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An Extension of the Theory of Three-Electrode Vacuum Tube Circuits

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The relations between input voltage and output current of the three-electrode vaccum tube are discussed when arbitrary feedback is present between grid and plate circuits. Fundamental assumptions are that the amplification factor is constant and conductive grid current absent. The relations developed in the present paper are generalizations of those given by J. R. Carson in I. R. E. Proc. of 1919, page 187. The use of the theory is illustrated by application to a simple modulator circuit. The numerical calculations in this case indicate that neglecting the effects of interelectrode tube capacitances may introduce serious errors.

INTRODUCTION

THE relations between input voltage and output current of the three-electrode vacuum tube when connected to impedances in both input and output circuits have been the subject of several papers. One of the first more extensive treatments of this problem was given by J. R. Carson,[1] using a method of successive approximations. The theory was further extended by F. B. Llewellyn,[2] E. Peterson and H. P. Evans,[3] and J. G. Brainerd.[4] The theories given by these authors did not take into account any feedback between input and output circuit except in the first approximation.

The aim of the present paper is to extend the theory of the three-electrode vacuum tube to include the effects of feedback between input and output circuits not only in the first but also in the second and higher approximations. The assumptions underlying Carson's treatment, constancy of the amplification factor and absence of conductive grid current, will be maintained. The extension of the present theory to such cases as treated by Llewellyn, Peterson-Evans and Brainerd still remains to be done.

[1] J. R. Carson: *I. R. E. Proc.*, April, 1919, page 187.
[2] F. B. Llewellyn: *B. S. T. J.*, July, 1926, page 433.
[3] E. Peterson and H. Evans: *B. S. T. J.*, July, 1927, page 442.
[4] J. G. Brainerd: *I. R. E. Proc.*, June, 1929, page 1006.

Theory

Let us consider the circuit arrangement shown in Fig. 1, where Z_1, Z_2 and Z_3 are linear impedances which may include interelectrode admittances. The impressed variable electromotive forces whose instantaneous values are denoted by E_g and E_p are in series with the impedances Z_g and Z_p, respectively. In the absence of these electromotive forces direct currents and voltages are established in the circuit due to constant grid and plate electromotive forces. With the variable electromotive forces impressed incremental currents and voltages are produced. The instantaneous values of these incremental voltages are indicated on Fig. 1 by g, e, v and p. The incremental plate current is J. The positive directions of these quantities are given by the directions of the arrows.

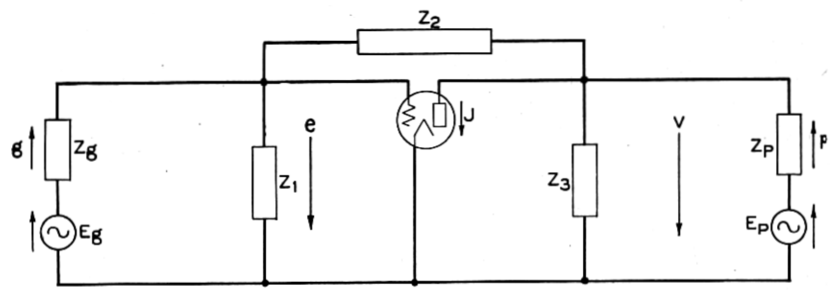

Fig. 1—Three-electrode vacuum tube and circuit.

We will now make two restrictive assumptions: first that the grid is never positive so that conductive grid current is absent, and second that the amplification factor μ is constant.

The basis for the analysis is given by the characteristic tube equation:

$$I = f\left(E_c + \frac{E_b}{\mu}\right), \tag{1}$$

where I is the total instantaneous current flowing from plate to filament; E_c is the total instantaneous potential difference between grid and filament and E_b the total instantaneous potential difference between plate and filament. μ is the amplification factor. The relation between the increments e, v and J is given by the following equation:

$$J = P_1(\mu e + v) + P_2(\mu e + v)^2 + \cdots + P_n(\mu e + v)^n + \cdots, \tag{2}$$

where

$$P_m = \frac{1}{m!}\frac{\partial^m I}{\partial E_b{}^m}$$

and has to be evaluated at the operating point.[1]

We have further:

$$E_g = g + e, \qquad E_p = p + v. \tag{3}$$

The equations (3) are obtained by applying the circuital laws to the network external to the tube.

We now proceed to a solution of equations (2) and (3) by means of a method of successive approximations. Let

$$J = \sum_1^\infty J_i, \qquad g = \sum_1^\infty g_i, \qquad e = \sum_1^\infty e_i, \tag{4}$$
$$p = \sum_1^\infty p_i, \qquad v = \sum_1^\infty v_i$$

and let us define the relations between the terms in the series (4) as follows:

$$J_1 = P_1(\mu e_1 + v_1), \qquad Eg = g_1 + e_1, \qquad E_p = p_1 + v_1, \tag{5}$$

$$J_2 = P_1(\mu e_2 + v_2) + P_2(\mu e_1 + v_1)^2, \tag{6}$$
$$0 = g_2 + e_2, \qquad 0 = p_2 + v_2,$$

$$J_3 = P_1(\mu e_3 + v_3) + 2P_2(\mu e_1 + v_1)(\mu e_2 + v_2) + P_3(\mu e_1 + v_1)^3, \tag{7}$$
$$0 = g_3 + e_3, \qquad 0 = p_3 + v_3,$$

$$J_4 = P_1(\mu e_4 + v_4) + P_2(\mu e_2 + v_2)^2 + 2P_2(\mu e_1 + v_1)(\mu e_3 + v_3)$$
$$+ 3P_3(\mu e_2 + v_2)(\mu e_1 + v_1)^2 + P_4(\mu e_1 + v_1)^4, \tag{8}$$
$$0 = g_4 + e_4, \qquad 0 = p_4 + v_4,$$

and so forth for subsequent terms.[5]

If we now let

$$R_0 = \frac{1}{P_1}, \tag{9}$$

[1] Loc. cit.

[5] The procedure of finding these equations is as follows: By substituting the first term in each of the series (4) into (2) and (3) and neglecting all terms higher than the first order equations (5) are obtained. By substituting the first two terms in each of the series (4) into (2) and (3), and neglecting terms of higher order than the second and by noting (5) equations (6) are found and so on for the remaining equations.

where R_0 is the internal resistance of the tube, equations (5), (6), (7) and (8) may be rewritten as:

$$R_0J_1 - v_1 = \mu e_1, \qquad E_g = g_1 + e_1, \qquad E_p = p_1 + v_1, \tag{10}$$

$$\begin{aligned} R_0J_2 - v_2 &= \mu e_2 + R_0P_2(\mu e_1 + v_1)^2, \\ 0 &= g_2 + e_2, \qquad 0 = p_2 + v_2, \end{aligned} \tag{11}$$

$$\begin{aligned} R_0J_3 - v_3 &= \mu e_3 + 2R_0P_2(\mu e_1 + v_1)(\mu e_2 + v_2) + R_0P_3(\mu e_1 + v_1)^3 \\ 0 &= g_3 + e_3, \qquad 0 = p_3 + v_3, \end{aligned} \tag{12}$$

$$\begin{aligned} R_0J_4 - v_4 &= \mu e_4 + R_0P_2(\mu e_2 + v_2)^2 + 2R_0P_2(\mu e_1 + v_1)(\mu e_3 + v_3) \\ &\quad + 3R_0P_3(\mu e_2 + v_2)(\mu e_1 + v_1)^2 + R_0P_4(\mu e_1 + v_1)^4, \\ 0 &= g_4 + e_4, \qquad 0 = p_4 + v_4, \end{aligned} \tag{13}$$

and so forth.

Equations (10) to (13) admit of simple physical interpretations. Referring first to equations (10) it is clear that the equivalent circuit corresponding to Fig. 1 for first order quantities is given by Fig. 2. Similarly Fig. 3 is the equivalent circuit of Fig. 1 for second order effects and Fig. 4 for third order effects. Higher order effects correspond to similar circuits.

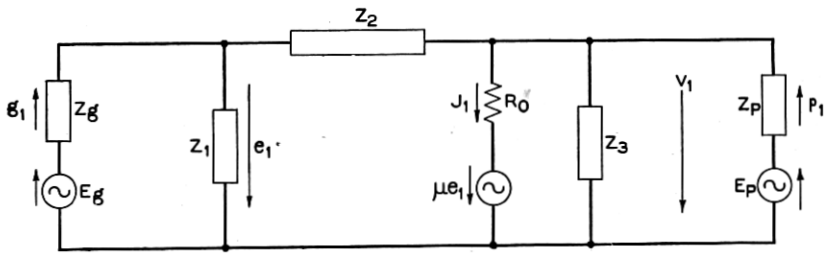

Fig. 2—Equivalent circuit, first order effects.

The equivalence expressed by Fig. 2 is the familiar circuit which has found such wide application, for instance, in amplifier and oscillator work; while the equivalent circuits in Figs. 3 and 4 represent the second and third order effects. With no feedback, that is when Z_2 is infinite, they reduce to the equivalences given by Carson.[1] Comparing now any two equivalent circuits for same order effects with and without feedback we find different values of the electromotive forces appearing in series with the internal tube resistance R_0. Otherwise

[1] Loc. cit., equations (23) and following.

the two circuits are identical except that for one the impedance Z_2 is finite and for the other infinite.

By the aid of the equivalent circuits given, that is by using equations (10), (11), (12), (13) and so forth, the terms in the series (4) can be calculated. These series formally satisfy equations (2) and (3) and are the solutions if they converge.

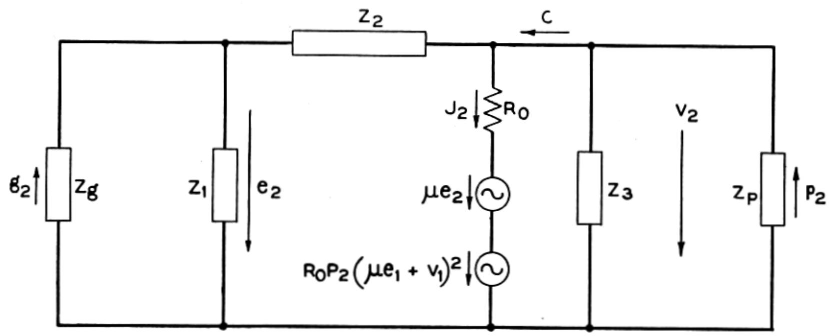

Fig. 3—Equivalent circuit, second order effects.

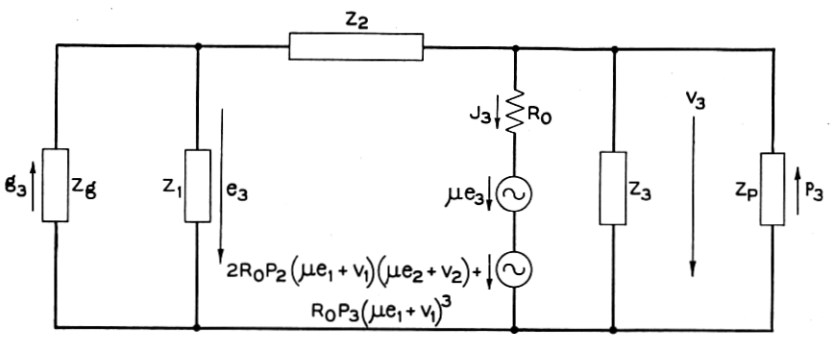

Fig. 4—Equivalent circuit, third order effects.

For the purpose of fixing our ideas we assumed at the start a definite circuit to which the tube was connected. It is obvious, however, that no matter how complicated the linear network is to which the input and output terminals of the tube are connected the procedure given above can be followed.

Application to a Modulator Circuit

As an illustration of the theory just presented we shall calculate the steady state second order effect assuming the circuit configuration to be that given in Fig. 1. In so doing we shall assume that no variable e.m.f. is impressed in the plate circuit and that the impressed e.m.f. in the grid circuit is given by [6]

$$e_g = K \cos \omega_1 t + S \cos \omega_2 t. \tag{14}$$

We now find the instantaneous value of $\mu e_1 + v_1$ by solving the mesh equations for the equivalent circuit of Fig. 2. The result is:

$$\mu e_1 + v_1 = R_0 \left[\left| \frac{F(\omega_1)}{Z(\omega_1)} \right| K \cos (\omega_1 t - \varphi(\omega_1)) + \left| \frac{F(\omega_2)}{Z(\omega_2)} \right| S \cos (\omega_2 t - \varphi(\omega_2)) \right], \tag{15}$$

where

$$\begin{aligned} F(\omega) &= \frac{Z_1(\mu Z_2 + \mu Z_p' + Z_p')}{(Z_g + Z_1)(Z_g' + Z_2)}, \\ Z(\omega) &= R_0 + Z_p' + \frac{Z_p'(R_0 + \mu Z_g')}{Z_2 + Z_g'}, \\ Z_p' &= \frac{Z_3 Z_p}{Z_3 + Z_p}, \qquad Z_g' = \frac{Z_1 Z_g}{Z_1 + Z_g}, \\ \frac{F(\omega)}{Z(\omega)} &= \left| \frac{F(\omega)}{Z(\omega)} \right| e^{-\varphi(\omega) i} (i = \sqrt{-1}). \end{aligned} \tag{16}$$

In equations (16) we note that Z_1, Z_2, Z_3, Z_g and Z_p all are complex impedances. The driving e.m.f. for the second approximation is $R_0 P_2(\mu e_1 + v_1)^2$. Letting

$$M = R_0^3 P_2, \tag{17}$$

we get from (15)

$$\begin{aligned} R_0 P_2(\mu e_1 + v_1)^2 \\ = M \Bigg[&\frac{1}{2} \left(\left| \frac{F(\omega_1)}{Z(\omega_1)} \right|^2 K^2 + \left| \frac{F(\omega_2)}{Z(\omega_2)} \right|^2 S^2 \right) \\ &+ \frac{1}{2} \left| \frac{F(\omega_1)}{Z(\omega_1)} \right|^2 K^2 \cos (2\omega_1 t - 2\varphi(\omega_1)) \\ &+ \frac{1}{2} \left| \frac{F(\omega_2)}{Z(\omega_2)} \right|^2 S^2 \cos (2\omega_2 t - 2\varphi(\omega_2)) \\ &+ \left| \frac{F(\omega_1) F(\omega_2)}{Z(\omega_1) Z(\omega_2)} \right| KS \cos ((\omega_1 - \omega_2) t - \varphi(\omega_1) + \varphi(\omega_2)) \\ &+ \left| \frac{F(\omega_1) F(\omega_2)}{Z(\omega_1) Z(\omega_2)} \right| KS \cos ((\omega_1 + \omega_2) t - \varphi(\omega_1) - \varphi(\omega_2)) \Bigg]. \end{aligned} \tag{18}$$

[6] The extension to any number of sinusoidal e.m.f.'s of arbitrary phases in both plate and grid circuit is obvious.

The driving e.m.f. given by (18) thus consists of a number of sinusoidal components including one of zero frequency. By means of the superposition theorem and the mesh equations we obtain the current and voltage distribution for our equivalent circuit in Fig. 3. Let us for instance calculate the instantaneous current flowing through the impedance consisting of Z_p and Z_3 in parallel and indicated by C in Fig. 3. The result is

$$
\begin{aligned}
C = M \Bigg[& \frac{\frac{1}{2}\left(\left| \frac{F(\omega_1)}{Z(\omega_1)} \right|^2 K^2 + \left| \frac{F(\omega_2)}{Z(\omega_2)} \right|^2 S^2 \right)}{Z(0)} + \\
& + \frac{\frac{1}{2}\left| \frac{F(\omega_1)}{Z(\omega_1)} \right|^2 K^2}{|Z(2\omega_1)|} \cos\,(2\omega_1 t - 2\varphi(\omega_1) - \psi(2\omega_1)) \\
& + \frac{\frac{1}{2}\left| \frac{F(\omega_2)}{Z(\omega_2)} \right|^2 S^2}{|Z(2\omega_2)|} \cos\,(2\omega_2 t - 2\varphi(\omega_2) - \psi(2\omega_2)) \\
& + \frac{\left| \frac{F(\omega_1)F(\omega_2)}{Z(\omega_1)Z(\omega_2)} \right| KS}{|Z(\omega_1 - \omega_2)|} \cos\,((\omega_1 - \omega_2)t - \varphi(\omega_1) \\
& \qquad\qquad\qquad + \varphi(\omega_2) - \psi(\omega_1 - \omega_2)) \\
& + \frac{\left| \frac{F(\omega_1)F(\omega_2)}{Z(\omega_1)Z(\omega_2)} \right| KS}{|Z(\omega_1 + \omega_2)|} \cos\,((\omega_1 + \omega_2)t - \varphi(\omega_1) \\
& \qquad\qquad\qquad - \varphi(\omega_2) - \psi(\omega_1 + \omega_2)) \Bigg],
\end{aligned}
\tag{19}
$$

where $\psi(\omega)$ is defined by

$$Z(\omega) = |Z(\omega)| e^{i\psi(\omega)} (i = \sqrt{-1}). \tag{20}$$

Let us now consider the peak value of the current of lower side-band frequency. This value is from (19)

$$MKS \left| \frac{F(\omega_1)F(\omega_2)}{Z(\omega_1)Z(\omega_2)Z(\omega_1 - \omega_2)} \right| . \tag{21}$$

If we write

$$|Z(\omega)| = \left| \frac{R_0}{1 + \frac{R_0 + \mu Z_g'}{Z_2 + Z_g'}} + Z_p' \right|_\omega \left| 1 + \frac{R_0 + \mu Z_g'}{Z_2 + Z_g'} \right|_\omega , \tag{22}$$

$$|F(\omega)| = (\mu + 1) \left| \frac{Z_1}{(Z_g + Z_1)(Z_g' + Z_2)} \right|_\omega \left| \frac{\mu}{\mu + 1} Z_2 + Z_p' \right|_\omega , \tag{23}$$

expression (21) becomes:

$$\frac{MKS(\mu+1)^2\left|\frac{Z_1}{(Z_g+Z_1)(Z_g'+Z_2)}\right|_{\omega_1}\times\left|\frac{Z_1}{(Z_g+Z_1)(Z_g'+Z_2)}\right|_{\omega_2}\times\left|\frac{\mu}{\mu+1}Z_2+Z_p'\right|_{\omega_1}\left|\frac{\mu}{\mu+1}Z_2+Z_p'\right|_{\omega_2}}{\left|1+\frac{R_0+\mu Z_g'}{Z_2+Z_g'}\right|_{\omega_1}\left|1+\frac{R_0+\mu Z_g'}{Z_2+Z_g'}\right|_{\omega_2}\left|1+\frac{R_0+\mu Z_g'}{Z_2+Z_g'}\right|_{(\omega_1-\omega_2)}\times\left|Z_p'+\frac{R_0}{1+\frac{R_0+\mu Z_g'}{Z_2+Z_g'}}\right|_{\omega_1}\left|Z_p'+\frac{R_0}{1+\frac{R_0+\mu Z_g'}{Z_2+Z_g'}}\right|_{\omega_2}\times\left|Z_p'+\frac{R_0}{1+\frac{R_0+\mu Z_g'}{Z_2+Z_g'}}\right|_{(\omega_1-\omega_2)}}. \quad (24)$$

With no feedback present, that is when $Z_2=\infty$, equation (24) reduces to

$$\frac{MKS\mu^2\left|\frac{Z_1}{Z_1+Z_g}\right|_{\omega_1}\left|\frac{Z_1}{Z_1+Z_g}\right|_{\omega_2}}{|Z_p'+R_0|_{\omega_1}|Z_p'+R_0|_{\omega_2}|Z_p'+R_0|_{(\omega_1-\omega_2)}}. \quad (25)$$

But in this case $K\left|\frac{Z_1}{Z_1+Z_g}\right|_{\omega_1}$ is the peak value K' of the grid voltage of frequency ω_1 and similarly $S\left|\frac{Z_1}{Z_1+Z_2}\right|_{\omega_2}$ is the peak value S' of the grid voltage of frequency ω_2. Expression (25) may thus be written:

$$\frac{M\mu^2K'S'}{|Z_p'+R_0|_{\omega_1}|Z_p'+R_0|_{\omega_2}|Z_p'+R_0|_{(\omega_1-\omega_2)}},$$

which is the well known expression given by Carson.[7]

For the purpose of getting an idea of the magnitudes involved let us consider a numerical example. A Western Electric No. 101D vacuum tube may have the following constants when used as a modulator: $R_0=9000$ ohms; $\mu=6$; grid-cathode capacitance $C_1=10.5$, plate-grid capacitance $C_2=4.8$, and plate-cathode capacitance $C_3=8.1$ micromicrofarads. The impedances Z_p and Z_g are assumed to be pure resistances at all frequencies with the values 9000 and 10,000 ohms, respectively. The impressed e.m.f. is of the form given by

[7] Carson: *I. R. E. Proc.*, June, 1921, page 243.

the equation (14) where $\omega_1/2\pi$ is equal to 250,000 and $\omega_2/2\pi$ is equal to 5,000,000 cycles per second. As a reference condition let us take that for which the effect of the interelectrode capacitances is neglected. The plate current for this condition is obtained from equation (25) when Z_1 and Z_3 as well as Z_2 are made infinite. As a next step we compute the plate current from equation (24) when Z_1, Z_2, and Z_3 are the impedances corresponding to the interelectrode capacitances C_1, C_2, and C_3, respectively. It is found that this plate current is 12 db below that obtained in the reference condition. Finally it is of some interest to compute the plate current when the grid-plate capacitance alone is effective. This plate current is obtained from equation (24) by assuming Z_1 and Z_3 to be infinite and Z_2 to be the impedance corresponding to the capacitance C_2. This current is found to be 24 db below that of the reference condition.