R_H I_H + i\omega L_H I_H + i\omega L_{HK} I_K, \tag{1a}$$
$$V_K = R_K I_K - U_K = R_K I_K + i\omega L_K I_K + i\omega L_{KH} I_H, \tag{1b}$$

where L_H denotes the self inductance of wire H, L_{HK} the mutual inductance[7] between H and K, $\omega = 2\pi$ times the frequency, and $i = \sqrt{-1}$. Further, on account of the choice of positive directions shown in Fig. 2, $I_K = -I_H$. Accordingly, replacing I_K by $-I_H$ and substituting the resulting values of V_H and V_K into equation (1) gives:

$$V_G = (Z_{HH} + Z_{KK} - 2Z_{HK}) I_H, \tag{1c}$$

where $Z_{HK} = i\omega L_{HK} = i\omega L_{KH} = Z_{KH}$, $Z_{HH} = R_H + i\omega L_H$, etc. It will be observed that while the "current voltages" have been eliminated (through the self and mutual inductances and the currents), the "charge voltages" remain and play the role of "applied voltages."

For wire H (Fig. 2), the equation (1a), when written in the form

$$R_H I_H = W_H = V_H + U_H = V_H - i\omega(L_H - L_{HK}) I_H, \tag{2}$$

[7] The first subscript designates the "disturbed" wire, the second the "disturbing" wire ("inducing" wire).

and its "vector diagram" (Fig. 3) both show that, in the limiting case of a perfectly conducting wire ($R_H = 0$), V_H and U_H would exactly balance each other, their values being exactly equal and opposite;

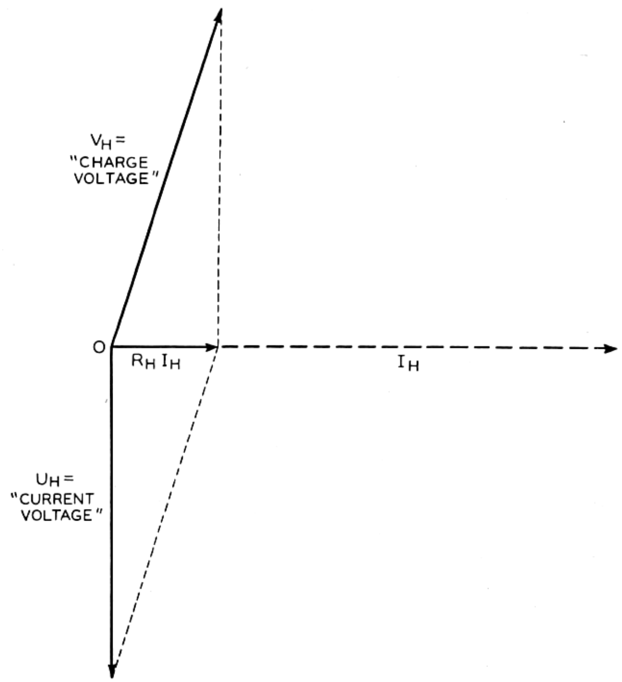

Fig. 3—Vector diagram relating to wire H of Fig. 2.

and that, in the case of an actual wire of low resistance, V_H and U_H nearly balance each other, their values being nearly equal and nearly opposite, so that their resultant $W_H = R_H I_H$ is a small residual voltage, although V_H and U_H individually may be very large compared with W_H.

The Mutual and Self Impedances

The mutual impedance (and similarly the inductive part of each self impedance) of the wires A and B (Fig. 1) cannot be defined as the negative of the voltage induced in either by unit current in the other, because the induced voltage necessarily has unequal values along the various filaments of which the disturbed wire may be regarded as composed. Thus in general this definition, which might be called the elementary definition, is applicable only to the individual filaments composing the wires, or to wires which themselves are fine enough to be regarded as filamentary.[8]

The self and mutual impedances of the wires could be formulated in terms of the self and mutual impedances of their filaments by eliminating the filamentary currents. However, without such elimination it is possible to obtain a type of formulation which is simpler, more compact, and in many ways more enlightening, as will now be shown by aid of the foregoing section:—Considering, in Fig. 1, any filament p of wire A, let R_p denote the resistance of that filament and I_p the current through it; then, by Ohm's law, the product R_pI_p is equal to the total voltage W_p along p. But W_p is the resultant of the "current voltage" U_p due to all of the wire currents, and the "charge voltage" V_p due to all of the charges; however, V_p is equal to V_A, the "charge voltage" along wire A as a whole, since the "charge voltages" along all of the various filaments in a wire must be equal. These various facts are expressed by the equation [9]

$$R_pI_p = W_p = U_p + V_p = U_p + V_A. \tag{3}$$

But, from the definitions of the filamentary self and mutual impedances,

$$U_p = -Z_pI_p - \sum_{q \neq p} Z_{pq}I_q - \sum_{\phi} Z_{p\phi}I_\phi, \tag{4}$$

where $Z_p = i\omega L_p$ denotes the inductive part of the self impedance $Z_{pp} = R_p + i\omega L_p$ of filament p, $Z_{pq} = i\omega L_{pq}$ the mutual impedance[7] between p and any other filament q of A, and $Z_{p\phi} = i\omega L_{p\phi}$ that between

[8] The case where one wire is "thick" and the other filamentary is on the border line, the elementary definition of the mutual impedance being applicable when the "thick" wire is the disturbing wire but not when it is the disturbed wire.

The generalized definition, to be formulated later herein, must of course be such that the mutual impedance between any two wires will have exactly equal values in the two directions.

[9] The distribution of V over the cross-section of the wire being uniform, equation (3) shows that if U is non-uniform I also must be, and conversely. This is exemplified in skin effect and proximity effect.

By averaging the whole set of equations, of which (3) is typical, relating to all of the filaments in wire A, and denoting the total current in this wire by I and its *direct current* resistance by R,^{0} we find that

$$R^0I = \overline{W} = \overline{U} + \overline{V} = \overline{U} + V,$$

a bar indicating an average value over the cross-section. The relation $R^0I = \overline{W}$ appears sufficiently useful and interesting to justify its enunciation in the form of a theorem, as follows: *When the varying current in a single piece of uniform wire, which may have any cross-sectional shape, has sensibly the same total value I throughout the length of the wire, whose direct current resistance is R^0, the product R^0I is equal to the cross-sectional average $\overline{W}$ of the total, or resultant, voltage W along the wire between its two ends.* For a wire which is fine enough to be regarded as filamentary, the above equation reduces to $RI = W = U + V$. For a wire carrying direct current, it reduces to $R^0I = \overline{V} = V$.

filament p of A and filament ϕ of B. On substituting (4) into (3) we get for filament p the "voltage equation:"

$$V_p = \sum_q Z_{pq} I_q + \sum_\phi Z_{p\phi} I_\phi = V_A. \tag{5}$$

Next we multiply this equation through by I_p, add together the m such resulting equations corresponding respectively to the m filaments of wire A, and introduce the condition that the sum of the filamentary currents in A is equal to I_A. Finally, we divide the resulting equation through by I_A and denote the current-ratios I_p/I_A, I_q/I_A, I_ϕ/I_B by J_p, J_q, J_ϕ respectively, each J thus denoting the ratio of a filament current to the total current in the wire to which that filament belongs, so that J may be called a "relative filamentary current." We thus get the equation

$$V_A = I_A \sum_p \sum_q Z_{pq} J_p J_q + I_B \sum_p \sum_\phi Z_{p\phi} J_p J_\phi. \tag{6}$$

Comparison of this with the equation

$$V_A = Z_{AA} I_A + Z_{AB} I_B, \tag{6a}$$

which is the "voltage equation" for wire A as a whole, yields the following formulas for the self impedance Z_{AA} of wire A and the mutual impedance Z_{AB} to A from B (Fig. 1):

$$Z_{AA} = \sum_p \sum_q Z_{pq} J_p J_q, \quad (7) \qquad Z_{AB} = \sum_p \sum_\phi Z_{p\phi} J_p J_\phi. \quad (8)$$

Similarly, for wire B,

$$Z_{BB} = \sum_\phi \sum_\theta Z_{\phi\theta} J_\phi J_\theta, \quad (9) \qquad Z_{BA} = \sum_\phi \sum_p Z_{\phi p} J_\phi J_p, \quad (10)$$

Z_{BA} denoting the mutual impedance to B from A. It will be recalled that p, q designate any two typical filaments of wire A, and ϕ, θ any two of B.

The presence of the relative filamentary currents (the J's) in these equations accounts for the fact that the self impedance of a wire depends on the current-distribution over its cross-section, and the mutual impedance between two wires on the current-distributions over their cross-sections. The self and mutual impedances of two wires, such as A and B, must thus depend on the currents in any other wires that may be present, because the voltages induced in A and B by these other currents will partly determine the current-distributions in A and B. Although the *values* of the summation expressions in equations (7) to (10) depend on the currents in any other wires that may be present, nevertheless the *forms* of these expressions do not. Thus, so far as the *forms* of the expressions are concerned, the two wires A and B need not be alone but may be any two of a system of parallel wires $A, \cdots, B, \cdots, D$ carrying arbitrary currents $I_A, \cdots, I_B, \cdots, I_D$ respectively; still further, A and B may even be any two of the parallel longitudinal parts of which any wire may arbitrarily be regarded as composed.

Equations (8) and (7) respectively show that the mutual and self impedances of "thick" wires have the following significance:

The mutual impedance between two wires is equal to the sum of the weighted mutual impedances from every filament in one wire to every filament in the other, the weighting factor of any filamentary mutual impedance being the product of the corresponding two relative filamentary currents (the J's).[10]

The self impedance of a wire is equal to the sum of the weighted mutual impedances from every filament to every other filament, including the weighted mutual impedance from every filament to itself, the weighting factor of any filamentary mutual impedance being the product of the corresponding two relative filamentary currents (the J's).[11]

Or, more briefly, *the self impedance of a wire is equal to the sum of the weighted mutual impedances from every filament to every other filament and to itself.*

Several matters of interest regarding the "thick" wires A and B (Fig.1) will next be discussed, mainly from the physical viewpoint corresponding to equations (7) to (10).

Reciprocity of the Two Mutual Impedances

Since $Z_{p\phi}$ and $Z_{\phi p}$ are unquestionably equal, because they relate to filaments, comparison of formulas (8) and (10) shows that the mutual impedances Z_{AB} and Z_{BA} between the wires A and B are equal. The same conclusion follows also from the first italicised paragraph above, which is based on formulas (8) and (10).

Complex Nature of the Mutual and Self Impedances

Although every mutual impedance between different filaments is a pure reactance which is directly proportional to the frequency, nevertheless the mutual impedance Z_{AB} between the wires A and B has in

[10] In other words, the mutual impedance of two wires is equal to the sum of the weighted mutual impedances between all of the various filaments taken in pairs each pair consisting of one filament from each wire.

[11] In other words, the self impedance of a wire is equal to twice the sum of the weighted mutual impedances between all of the various filaments taken in pairs, plus the sum of the weighted self impedances of the filaments. (The weighting factor of the self impedance of any filament is evidently the square of its relative filamentary current.)

general not only a reactance component which is not quite proportional to the frequency, but also a resistance component which does not vary in any simple way with the frequency. On the basis of formula (8) these facts are to be accounted for by the consideration that in general the various filamentary currents in a wire are not only not in phase but have no simple phase relations. Thus if (8) is written in the form

$$Z_{AB} = i\omega\sum_{p}\sum_{\phi} L_{p\phi}J_pJ_\phi = R_{AB} + i\omega L_{AB}, \tag{11}$$

then R_{AB} is not zero, and R_{AB} and L_{AB} vary with ω although $L_{p\phi}$ does not.

That the self impedance Z_{AA} of the wire A is not a pure reactance can be accounted for similarly, with the additional reason that the self impedance of each filament is complex, because of its resistance. Thus if (7) is written in the form

$$Z_{AA} = \sum_{p}(R_p + i\omega L_p)J_p^2 + i\omega\sum_{p}\sum_{q\neq p} L_{pq}J_pJ_q = R_A + i\omega L_A, \tag{12}$$

then R_A is not zero, and R_A and L_A vary with ω although R_p, L_p, L_{pq} do not.

It may be noted that in the idealized case of perfect conductivity the mutual and self impedances of the wires would be pure reactances and directly proportional to the frequency; for in this case the filamentary currents in any wire would all be in phase and their distribution would be independent of the frequency. (The current distribution would be the same as the charge distribution and hence purely superficial.)

Case of Negligible Proximity Effect

The case here considered is that in which the wires A and B are of circular or of annular cross-section (but external to each other) and are far enough apart so that the proximity effect [1] is negligible and so that therefore the current distribution over the cross-section of each wire is sensibly axially symmetrical.

For this particular case the mutual impedance $Z_{AB} = Z_{BA}$ is *not* complex but is pure reactance, being equal to the mutual impedance $Z_{A'B'} = Z_{B'A'}$ between two filamentary wires A' and B' having the same interaxial spacing as the given "thick" wires A and B. Although this statement is clearly true when only one of the wires is "thick," it really needs a proof in the general case where *both* are "thick." The following simple proof depends on the fact that everywhere (except near its ends) outside of a long straight wire, of circular or of annular section, carrying an axially symmetrical current the magnetic field produced by that current is the same as though the current were concentrated in the axis, and the proof also utilizes the reciprocity relation for the mutual impedances in the two directions between the two wires involved; thus,[7]

$$Z_{AB} = Z_{AB'} = Z_{B'A} = Z_{B'A'} = Z_{A'B'},$$
$$Z_{BA} = Z_{BA'} = Z_{A'B} = Z_{A'B'} = Z_{B'A'}.$$

The statement at the beginning of this paragraph is thus proved.

PART II

Mathematical Theory and Calculations

The theoretical investigation of the self and mutual impedances per unit length of two long parallel pairs of wires in space is an application

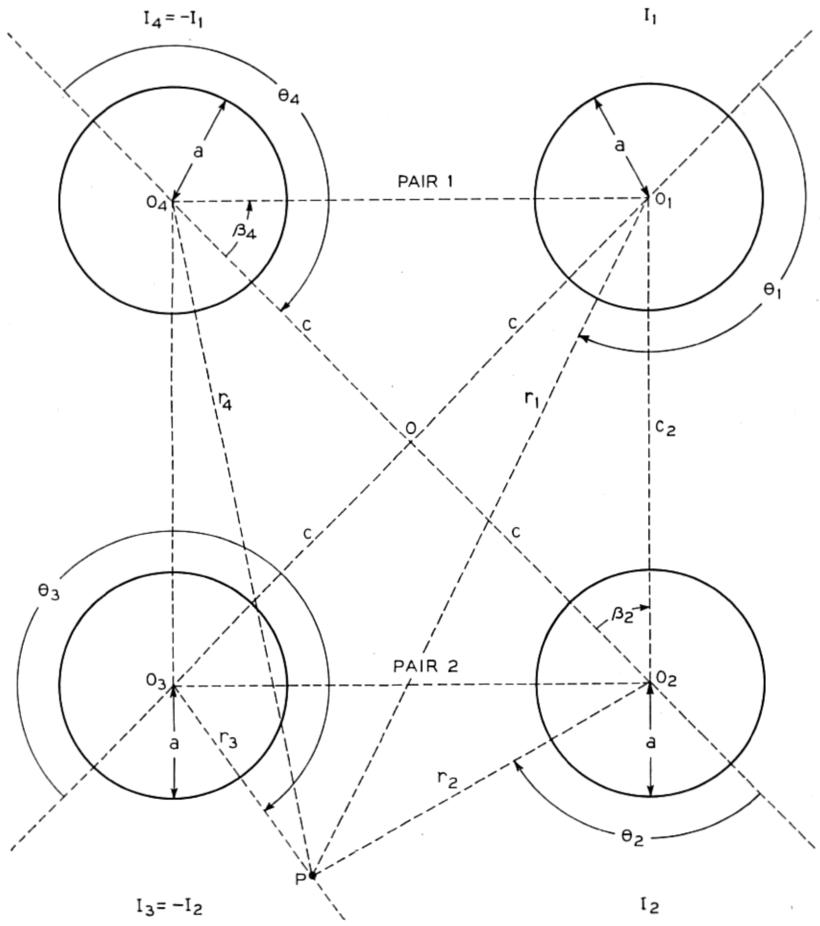

Fig. 4—Cross-sectional diagram of 4-wire system.

of two dimensional wave propagation theory. The specific case of four wires in a square array was selected as the basis of a comparison of measured and theoretical values of mutual inductance. The configuration with four equal wires is shown in cross section in Fig. 4 where wires No. 1 and No. 4, centered at O_1 and O_4 and carrying currents I_1 and $-I_1$, respectively, form the first pair or primary and wires No. 2 and No. 3 at O_2 and O_3 and carrying currents I_2 and $-I_2$, respectively,

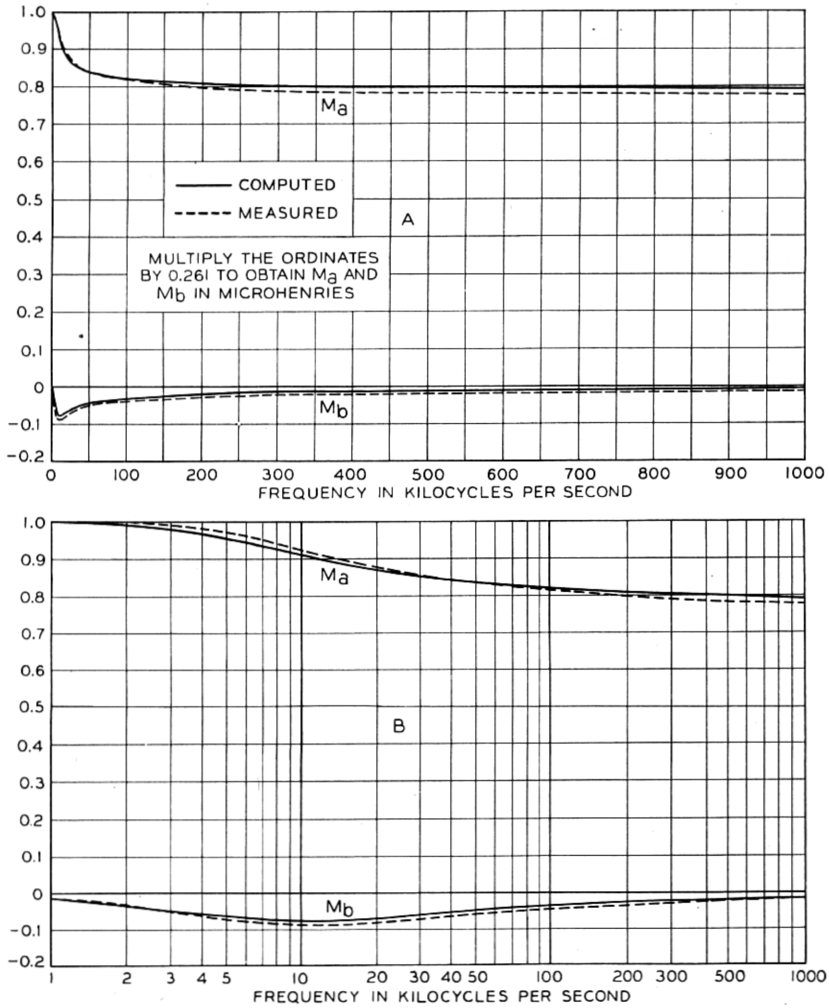

Fig. 5—Real and imaginary components, M_a and M_b, respectively, of the mutual inductance between a pair of No. 10 A.W.G. wires and a pair of filaments. Length of wires = 74 inches; interaxial separation = 0.14 inch.

form the second pair or secondary. We have

$$O_1O_3 = O_2O_4 = 2c$$

and

$$O_1O_2 = O_1O_4 = \sqrt{2}c.$$

The notation for the dimensions and coordinate systems is shown in Fig. 4. The theoretical values of mutual inductance are calculated from the geometry and electrical constants of this system by means of the formulas which will be derived herein, while the measured values are those obtained for this system by R. N. Hunter and R. P. Booth.[2]

Numerical Results

A close agreement between the values of mutual inductance computed on the basis of the approximate formulas derived below and the experimental results is shown by the curves in Figs. 5 and 6. In fact, for No. 18 gauge wires, in which case the proximity effect is comparatively small, the computed and measured values are indistinguishable in Figs. 6A and 6B. Evidently the error introduced by the fact that actually the line is comparatively short while theoretically we assume it of doubly infinite length, is inappreciable. The drawings give relative values of the real and imaginary components of the complex mutual inductance $M = M_a + iM_b$, for 74 inch lengths of wires with vertical and horizontal interaxial spacing of 0.14 inch over a frequency range of 1 to 1000 kilocycles per second. (The value 0.565×10^{-3} emu. is assumed for the conductivity of the wires and unit permeability for both wires and dielectric.) The solid curves represent computed values and the dotted curves measured values. The values shown are the ratios of M_a and M_b to the value of M_a at 1 kilocycle. In Figs. 5A and 6A the frequency scale is linear, while in Figs. 5B and 6B it is logarithmic. The computed curves of Fig. 5 (obtained from formula (13) below) assume a pair of No. 10 A.W.G. wires (0.102 inch in diameter) as the primary and a filamentary secondary. Actually the secondary was a pair of No. 28 A.W.G. wires. In the two cases shown in Fig. 6, computed from formula (14) below, both pairs of conductors are of the same size; namely, No. 10 and No. 18 A.W.G. wires, respectively (the latter being actually 0.0410 inch in diameter).

It will be observed that we have the relation

$$M = Z_m/i\omega,$$

Z_m denoting the mutual impedance, $\omega/2\pi$ the frequency and i the

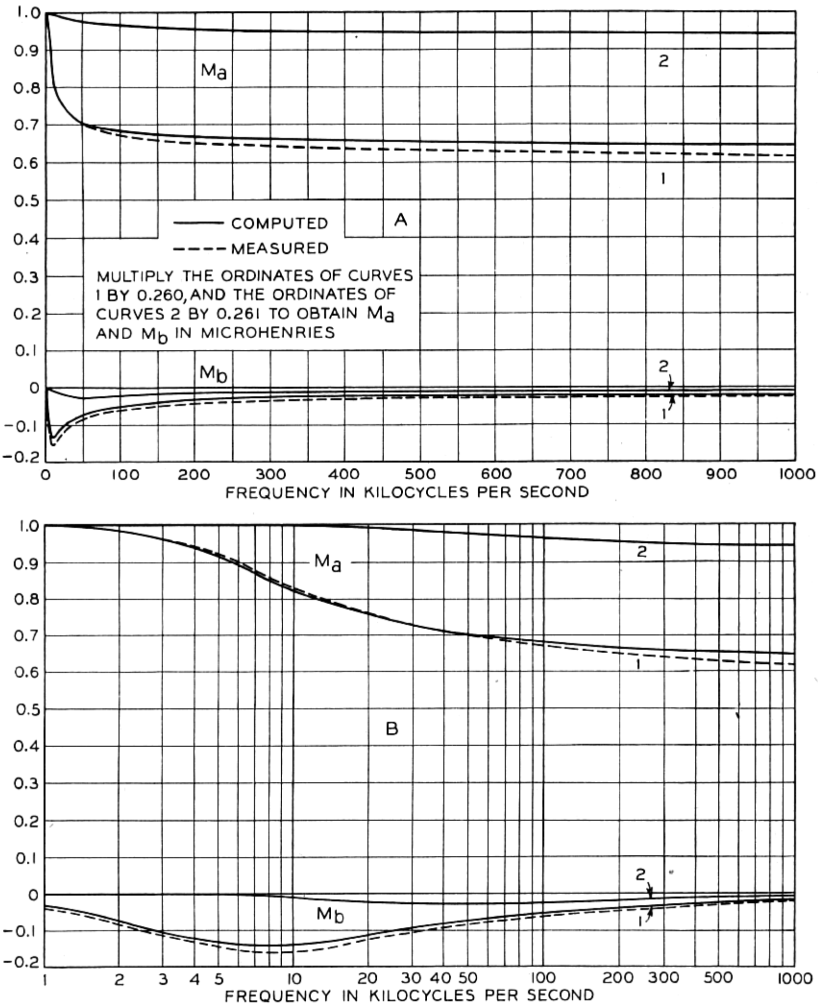

Fig. 6—Real and imaginary components, M_a and M_b, respectively, of the mutual inductance between (1) two pairs of No. 10 A.W.G. wires and (2) two pairs of No. 18 A.W.G. wires. Length of wires = 74 inches; interaxial separation = 0.14 inch.

imaginary. In the last step of the analysis below the mutual impedance of two circuits of wires of large [12] cross-section is derived by a method of successive approximations. A first approximation is obtained by assuming the field due to each wire of the second pair cir-

[12] The wires of a pair are to be considered "large" for a given frequency (f), provided the values of the radius (a), interaxial separation (d), conductivity (σ) and permeability (μ_c) are such that the magnitude

$$|(2a/d)(J_1(z)/J_0(z))^2|$$

is not small compared to unity. Here J_0 and J_1 are Bessel functions of the first kind of zeroth and first orders, respectively, and of complex argument, $z = ia\sqrt{4\pi\sigma\mu_c i\omega}$.

cularly symmetrical. Physically this is equivalent to assuming the concentration of the current on the axes of the secondary as if it were filamentary so that the proximity effect in this pair is eliminated. Thus equation (13) formulates the solution of the case which is represented in Fig. 5. Regarded as an approximation to the solution when both pairs of wires are of large cross-section, it will be seen that these values account for about 50 per cent of the departure of the final results from the d.c. value. (This is 0.261 microhenry for the square arrangement.) The second approximation (formula (14)) takes into account the circularly unsymmetrical components of the field due to the unsymmetrical distribution of current density in the wires of the secondary as well as of the primary and so adds the proximity effect due to the thickness of the secondary. A summary of the formulas for mutual inductance follows:

Formulas

With the notation

$\lambda = a/2c$

$a =$ radius of wires in centimeters

$2c =$ diagonal interaxial separation of wires in centimeters

$\sigma =$ conductivity of wires in emu.

$f = \omega/2\pi =$ frequency

and denoting by $M^{(0)}$ the complex mutual inductance per unit length of two circuits, one of wires of large cross-section and one filamentary or, from the other point of view, a first approximation to the mutual inductance of two circuits of wires of large cross-section and, by M, a second approximation to the latter, we have

$$M^{(0)} = 4(\log_e \sqrt{2} - k_1), \tag{13}$$

$$M = M^{(0)} - 4(k_1 - 7k_1^2 + 4k_2), \tag{14}$$

where

$$k_1 = \frac{\lambda^2\tau_1}{1 - 2\lambda^2\tau_1},$$

$$k_2 = \frac{\lambda^4\tau_2}{1 - 12\lambda^4\tau_2},$$

$$\tau_1 = 1 + i\frac{2}{x}\frac{\operatorname{ber}' x + i \operatorname{bei}' x}{\operatorname{ber} x + i \operatorname{bei} x},$$

$$\to 1 - \nu + i\nu \text{ when } x \to \infty,$$

$$\tau_2 = 1 - i\frac{8}{x^2} - \frac{4}{x}\frac{\operatorname{ber} x + i \operatorname{bei} x}{\operatorname{ber}' x + i \operatorname{bei}' x},$$

$$\to 1 - 2\nu + i2\nu \text{ when } x \to \infty,$$

$$x = a\sqrt{4\pi\sigma\omega} = \sqrt{2}/\nu,$$

$$\nu = 1/(2\pi a\sqrt{f\sigma}).$$

For values of ber x, bei x, ber′ x and bei′ x see Jahnke u. Emde, "Funktionentafeln." $M^{(0)}$ and M are given above in emu. per centimeter. We assume $M^{(0)}$ and M proportional to the length and multiply by 0.1880 to obtain michrohenries per 74 inches.

No assumptions with respect to frequency are made in formulas (13) and (14) but terms of the order of magnitude with respect to unity of $9k_1^3 \approx 9\lambda^6$ or smaller are neglected. That is, the accuracy of (13) and (14) is limited by the dimensions rather than by the frequency. But for frequencies of about 100 kilocycles or higher and not too small wires (that is, when $x \geqq$ about 10) formulas (13) and (14) may be expressed in interpretable form; namely,

$$M^{(0)} = M_a^{(0)} + iM_b^{(0)} \tag{15}$$

and

$$M = M_a + iM_b, \tag{16}$$

where

$$M_a^{(0)} = 4\left(\log_e \sqrt{2} - \frac{\lambda^2}{1 - 2\lambda^2} + \frac{\lambda^2}{1 - 4\lambda^2}\nu\right),$$

$$M_b^{(0)} = -4\frac{\lambda^2}{1 - 4\lambda^2}\nu,$$

$$M_a = M_a^{(0)} - 4\left(\frac{\lambda^2}{1 - 2\lambda^2} - 3\lambda^4\right) + 4\nu\left(\frac{\lambda^2}{1 - 4\lambda^2} - 6\lambda^4\right),$$

$$M_b = M_b^{(0)} - 4\nu\left(\frac{\lambda^2}{1 - 4\lambda^2} - 6\lambda^4\right).$$

The asymptotic values (when f or σ or both approach infinity) are, therefore,

$$M^{(0)} = 4\left(\log_e \sqrt{2} - \frac{\lambda^2}{1 - 2\lambda^2}\right), \tag{17}$$

$$M = M^{(0)} - 4\left(\frac{\lambda^2}{1 - 2\lambda^2} - 3\lambda^4\right) \tag{18}$$

and the d.c. value (when f approaches zero) is, of course,

$$M = 4 \log_e \sqrt{2}.$$

Thus M is real (i.e., $M_b = 0$) when the frequency is either zero or infinite.

Formal Solution

The following derivation of these results is an application of the general method of calculating the self and mutual impedances in a system of parallel wires which is outlined in Section V of John R. Carson's paper "Rigorous and Approximate Theories of Wave Transmission along Wires," *B. S. T. J.*, Jan., 1928. This method of solution has proved valuable in investigating a variety of problems in wave propagation along parallel conductors. Reference may be made to the paper itself for a fuller exposition of the underlying theory, much of which is omitted from the present analysis.

In terms of a vector potential A and a scalar potential V, the electric and magnetic forces E and H per unit length in the dielectric are given by the relations,

$$E = -\text{ grad } V - i\omega A$$

and (19)

$$\mu H = \text{curl } A,$$

where μ is the permeability of the dielectric. Assuming that the wave varies as exp $(i\omega t - \gamma z)$ and putting $F = i\omega A_z$, the axial electric force E_z (omitting the subscript z) may be written

$$E = \gamma V - F, \tag{20}$$

where γ is the propagation constant per unit length of the system.

Now, as we shall show below, the electric force E inside of the conductors and the wave function F in the dielectric may be expressed as linear functions of the conductor currents. That is, at the surface of the jth conductor, for example, we may write

$$E_j = e_{1j}I_1 + e_{2j}I_2 + e_{3j}I_3 + e_{4j}I_4$$

and (21)

$$F_j = f_{1j}I_1 + f_{2j}I_2 + f_{3j}I_3 + f_{4j}I_4,$$

where e_{jk} and f_{jk} are determined by the geometry and electrical constants of the system. (V_j, the potential at the surface of the jth conductor, is determinable from the geometry of the conductors as a linear function of the conductor *charges*, Q_1, Q_2, $\cdots$; that is, V_j may be written

$$V_j = p_{1j}Q_1 + p_{2j}Q_2 + p_{3j}Q_3 + p_{4j}Q_4,$$

the p coefficients being the Maxwell potential coefficients of the system. These, however, are not required in the present problem.)

But relation (20) must hold at the surface of the conductors. Thus, since the electric force is continuous at the surface of the conductors, relations (20) and (21) give

$$\begin{aligned} Z_{11}I_1 + Z_{21}I_2 + Z_{31}I_3 + Z_{41}I_4 &= \gamma V_1 = E_1 + F, \quad r_1 = a, \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \\ Z_{14}I_1 + Z_{24}I_2 + Z_{34}I_3 + Z_{44}I_4 &= \gamma V_4 = E_4 + F, \quad r_4 = a, \end{aligned} \tag{22}$$

where

$$Z_{jk} = e_{jk} + f_{jk} = Z_{kj},$$

and

$$Z_{jj} = Z_{kk},$$

the Z coefficients being the self and mutual impedances of the individual conductors. The required self and mutual impedances, Z_s and Z_m, respectively, however, are the impedances of the circuits 1–4 and 2–3. Owing to the relations, $I_1 = -I_4$ and $I_2 = -I_3$, of the currents, Z_s and Z_m are given by

$$Z_s = 2(Z_{11} - Z_{41})$$

and

$$Z_m = 2(Z_{21} - Z_{31}). \tag{23}$$

Thus, from equations (22) and (23), we have

$$Z_sI_1 + Z_mI_2 = \gamma(V_1 - V_4) = 2(E_1 + F), \quad r_1 = a. \tag{24}$$

The problem is then reduced to the determination of E and F in terms of I_1 and I_2.

The function F must satisfy Laplace's equation in two dimensions and may be resolved into four waves centered respectively on the axes of the four wires, each satisfying Laplace's equation. Thus, at any point (r_j, θ_j) in the dielectric, F may be written

$$F = F_1 + F_2 + F_3 + F_4, \tag{25}$$

where

$$F_j = A_{0j} \log r_j + \sum_{n=1}^{\infty} \left(A_{nj} \frac{\cos n\theta_j}{r_j^n} + B_{nj} \frac{\sin n\theta_j}{r_j^n} \right), \qquad j = 1, 2, 3, 4.$$

The arbitrary constants A_{0j} are determined by the relations

$$4\pi\mu i\omega I_j = -\int_0^{2\pi} \left(\frac{\partial F}{\partial r_j} \right)_{r_j=a} a d\theta_j. \tag{26}$$

But owing to the specific configuration and to the conditions

$$I_4 = -I_1 \quad \text{and} \quad I_3 = -I_2, \tag{27}$$

the $8n$ arbitrary constants A_{nj}, B_{nj} may be reduced to $4n$. Thus, we have

$$\begin{aligned} A_{n1} &= -A_{n4} = A_n, & B_{n1} &= B_{n4} = B_n, \\ A_{n2} &= -A_{n3} = C_n, & B_{n2} &= B_{n3} = D_n, \end{aligned} \tag{28}$$

and also

$$A_{01} = -A_{04} = -2\mu i\omega I_1, \qquad A_{02} = -A_{03} = -2\mu i\omega I_2.$$

Inside of the conductors the axial electric force E_j must satisfy the wave equation in two dimensions. It may, therefore, be expressed as the Fourier-Bessel series,

$$E_j = g_{0j}J_o(r_jz/a) + \sum_{n=1}^{\infty} J_n(r_jz/a)(g_{nj}\cos n\theta_j + h_{nj}\sin n\theta_j). \quad (29)$$

The constants g_{0j} are given by the relations

$$4\pi\mu_c i\omega I_j = \int_0^{2\pi}\left(\frac{\partial E_j}{\partial r_j}\right)_{r_j=a} a\,d\theta_j,$$

or

$$g_{0j} = Z_jI_j\frac{J_o(r_jz/a)}{J_o(z)}, \quad (30)$$

Z_j being the internal impedance per unit length of the jth conductor with concentric return. Here μ_c is the permeability of the conductor, $J_n(z)$ is the Bessel function of the first kind of nth order and argument $z = ai\sqrt{4\pi\sigma\mu_c i\omega}$ and the arbitrary constants g_{nj} and h_{nj} are to be determined by boundary conditions; it is evident, however, that we must have

$$\begin{aligned} g_{n1} &= -g_{n4}, & h_{n1} &= h_{n4}, \\ g_{n2} &= -g_{n3}, & h_{n2} &= h_{n3}. \end{aligned} \quad (31)$$

At the surfaces $r_j = a$, the boundary relations are

$$\frac{\partial F}{\partial r_j} = -\frac{\mu}{\mu_c}\frac{\partial E}{\partial r_j}$$

and

$$\frac{\partial F}{\partial \theta_j} = -\frac{\partial E}{\partial \theta_j}. \quad (32)$$

Hence, introducing (25) and (29) in (32), applying (32) at the two surfaces $r_1 = a$ and $r_2 = a$ and equating harmonic coefficients, gives $8n$ equations in the $8n$ arbitrary constants A_n, B_n, C_n, D_n, g_{n1}, g_{n2}, h_{n1}, h_{n2}. This procedure requires that F be expressed in terms of r_1, θ_1 and of r_2, θ_2 by suitable transformations of coordinates.[13] Thus, for all points in the neighborhood of $r_1 = a$, for example, F may be written

$$\begin{aligned} F = {} & 2\mu i\omega I_1 \log\frac{c_2}{r_1} + 2\mu i\omega I_2\log\frac{2c}{c_2} - \sum_{n=1}^{\infty}\left(-\frac{1}{2c}\right)^n C_n \\ & - \sum_{n=1}^{\infty}\left(-\frac{1}{c_2}\right)^n\left[\left(\cos\frac{n\pi}{4}\right)(A_n - C_n) + \left(\sin\frac{n\pi}{4}\right)(B_n - D_n)\right] \\ & + \sum_{n=1}^{\infty}(\cos n\theta_1)(A_n' r_1^n + B_n' r_1^{-n}) + \sum_{n=1}^{\infty}(\sin n\theta_1)(C_n' r_1^n + D_n' r_1^{-n}), \end{aligned} \quad (33)$$

[13] The necessary formulas for these transformations are derived in Note II of the paper "Transmission Characteristics of the Submarine Cable" by John R. Carson and J. J. Gilbert, *Journal Franklin Institute*, December, 1921.

where A_n', B_n', C_n' and D_n' are expressible in terms of A_n, B_n, C_n and D_n and of the currents, electrical constants and dimensions of the system. In the neighborhood of $r_2 = a$, F is given by a similar expression in the coordinates r_2, θ_2. The application of the boundary relations at $r_1 = a$ and $r_2 = a$ then, as explained above, leads to a set of equations which determine the arbitrary constants in terms of the currents, electrical constants and dimensions of the system. When these equations are solved and the arbitrary constants are known, equation (24) becomes

$$Z_sI_1 + Z_mI_2 = 2\left(Z_1 + 2\mu i\omega \log \frac{c_2}{a} + \Delta_s \right) I_1 + 2\left(2\mu i\omega \log \frac{2c}{c_2} + \Delta_m \right) I_2, \quad (34)$$

where

$$\Delta_sI_1 + \Delta_mI_2 = -\sum_{n=1}^{\infty}\left(-\frac{1}{2c}\right)^n C_n - \sum_{n=1}^{\infty}\left(-\frac{1}{c_2}\right)^n \left[\left(\cos\frac{n\pi}{4}\right)(A_n - C_n) + \left(\sin\frac{n\pi}{4}\right)(B_n - D_n)\right].$$

The formal solution is then complete, $\Delta_sI_1 + \Delta_mI_2$ representing the correction in the series voltage drop of the primary circuit due to the proximity effect.

Solution by Successive Approximations

As the set of simultaneous equations, upon which depends the determination of the arbitrary constants A_n, B_n, C_n and D_n, involves an infinite number of unknowns, a direct solution is, in general, impossible. Consequently, some method of successive approximation is required. The convergence of the harmonic sequences indicates the practicability of the following procedure in the present problem.

(1) Determine first approximations $A_n^{(0)}$ and $B_n^{(0)}$ by boundary conditions at $r_1 = a$, neglecting the summations in C_n and D_n. For the first approximation only $A_1^{(0)}$ and $B_1^{(0)}$ will be required and the series may be represented by their leading terms.

(2) Determine $C_n^{(1)}$ and $D_n^{(1)}$ in terms of $A_n^{(0)}$ and $B_n^{(0)}$ by conditions at $r_2 = a$.

(3) Determine $A_n^{(1)}$ and $B_n^{(1)}$ in terms of $C_n^{(1)}$ and $D_n^{(1)}$ by conditions at $r_1 = a$.

. . . .

Then, we have, for example,

$$\sum_{n=1}^{\infty}\left(-\frac{1}{2c}\right)^n C_n = \sum_{n=1}^{\infty}\left(-\frac{1}{2c}\right)^n (C_n^{(0)} + C_n^{(1)} + C_n^{(2)} + \cdots) \quad (35)$$

and similar expressions for the other summations. Now, putting $C_n^{(0)} = D_n^{(0)} = 0$, the first approximation to the proximity effect is given by

$$\Delta_s^{(0)} I_1 + \Delta_m^{(0)} I_2 = \frac{A_1^{(0)}}{2c} + \frac{B_1^{(0)}}{2c}. \quad (36)$$

Next, since, for example, $A_2^{(0)}/(2c)^2$ is of the same order of magnitude as $A_1^{(1)}/(2c)$, the increment due to $C_n^{(1)}$ and $D_n^{(1)}$ will be

$$\Delta_s^{(1)} I_1 + \Delta_m^{(1)} I_2 = \frac{A_1^{(1)}}{2c} + \frac{B_1^{(1)}}{2c} - \frac{D_1^{(1)}}{2c} - 2\frac{B_2^{(0)}}{(2c)^2} - \frac{C_2^{(1)}}{(2c)^2} + 2\frac{D_2^{(1)}}{(2c)^2}. \quad (37)$$

Then a second approximation to $\Delta_s I_1 + \Delta_m I_2$ will be

$$(\Delta_s^{(0)} + \Delta_s^{(1)}) I_1 + (\Delta_m^{(0)} + \Delta_m^{(1)}) I_2$$

and, in general,

$$\Delta_s I_1 + \Delta_m I_2 = \sum_{n=0}^{\infty} (\Delta_s^{(n)} I_1 + \Delta_m^{(n)} I_2). \quad (38)$$

Applying this method we assume unit permeability for wires and dielectric. Then putting the first approximation in equation (34) gives equation (13) above for $M^{(0)}$. Neglecting terms containing λ^6, we find $C_1^{(2)}/2c$ and $D_1^{(2)}/2c$ ignorable. In $B_2^{(0)}/(2c)^2$, $C_2^{(1)}/(2c)^2$ and $D_2^{(1)}/(2c)^2$ we require the first terms. We then have

$$\begin{aligned}\Delta_s^{(1)} I_1 + \Delta_m^{(1)} I_2 = &- 2i\omega I_1(k_1 - 6k_1^2 + \cdots + \tfrac{9}{2}k_2 + \cdots) \\ &- 2i\omega I_2(k_1 - 7k_1^2 + \cdots + 4k_2 + \cdots).\end{aligned} \quad (39)$$

Hence

$$Z_m = 4i\omega(\log\sqrt{2} - 2k_1 + 7k_1^2 - \cdots - 4k_2 + \cdots) \quad (40)$$

and

$$Z_s = 2Z_1 + 4i\omega\left(\log\frac{\sqrt{2}c}{a} - 3k_1 + 6k_1^2 - \cdots - \frac{9}{2}k_2 + \cdots\right), \quad (41)$$

where

$$k_1 = \frac{\lambda^2 \tau_1}{1 - 2\lambda^2 \tau_1},$$

$$k_2 = \frac{\lambda^4 \tau_2}{1 - 12\lambda^4 \tau_2}$$

and

$$\tau_n = 1 - \frac{2n}{z} \frac{J_n(z)}{J_{n-1}(z)}.$$

The relations

$$J_o(z) = \text{ber } x + i \text{ bei } x$$

and

$$J_1(z) = \frac{1}{\sqrt{-i}} (\text{ber}' \, x + i \text{ bei}' \, x),$$

where

$$z = x\sqrt{-i}$$

give the expressions in equations (13) and (14) for τ_1 and τ_2.

For the asymptotic values we have

$$\frac{J_n(z)}{J_{n-1}(z)} \to -i, \text{ when } x \to \infty,$$

so that

$$\tau_n \to 1 + i \frac{2n}{z}$$

or

$$\tau_1 \to 1 - \nu + i\nu$$

and

$$\tau_2 \to 1 - 2\nu + i2\nu,$$

where

$$\nu = \sqrt{2}/x = 1/(2\pi a \sqrt{f\sigma}).$$

Also,

$$k_1 \to \frac{\lambda^2}{1 - 2\lambda^2}\left(1 - \frac{\nu}{1 - 2\lambda^2}\right) + i \frac{\lambda^2 \nu}{(1 - 2\lambda^2)^2},$$

$$k_1^2 \to \frac{\lambda^4}{(1 - 2\lambda^2)^2}\left(1 - \frac{2\nu}{1 - 2\lambda^2}\right) + i \frac{2\lambda^4 \nu}{(1 - 2\lambda^2)^3},$$

and

$$k_2 \to \frac{\lambda^4}{1 - 12\lambda^4}\left(1 - \frac{2\nu}{1 - 12\lambda^4}\right) + i \frac{2\lambda^4 \nu}{(1 - 12\lambda^4)^2}.$$

Thus equations (15) and (16) readily follow.

In addition, the high frequency value of the self *impedance*, Z_s, is given by

$$Z_s^{(0)} = 2Z_1\left(1 + \frac{4\lambda^2}{1 - 4\lambda^2}\right) + 4i\omega\left(\log \frac{\sqrt{2}c}{a} - \frac{2\lambda^2}{1 - 2\lambda^2} + \frac{2\lambda^2 \nu}{1 - 4\lambda^2}\right)$$

and

$$Z_s = Z_s^{(0)} + 2Z_1\left(\frac{2\lambda^2}{1-4\lambda^2} - 6\lambda^4\right)$$
$$- 4i\omega\left(\frac{\lambda^2}{1-2\lambda^2} - \frac{3}{2}\lambda^4 - \nu\left(\frac{\lambda^2}{1-4\lambda^2} - 3\lambda^4\right)\right),$$

where

$$Z_1 \to \frac{1}{a}\sqrt{\frac{f}{\sigma}}(1+i) = \omega\nu(1+i).$$

APPENDIX *

Production and Properties of Electric Field Intensities and Voltages

This appendix gives a summary of certain points in fundamental electromagnetic theory which are necessary for a thorough understanding of some portions of Part I of the paper.

Precisely defined, "voltage" (W) means the line-integral of the electric field intensity (E) along a specified path (s) between two specified points.[14] Thus

$$W = \int_{(s)} E_s ds = \int_{(s)} E \cdot ds. \tag{1}$$

At any point, in a dielectric or in a conductor, the total electric field intensity E is the resultant of a part E_q due to all charges and a part E_u due to all currents; thus $E = E_q + E_u$. (E_q and E_u might be called the "charge electric intensity" and the "current electric intensity" respectively.)

Precisely stated, the phrases "due to all charges" and "due to all currents" have the same meanings respectively as in the formulations of the "retarded scalar potential" Ψ and the "retarded vector potential" A of electromagnetic theory, as summarized in the following paragraphs. "All charges" and "all currents," respectively, include polarization charges and polarization currents in a dielectric, thus allowing (indirectly) for a specific inductive capacity of any specified value. Furthermore, "all currents" include also such additional currents (current whirls) as would account for a magnetic permeability of any specified value. On the other hand, displacement currents are *not* included and should not be, for they do not play the role of true

* This appendix relates to Part I.

[14] The "electric field intensity" (or, briefly, "electric intensity") is often called the "electric force."

physical "causes" when "retardation" is allowed for in the formulation of the effects.[15]

It is of course possible to give, in a single step, formulas for E_q and E_u in terms explicitly of the charges and currents to which they are respectively due. However, it is much preferable, both mathematically and physically, to proceed in two steps, of which the first consists in giving the formulas for the two potential functions, Ψ and A, and the second in giving the formulas expressing E_q and E_u in terms of Ψ and A respectively. For convenience these four formulas will now be given together. For completeness the formula for the magnetic field intensity H will be added, although it is of only secondary interest here and in Part I of this paper; further, the formula for the relation between Ψ and A will be included, since it underlies the formulas for Ψ and A. These six formulas, which are classical, follow. The functional notation $q(t - r/c)$, in formula (2), indicates that the charge-density q is to be evaluated at the time $t - r/c$, as discussed in the next paragraph; similarly for the current-density u in (3).

$$\Psi = \int \frac{q(t - r/c)}{r} dv, \qquad (2) \qquad\qquad A = \frac{1}{c}\int \frac{u(t - r/c)}{r} dv, \qquad (3)$$

$$E_q = -\text{grad}\,\Psi, \qquad (4) \qquad\qquad E_u = -\frac{1}{c}\frac{\partial A}{\partial t}, \qquad (5)$$

$$H = \text{curl}\,A, \qquad (6) \qquad\qquad \text{div}\,A + \frac{1}{c}\frac{\partial \Psi}{\partial t} = 0. \qquad (7)$$

Although usually the application of the first two of these formulas to specific cases is difficult and laborious, their physical meaning is rather simple, as will shortly appear in the following description and discussion of them.

The six formulas in the above set constitute a complete explicit solution of Maxwell's differential equations of the electromagnetic field, and form the connecting link between those differential equations and electric circuit theory. They express the potentials (Ψ, A), and thence the field intensities (E_q, E_u, H), at a specified point P and time t, due to all of the distributed charges and currents contemplated. The point P may be anywhere, in a dielectric or in a conductor; and the time t is that observed at P. dv is a *fixed* element of volume or of surface (as the case may be[16]) at any typical point in the contemplated

[15] For a mathematical treatment relating to the various matters touched on in this paragraph reference may be made to the appendix of the paper by John R. Carson cited at the end of footnote 5.

[16] For brevity the term "volume-element" will throughout be used generically to include "surface-element" as a limiting case, with "charge-density" being interpreted as "volume charge-density" and "surface charge-density" respectively.

system of charges and currents; r is the distance between dv and P; and c is the velocity of light in free space. $q(t - r/c)$ and $u(t - r/c)$ are the charge-density and the vector current-density, respectively, in dv, not at the time t but at the slightly earlier time $t - r/c$, allowance thus being made for the time of propagation of the effect from dv to P. Thus in (2) the integration, made at the time t, which is that observed at P, must include every volume-element dv which contained any charges at the time $t - r/c$, whatever the motions of those charges; and in (3) the integration must include every volume-element dv which contained any current (moving charges) at the time $t - r/c$; moreover, associated with each volume-element dv is a corresponding value of r.

r denoting distance, Ψ and A are called "potentials" because of their inverse dependence on r and their direct dependence on the charge-density q and the current-density u respectively. Ψ is called the "scalar potential" because it does not have direction in space; A the "vector potential" because it has direction. These potentials are qualified as being "retarded" potentials[17] because the values to be taken for the charge-elements and current-elements are not their actual values at the contemplated instant t but their "retarded" values, that is, their values at the earlier instants $t - r/c$. (It is to be remembered that the time t is that observed at the point P where Ψ and A are to be calculated.)

In the way of a summary statement regarding the set of formulas (2) to (7), we may say that electric charges, whether stationary or moving, produce a scalar potential Ψ calculable from (2), and thence an electric field intensity E_q calculable from (4); and that if the charges are in motion, thus constituting currents, they produce also a vector potential A calculable from (3), and thence an additional electric field intensity E_u calculable from (5) and a magnetic field intensity H calculable from (6). Thus the total, or resultant, electric field intensity $E = E_q + E_u$ is calculable from

$$E = -\text{grad } \Psi - \frac{1}{c}\frac{\partial A}{\partial t}. \tag{8}$$

If the contemplated point P for which E is calculated is in a conductor, of resistivity ρ, where the current-density is u', there exists the additional relation $E = \rho u'$, in accordance with Ohm's law.

Of the important contrasting principles enunciated in the section entitled "The Two Parts of a Voltage, and Their Resultant," principle "1" is an immediate consequence of equation (4) of this appendix, and "2" is a consequence of (5) and (6) together.

[17] Sometimes called "propagated" potentials.