Ferromagnetic Distortion of a Two-Frequency Wave

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Frequency components are found for the ferromagnetic induction produced by a small magnetizing force of two incommensurable frequencies. Because of hysteresis the results depend intimately upon the ratios of these frequencies and of their amplitudes. With these ratios as criteria, two solutions are provided, adequate for most modulation problems of this character occurring in the field of communications.

The development is based on Madelung's empirical propositions. From these are deduced the forms of complex hysteresis loops occasioned by two-frequency magnetomotive forces, and from the loops sinusoidal components of the flux wave are derived by means of Fourier's series. The various voltages generated in a coil by such a flux are then calculated and next correlated with analyses for a single applied frequency. The resulting changes in the impedances to the two fundamental frequencies are also evaluated. The most important results are given in graphs and tables.

Experimental data on a number of specimens show close agreement with curves computed by the theory.

The analysis discloses several interesting features. It is shown that Madelung's conclusions imply Rayleigh's law of loop similarity; as a consequence the parameters of a Rayleigh loop suffice to describe a complex loop to the extent that it conforms to Madelung's results. Hysteresis suppression is found not to occur at low fields, although harmonic suppression may. The generated side frequencies of the flux appear in unequal pairs, the lower one being the stronger in each instance. Such inequality is a general property ascribable to the multivaluedness of the loop.

FOR precisely evaluating the performance of communication circuits containing ferromagnetic materials, methods for taking into account the non-linear effects of these materials are needed. To this end there have been devoted certain investigations of the behavior of such materials at the low flux densities usual in communication. These early disclosed that hysteresis is a governing factor for weak fields and led to attempts to solve the problem of its bearing on speech.

The complexity of speech and of hysteretic phenomena has made desirable the use of simple testing methods, which in turn require for their interpretation a quantitative theory. Since tests of this sort are usually made with one or more sinusoidal test waves, a theory of single-frequency magnetic performance has already been evolved as a first step toward fulfilling this need. For many purposes single-frequency tests are inadequate, and two-frequency waves are often used to obtain better information bearing on the design or performance of communication systems. It is the purpose of the present paper to take a further step by furnishing the theory of magnetic behavior under a two-frequency force.

The very simplicity which sometimes makes sinusoidal waves valuable for analyzing or testing the non-linear properties of channels of communication makes such waves worthless when applied in other instances, and more complex test waves must then be employed. The harmonics produced by a sine wave furnish an index of the distorting properties of a system, but the side frequencies produced by two such waves are needed to indicate its modulating properties or to give a measure of the interference between carrier channels. When two waves are used, one may be thought of as the carrier, modulated by the other, whose amplitude is chosen proportionate to the square root of the energy in the more complex modulating wave it represents, or both may be thought of as component waves in the same channel, or as carriers in different channels, producing interference in certain other channels. The amplitudes of the several product frequencies then give a measure of the energy falling in their respective regions of the spectrum under actual operation. The effect of the presence of one fundamental upon the transmission of the other can also be ascertained. Increasing the complexity of the test wave by the superposition of additional frequencies can be seen to afford little added advantage at the cost of much complication, unless the character of the waves actually transmitted is simulated, in which case statistical methods of study can perhaps be applied.

From the foregoing circumstances the utility of information pertaining to the application of two-frequency inputs as well as single-frequency inputs to non-linear circuit elements is apparent; many investigations have been conducted in this field to provide such information. When the current-voltage relation is not single-valued a more intricate treatment is necessary in carrying out the analysis. A general method of attack for double-valued characteristics has been provided and applied to hysteresis loops by E. Peterson [1] to determine the flux in ferromagnetic materials under single-frequency magnetizing forces. The fundamental dependence of loop form upon wave shape precludes immediate extension of Peterson's results to the case of a multi-frequency force except for certain harmonic combinations, one of which he considers.[2] A study of flutter effect has been published by Walter Deutschmann,[3] who analyzed a complex loop made up of straight lines. Both instances serve to emphasize the desirability of a broader investigation of the theoretical aspects of two-frequency magnetization including hysteresis. While no general and rigorous

[1] *B. S. T. J.*, Vol. 7, pp. 762–796, Oct. 1928.
[2] *Ibid.*, p. 773.
[3] *Wiss. Ver.* a.d. Siemens-Konzern, Vol. 8, No. 2, pp. 22–44, 1929; *E. N. T.*, Vol. 6, pp. 80–86, Feb. 1929.

method has yet been developed for handling this problem in a manner analogous to that applicable for a single frequency, practically important cases are solved here by means extensible to more complicated ones.

Considerations of Wave Shape and Loop Forms

Scope of the Two-Frequency Analysis

The envelope of a wave affords a means for classification. Waves whose envelopes change gradually during any oscillation form a class apart from those waves which have envelopes subject to abrupt changes. Members of each class can be segregated into those with envelopes nearly symmetrical with respect to the average magnetizing force, those with envelopes of almost uniform width, and so on. One of the two last-mentioned properties in a wave with a gradually changing envelope is essential to successful analysis by the methods about to be detailed.

The cases of magnetization analyzed, which include all the two-frequency wave shapes that can qualify under the foregoing criterion of tractability to analysis, are the following:

Case 1. The ratio between the geometric and arithmetic means of the amplitudes much smaller than the ratio between the sum and difference of the frequencies.

Case 2. One fundamental frequency high relative to the other; the product of the higher frequency with its amplitude large relative to the product of the lower frequency with its amplitude.

Between the two cases there exist intermediate ratios of frequencies and amplitudes over which the theory does not extend; however, the most frequent problems are usually entirely within the domain of a single case. The inequalities involved in the foregoing case definitions are not susceptible to simple explicit statement as limiting numerical ratios, in advance of the development of the theory. Much depends upon the accuracy required in predicting performance. From data supplied in the paper, following the theory, it is possible to determine the practical limitations of the mathematical treatment.

Formation of Complex Loops

Considering the complex hysteresis loops arising from multi-frequency magnetizing forces to be many-valued characteristics for determining the flux density, it is pertinent to study their formation and to correlate their parameters in so far as may be possible with those of single-frequency loops. The latter at low fields are known to consist approximately of two parabolic branches, the exact shape of each dependent upon its point of origin. For fields confined somewhat below maximum permeability, the situation customarily obtaining in communication circuits, this representation has proved to be sufficiently exact to warrant the neglect of higher order terms in analyses. In conformity, terms of higher than second order will not be retained in equations used here; the development so presented can be extended to include them without other change in procedure. The induction at any point on a simple loop centered on the origin is expressed as a function of the instantaneous magnetizing force h by means of a formula developed by Peterson,

$$B = (a_{10} + a_{11}H)h \pm a_{02}(H^2 - h^2). \tag{1}$$

The upper sign of the double sign is used for the descending (upper) branch of the loop, and the lower sign for the ascending (lower) branch. H is the maximum magnetizing force and the coefficients are constants of the ferromagnetic material, determinable by single-frequency measurements. They have the following significance: a_{10} is the initial permeability, a_{11} the rate of change of permeability with magnetizing force, and a_{02} a factor of proportionality between the hysteresis loss and the cube of the maximum magnetizing force. The concepts in terms of which these parameters are defined acquire extended meanings for complex loops.

In the absence of an adequate theory of ferromagnetism the question of whether branches of complex loops and of simple loops have similar forms must be answered by experiment. The steady state of retracing alternately the two branches of a simple loop may eventuate in a different relation of B versus h than results from the first cycle; such a condition would mean that transient branches compose the complex loop, inasmuch as it is not retraced. It is also possible that the biasing effect of one sinusoidal component of the magnetizing force upon the other might cause the branches of the two types of loops to be dissimilar. The coefficients which specify the branches of the simple loop are evaluated with it centered at the origin of the B–h plane, using single-frequency methods, and cannot be assumed *a priori* to apply to to other situations, or to an unrepeated branch.

According to experiments by R. Goldschmidt [4] the superposed field necessary to cause much change of either the shapes or axial slopes of loops exceeds the weak fields to which this development is limited. Likewise, Lord Rayleigh [5] in his original investigation found small super-

[4] *Zeits. f. Techn. Physik*, Vol. 11, pp. 8–12, 1930.
[5] *Phil. Mag.*, Vol. 23, 1887.

posed fields to have no measurable effect upon the form of the loops for the specimens he investigated. Based upon these results, the branches of simple loops can be taken to be independent of their location in the B–h plane, and in so far as the complex magnetizing force gives rise to branches of simple loops, they have the same shape they would have in such a loop centered at the origin.

The formation of complex loops has been determined experimentally by E. Madelung.[6] He found that after reversing the magnetizing force at any point along the branch of a hysteresis loop a new curve is traced which, if continued, passes through the tip of the loop. A second reversal before the tip is reached causes another new curve to be traced back to the point of reversal on the original branch, which is followed thenceforth as if the two reversals have not occurred. The return to a reversal point makes all subsequent traces of the loop the same as if no changes of magnetizing force intervened between the two transits through that point. Any branch of the complex loop is then, in accordance with Madelung's determinations, completely specified by two points of reversal—the one from which it starts and the one through which it must pass if continued far enough; after passing the latter point it becomes the continuation of another branch similarly specified by different points of reversal.

The foregoing principles furnish sufficient information for deducing the form of the branches of complex loops. If such a branch be extended to one of the reversal points defining it and a trace then be carried back to the other, the loop so formed will be retraced by repeating the cycle. As these repetitions can be carried on indefinitely, the path must comprise a simple loop, the branch of the complex loop forming a portion of it. Every complex loop can therefore be considered as composed of adjoined sections of simple loops. Each branch of the complex loop is representable on suitably transformed axes by formula (1) with H taken as half the change in magnetizing force between the reversal points which specify the branch. In general, a different pair of axes will be required for each branch; they can subsequently be referred to a common origin.

The application of this analysis to the complex loop discloses the requirement that the relation $a_{11} = 2a_{02}$ must be true if Madelung's propositions are to hold, because the values of H differ for the two branches of a subsidiary loop. If this equality is not satisfied, the return to the original branch does not take place at the point of departure. Madelung's observations do not include this possibility,

[6] *Annalen der Physik*, Vol. 17, pp. 861–890, 1905. See also *Handbuch der Physik*, Vol. 15, pp. 106–107, Berlin, 1927.

and sufficient evidence of behavior for extending them in such circumstances is not available at present. Some experiments by Lehde [7] and others indicate that subsidiary loops do not quite close at their junctions with the major loop and often show departures both ways on different parts of the same complex loop. This sort of behavior is not explainable by inequality between a_{11} and $2a_{02}$ for the subsidiary loops, as it would cause the departure of one specimen to be always the same way depending upon which quantity was the larger. In all cases, even near saturation, Lehde's results show this departure to be small and the connecting branches between successive subsidiary loops to form approximately a simple loop. On this basis Madelung's results can be considered confirmed to a sufficient degree of approximation.

The ratio a_{11}/a_{02}, which has been taken as a measure of the validity of Rayleigh's relation in single-frequency theory, becomes a criterion of the usefulness of Madelung's propositions concerning loop form in multi-frequency theory. Those substances which most closely accord with Rayleigh's analysis can also be expected to be in best agreement with Madelung's results. The relation between coefficients required on the basis of Madelung's and Rayleigh's experiments will be used hereafter to simplify the analysis. The simplification will be evidenced by the customary nomenclature, in which

$$\mu_0 \equiv a_{10}, \quad \nu \equiv a_{02} = \tfrac{1}{2}a_{11}.$$

Any attempt to distinguish here between the two latter constants would be meaningless because beyond the scope of Madelung's empirical rules. Fortunately the constants of most commercial materials conform closely to the above equality.

Types of Two-Frequency Loops

The aspect of a hysteresis loop formed by a two-frequency wave changes greatly with the frequencies and their amplitudes. Different pairs of frequencies having equal ratios give rise to families of loops which are identical except as affected by eddy currents. These, for the purpose of this study, are supposed to be so small that the flux is substantially uniform over a cross-section of the magnetic circuit. If the two frequencies have a common source or are synchronized, the hysteretic phenomena are singly periodic and subject to simpler treatment than developed here for independent sources.

For detailed analysis of the effects of hysteresis with two applied frequencies from independent sources, the phase angles of both may

[7] *Rev. of Sci. Instr.*, Vol. 2, pp. 16–43, Jan. 1931.

be taken as zero, equivalent to measuring the time from a point where the two components of the magnetizing force become maximum simultaneously. Then

$$h = P \cos pt + Q \cos qt \tag{2}$$

is the instantaneous magnetizing force. Phase angles can be made arbitrary by replacing pt and qt by $pt + \theta_p$ and $qt + \theta_q$, respectively, in this equation and the subsequent results.

The configurations of complex loops can be followed by altering single-frequency loops to accord with the results of investigations of loop formation. If to a single-frequency magnetizing force a relatively small one of slightly lower frequency is added, the resultant is an oscillatory force whose peaks undulate around the value reached by those of the original wave. Their maximum will be the sum of the amplitudes of the two components and their minimum the difference. On a hysteresis loop (Fig. 1*a*) this means that portions between successive reversal points will differ slightly from one another, being formed approximately as if belonging to successively smaller loops until a minimum peak is passed, thenceforth as if belonging to successively larger loops, and so on cyclically. Such behavior is sketched in the figure.

As the amplitude of the lower frequency component of the magnetizing force is increased the undulations become more pronounced and the preceding picture more inexact. When both amplitudes are equal the envelope of the resultant magnetizing force vanishes periodically and the portion of the hysteresis loop formed while this envelope goes from its maximum to zero is in the nature of a spiral (Fig. 1*b*), a similar curve being developed outwardly as the envelope increases again to its maximum. Provided only that successive peaks of the magnetizing force do not differ greatly in magnitude, each portion of such a loop between adjacent reversal points may be assumed to have the form of a branch of a single-frequency hysteresis loop having its point of origin coincident with that of the portion of the complex loop. Then the induction may be derived from the magnetizing force by the use of single-frequency data in accordance with the known manner of formation of complex loops.

If now the amplitude of the higher frequency component be decreased to a relatively small value, the undulations in the envelope subside, and a condition similar to the original one is seen to obtain. This time, however, the amplitude of the lower frequency will be found to be the one about which these undulations occur, and the characteristic will again look like that in Fig. 1*a*.

As long as one frequency is less than twice the other the undulations in the peaks of the magnetizing force will be regular and gradual. If the higher frequency be raised to more than twice the lower, the undulations become more abrupt and complex, and increasingly so

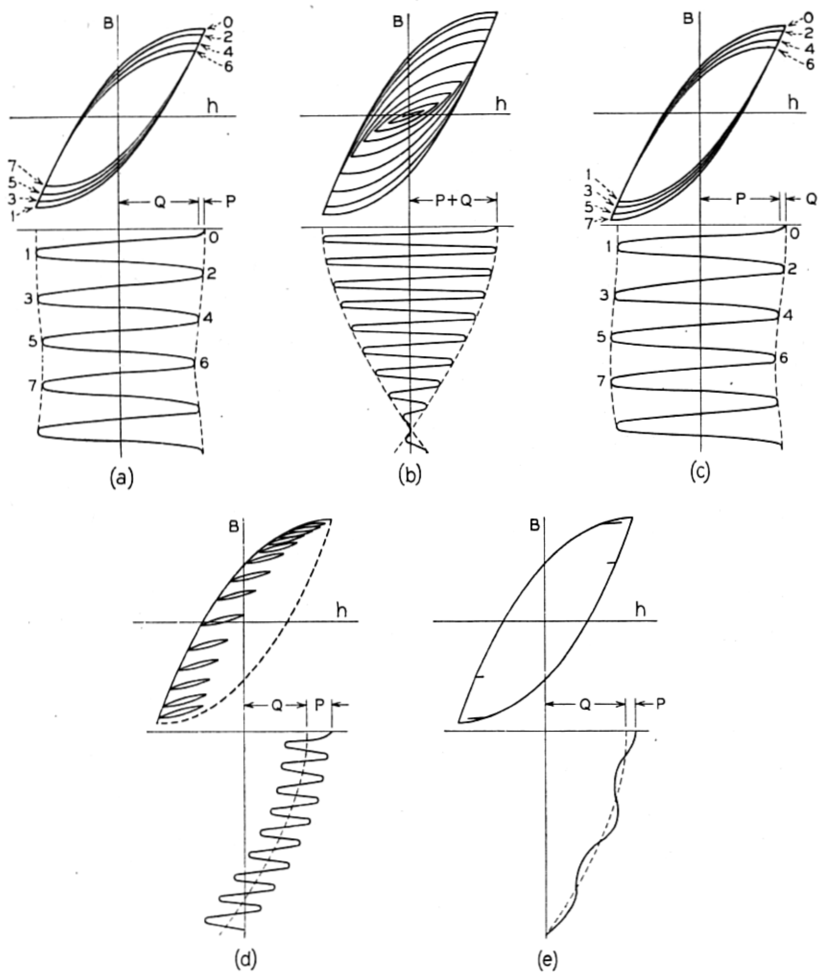


Fig. 1—Types of hysteresis loops characteristic of a two-frequency magnetizing force.

as it is raised still more. The complexity attendant upon the formation of the hysteresis loops becomes greater and the simplifying artifices heretofore suggested no longer apply.

When one frequency has become several times the other, the higher frequency component must have one of its maxima very near each maximum of the lower one so that a maximum value practically equal to the sum of the amplitudes of the two components is reached every cycle of the lower one; likewise for the minima. Between these two extremes other reversals in the magnetizing force would be expected to cause the hysteresis loop to comprise additional small loops not necessarily closed. Experiments confirm this conjecture and indicate that not only for weak fields but even for fields near saturation the small loops nearly close and the paths between them are traced nearly the same as portions of a large loop.

If the amplitude of the higher frequency component is considerably the larger, all the loops composing the characteristic are virtually the same size and shifted slightly with respect to each other on account of the lower frequency component. The characteristic will be that depicted in Fig. 1*c*.

When the amplitudes of the two components are not grossly unequal, the hysteresis loop is of the type represented by Fig. 1*d*. Small loops formed when the magnetizing force is near an extreme value are longer than those formed when it is near zero, aside from any effects of superposition, because of its different rates of change in the two positions. In general these loops will not occur in the same place for different cycles and the distances between them will be lessened by increasing the higher frequency. By considering the complex loop to consist of a major loop, such as a single frequency would generate, encompassing a number of minor loops, the induction may be derived from the magnetizing force since the form of the minor loops is known. When these are not too widely spaced, each may be assigned a mean position in a loop fixed for all time and the induction calculated therefrom. It is evident that such an undertaking is vastly more complicated than the ones suggested heretofore and that unlike them it requires information in addition to that obtained from single-frequency measurements performed with no superposed field.

By decreasing the amplitude of the higher frequency component of the magnetizing force, the amplitudes of the minor loops may be reduced until those in the neighborhood of zero magnetizing force vanish entirely. Further decrease of the same component causes more and more of the minor loops to disappear, so that finally only a few small ones remain at each end of the major loop. This condition is shown in Fig. 1*e*. As the wave form of the induction is only slightly affected by the presence of these loops, they may safely be omitted and the characteristic simply taken as the major loop, determined from single-frequency results alone.

The point where the minor loops first vanish and the requirement that they vanish at all may be readily determined. Two adjacent extremes of the magnetizing force approach a common value as the amplitude of the higher frequency component is decreased. When they attain this value, the slope is no longer reversed between them and consequently no minor loop is formed. The instantaneous magnetizing force is expressed by equation (2), and in this instance $p \gg q$. The minor loops first appear where $dh/dt = 0$ when $pt = -\frac{\pi}{2}$. Solved simultaneously, these equations yield

$$qt = \sin^{-1} \frac{Pp}{Qq}, \tag{3}$$

where the minor loops vanish. They reappear at an equal negative value of sin qt, and other intervals during which they vanish are apparent from symmetry. If $(Pp/Qq) > 1$, no real solution of equation (3) exists, so the minor loops do not vanish anywhere. The appearance and non-appearance of minor loops is seen to be governed by the ratio of amplitudes in the same way as by the ratio of frequencies as long as the restriction on the latter is observed, and the product of these ratios

$$\kappa = \frac{Qq}{Pp}$$

determines the type of hysteresis loop (1*d* or 1*e*, or an intermediate form) obtained.

Induction with a Two-Frequency Magnetizing Force

General Expression for the Induction

As a function of time, the induction for any type of loop described will consist of intermodulation products of the two fundamental frequencies. Because of the kind of symmetry the characteristic has these products will include only the odd orders, but because it is a loop, quadrature terms must be expected. The induction then will ultimately be in the form

$$B = \sum_{m=0}^{\infty} \sum_{n-\infty}^{\infty} [c_{mn} \sin (mp + nq)t + d_{mn} \cos (mp + nq)t] \tag{4}$$

with all even order coefficients zero. The odd order coefficients remain to be determined from the hysteresis loop. In doing this only third order products in addition to the fundamentals will be evaluated explicitly, as they are stronger than the higher orders and therefore of most importance in measurements of distortion. Any higher order products would be less precisely evaluated, more numerous, and probably of less interest; their computation is clearly evident as an extension of the processes later carried out.

The Multi-Branched Hysteresis Loop

Under certain circumstances mentioned earlier, the portions of a complex loop between adjacent reversal points are representable by

$$B = (\mu_0 + 2\nu H)h \pm \nu(H^2 - h^2). \tag{5}$$

Referred to the origin of the B–h plane, the induction on such a portion originating at the jth reversal point from $t = 0$ is

$$B_j = G_j + \mu_0(h - H_j) - (-1)^j \nu(h - H_j)^2. \tag{6}$$

Here H_j is the magnetizing force and G_j the induction at the jth reversal point. The latter quantity satisfies the difference equation

$$G_j = G_{j-1} + \mu_0(H_j - H_{j-1}) + (-1)^j \nu(H_j - H_{j-1})^2,$$

arrived at by evaluating the induction on the $(j - 1)$st branch at the jth reversal point. Subject to the initial condition

$$G_0 = (\mu_0 + 2\nu H_0)H_0,$$

this can be solved by the method of successive substitutions; the solution is

$$G_j = \mu_0 H_0 + 2\nu H_0^2 + \mu_0 \sum_{i=1}^{j} (H_i - H_{i-1}) + \nu \sum_{i=1}^{j} (-1)^i (H_i - H_{i-1})^2. \tag{7}$$

The foregoing expressions define the induction everywhere on the complex loop. Equations (7) and (6) combined to eliminate G_j give

$$B_j = \mu_0 h + (-1)^j 2\nu H_j h + (-1)^j \nu(H_j^2 - h^2). \tag{8}$$

The problem remaining is to develop this equation into the equivalent of equation (4).

The instantaneous magnetizing force is

$$h = P \cos pt + Q \cos qt. \tag{2}$$

By a trigonometric transformation this may be put in the form

$$h = \sqrt{(P+Q)^2 - 4PQ\sin^2\frac{p-q}{2}t}$$
$$\times \cos\left[\frac{p+q}{2}t + \tan^{-1}\left(\frac{P-Q}{P+Q}\tan\frac{p-q}{2}t\right)\right], \quad (9)$$

which is sometimes more convenient. The envelope of the wave is represented by the two branches of the radical. If its magnitude does not change much between adjacent maxima and minima of the wave, these extremes lie close to the points of tangency between the wave and its envelope; the latter condition is the one necessary in order that the envelope may be used to evaluate the extreme magnetizing force H_j.

This force acquires its values at the reversal points, which are situated at the zeros of dh/dt. Put into a form like equation (9) by the same transformations as were used above, this derivative is

$$\frac{dh}{dt} = -\sqrt{(Pp+Qq)^2 - 4PpQq\sin^2 st}$$
$$\times \sin\left[rt + \tan^{-1}\left(\frac{Pp-Qq}{Pp+Qq}\tan st\right)\right], \quad (10)$$

where $2r = p + q$, $2s = p - q$. Except when $\kappa = 1$, this vanishes only at

$$rt + \tan^{-1}\left[\frac{1-\kappa}{1+\kappa}\tan st\right] = j\pi, \quad (11)$$

j integral or null. Substituting equation (11) into equation (9) yields the magnetizing force at the jth reversal from $t = 0$:

$$H_j = \sqrt{(P+Q)^2 - 4PQ\sin^2 st}$$
$$\times \cos\left[j\pi + \tan^{-1}\left(\frac{1-k}{1+k}\tan st\right) - \tan^{-1}\left(\frac{1-\kappa}{1+\kappa}\tan st\right)\right],$$

letting $k = Q/P$. Upon combining the arc tangents this becomes

$$H_j = P\sqrt{(1+k)^2 - 4k\sin^2 st}$$
$$\times\left[\cos j\pi - \tan^{-1}\frac{2(k-\kappa)\tan st}{(1+k)(1+\kappa) + (1-k)(1-\kappa)\tan^2 st}\right]. \quad (12)$$

According to equation (12) the magnetizing force and its envelope are tangent at reversal points provided the arc tangent is always zero. This it is if $k = \kappa$, a trivial solution. By certain choices of these two parameters, however, it is possible to keep the angle between any reversal point and the nearest extreme of the magnetizing force from exceeding any prescribed limit. The maximum value of the angle in equation (12) is

$$2\tan^{-1}\sqrt{\frac{(1-k)(1+\kappa)}{(1+k)(1-\kappa)}}-\frac{\pi}{2},$$

which can be made small by making k and κ each small compared to unity, or each large compared to unity, or both approximately equal. For the previously excluded instance $\kappa = 1$, this angle can be limited and the last condition fulfilled by keeping k nearly equal to one, as is obvious from the equations. So for each of these three conditions on the parameters, to a definite degree of approximation the envelope at each point of tangency becomes the magnetizing force for the nearest reversal point, a feature useful for the transformation of equation (8) into a function of time.

Calculation of the Induction—Case 1

The three conditions are confluent and will be seen to set the limiting bounds for case 1. When the fundamental frequencies lie close together the ratio of their amplitudes is practically unrestricted; as more widely spaced frequencies are chosen it becomes necessary to require an increased (or diminished) amplitude ratio in order that the phase angle in equation (12) does not exceed the chosen limit. This limit must be such that the cosine of that phase angle is substantially unity.

The maximum magnetizing force is thereupon

$$H_j = (-1)^j P\sqrt{(1+k)^2 - 4k\sin^2 st}. \tag{13}$$

As a periodic even function of st, H_j may be expanded in a Fourier's series

$$H_j = (-1)^j (P+Q)\left[\frac{A_0}{2} + \sum_{\eta=1}^{\infty} A_\eta \cos\eta st\right], \tag{14}$$

where

$$A_\eta = \frac{4}{\pi}\int_0^{\pi/2} \sqrt{1-k_1^2\sin^2\lambda}\cos\eta\lambda d\lambda \tag{15}$$

in terms of the parameter

$$k_1 = \frac{2\sqrt{PQ}}{(P+Q)} = \frac{2\sqrt{k}}{(1+k)} = \frac{2\sqrt{1/k}}{(1+1/k)},$$

which never exceeds unity and which diminishes as k is either increased or decreased from unity. The integral (15) reduces immediately to elliptic form by the substitution $z = \sin\lambda$. All odd order coefficients are zero. For the first three significant values of η:

$$\left.\begin{aligned} A_0 &= \frac{4}{\pi} E_1 \\ A_2 &= \frac{4}{3\pi k_1^2}[(2 - k_1^2)E_1 - 2(1 - k_1^2)K_1] \\ A_4 &= \frac{4}{15\pi k_1^4}[8(2 - k_1^2)(1 - k_1^2)K_1 - (16 - 16k_1^2 + k_1^4)E_1] \end{aligned}\right\} \qquad (16)$$

Here K_1 and E_1 are complete elliptic integrals of the first and second kind, respectively, with modulus k_1.

The series (14) may be used to evaluate the variable permeability anywhere on the loop, for upon substitution in equation (8) reference to a particular branch is eliminated by the disappearance of the double sign on the second term.

The square of the maximum magnetizing force needed in the final term of equation (8) comes directly from equation (13); to determine the sign of this term at any instant remains the only problem. Interpretation of $(-1)^i$ seems simple when it is remembered to be positive for decreasing h and negative for increasing h, and therefore an odd function of time. The rate of change of the magnetizing force is

$$\frac{dh}{dt} = -Pp \sin pt - Qq \sin qt,$$

so it follows that

$$\left.\begin{aligned} (-1)^i &= +1, \quad \sin pt + \kappa \sin qt > 0 \\ &= -1, \quad \sin pt + \kappa \sin qt < 0 \end{aligned}\right\}. \qquad (17)$$

The solution may be completed by expanding this quantity in a Fourier's series:[8]

$$(-1)^i = \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} A_{mn} \sin (mp + nq)t. \qquad (18)$$

When $m = 0$ the summation is to be extended over only positive values of n. With this convention the coefficients are

$$A_{mn} = \frac{1}{2\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} (-1)^i \sin (mx + ny) dx\, dy \qquad (19)$$

or, with the use of equation (17),

$$A_{mn} = (-1)^{\frac{m+n-1}{2}} \frac{4}{\pi^2} \int_0^{\pi} dy \int_0^{\cos^{-1}(-\kappa \cos y)} \cos mx \cos ny\, dx, \qquad (20)$$

[8] W. R. Bennett, "New Results in the Calculation of Modulation Products," *B. S. T. J.*, Vol. 12, pp. 228–243, April, 1933.

where p and q are to be so assigned that $\kappa \leq 1$. Then the coefficients are all expressible in terms of complete elliptic integrals with modulus κ; these will be designated as K_2 and E_2. Coefficients of even order vanish, while those of the first three odd orders are found to be

$$\left.\begin{aligned}
A_{10} &= \frac{8}{\pi^2} E_2 \\
A_{01} &= \frac{8}{\pi^2 \kappa}[E_2 - (1-\kappa^2)K_2] \\
A_{2\underline{1}} = -A_{21} &= \frac{8}{3\pi^2\kappa}[(1-2\kappa^2)E_2 - (1-\kappa^2)K_2] \\
A_{1\underline{2}} = A_{12} &= \frac{8}{3\pi^2\kappa^2}[(2-\kappa^2)E_2 - 2(1-\kappa^2)K_2] \\
A_{30} &= \frac{8}{9\pi^2}[(7-8\kappa^2)E_2 - 4(1-\kappa^2)K_2] \\
A_{03} &= \frac{8}{9\pi^2\kappa^3}[(8-3\kappa^2)(1-\kappa^2)K_2 - (8-7\kappa^2)E_2] \\
A_{4\underline{1}} = -A_{41} &= \frac{8}{15\pi^2\kappa}[(1-16\kappa^2+16\kappa^4)E_2 \\
&\qquad - (1-8\kappa^2)(1-\kappa^2)K_2] \\
A_{1\underline{4}} = A_{14} &= \frac{8}{15\pi^2\kappa^4}[8(2-\kappa^2)(1-\kappa^2)K_2 \\
&\qquad - (16-16\kappa^2+\kappa^4)E_2] \\
A_{3\underline{2}} = A_{32} &= \frac{8}{15\pi^2\kappa^2}[2(1+2\kappa^2)(1-\kappa^2)K_2 \\
&\qquad - (2+3\kappa^2-8\kappa^4)E_2] \\
A_{2\underline{3}} = -A_{23} &= \frac{8}{15\pi^2\kappa^3}[(8+\kappa^2)(1-\kappa^2)K_2 - (8-3\kappa^2-2\kappa^4)E_2] \\
A_{50} &= \frac{8}{75\pi^2}[(43-168\kappa^2+128\kappa^4)E_2 \\
&\qquad - 4(7-16\kappa^2)(1-\kappa^2)K_2] \\
A_{05} &= \frac{8}{75\pi^2\kappa^5}[(128-168\kappa^2+43\kappa^4)E_2 \\
&\qquad -(128-104\kappa^2+15\kappa^4)(1-\kappa^2)K_2]
\end{aligned}\right\} \quad (21)$$

Negative digits of the subscripts are underscored in the coefficients for lower side frequencies.

Upon putting the various quantities into equation (8) from equations (2), (13), (14), and (18) it thus becomes

$$\left.\begin{aligned}
B = {} & \mu_0 P \cos pt + \mu_0 Q \cos qt \\
& + \nu P(P+Q) \sum_{m=0}^{\infty} (A_{2m} + kA_{2m+2}) \cos [(m+1)p - mq]t \\
& + \nu P(P+Q) \sum_{m=0}^{\infty} (kA_{2m} + A_{2m+2}) \cos [mp - (m+1)q]t \\
& - \tfrac{1}{4}\nu P^2 \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (A_{m+2,\,n} - 2A_{mn} + A_{m-2,\,n}) \\
& \qquad\qquad \times \sin [mp + nq]_t \\
& - \tfrac{1}{4}\nu Q^2 \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (A_{m,\,n-2} - 2A_{mn} + A_{m,\,n+2}) \\
& \qquad\qquad \times \sin [mp + nq]^t \\
& + \tfrac{1}{2}\nu PQ \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (A_{m+1,\,n-1} - A_{m+1,\,n+1} \\
& \qquad\qquad + A_{m-1,\,n+1} - A_{m-1,\,n-1}) \sin [mp + nq]t
\end{aligned}\right\} \quad (22)$$

with the understanding that $A_{rs} \equiv 0$ for $r < 0$ and for $r = 0,\ s < 0$. The first line is the linear portion of the induction, given by a permeability constant at its initial value; the first two summations arise from the variation of the permeability with the maximum magnetizing force; the remaining terms comprise the results of distortion attributable to hysteresis *per se*. The coefficients of the induction, c_{mn} and d_{mn} of equation (4), may now be evaluated by selecting the necessary quantities from equation (22).

General expressions for the coefficients can be evolved in series known as hypergeometric functions. These are all of the type

$$F(\alpha, \beta; \gamma; z) = 1 + \frac{\alpha\beta}{\gamma}\frac{z}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{z^2}{2!} + \cdots; \quad (23)$$

a particular one is chosen by specifying the parameters. The coefficients needed here are

$$A_{mn} = \frac{2}{\pi} \frac{\Gamma\left(\frac{m+n}{2}\right)\kappa^n}{\Gamma(n+1)\,\Gamma\left(\frac{m-n}{2}+1\right)} F\left(\frac{m+n}{2}, \frac{n-m}{2}; n+1; \kappa^2\right) \quad (24)$$

for m and n both positive and $m + n$ odd, and

$$A_\eta = \frac{\Gamma\left(\frac{\eta+1}{2}\right) k_1{}^\eta}{\Gamma(\eta+1)\,\Gamma\left(\frac{1-\eta}{2}+1\right)} F\left(\frac{\eta+1}{2}, \frac{\eta-1}{2}; \eta+1; k_1{}^2\right) \quad (25)$$

for η even. When n is negative, the coefficient may be found by using the relation

$$A_{m\underline{n}} = (-1)^n A_{mn}; \tag{26}$$

for all other values of the subscripts excluded the coefficients are zero. A recurrence formula for computing products of higher order is

$$A_{mn} = \frac{1}{m+n}\left[2\left(\frac{n-1}{\kappa} + (m-1)\kappa\right)A_{m-1,\,n-1} - (m+n-4)A_{m-2,\,n-2}\right] \tag{27}$$

with $m - 2$ positive. Comparison of equation (24) with equation (25) reveals that if in the former κ is replaced by k_1,

$$A_\eta = \frac{\pi}{2} A_{1,\eta}, \tag{28}$$

so the equations (21) and the recurrence formula (27) suffice for computing all the coefficients in the series for the induction.

Calculation of the Induction—Case 2

For case 2 the two branches of a minor loop and an adjoined portion of the major loop are combined into a Fourier's series whose coefficients are functions of position in the major loop. By developing these coefficients into Fourier's series, a double series in time is obtained. For this case, as for the other just considered, the induction thus is developed in the form of equation (4) and the coefficients are determined through the third order.

When minor loops are formed throughout the lower frequency cycle, an expression for each minor loop and the portion of the major loop joining it to the next one is found relative to an origin at the junction of the major and minor loops. A succession of such loops is then referred to the origin of the major loop by transforming the coordinates of a typical minor loop, the transformation being a function of the position of the minor loop.

Attention first will be devoted to a single high-frequency cycle occurring while the lower-frequency component of the magnetizing force is decreasing. Let the time of occurrence of the maximum in the higher frequency component of the magnetizing force during this cycle be designated by τ. By restricting application to characteristics with sizeable minor loops, i.e. $\kappa \ll 1$ when $p > q$, consistent with the stipulation that the minor loops do not vanish anywhere, this maximum can be made practically coincident with the corresponding maximum in the magnetizing force.

Writing

$$t = \tau + \lambda, \tag{29}$$

the lower-frequency component in the vicinity is expressible by the Taylor's series

$$Q \cos qt = Q\left[1 - \frac{q^2\lambda^2}{2!} + \frac{q^4\lambda^4}{4!} - \cdots\right] \cos q\tau - Q\left[q\lambda - \frac{q^3\lambda^3}{3!} + \frac{q^5\lambda^5}{5!} - \cdots\right] \sin q\tau. \tag{30}$$

Over one cycle of the higher frequency this component is very nearly linear, so its variation in this range is

$$-4\Delta = -2\pi\kappa P \sin q\tau. \tag{31}$$

Since τ has been so chosen as to be an integral multiple of $2\pi/p$, when equations (29) and (30) are substituted in the magnetizing force given by equation (2) it becomes

$$h = P \cos p\lambda + Q \cos q\tau - \lambda Qq \sin q\tau. \tag{32}$$

Its value referred to a new set of coordinates, B', h', with their origin at the junction of the major and minor loops is

$$h' = P(1 + \cos p\lambda) - 2\Delta\left(1 + \frac{p\lambda}{\pi}\right)\cdot \tag{33}$$

According to Madelung's findings previous minor loops will not influence the one under consideration, so its lower branch will proceed toward the upper tip of the major loop as indicated in Fig. 2. A transformation of equation (1), simplified by the use of Rayleigh's relation, then gives for this branch

$$B_2' = \mu_0 h' + \nu h'^2. \tag{34}$$

The upper branch is

$$B_1' = [\mu_0 + 4\nu(P - \Delta)]h' - \nu h'^2. \tag{35}$$

The small portion of the major loop traversed during the last of the cycle may be expanded in a Taylor's series

$$B' = B'(0) + h' \left.\frac{\partial B_1}{\partial h}\right|_{h=h_r} + \frac{h'^2}{2!} \left.\frac{\partial^2 B_1}{\partial h^2}\right|_{h=h_r} + \cdots, \tag{36}$$

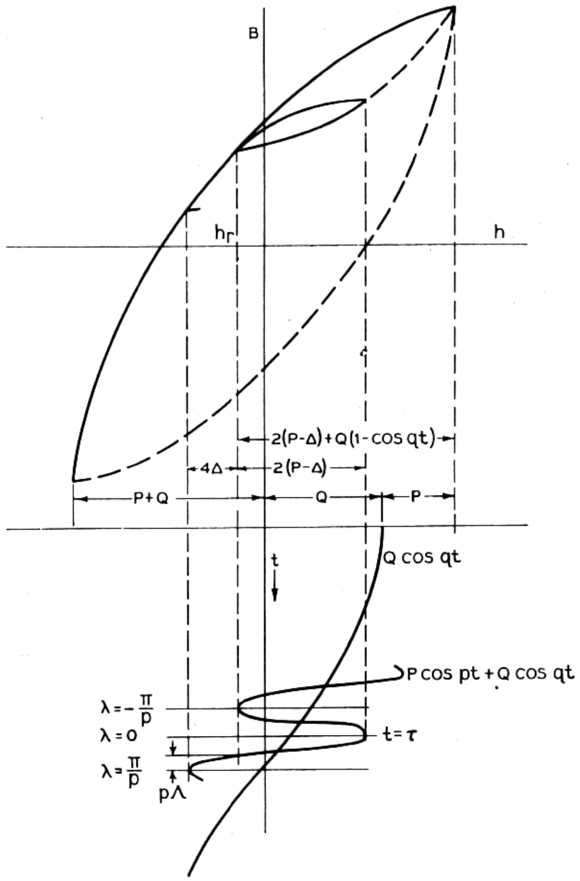


Fig. 2—Tracing of a subsidiary loop according to Madelung.

denoting by h_r the value of h at $h' = 0$, which is

$$h_r = 2\Delta - P + Q \cos q\tau. \tag{37}$$

The upper branch of the major loop as a function of h (not h') is

$$B_1 = (\mu_0 + 2\nu H)h + \nu(H^2 - h^2) \tag{38}$$

with

$$H = P + Q.$$

The expansion (36) is now found to be

$$B' = \{\mu_0 + 2\nu[2(P - \Delta) + Q(1 - \cos q\tau)]\}h' - \nu h'^2. \tag{39}$$

Equations (34), (35), and (39) define the induction in terms of h' over an entire high-frequency cycle; the first is valid for $-\pi < p\lambda < 0$, the second for $0 < p\lambda < \pi - p\Lambda$, and the last for $\pi - p\Lambda < p\lambda < \pi$.

For combining these three expressions a Fourier's series may be developed applicable over the entire interval $-\pi < p\lambda < \pi$, and the induction expressed by this series can be referred to the central axes, B, h, of the major loop by including in its constant term the value of the induction at the junction of the major and minor loops. The series is

$$B_1 = \tfrac{1}{2}b_0' + b_1' \cos p\lambda + b_2' \cos 2p\lambda + b_3' \cos 3p\lambda \\ + a_1' \sin p\lambda + a_2' \sin 2p\lambda + a_3' \sin 3p\lambda. \quad (40)$$

The coefficients are determined by the integrals

$$a_n' = \frac{1}{\pi}\int_{-\pi}^{\pi} B \sin np\lambda \, d(p\lambda), \qquad b_n' = \frac{1}{\pi}\int_{-\pi}^{\pi} B \cos np\lambda \, d(p\lambda), \quad (41)$$

where

$$B = B' + B_r.$$

B_r is the induction at the junction of the major and minor loops, found by inserting equation (37) in equation (38). The resulting quantity together with expressions previously found for B' over various parts of the cycle are substituted in the integrands of equation (41), and the integrations are performed, using h' given by equation (33). All terms of order higher than the third and those containing the square of the frequency ratio as a factor are rejected as they occur. The resulting coefficients are functions of $q\tau$ both explicitly and also through Δ and $p\Lambda$.

To determine $p\Lambda$ as a function of $q\tau$, the vanishing of h' at the tip of the minor loop on the major loop gives an equation for use along the descending branch of the major loop. By equation (33)

$$P(1 - \cos p\Lambda) = 4\Delta\left(1 - \frac{p\Lambda}{2\pi}\right),$$

$$(1 - \cos p\Lambda) = (2\pi - p\Lambda)\kappa \sin q\tau. \quad (42)$$

An approximation to the general solution can be got by transforming equation (42) into a quadratic algebraic equation and solving. This is done by means of the first two terms of the power series expansion for $\cos p\Lambda$. The approximation will be best for small values of the angle, but very good over all its admissible range. This reduction gives

$$(p\Lambda)^2 + 2\kappa \sin q\tau\,(p\Lambda) - 4\pi\kappa \sin q\tau = 0, \quad (43)$$

the roots of which are

$$p\Lambda = -\kappa \sin q\tau \pm \sqrt{\kappa^2 \sin^2 q\tau + 4\pi\kappa \sin q\tau},$$

the positive one being that sought. By expanding the quantity under the radical according to the binomial theorem this root reduces to

$$p\Lambda \mp 2\sqrt{\pi}\,\sqrt{\kappa \sin q\tau} - \kappa \sin q\tau \tag{44}$$

when higher powers of κ are dropped.

The coefficients of the series then become, using the value of Δ from equation (31) and the value of $p\Lambda$ from equation (44),

$$\begin{aligned}
\frac{1}{2} b_0' &= \mu_0 Q \cos q\tau + 2\nu(2P + Q)Q \cos q\tau + \left(\pi - \frac{4}{\pi}\right) \nu PQ \frac{q}{p} \sin q\tau \\
&\qquad + 2\pi\nu Q^2 \frac{q}{p} \sin q\tau\,(1 - \cos q\tau) + \nu Q^2 \sin^2 q\tau, \\
a_1' &= -\,2\mu_0 Q \frac{q}{p} \sin q\tau - 8\nu PQ \frac{q}{p} \sin q\tau + \frac{8}{3\pi} \nu P^2, \\
b_1' &= \mu_0 P + 2\nu P^2, \\
a_2' &= \mu_0 Q \frac{q}{p} \sin q\tau + \frac{14}{3} \nu PQ \frac{q}{p} \sin q\tau, \\
b_2' &= -\frac{40}{9\pi} \nu PQ \frac{q}{p} \sin q\tau, \\
a_3' &= -\frac{2}{3} \mu_0 Q \frac{q}{p} \sin q\tau - \frac{8}{15\pi} \nu P, \\
b_3' &= 0.
\end{aligned} \tag{45}$$

The relation (29) can be used to return equations (45) to the general time-coordinate. Replacing τ by $t - \lambda$ gives

$$\begin{aligned}
\cos q\tau &= \cos q\lambda \cos qt + \sin q\lambda \sin qt, \\
\sin q\tau &= \cos q\lambda \sin qt - \sin q\lambda \cos qt.
\end{aligned}$$

As $|\lambda|$ never exceeds π/p these equations can be simplified to

$$\begin{aligned}
\cos q\tau &= \cos qt + \frac{q}{p}\,(2 \sin p\lambda - \sin 2p\lambda) \sin qt, \\
\sin q\tau &= \sin qt - \frac{q}{p}\,(2 \sin p\lambda - \sin 2p\lambda) \cos qt,
\end{aligned} \tag{46}$$

using the first terms of the Fourier's series for $\sin q\lambda$ in multiples of $p\lambda$. Upon substitution of equation (29) trigonometric functions of $p\lambda$ become the same functions of $(pt - 2j\pi)$, j an integer, since τ is defined as an even integral multiple of $2\pi/p$. The phase angle is therefore immaterial, so $p\lambda$ can be replaced by pt in equations (40) and (46). Combination of equations (45) and (46) with equation (40) results in an expression of the induction on the upper half of the complex loop in terms of time.

For the lower half of the loop a mean position must be found. Having started with incommensurable frequencies at zero phase angles, the reversals at the lower tip of the major loop will occur at

$$pt = 2m\pi + R,$$

where $0 < R < 2\pi$ and all values of R within these limits are equally probable. The lower frequency component at the instant of reversal will have a time angle

$$qt = (2n + 1)\pi + S,$$

where $-(q/p)\pi < S < (q/p)\pi$ and all values of S within the limits are equally probable. The expected medians of the time angles are therefore $(2m + 1)\pi$ and $(2n + 1)\pi$ for the higher and lower frequency components respectively. These values and the point symmetry of the characteristic specify that the induction during the ascendency of the lower-frequency component of the magnetizing force will be equal in magnitude and opposite in sign to the induction for the descent with the phase of each component increased by its expected median. The induction for an increasing lower-frequency component is therefore given by

$$B_2[pt, qt] = -B_1[pt + \pi, qt + \pi], \tag{47}$$

where the right-hand member is evaluated for a decreasing lower-frequency component.

The coefficients of equation (40) may be altered accordingly to furnish a set for use when the lower-frequency component is increasing by replacing qt by $(qt + \pi)$ and pt by $(pt + \pi)$. In series form, then, the induction on the lower half of the loop is

$$\begin{aligned} B_2 = \tfrac{1}{2}b_0'' + b_1'' \cos pt + b_2'' \cos 2pt + b_3'' \cos 3pt \\ + a_1'' \sin pt + a_2'' \sin 2pt + a_3'' \sin 3pt. \end{aligned} \tag{48}$$

One pair of coefficients is necessary to specify completely the amplitude of each component of the induction when it is split into in-phase and quadrature terms harmonic in pt. Coefficients of corresponding terms in equations (40) and (48) are all functions of qt, each series applying over one half of a lower-frequency cycle. Each pair of coefficients can therefore be developed into a Fourier's series in qt, so that the single expression

$$B = \tfrac{1}{2}b_0 + b_1 \cos pt + b_2 \cos 2pt + b_3 \cos 3pt + a_1 \sin pt + a_2 \sin 2pt + a_3 \sin 3pt \quad (49)$$

defines the induction everywhere.

The coefficients of equation (49) are given by the expressions

$$\begin{aligned} b_n &= \frac{1}{2}[b_n' + b_n''] + \frac{2}{\pi}[b_n' - b_n''] \sum_{m=0}^{\infty} \frac{\sin(2m+1)q\tau}{(2m+1)}, \\ a_n &= \frac{1}{2}[a_n' + a_n''] + \frac{2}{\pi}[a_n' - a_n''] \sum_{m=0}^{\infty} \frac{\sin(2m+1)q\tau}{(2m+1)}. \end{aligned} \quad (50)$$

After putting the values of the primed coefficients into the respective terms, changing the arguments of functions of $q\tau$ to qt by using equations (46), and expanding the result into multiple angles, there remains when terms beyond the third order are dropped

$$\left.\begin{aligned} B = {} & [\mu_0 + 2\nu P]P \cos pt + \frac{8}{3\pi}\nu[P + 3\kappa Q]P \sin pt \\ & + \left[\mu_0 + 2\nu\left(2 + k - \frac{4}{3}\kappa\right)P\right] Q \cos qt \\ & + \frac{8}{3\pi}\nu\left[k + \frac{3\pi^2}{4}\kappa + \frac{3}{2}\left(\frac{\pi^2}{4} - 1\right)\frac{\kappa}{k}\right] PQ \sin qt \\ & - \frac{1}{3}\nu[1 - 6k]\kappa P^2 \cos(2p + q)t \\ & - \frac{40}{9\pi}\nu\left[1 + \frac{3}{10}k\right]\kappa P^2 \sin(2p + q)t \\ & + \frac{1}{3}\nu[1 - 6k]\kappa P^2 \cos(2p - q)t \\ & + \frac{40}{9\pi}\nu\left[1 - \frac{3}{10}k\right]\kappa P^2 \sin(2p - q)t \\ & + \frac{32}{5\pi^2}\nu\kappa PQ \cos(p + 2q)t \\ & - \frac{2}{3\pi}[\mu_0 + 2\nu(2 + 3k)P]\kappa P \sin(p + 2q)t \\ & - \frac{32}{5\pi^2}\nu\kappa PQ \cos(p - 2q)t \\ & - \frac{2}{3\pi}[\mu_0 + 2\nu(2 + 3k)P]\kappa P \sin(p - 2q)t \\ & - \frac{4}{3\pi}\left[\mu_0\kappa - \frac{2}{5}\nu P\right] P \sin 3pt \\ & + \frac{8}{5}\nu\kappa PQ \cos 3qt - \frac{8}{15\pi}\nu Q^2 \sin 3qt. \end{aligned}\right\} \quad (51)$$

Recapitulation of Principal Results

General formulæ have now been made available for calculating the flux density over a wide range of conditions of two-frequency magnetization. For many ordinary purposes a table or graph of some of the results is convenient; useful ones are therefore included.

The hypergeometric expansions in the coefficients of case 1 can be put to further use to examine the behavior of the induction for special ratios of fundamental amplitudes and frequencies. When $k \ll 1$ and $\kappa \ll 1$ the coefficients of the several frequencies in the induction reduce to simple, rational, algebraic expressions of the amplitudes. These coefficients likewise reduce when $k \doteqdot 1$ and $\kappa \doteqdot 1$, since

$$F(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma - \beta - \alpha)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}.$$

Third order coefficients for case 1 with restricted parameters, and also those for case 2 are tabulated in the accompanying table through terms in the lowest power of the smaller amplitude. Underlined subscripts distinguish lower side frequencies; a bar under a digit indicates it is to be taken with a negative sign.

When the ratio of amplitudes is unrestricted, graphs of the coefficients which specify the induction enable their magnitudes to be determined most readily. The strongest products are found to be the third order lower side frequencies; Fig. 3, calculated by A. G. Tynan, may be used to get both components of either of these. The corresponding upper side frequencies are almost as strong; their amplitudes can be found from the figure by virtue of the relations $c_{12} = c_{1\underline{2}}$ and $c_{21} = c_{2\underline{1}}$, since their other components are zero.

By interchanging P and Q and likewise p and q in either the table or the graphs, the subscripts are also reversed. In the table the inequalities restricting the columns are reversed thereby, since the interchanged quantities are arbitrarily assignable. This procedure does not extend the applicability of either case, but does permit the application of case 1 directly to both extremes of the amplitude ratio and the use of the curves to evaluate the amplitudes of both lower side frequencies. The scope of the table and curves given has been extended in this manner. A field of usefulness sufficiently extensive for most purposes of present-day communication is thereby achieved.

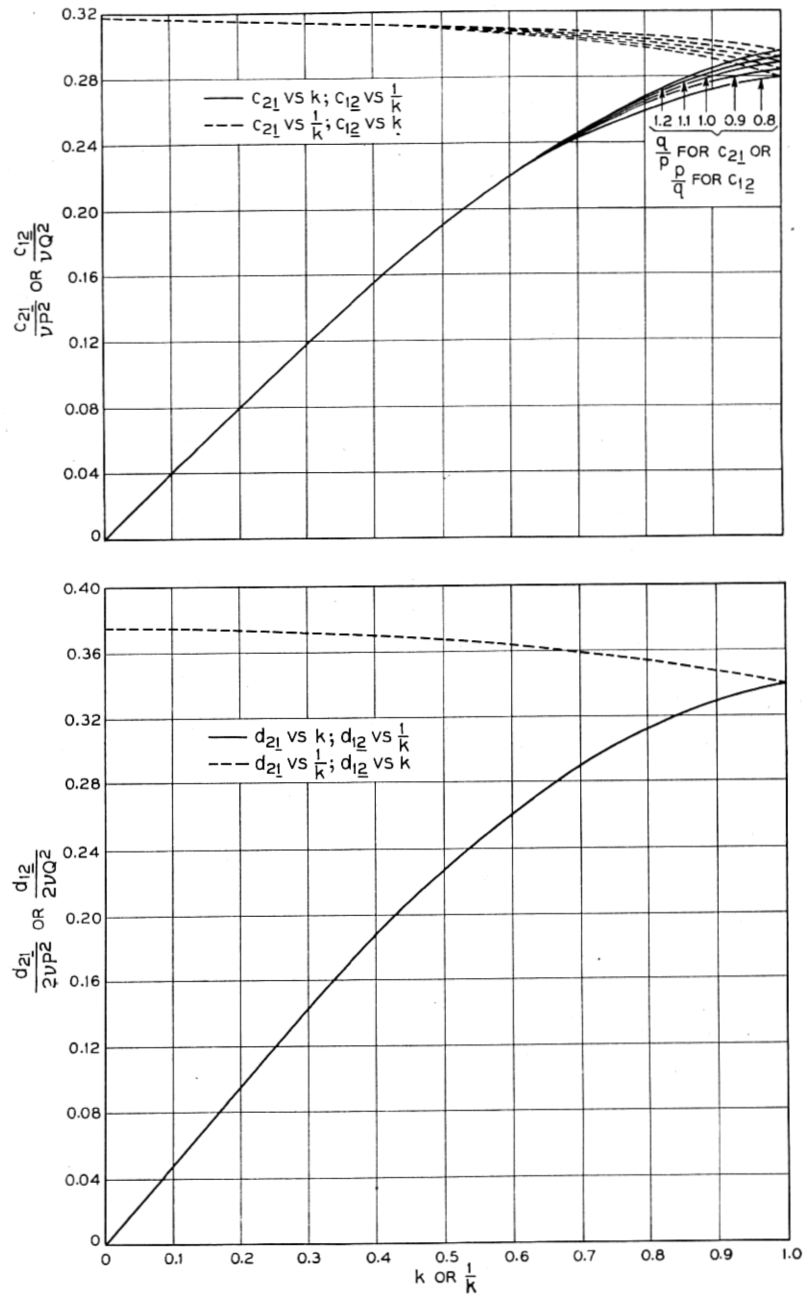


Fig. 3—Components of third order lower side frequencies—case 1.

TABLE I

	Case 1a $\kappa \ll 1$ $k \ll 1$	Case 1b $\kappa \doteqdot 1$ $k \doteqdot 1$	Case 1c $\kappa \gg 1$ $k \gg 1$	Case 2 $\kappa \ll 1$ $k \gg \kappa$
c_{10}	$\frac{8}{3\pi}\nu P^2$	$\frac{128}{9\pi^2}\nu PQ$	$\frac{4}{\pi}\nu PQ$	$\frac{8}{3\pi}\nu P^2$
d_{10}	$[\mu_0 + 2\nu P]P$	$\left[\mu_0 + \frac{32}{3\pi}\nu P\right]P$	$[\mu_0 + 3\nu Q]P$	$[\mu_0 + 2\nu P]P$
c_{01}	$\frac{4}{\pi}\nu PQ$	$\frac{128}{9\pi^2}\nu PQ$	$\frac{8}{3\pi}\nu Q^2$	$\frac{8}{3\pi}\nu Q^2$
d_{01}	$[\mu_0 + 3\nu P]Q$	$\left[\mu_0 + \frac{32}{3\pi}\nu Q\right]Q$	$[\mu_0 + 2\nu Q]Q$	$[\mu_0 + 2\nu(2P + Q)]Q$
$c_{1\underline{2}}$	$-\frac{1}{\pi}\nu Q^2$	$-\frac{128}{45\pi^2}\nu Q^2$	$-\frac{4}{3\pi}\nu PQ$	$\frac{2}{3\pi}[\mu_0 + 2\nu(2P + 3Q)]\kappa P$
$d_{1\underline{2}}$	$\frac{3}{4}\nu(1 + k)Q^2$	$\frac{32}{15\pi}\nu Q^2$	νPQ	$-\frac{32}{5\pi^2}\nu\kappa PQ$
c_{12}	$-\frac{1}{\pi}\nu Q^2$	$-\frac{128}{45\pi^2}\nu Q^2$	$-\frac{4}{3\pi}\nu PQ$	$\frac{2}{3\pi}[\mu_0 + 2\nu(2P + 3Q)]\kappa P$
d_{12}	0	0	0	$\frac{32}{5\pi^2}\nu\kappa PQ$
$c_{2\underline{1}}$	$\frac{4}{3\pi}\nu PQ$	$\frac{128}{45\pi^2}\nu P^2$	$\frac{1}{\pi}\nu P^2$	$-\frac{4}{9\pi}\nu[5P - 6Q]\kappa P$
$d_{2\underline{1}}$	νPQ	$\frac{32}{15\pi}\nu P^2$	$\frac{3}{4}\nu(1 + 1/k)P^2$	$\frac{1}{3}\nu[P - 6Q]\kappa P$
c_{21}	$-\frac{4}{3\pi}\nu PQ$	$-\frac{128}{45\pi^2}\nu P^2$	$-\frac{1}{\pi}\nu P^2$	$-\frac{4}{9\pi}\nu[5P + 6Q]\kappa P$
d_{21}	0	0	0	$-\frac{1}{3}\nu[P - 6Q]\kappa P$
c_{30}	$-\frac{8}{15\pi}\nu P^2$	$-\frac{128}{225\pi^2}\nu P^2$	0	$-\frac{4}{3\pi}\left[\mu_0\kappa - \frac{2}{5}\nu P\right]P$
d_{30}	0	0	0	0
c_{03}	0	$-\frac{128}{225\pi^2}\nu Q^2$	$-\frac{8}{15\pi}\nu Q^2$	$-\frac{8}{15\pi}\nu Q^2$
d_{03}	0	0	0	$\frac{8}{5}\nu\kappa PQ$

Intermodulation Products

Generated Modulation Voltages

From the foregoing results the voltage generated in a coil of N turns on a closed ferromagnetic core of cross-sectional area A can be found by the use of

$$e(t) = NA\, 10^{-8} \frac{dB}{dt} \cdot \tag{52}$$

Components of this voltage segregated according to frequency are of the type

$$e_{mn}(t) = (mp + nq)NA\, 10^{-8} [c_{mn} \cos (mp + nq)t - d_{mn} \sin (mp + nq)t], \tag{53}$$

each proportional to its frequency and in general having two components in quadrature. The amplitudes of these will be designated by

$$E_{mn}' = (mp + nq)NA\, 10^{-8}\, c_{mn}, \tag{54}$$

$$E_{mn}'' = (mp + nq)NA\, 10^{-8} d_{mn}. \tag{55}$$

The amplitude of their resultant is

$$E_{mn} = (mp + nq)NA\, 10^{-8}\sqrt{c_{mn}^2 + d_{mn}^2}. \tag{56}$$

One component, if it greatly exceeds the other, may be taken as the generated distortion voltage. The various coefficients to which the components of the voltage are proportional have already been calculated, and also given in tabular or graphical form for specific instances. The relations

$$P = \frac{0.4\pi NI}{l}, \quad Q = \frac{0.4\pi NJ}{l},$$

where l is the mean length of magnetic path, may be used to convert these results into terms of the current amplitudes I and J. Where r-m-s quantities are used, they will be distinguished by bars over them.

Several features of the distortion are particularly outstanding. Perhaps chief among these is the dissimilarity of corresponding upper and lower side-frequency voltages. Inasmuch as these are products of a reactance modulator, they might be expected to be in the ratio of their frequencies, as they are found to be for one component. Often, however, a predominating component appears at each lower side frequency with no counterpart at the upper side frequency. This component can be traced to the different axial slopes of the several branches of the loop, caused by their different points of origination. The slopes are fixed by the envelope of the magnetizing force; since this envelope is periodic in the difference of the fundamental frequencies, difference products will appear without corresponding sum products in the induction. Such a phenomenon is a fundamental property of the multivalued characteristic, and will occur wherever the envelope of a complex wave is instrumental in selecting the branch to be traversed.

No simple yet general rule seems to embrace the behavior of the various products in an iron core coil as governed by the amplitudes of the fundamental currents. Each voltage component is proportional to its frequency and to the product of two amplitudes, but these often enter in a complicated way. For fundamental frequencies close together all the higher order voltages vary directly with the hysteretic coefficient ν; for widely separated frequencies the distortion may depend also on the permeability through its effect on the axial slope of the minor loops. At the extremes of the amplitude ratio certain products or their components are found to be independent of one of the fundamental currents, the stronger one in some instances. When case 1 is applicable the third harmonic of the weaker fundamental current is suppressed below the value it would have without the stronger current superposed, while the third harmonic of the stronger fundamental is affected only slightly by the presence of a second frequency. Perusal of the table will disclose more detailed relations.

Distortion Measured in Coils

Voltages calculated by the theory have been compared with measured values for several coils using two common core materials. The agreement found provides a check of the theoretical predictions.

The two third order lower side frequencies of fundamental frequencies $p/2\pi = 760$ cycles per second and $q/2\pi = 600$ cycles per second are plotted in Fig. 4 for a higher frequency current of ten milliamperes in an iron dust coil of special design. The frequencies of the products are 920 and 440 cycles per second. These data were taken by I. E. Wood and the calculations were made by A. G. Tynan. These curves show the product as a function of the amplitude ratio more directly than the curves of Fig. 3.

Both upper and lower third order side frequencies have been measured by A. G. Landeen. The results are given in Figs. 5 and 6 for an annular core of iron dust. It is so wound that the magnetizing force is 0.04 times the current in milliamperes. The figures show two third

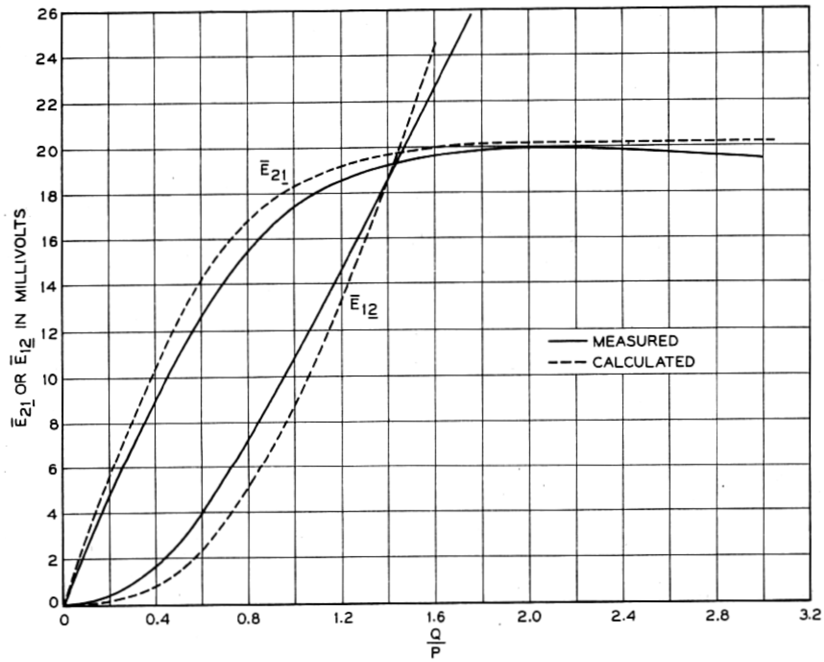


Fig. 4—Lower side-frequency voltages in an iron dust coil.

order products plotted against the fundamental current of higher frequency in each instance. For these measurements the current of one frequency was maintained at some fixed value and the amplitude of the other one varied. Both sum and difference products, but with different fundamental frequencies, are exhibited. The upper side

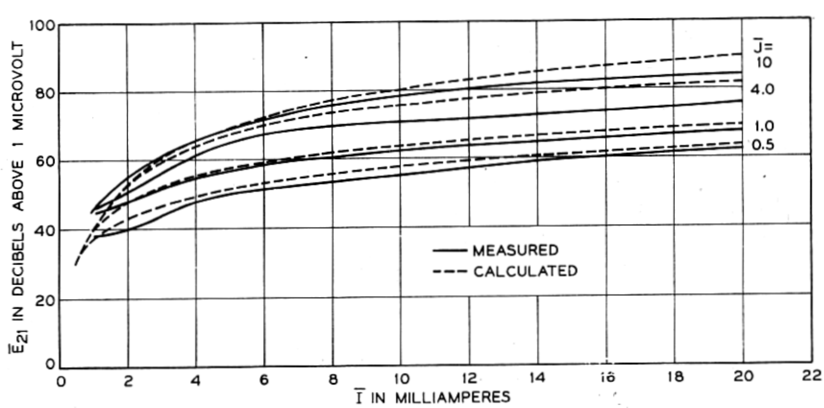


Fig. 5—Twenty-five kc. third order product of 9 and 7 kc., iron dust core.
$\mu_0 = 24.5$, $\nu = 0.18$, $L_0 = 4.54$ mh., $H = 38.9\ I$.

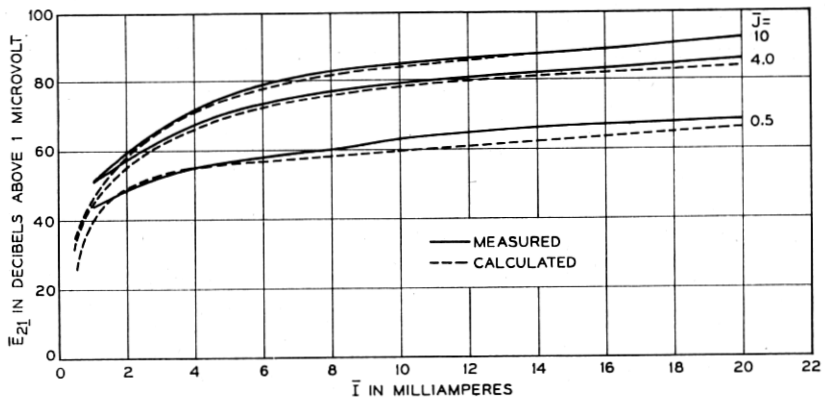


Fig. 6—Seventeen-kc. third order product of 13 and 9 kc., iron dust core. $\mu_0 = 24.5$, $\nu = 0.18$, $L_0 = 4.54$ mh., $H = 38.9\ I$.

frequency is the 25-kilocycle product of 9 and 7 kilocycles; the lower side frequency is the 17-kilocycle product of 13 and 9 kilocycles.

Some measurements to which subcase 1*b* is applicable are given in Figs. 7 and 8. Each curve gives a third order product for a permalloy dust core. The upper side frequency is 25 kilocycles generated by 9 and 7 kilocycle fundamental frequencies. The lower side frequency

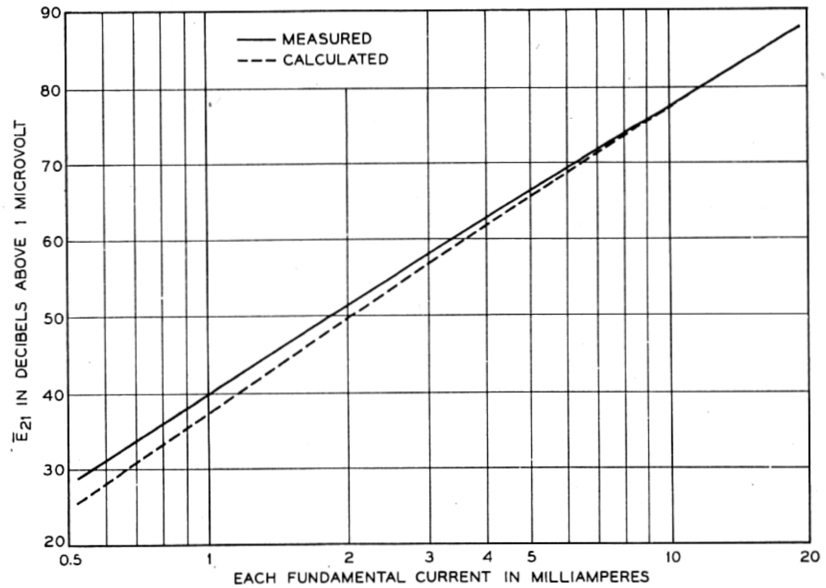


Fig. 7—Twenty-five kc. third order product of equal currents at 9 and 7 kc., permalloy dust core. $\mu_0 = 75$, $\nu = 0.41$, $L_0 = 5.45$ mh., $H = 36.0\ I$.

is 13 kilocycles from fundamental frequencies at 21 and 17 kilocycles. In these measurements both fundamental currents were changed simultaneously so as to be kept equal throughout. The approach to saturation at currents above ten milliamperes is apparent on these curves; below, the distortion voltage is proportional to the square of the current.

The measured curves are seen to agree well with the calculated ones in every instance, confirming theoretical values of the products within close limits. Eddy currents were negligible in all these coils because of the dust cores. The use of the formulae to determine important

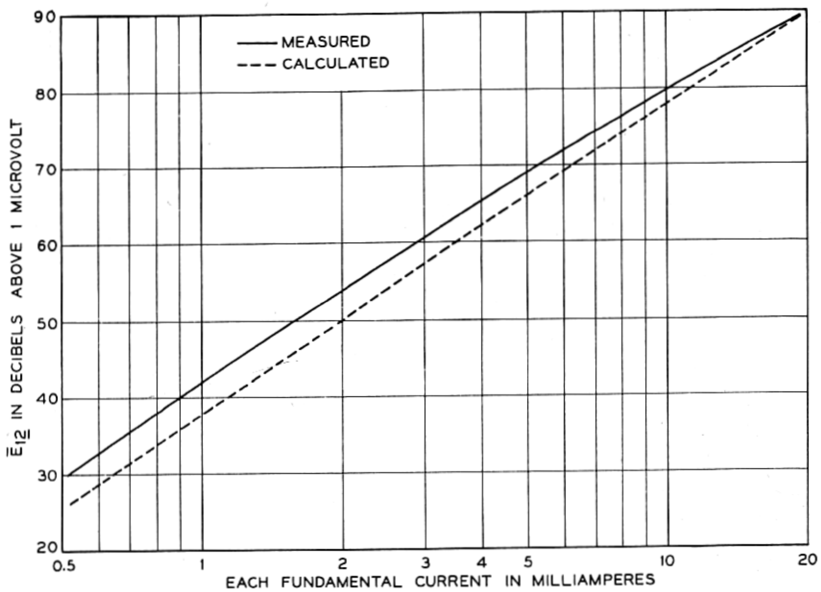


Fig. 8—Thirteen-kc. third order product of equal currents at 21 and 17 kc., permalloy dust core. $\mu_0 = 75$, $\nu = 0.41$, $L_0 = 5.45$ mh., $H = 36.0\ I$.

intermodulation products from the constants of the coils therefore seems to be justified.

Correlation with Single-Frequency Results

In the single-frequency case Peterson found the resistance and reactance at a fundamental frequency $\omega/2\pi$ to be increased by

$$\Delta R(\omega, H) = \frac{8}{3\pi}\,\omega L_0 \frac{\nu}{\mu_0} H,$$

$$\Delta X(\omega, H) = 2\omega L_0 \frac{\nu}{\mu_0} H,$$

respectively, on account of hysteresis and variable permeability, H being the maximum of the applied magnetizing force. The total non-linear reactance is then

$$X(\omega, H) = X_0 + \Delta X(\omega, H)\cdot$$

with

$$X_0 = \omega L_0,$$

the constant part, representing the reactance the coil would have if the permeability remained constant at its initial value.

The distortion voltages for a two-frequency input may be written in terms of these non-linear impedances. With some simplication this is done in Table II for special cases, using equation (56) and the relations

$$P = \frac{0.4\pi NI}{l}, \quad Q = \frac{0.4\pi NJ}{l}\cdot$$

TABLE II

	Case 1a $\kappa \ll 1$ $k \ll 1$	Case 1b $\kappa \doteq 1$ $k \doteq 1$
$E_{1\underline{2}}$	$0.960\ \Delta R(p - 2q, Q)J$	$0.870\ \Delta R(p - 2q, Q)J$
E_{12}	$0.375\ \Delta R(p + 2q, Q)J$	$0.340\ \Delta R(p + 2q, Q)J$
$E_{\underline{2}1}$	$1.28\ \Delta R(2p - q, Q)I$	$0.870\ \Delta R(2p - q, P)I$
E_{21}	$0.500\ \Delta R(2p + q, Q)I$	$0.340\ \Delta R(2p + q, P)I$
E_{30}	$0.200\ \Delta R(3p, P)I$	$0.068\ \Delta R(3p, P)I$
E_{03}	—	$0.068\ \Delta R(3q, Q)J$
	Case 1c $k \gg 1$ $\kappa \gg 1$	**Case 2** $\kappa \ll 1$ $k \gg \kappa$
$E_{1\underline{2}}$	$1.28\ \Delta R(p - 2q, P)J$	$0.212\ \kappa X(p - 2q, 2P + 3Q)I$
E_{12}	$0.500\ \Delta R(p + 2q, P)J$	$0.212\ \kappa X(p + 2q, 2P + 3Q)I$
$E_{\underline{2}1}$	$0.960\ \Delta R(2p - q, P)I$	$1.06\ \kappa\Delta R(2p - q, P - 0.944Q)I$
E_{21}	$0.375\ \Delta R(2p + q, P)I$	$1.23\ \kappa\Delta R(2p + q, P + 2.73Q)I$
E_{30}	—	$[0.425\ \kappa X_0(3p) - 0.200\ \Delta R(3p, P)]I$
E_{03}	$0.200\ \Delta R(3p, Q)J$	$0.200\ \Delta R(3p, Q)J$

The general formulæ of case 1 can be similarly represented, but not as concisely. Besides exhibiting the connection between intermodulation products and impedance changes, this table provides a convenient means for computing voltage components directly from data obtainable by single-frequency bridge measurements.

Hysteretic Impedances

The fundamental voltages are not included in the table. For them each component is separately significant. The singly primed E's are in phase with the corresponding magnetizing current and the doubly primed E's in quadrature. Hence, the former are resistance drops and the latter reactance drops, defining incremental components of impedance analogous to those mentioned for a single-frequency input.

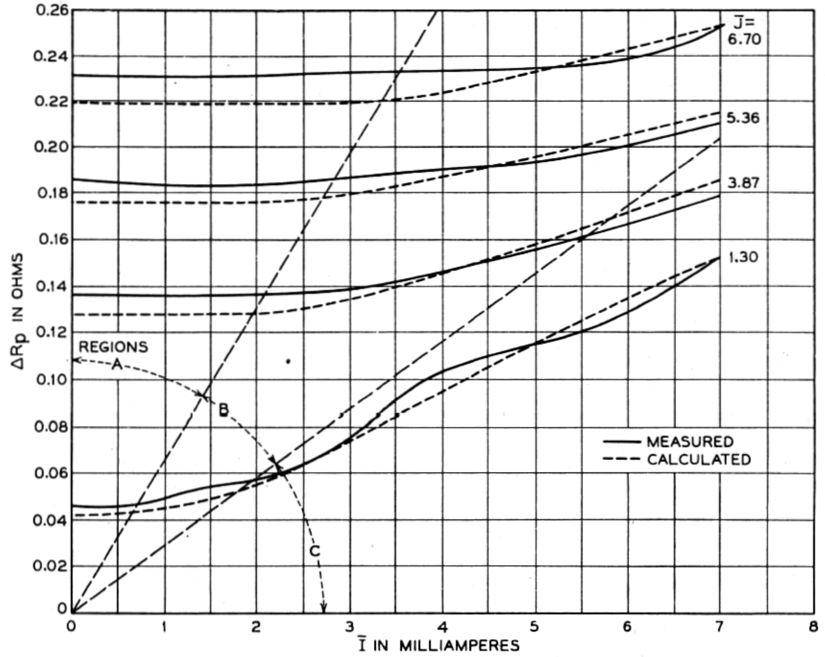
For each of the fundamental frequencies the results developed previously may be used to determine these components. They represent the hysteretic resistance and the hysteretic reactance to the fundamental at hand, specified by a subscript p or q. They are tabulated in Table III for the special cases considered. The total resistance of the coil

TABLE III

	Case 1a $\kappa \ll 1$ $k \ll 1$	Case 1b $\kappa \doteq 1$ $k \doteq 1$	Case 1c $k \gg 1$ $\kappa \gg 1$	Case 2 $\kappa \ll 1$ $k \gg \kappa$
ΔR_p	$\frac{16}{15}\frac{N}{l}\frac{\nu}{\mu_0}pL_0I$	$\frac{256}{45\pi}\frac{N}{l}\frac{\nu}{\mu_0}pL_0J$	$\frac{8}{5}\frac{N}{l}\frac{\nu}{\mu_0}pL_0J$	$\frac{8}{5}\frac{N}{l}\frac{\nu}{\mu_0}pL_0I$
ΔX_p	$\frac{4\pi}{5}\frac{N}{l}\frac{\nu}{\mu_0}pL_0I$	$\frac{64}{15}\frac{N}{l}\frac{\nu}{\mu_0}pL_0I$	$\frac{6\pi}{5}\frac{N}{l}\frac{\nu}{\mu_0}pL_0J$	$\frac{4\pi}{5}\frac{N}{l}\frac{\nu}{\mu_0}pL_0I$
ΔR_q	$\frac{8}{5}\frac{N}{l}\frac{\nu}{\mu_0}qL_0I$	$\frac{256}{45\pi}\frac{N}{l}\frac{\nu}{\mu_0}qL_0I$	$\frac{16}{15}\frac{N}{l}\frac{\nu}{\mu_0}qL_0J$	$\frac{16}{15}\frac{N}{l}\frac{\nu}{\mu_0}qL_0J$
ΔX_q	$\frac{6\pi}{5}\frac{N}{l}\frac{\nu}{\mu_0}qL_0I$	$\frac{64}{15}\frac{N}{l}\frac{\nu}{\mu_0}qL_0J$	$\frac{4\pi}{5}\frac{N}{l}\frac{\nu}{\mu_0}qL_0J$	$\frac{4\pi}{5}\frac{N}{l}\frac{\nu}{\mu_0}qL_0[2I + J]$

to either fundamental current can be calculated by adding to the value of ΔR from the table the resistance of the windings, the eddy current resistance, and the initial (viscosity) resistance, all evaluated for the frequency of the fundamental. The eddy currents must be so small that the flux density is substantially uniform across the cross section of the core. The reactance X_0 of the coil can be diminished by the eddy current reduction factor for the fundamental frequency and added to the hysteretic reactance to give the net reactance of the coil under these conditions.

The table is helpful in evaluating the effect of one fundamental current upon the other. Within the limits of the analysis, which in substance limits the permeability to linear variation with the field intensity, the hysteresis loss at any frequency is either increased or unchanged by the superposition of a second frequency. Nearly equal input currents whose frequencies do not differ greatly share equally the hysteresis loss. This amounts to about twice what it would if either fundamental flowed alone. If the frequencies differ greatly, the



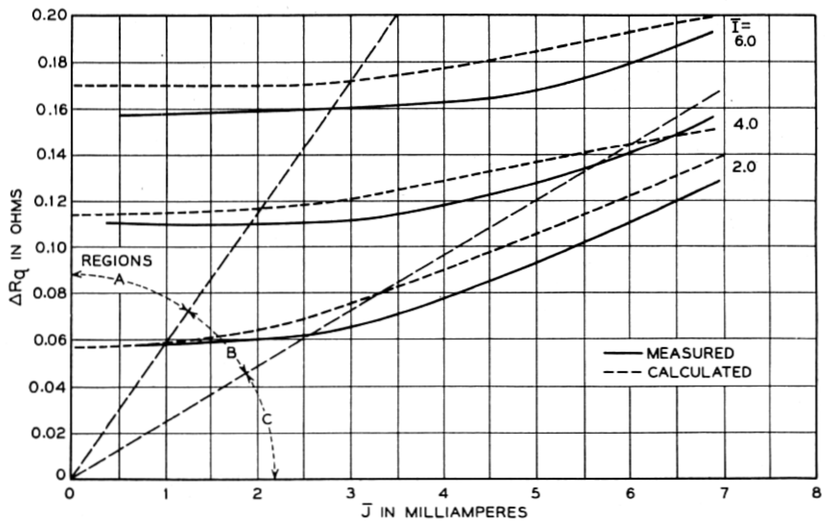


Fig. 9—Hysteretic resistance to one current in the presence of another.

loss to either is not affected by the other; if the amplitudes differ greatly, the loss to either is governed by the stronger current. In no case, at these weak fields, does the increased loss at one frequency reduce the loss at another frequency, contrary to well established experiments at considerably higher fields, for which the hysteresis loss at one frequency may be reduced by superposing a magnetizing force at a different frequency. The inductance is the same to both funda-

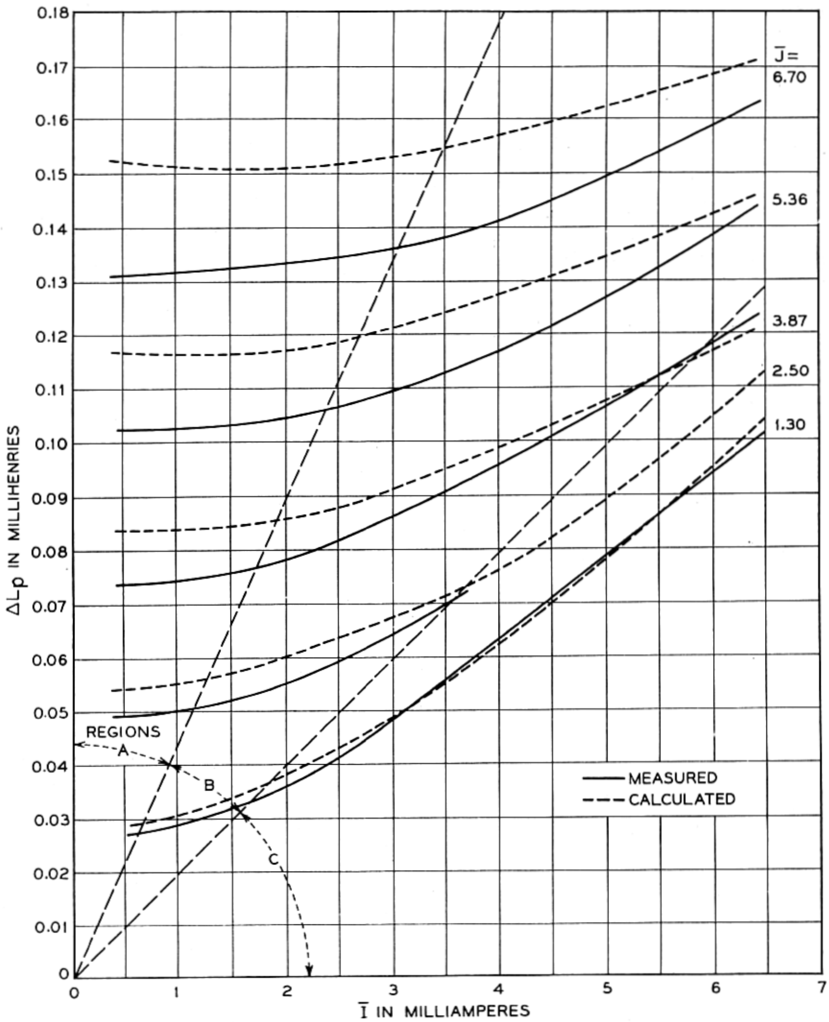


Fig. 10a—Hysteretic inductance to one current in the presence of another.

mentals in all cases except the second, for which a small difference exists. The effect of a superposed alternating current is always apparent through an increased inductance, although sometimes the increase may be slight; it is determined by the larger current.

The influence of one fundamental current upon another has been termed mutual crowding. Because of the increased attenuation, and at times because of resulting unbalance or phase shift, crowding becomes important when different frequencies or bands of frequencies

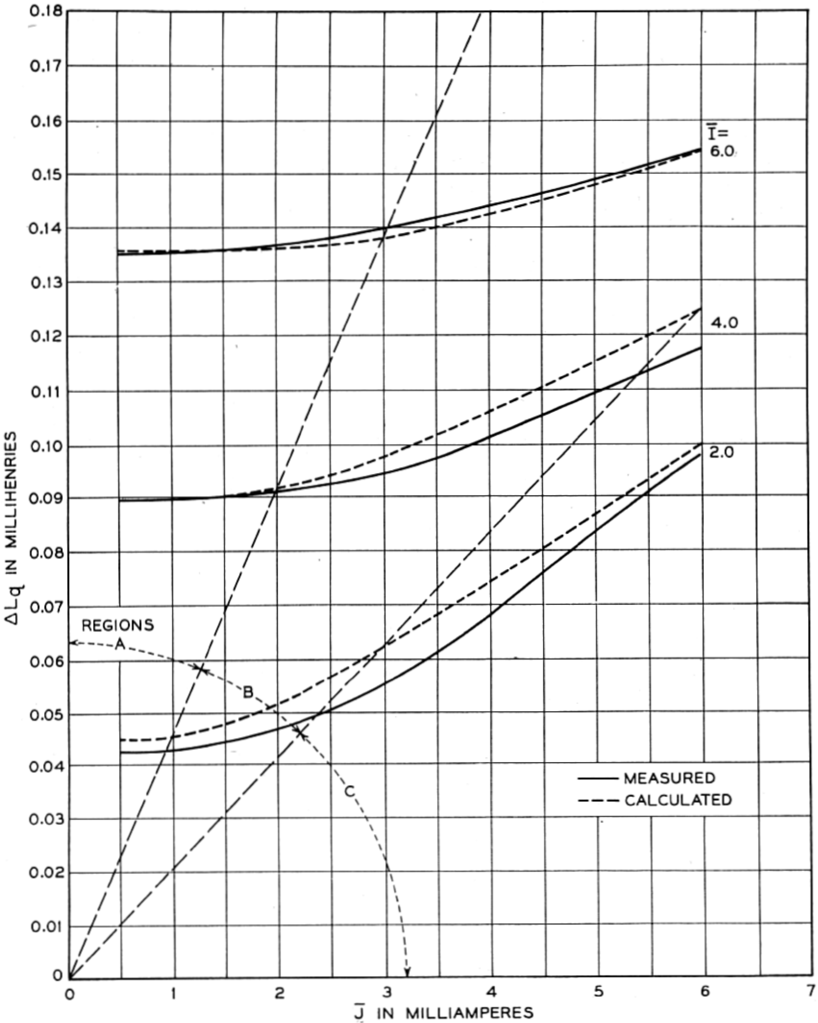


Fig. 10b—Hysteretic inductance to one current in the presence of another.

are transmitted simultaneously through a circuit including ferromagnetic material.

Incremental impedances for a twenty-two millihenry permalloy dust loading coil are given in Figs. 9 and 10. The current I had a frequency of 550 cycles per second, and J 475 cycles per second. Measured and calculated values are plotted for comparison, with regions of applicability of the special subcases indicated. Portions of calculated curves falling in regions A or C were computed by the two sets of formulæ for subcases $1a$ and $1c$, portions in the middle of region B by the formulæ for subcase $1b$. The trend of the quantities measured is accurately portrayed by the calculations and agreement of the values is good.

All the curves commence at those values of resistance or inductance which would obtain in a single-frequency case with a current having the magnitude of the one here superposed in fixed amount. Upon increasing the variable current the measured quantities show an increase as it begins to preponderate, and eventually they approach asymptotically the values they would have if it flowed alone.

The measurements were made by L. R. Wrathall using a Maxwell inductance bridge with two imputs and a tuned detector. Eddy currents being of no consequence at the low frequencies employed, the chief sources of possible error are calibrations of the standards used and variation in the temperature of the coil during taking of the data. Changes in winding resistance caused by the latter are of the same order of magnitude as the changes in hysteretic resistance being observed. Precautions against both possibilities were taken.

Conclusion

The multiplicity of forms of complex hysteresis loops makes their analysis in general a complicated and difficult matter if indeed possible at all. Extensive experiments with two frequencies must be completed and the results classified according to the types of loops before an acceptable method of taking their form into account can be formulated. The parameters k and κ seem to be effective quantities for denoting concisely a particular form of loop in many instances.

A way of representing the behavior of complex loops more exactly than do Madelung's propositions is needed, and might be the fruit of precise experiments designed to clear up also the early closure and lack of closure which Lehde apparently found in minor loops. The tracing of complex loops is not simply cyclic, and only when a nearly complete magnetic cycle is executed between successive maxima in the magnetizing force can conditions approaching a cyclic state be expected to exist. Some experimental evidence of performance in other conditions is a present need which can perhaps be met by a thorough investigation of spiral characteristics. These seem to have been ignored entirely in the past, the literature dealing with subsidiary loops only along a magnetization curve or a branch of a major loop.

Correlation of certain magnetic phenomena to a degree not heretofore attainable is made possible by the preceding development. Among at least some of these a qualitative connection has been well recognized. Flutter and allied effects are known to be aspects of modulation, and crowding is observed to accompany non-linear distortion quite generally. These features are related quantitatively by means of the theory, and are linked with their single-frequency counterparts. It thus becomes feasible to evaluate some of the more abstruse occurrences in terms of readily understandable effects simpler in nature; ultimate in this direction is the use of steady state results to forecast the behavior of transmitted speech or music.