Mutual Impedance of Grounded Wires Lying On or Above the Surface of the Earth *

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This paper presents a formula for the mutual impedance of any insulated wires of negligible diameter lying in horizontal planes above the surface of the earth and grounded by vertical wires at their four end-points. The formula holds for frequencies which are not too high to allow all displacement currents to be neglected. Tables and curves are given to facilitate numerical computation by means of the formula.

In the expansion of this formula for low frequencies and for any heights the first two terms give the direct-current mutual impedance; the third term is independent of the heights, thus being identically the same as that previously found for wires on the surface. The mutual impedance for wires at any heights H and h, with separations large in comparison with these heights, is found to be approximately equal to the mutual impedance for wires on the surface multiplied by the complex factor $[1 + \Gamma(H + h)]$, where Γ is the propagation constant in the earth.

THE formula established in a previous paper [1] for the mutual impedance of any grounded thin wires lying on the surface of the earth has now been extended to include wires lying in horizontal planes above the surface of the earth and grounded by vertical wires at their four end-points. As before, we assume the earth to be flat, semi-infinite in extent, of negligible capacitivity, of uniform resistivity ρ, and of inductivity ν equal to that of free space. The air is also assumed to be of negligible capacitivity and of inductivity ν equal to that of free space. All displacement currents are thus neglected both in the earth and in the air; this is the assumption which is ordinarily made as a first approximation at power frequencies for the shorter transmission lines.

By the same general method of derivation as before, the extended formula is found to be:

$$Z_{12} = \int\int\left[\frac{d^2P(r,H,h)}{dSds} + \cos\epsilon\, M(r,H,h)\right]dSds, \qquad (A)$$

where

$$P(r,H,h) = P_0(r) + P_1(r,H+h) - P_2(r,|H-h|),$$

* A brief report of the principal theoretical result obtained in this paper was recently published in the *Physical Review* (2), *41*, 536–537 (August 15, 1932). An error in the formula for $N_0(r)$, as printed there, should be corrected: in the denominator of the fraction, r^2 should be r^3.

[1] R. M. Foster, "Mutual Impedance of Grounded Wires Lying on the Surface of the Earth," *Bell System Technical Journal*, *10*, 408–419 (July, 1931); see also *Bulletin of the American Mathematical Society*, *36*, 367–368 (May, 1930).

$$M(r,H,h) = M_0(r) + M_1(r,H+h) - M_2(r,|H-h|),$$

$$P_0(r) = \frac{\rho}{2\pi r},$$

$$P_1(r,s) = \frac{i\omega\nu}{4\pi}\int_0^\infty \left\{\frac{s}{\mu} - \frac{1-e^{-s\mu}}{\mu^2}\left[\frac{(\mu^2+\Gamma^2)^{1/2}-\mu}{(\mu^2+\Gamma^2)^{1/2}+\mu}\right]\right\} J_0(r\mu)d\mu,$$

$$P_2(r,d) = \frac{i\omega\nu}{4\pi}\left[d\log\frac{(r^2+d^2)^{1/2}+d}{r} - (r^2+d^2)^{1/2} + r\right],$$

$$M_0(r) = \frac{\rho}{2\pi r^3}[1-(1+\Gamma r)e^{-\Gamma r}],$$

$$M_1(r,s) = \frac{i\omega\nu}{4\pi}\int_0^\infty (1-e^{-s\mu})\left[\frac{(\mu^2+\Gamma^2)^{1/2}-\mu}{(\mu^2+\Gamma^2)^{1/2}+\mu}\right] J_0(r\mu)d\mu,$$

$$M_2(r,d) = \frac{i\omega\nu}{4\pi}\left[\frac{1}{r} - \frac{1}{(r^2+d^2)^{1/2}}\right].$$

The integrations in the iterated integral are extended over the two wires S and s, lying in planes at heights H and h, respectively. The elements dS and ds are separated by the horizontal distance r and include the angle ϵ between their directions. The propagation constant of plane electromagnetic waves in the earth, varying with the time as $e^{i\omega t}$, is Γ, which equals $(i\omega\nu/\rho)^{1/2}$. All distances are measured in meters, Z_{12} in ohms and ρ in meter-ohms; ν has the value 1.256 $\times 10^{-6}$ henries per meter; ω is equal to 2π times the frequency; J_0 is the Bessel function of order zero. The derivation of the formula is outlined in the latter part of this paper.

The functions P and M are divided into three parts: first, P_0 and M_0, which are functions only of the horizontal distance r; secondly, P_1 and M_1, which are functions of r and of the sum of the two heights H and h; and thirdly, P_2 and M_2, which are functions of r and of the numerical difference of the two heights. These three parts are arranged in the order of relative importance when the heights are reasonably small. For zero heights, the functions P and M reduce to P_0 and M_0, which are the values previously obtained for wires on the surface. For small values of the heights, P_1 and M_1 are of the order of magnitude of the sum of the heights, whereas P_2 and M_2 are of the order of magnitude of the square of the difference.

For some purposes it is convenient to transform formula (A) into the alternative expression:

$$Z_{12} = \int\int\left[\frac{d^2P(r,H,h)}{dSds} + \cos\epsilon\, M(r,H,h)\right]dSds, \qquad (B)$$

where

$$P(r,H,h) = P_0(r) + P^0(r,H,h) + P_3(r,H+h),$$

$$M(r,H,h) = M^0(r,H,h) + M_3(r,H+h),$$

$$P_0(r) = \frac{\rho}{2\pi r},$$

$$P^0(r,H,h) = \frac{i\omega\nu}{4\pi}\left\{ H \log \frac{[r^2+(H+h)^2]^{1/2}+H+h}{[r^2+(H-h)^2]^{1/2}+H-h} \right.$$
$$+ h \log \frac{[r^2+(H+h)^2]^{1/2}+H+h}{[r^2+(H-h)^2]^{1/2}-H+h}$$
$$\left. + [r^2+(H-h)^2]^{1/2} - [r^2+(H+h)^2]^{1/2} \right\}$$

$$P_3(r,s) = \frac{i\omega\nu}{2\pi}\int_0^\infty \frac{1-e^{-s\mu}}{\mu[(\mu^2+\Gamma^2)^{1/2}+\mu]} J_0(r\mu)d\mu,$$

$$M^0(r,H,h) = \frac{i\omega\nu}{4\pi}\left\{ \frac{1}{[r^2+(H-h)^2]^{1/2}} - \frac{1}{[r^2+(H+h)^2]^{1/2}} \right\},$$

$$M_3(r,s) = \frac{i\omega\nu}{2\pi}\int_0^\infty \frac{\mu e^{-s\mu}}{(\mu^2+\Gamma^2)^{1/2}+\mu} J_0(r\mu)d\mu.$$

The functions P and M are again divided into three parts: first, P_0, the term giving the direct-current mutual resistance; secondly, P^0 and M^0, terms giving the mutual impedance on the assumption of a perfectly conducting earth; and thirdly, P_3 and M_3, the correction terms for the finite conductivity of the earth. The P^0 and M^0 terms thus give $i\omega\nu/8\pi$ times the mutual Neumann integral of the two complete circuits formed from the actual wire circuits by adding to them their reflections in the surface of the earth.

For small values of Γ, the P_3 and M_3 terms can be expanded as follows:

$$\left.\begin{aligned} P_3(r,s) &= \frac{i\omega\nu}{4\pi}\{- s\log\Gamma - s\log[(r^2+s^2)^{1/2}+s] - r \\ &\qquad + (r^2+s^2)^{1/2} + [2\log 2 + \psi(1) + \tfrac{1}{2}]s \\ &\qquad + \tfrac{1}{3}s^2\Gamma - \tfrac{1}{48}s(3r^2-2s^2)\Gamma^2\log\Gamma + \cdots\}, \\ M_3(r,s) &= \frac{i\omega\nu}{4\pi}\left\{ \frac{1}{(r^2+s^2)^{1/2}} - \tfrac{2}{3}\Gamma - \tfrac{1}{4}s\Gamma^2\log\Gamma + \cdots \right\}. \end{aligned}\right\} \quad (1)$$

By means of these expansions, the complete P and M functions, as given by formula (B), can be put into the form:

$$\left.\begin{aligned} P(r,H,h) &= \frac{\rho}{2\pi r} + \frac{i\omega\nu}{4\pi}\{-H\log\{[r^2+(H-h)^2]^{1/2}+H-h\} \\ &\qquad - h\log\{[r^2+(H-h)^2]^{1/2}-H+h\} \\ &\qquad + [r^2+(H-h)^2]^{1/2} - r \\ &\qquad + F(H,h,\Gamma) + O(\Gamma^2\log\Gamma)\}, \\ M(r,H,h) &= \frac{i\omega\nu}{4\pi}\left\{\frac{1}{[r^2+(H-h)^2]^{1/2}} - \tfrac{2}{3}\Gamma + O(\Gamma^2\log\Gamma)\right\}. \end{aligned}\right\} \tag{2}$$

The function $F(H,h,\Gamma)$ is of no consequence, since it does not involve r; it contributes nothing to the value of the impedance. The remaining terms are infinitesimals of order $(\Gamma^2 \log \Gamma)$ for infinitesimal values of Γ; they are thus of higher order than Γ itself.

By means of equation (2) we can now show that the first three terms in the expansion of Z_{12} for low frequencies and for any heights are given by

$$Z_{12} = \frac{\rho}{2\pi}\left(\frac{1}{Aa} - \frac{1}{Ab} - \frac{1}{Ba} + \frac{1}{Bb}\right) + \frac{i\omega\nu}{4\pi}N_{(S-E)(s-e)} + \frac{1-i}{6\pi}\left(\frac{\omega^3\nu^3}{2\rho}\right)^{1/2} AB\,ab\cos\theta + \cdots, \tag{3}$$

where $N_{(S-E)(s-e)}$ is the mutual Neumann integral between the two circuits formed by the wires S and s, lying in planes at heights H and h above the earth, grounded by vertical wires at their four end-points, and with earth returns,—the four grounding points on the surface of the earth being A,B and a,b, respectively. The angle between the straight lines AB and ab is designated by θ. $N_{(S-E)(s-e)}$ is equal to N_{Ss}, the mutual Neumann integral between the two wires S and s, augmented by terms which depend only on the arithmetical distances between eight points,—the four end-points and the four grounding points.

The first two terms in the expansion (3) are precisely the direct-current mutual impedance as given ten years ago by G. A. Campbell.[2] The third term is independent of the heights of the wires; it is thus identically the same as the third term previously found for wires on the surface.

The leading term in the expansion of Z_{12} for a long straight wire S and any wire s located near the midpoint of S, for any heights, is

$$\int\left\{\frac{i\omega\nu}{2\pi}\log\frac{[x^2+(H+h)^2]^{1/2}}{[x^2+(H-h)^2]^{1/2}} + \frac{i\omega\nu}{\pi}\int_0^\infty \frac{e^{-(H+h)\mu}}{(\mu^2+\Gamma^2)^{1/2}+\mu}\cos x\mu\,d\mu\right\}\cos\epsilon\,ds, \tag{4}$$

[2] G. A. Campbell, "Mutual Impedances of Grounded Circuits," *Bell System Technical Journal*, 2, (no. 4), 1–30 (October, 1923).

x being the positive horizontal distance from ds to S, and ϵ the angle between ds and S.

This result is derived immediately from formula (B) upon assuming S to be doubly infinite and then integrating over its entire length. The first part of the expression comes from the M^0 function, the second part from the M_3 function. The other functions contribute nothing to the leading term in the expansion.

The expression enclosed in braces in (4) is the mutual impedance gradient parallel to an infinite wire at a positive horizontal distance x from the wire. It agrees with the results published independently by F. Pollaczek,[3] J. R. Carson,[4] and G. Haberland.[5] Pollaczek has also investigated the case of two grounded circuits of finite length, with certain modifications.[6]

For purposes of computation, however, formula (A) is better, in general, than formula (B). A distinct improvement is effected by multiplying all distances which occur in (A) by the attenuation constant, that is, by $(\omega\nu/2\rho)^{1/2}$. The numerical value thus obtained for any one distance is indicated by a prime accent on the corresponding letter. We then find the mutual impedance expressed in the following form:

$$Z_{12} = \frac{(\omega\nu\rho/2)^{1/2}}{2\pi}\int\int\left[\frac{d^2Q(r',H',h')}{dS'ds'} + \cos\epsilon\, N(r',H',h')\right]dS'ds', \quad (C)$$

where

$$Q(r',H',h') = Q_0(r') + Q_1(r',H'+h') - Q_2(r',|H'-h'|),$$

$$N(r',H',h') = N_0(r') + N_1(r',H'+h') - N_2(r',|H'-h'|),$$

$$Q_0(r') = \frac{1}{r'},$$

$$Q_1(r',s') = i\int_0^\infty\left\{\frac{s'}{\mu} - \frac{1-e^{-s'\mu}}{\mu^2}\left[\frac{(\mu^2+2i)^{1/2}-\mu}{(\mu^2+2i)^{1/2}+\mu}\right]\right\}J_0(r'\mu)d\mu,$$

$$Q_2(r',d') = i\left[d'\log\frac{(r'^2+d'^2)^{1/2}+d'}{r'} - (r'^2+d'^2)^{1/2} + r'\right],$$

[3] F. Pollaczek, "Über das Feld einer unendlich langen wechselstromdurchflossenen Einfachleitung," *Elektrische Nachrichten-technik*, *3*, 339–359 (September, 1926).

[4] J. R. Carson, "Wave Propagation in Overhead Wires with Ground Return," *Bell System Technical Journal*, *5*, 539–554 (October, 1926).

[5] G. Haberland, "Theorie der Leitung von Wechselstrom durch die Erde," *Zeitschrift für angewandte Mathematik und Mechanik*, *6*, 366–379 (October, 1926).

[6] F. Pollaczek, "Gegenseitige Induktion zwischen Wechselstromfreileitungen von endlicher Länge," *Annalen der Physik* (4), *87*, 965–999 (December, 1928). His assumptions regarding conditions at the ground connections seem to depart considerably from the conditions assumed in the present paper, and moreover his results are not expressed in convenient form for direct comparison with the formula given above for Z_{12}.

$$N_0(r') = \frac{1}{r'^3}\{1 - [1 + (1+i)r']e^{-(1+i)r'}\},$$

$$N_1(r',s') = i\int_0^\infty (1 - e^{-s'\mu})\left[\frac{(\mu^2+2i)^{1/2} - \mu}{(\mu^2+2i)^{1/2} + \mu}\right] J_0(r'\mu)d\mu,$$

$$N_2(r',d') = i\left[\frac{1}{r'} - \frac{1}{(r'^2 + d'^2)^{1/2}}\right];$$

the prime accent applied to any length L indicating the corresponding modified length $L' = (\omega\nu/2\rho)^{1/2}L$.

As in formula (A), the six constituent functions involved in formula (C) are arranged in order of importance: first, Q_0 and N_0, functions only of the modified horizontal distance r'; secondly, Q_1 and N_1, functions of r' and of the sum of the two modified heights H' and h'; and thirdly, Q_2 and N_2, functions of r' and of the numerical difference of the two modified heights.

To assist in the numerical application of this formula, a table of values of the real and imaginary parts of N_0 has been computed, for all values of r' from 0 to 10, in steps of 0.1. Beyond this range, the function is practically equal to the leading term in its asymptotic expansion, namely, $1/r'^3$. These computed values are also shown graphically in Fig. 1. The imaginary part changes sign at approximately $r' = 3.8$, and again at 7.0, oscillating for increasing values of r', although approaching zero very rapidly indeed.

The real and imaginary parts of the functions $Q_1(r',s')$ and $N_1(r',s')$ are shown in Figs. 2, 3, 4, and 5, for the range of r' from 0 to 10, and for the set of values of s' from 0 to 0.2 in steps of 0.02. It is believed that this will cover the range of heights likely to be encountered in ordinary problems. To cover this range adequately it was necessary to show portions of the N_1 curves with the horizontal scale enlarged two and a half times, and with a greatly reduced vertical scale, in Figs. 4-A and 5-A.

For actual computation, the Q_2 and N_2 functions are already expressed in (A) in convenient, closed form, but for purposes of comparison with Q_1 and N_1, the corresponding values of Q_2 and N_2 are shown in Figs. 6 and 7, which are drawn to the same scales as Figs. 3 and 5.

Tables II and III give the corresponding numerical values of Q_1 and N_1 for the range of r' from 0 to 1, in steps of 0.1, as well as the values for 1.5 and 2.

TABLE I

Real and Imaginary Parts of $N_0(r')$

r'	Real	Imag.	r'	Real	Imag.
0	0.66667	∞	5.0	0.0081667	−0.00038659
0.1	0.61800	9.33461	5.1	0.0076496	−0.00034816
0.2	0.57199	4.33824	5.2	0.0071782	−0.00031048
0.3	0.52860	2.67724	5.3	0.0067477	−0.00027431
0.4	0.48778	1.85135	5.4	0.0063538	−0.00024017
0.5	0.44949	1.36031	5.5	0.0059927	−0.00020839
0.6	0.41363	1.03722	5.6	0.0056609	−0.00017918
0.7	0.38014	0.81043	5.7	0.0053554	−0.00015262
0.8	0.34892	0.64405	5.8	0.0050736	−0.00012872
0.9	0.31987	0.51804	5.9	0.0048131	−0.00010740
1.0	0.29291	0.42035	6.0	0.0045717	−0.000088557
1.1	0.26792	0.34327	6.1	0.0043477	−0.000072047
1.2	0.24480	0.28161	6.2	0.0041392	−0.000057706
1.3	0.22346	0.23177	6.3	0.0039449	−0.000045358
1.4	0.20379	0.19116	6.4	0.0037634	−0.000034822
1.5	0.18568	0.15785	6.5	0.0035936	−0.000025919
1.6	0.16905	0.13041	6.6	0.0034344	−0.000018472
1.7	0.15379	0.10770	6.7	0.0032850	−0.000012315
1.8	0.13981	0.088878	6.8	0.0031444	−0.0000072892
1.9	0.12703	0.073237	6.9	0.0030120	−0.0000032482
2.0	0.11535	0.060227	7.0	0.0028872	−0.0000000570
2.1	0.10470	0.049402	7.1	0.0027693	0.0000024076
2.2	0.095002	0.040395	7.2	0.0026578	0.0000042564
2.3	0.086174	0.032905	7.3	0.0025522	0.0000055886
2.4	0.078152	0.026685	7.4	0.0024522	0.0000064921
2.5	0.070871	0.021526	7.5	0.0023573	0.0000070444
2.6	0.064268	0.017257	7.6	0.0022672	0.0000073128
2.7	0.058287	0.013734	7.7	0.0021816	0.0000073559
2.8	0.052874	0.010835	7.8	0.0021001	0.0000072236
2.9	0.047980	0.0084577	7.9	0.0020226	0.0000069586
3.0	0.043558	0.0065174	8.0	0.0019488	0.0000065967
3.1	0.039567	0.0049415	8.1	0.0018785	0.0000061678
3.2	0.035966	0.0036689	8.2	0.0018114	0.0000056965
3.3	0.032719	0.0026483	8.3	0.0017474	0.0000052028
3.4	0.029792	0.0018364	8.4	0.0016863	0.0000047027
3.5	0.027156	0.0011967	8.5	0.0016280	0.0000042086
3.6	0.024782	0.00069852	8.6	0.0015722	0.0000037302
3.7	0.022645	0.00031616	8.7	0.0015190	0.0000032745
3.8	0.020720	0.00002803	8.8	0.0014680	0.0000028466
3.9	0.018987	−0.00018390	8.9	0.0014193	0.0000024498
4.0	0.017427	−0.00033467	9.0	0.0013727	0.0000020858
4.1	0.016021	−0.00043678	9.1	0.0013280	0.0000017555
4.2	0.014754	−0.00050056	9.2	0.0012852	0.0000014587
4.3	0.013612	−0.00053454	9.3	0.0012443	0.0000011945
4.4	0.012582	−0.00054572	9.4	0.0012050	0.00000096159
4.5	0.011652	−0.00053979	9.5	0.0011673	0.00000075815
4.6	0.010811	−0.00052138	9.6	0.0011312	0.00000058219
4.7	0.010050	−0.00049420	9.7	0.0010966	0.00000043156
4.8	0.0093603	−0.00046121	9.8	0.0010633	0.00000030402
4.9	0.0087349	−0.00042473	9.9	0.0010313	0.00000019732
5.0	0.0081667	−0.00038659	10.0	0.0010007	0.00000010925

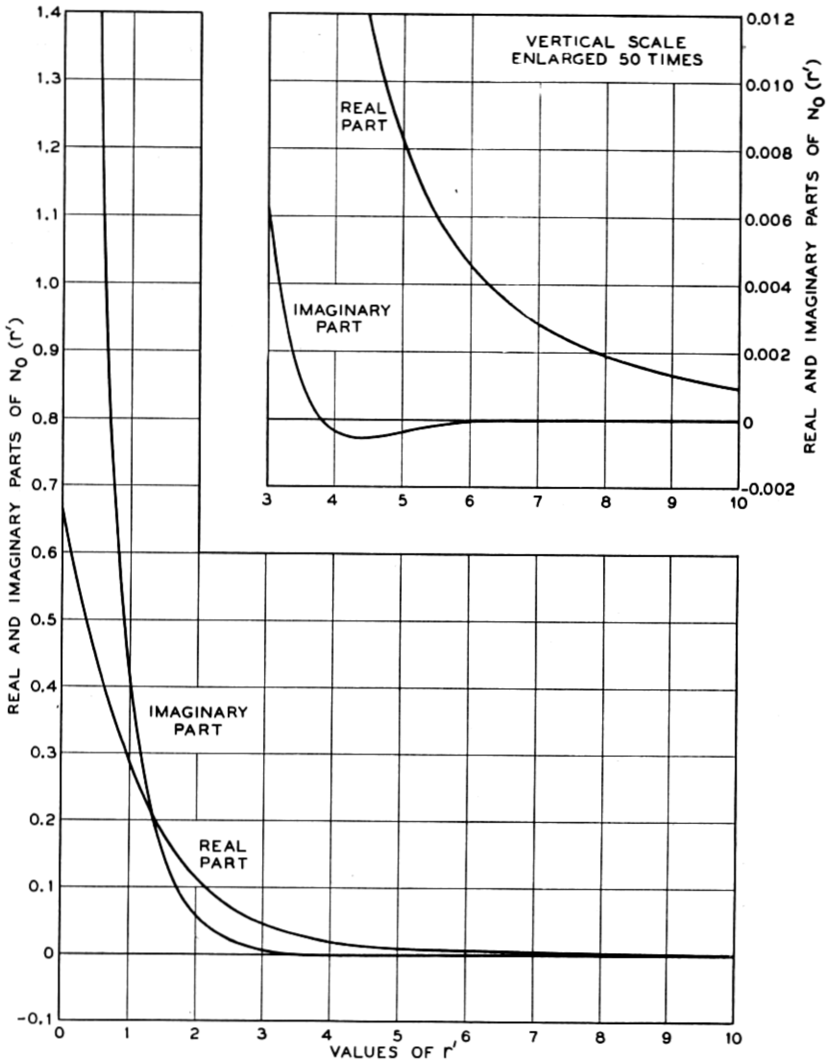

Fig. 1—Real and imaginary parts of $N_0(r')$.

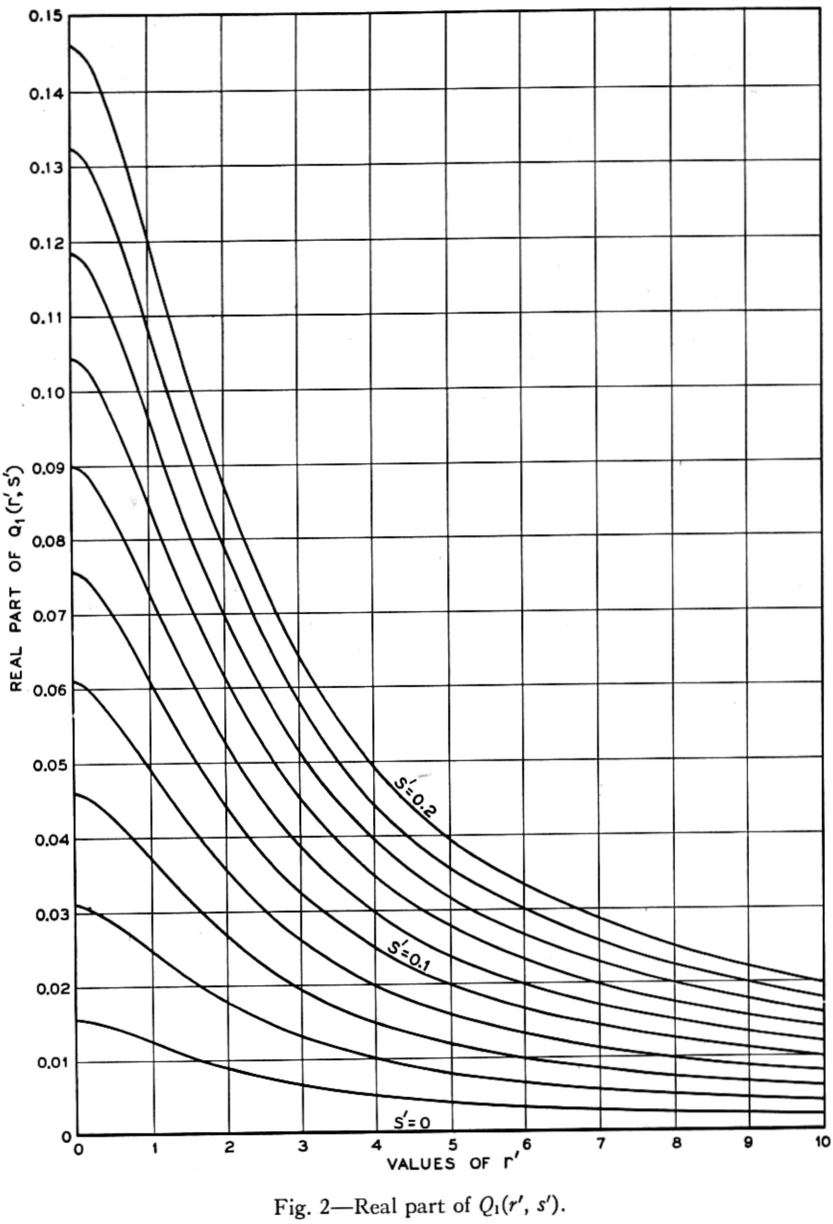

Fig. 2—Real part of $Q_1(r', s')$.

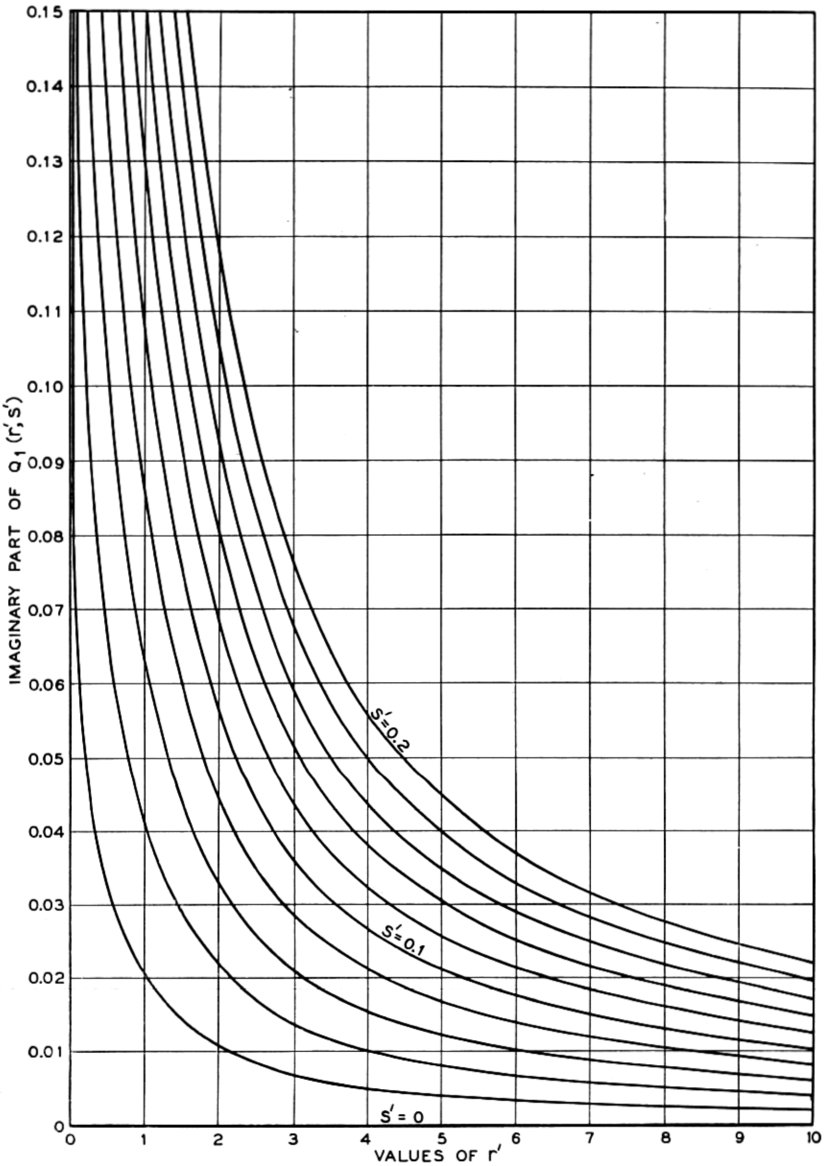

Fig. 3—Imaginary part of $Q_1(r', s')$.

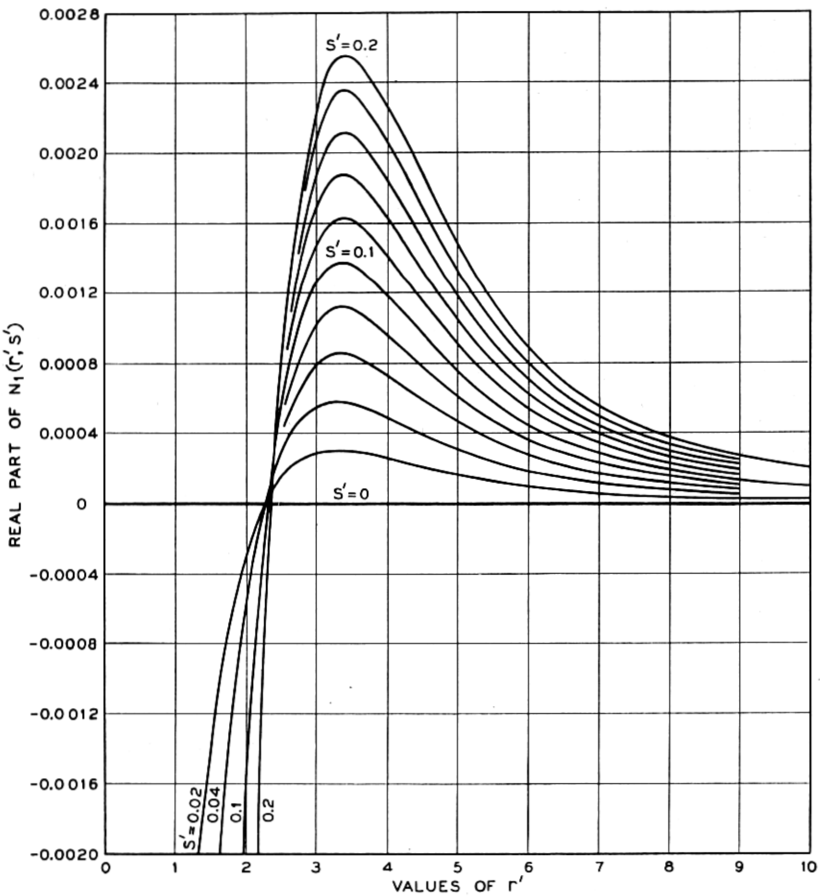

Fig. 4—Real part of $N_1(r', s')$.

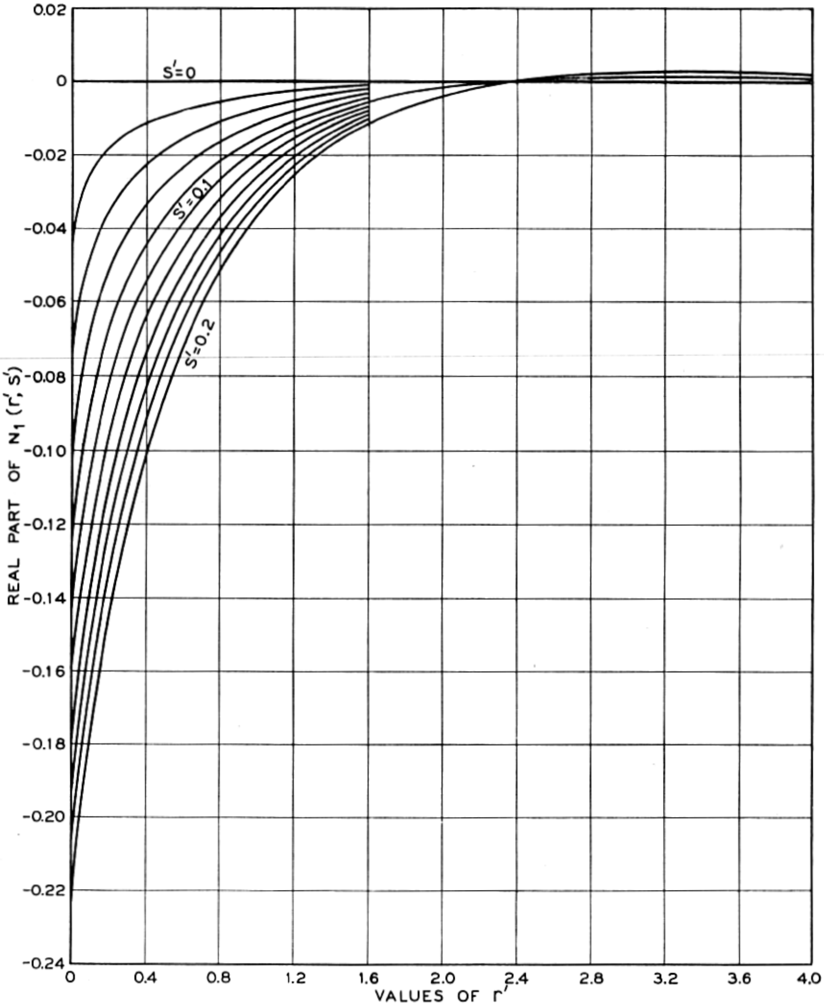

Fig. 4-A—Real part of $N_1(r', s')$, enlarged horizontal scale.

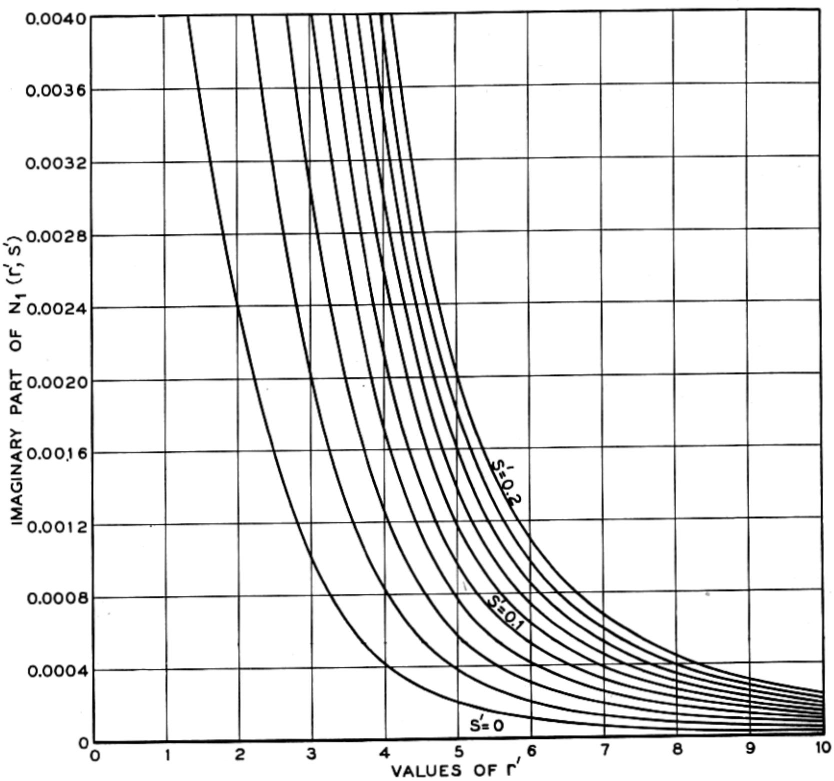

Fig. 5—Imaginary part of $N_1(r', s')$.

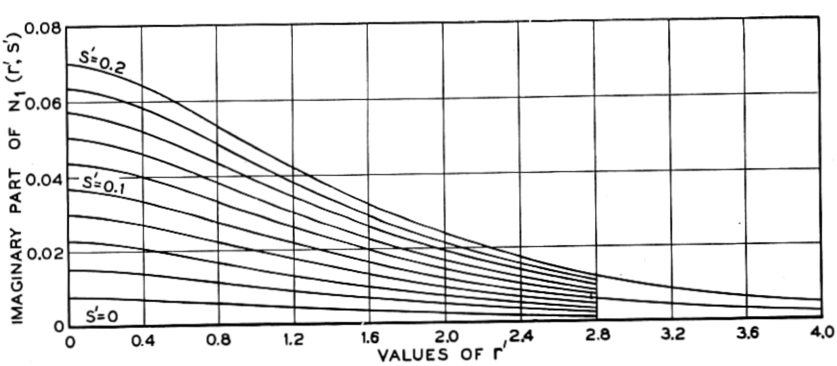

Fig. 5-A—Imaginary part of $N_1(r', s')$, enlarged horizontal scale.

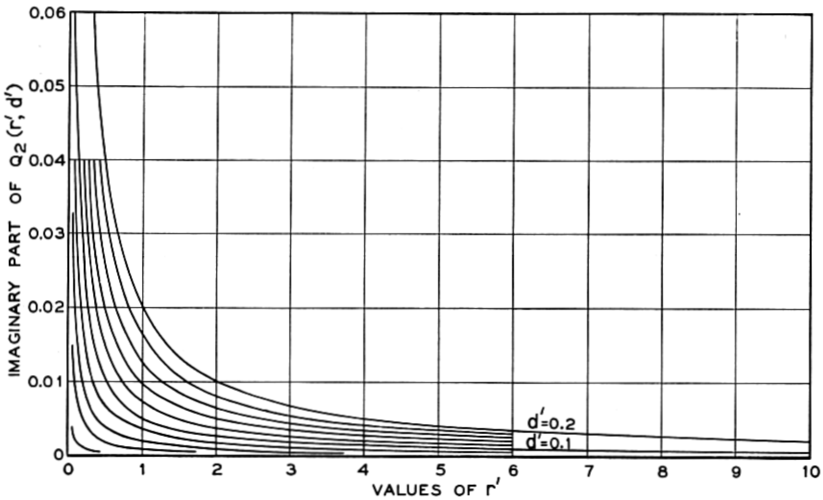

Fig. 6—Imaginary part of $Q_2(r', d')$.

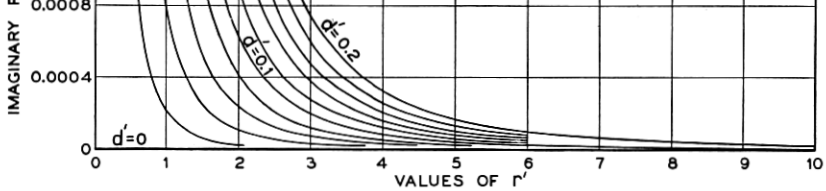

Fig. 7—Imaginary part of $N_2(r', d')$.

TABLE II

REAL PART OF $Q_1(r',s')$

r'	$s' = 0.02$	$s' = 0.04$	$s' = 0.06$	$s' = 0.08$	$s' = 0.10$	$s' = 0.12$	$s' = 0.14$	$s' = 0.16$	$s' = 0.18$	$s' = 0.20$
0	0.0156	0.0309	0.0460	0.0609	0.0755	0.0900	0.1042	0.1182	0.1321	0.1458
0.1	0.0155	0.0308	0.0458	0.0606	0.0752	0.0896	0.1038	0.1178	0.1316	0.1453
0.2	0.0153	0.0304	0.0453	0.0599	0.0744	0.0887	0.1028	0.1167	0.1304	0.1439
0.3	0.0151	0.0299	0.0446	0.0590	0.0733	0.0874	0.1013	0.1150	0.1286	0.1419
0.4	0.0148	0.0293	0.0437	0.0579	0.0719	0.0858	0.0994	0.1130	0.1263	0.1395
0.5	0.0144	0.0287	0.0427	0.0566	0.0704	0.0840	0.0974	0.1106	0.1237	0.1367
0.6	0.0141	0.0280	0.0417	0.0553	0.0687	0.0820	0.0951	0.1081	0.1209	0.1336
0.7	0.0137	0.0272	0.0406	0.0538	0.0669	0.0799	0.0927	0.1054	0.1180	0.1304
0.8	0.0133	0.0265	0.0395	0.0524	0.0651	0.0777	0.0902	0.1026	0.1149	0.1270
0.9	0.0129	0.0257	0.0383	0.0509	0.0633	0.0755	0.0877	0.0998	0.1117	0.1235
1.0	0.0125	0.0249	0.0372	0.0493	0.0614	0.0733	0.0852	0.0969	0.1085	0.1200
1.5	0.0106	0.0211	0.0316	0.0419	0.0523	0.0625	0.0727	0.0828	0.0928	0.1027
2.0	0.0089	0.0178	0.0267	0.0355	0.0442	0.0529	0.0616	0.0702	0.0788	0.0873

IMAGINARY PART OF $Q_1(r',s')$

r'	$s' = 0.02$	$s' = 0.04$	$s' = 0.06$	$s' = 0.08$	$s' = 0.10$	$s' = 0.12$	$s' = 0.14$	$s' = 0.16$	$s' = 0.18$	$s' = 0.20$
0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
0.1	0.0655	0.1312	0.1971	0.2634	0.3299	0.3966	0.4636	0.5308	0.5983	0.6660
0.2	0.0516	0.1036	0.1557	0.2082	0.2608	0.3138	0.3669	0.4204	0.4740	0.5279
0.3	0.0436	0.0875	0.1317	0.1761	0.2207	0.2656	0.3108	0.3562	0.4018	0.4477
0.4	0.0380	0.0763	0.1148	0.1536	0.1926	0.2319	0.2714	0.3111	0.3511	0.3913
0.5	0.0337	0.0677	0.1019	0.1363	0.1711	0.2060	0.2412	0.2766	0.3123	0.3482
0.6	0.0303	0.0608	0.0915	0.1225	0.1538	0.1852	0.2169	0.2489	0.2811	0.3135
0.7	0.0274	0.0550	0.0829	0.1110	0.1394	0.1680	0.1968	0.2259	0.2552	0.2847
0.8	0.0250	0.0502	0.0756	0.1013	0.1273	0.1534	0.1798	0.2064	0.2332	0.2603
0.9	0.0229	0.0460	0.0694	0.0930	0.1168	0.1409	0.1651	0.1896	0.2143	0.2393
1.0	0.0211	0.0424	0.0639	0.0857	0.1077	0.1299	0.1523	0.1750	0.1979	0.2209
1.5	0.0147	0.0296	0.0447	0.0601	0.0756	0.0913	0.1062	0.1233	0.1396	0.1561
2.0	0.0110	0.0221	0.0334	0.0449	0.0566	0.0684	0.0804	0.0926	0.1049	0.1174

TABLE III

Real Part of $N_1(r',s')$

r'	$s' = 0.02$	$s' = 0.04$	$s' = 0.06$	$s' = 0.08$	$s' = 0.10$	$s' = 0.12$	$s' = 0.14$	$s' = 0.16$	$s' = 0.18$	$s' = 0.20$
0	−0.0444	−0.0752	−0.1009	−0.1235	−0.1437	−0.1621	−0.1790	−0.1947	−0.2094	−0.2231
0.1	−0.0243	−0.0468	−0.0677	−0.0871	−0.1051	−0.1219	−0.1376	−0.1523	−0.1662	−0.1793
0.2	−0.0179	−0.0350	−0.0514	−0.0669	−0.0818	−0.0960	−0.1095	−0.1223	−0.1346	−0.1463
0.3	−0.0141	−0.0278	−0.0409	−0.0537	−0.0660	−0.0778	−0.0893	−0.1003	−0.1109	−0.1211
0.4	−0.0114	−0.0226	−0.0334	−0.0440	−0.0542	−0.0642	−0.0739	−0.0833	−0.0924	−0.1012
0.5	−0.0094	−0.0186	−0.0277	−0.0365	−0.0451	−0.0535	−0.0617	−0.0697	−0.0775	−0.0851
0.6	−0.0078	−0.0155	−0.0230	−0.0304	−0.0377	−0.0448	−0.0518	−0.0586	−0.0653	−0.0718
0.7	−0.0065	−0.0129	−0.0193	−0.0255	−0.0316	−0.0376	−0.0436	−0.0494	−0.0551	−0.0607
0.8	−0.0054	−0.0108	−0.0161	−0.0214	−0.0265	−0.0316	−0.0367	−0.0416	−0.0465	−0.0513
0.9	−0.0045	−0.0090	−0.0135	−0.0179	−0.0223	−0.0266	−0.0308	−0.0350	−0.0392	−0.0433
1.0	−0.0038	−0.0075	−0.0113	−0.0150	−0.0186	−0.0223	−0.0259	−0.0294	−0.0330	−0.0365
1.5	−0.0014	−0.0028	−0.0042	−0.0056	−0.0070	−0.0084	−0.0099	−0.0113	−0.0127	−0.0142
2.0	−0.0003	−0.0006	−0.0010	−0.0013	−0.0017	−0.0021	−0.0025	−0.0029	−0.0033	−0.0038

Imaginary Part of $N_1(r',s')$

r'	$s' = 0.02$	$s' = 0.04$	$s' = 0.06$	$s' = 0.08$	$s' = 0.10$	$s' = 0.12$	$s' = 0.14$	$s' = 0.16$	$s' = 0.18$	$s' = 0.20$
0	0.0078	0.0153	0.0227	0.0299	0.0369	0.0438	0.0505	0.0571	0.0635	0.0698
0.1	0.0077	0.0152	0.0225	0.0296	0.0366	0.0434	0.0501	0.0566	0.0630	0.0693
0.2	0.0075	0.0148	0.0220	0.0290	0.0359	0.0426	0.0492	0.0556	0.0619	0.0681
0.3	0.0073	0.0144	0.0213	0.0282	0.0348	0.0414	0.0478	0.0541	0.0603	0.0663
0.4	0.0070	0.0139	0.0206	0.0272	0.0336	0.0400	0.0462	0.0523	0.0583	0.0641
0.5	0.0067	0.0133	0.0197	0.0260	0.0322	0.0383	0.0443	0.0502	0.0560	0.0617
0.6	0.0064	0.0126	0.0188	0.0248	0.0308	0.0366	0.0424	0.0480	0.0536	0.0590
0.7	0.0061	0.0120	0.0179	0.0236	0.0293	0.0348	0.0403	0.0457	0.0510	0.0562
0.8	0.0057	0.0114	0.0169	0.0224	0.0277	0.0330	0.0382	0.0434	0.0484	0.0534
0.9	0.0054	0.0107	0.0159	0.0211	0.0262	0.0312	0.0361	0.0410	0.0458	0.0505
1.0	0.0051	0.0101	0.0150	0.0198	0.0246	0.0294	0.0340	0.0386	0.0432	0.0476
1.5	0.0036	0.0071	0.0106	0.0141	0.0176	0.0210	0.0243	0.0277	0.0310	0.0343
2.0	0.0024	0.0048	0.0072	0.0096	0.0119	0.0143	0.0166	0.0190	0.0213	0.0235

These tabulated values were computed from the corresponding convergent series, the first few terms of which are:

$$\left.\begin{aligned} Q_1(r',s') &= \tfrac{1}{4}\pi s' - \tfrac{1}{3}s'^2 + \cdots \\ &\quad + i\{- s' \log r' + [\tfrac{3}{2}\log 2 + \psi(1) + \tfrac{1}{2}]s' \\ &\qquad + \tfrac{1}{3}s'^2 + \cdots\}, \\ N_1(r',s') &= \tfrac{1}{2}s' \log [(r'^2 + s'^2)^{1/2} + s'] \\ &\quad - \tfrac{1}{2}[\tfrac{3}{2}\log 2 + \psi(1) - \tfrac{1}{4}]s' \\ &\quad + \tfrac{1}{2}r' - \tfrac{1}{2}(r'^2 + s'^2)^{1/2} - \tfrac{4}{15}s'^2 + \cdots \\ &\quad + i\{\tfrac{1}{8}\pi s' - \tfrac{4}{15}s'^2 + \cdots\}. \end{aligned}\right\} \tag{5}$$

For values of r' greater than 2, sufficient accuracy for ordinary purposes is obtained by using the first two terms in the expansions in terms of s':

$$\left.\begin{aligned} Q_1(r',s') &= s'Q_1^{(1)}(r') + s'^2Q_1^{(2)}(r') + \cdots, \\ N_1(r',s') &= s'N_1^{(1)}(r') + s'^2N_1^{(2)}(r') + \cdots, \end{aligned}\right\} \tag{6}$$

where

$$\left.\begin{aligned} Q_1^{(1)}(r') &= i[I_0(u)K_0(u) + I_1(u)K_1(u)], \\ Q_1^{(2)}(r') &= \frac{1-i}{8u^3}[(1 - 2u^2) - (1 + 2u)e^{-2u}], \\ N_1^{(1)}(r') &= \frac{1}{u^2}[1 - 2uI_1(u)K_0(u) - 2I_1(u)K_1(u)], \\ N_1^{(2)}(r') &= \frac{1+i}{16u^5}[(9 - 2u^2) - (9 + 18u + 16u^2 + 8u^3)e^{-2u}], \\ u &= \tfrac{1}{2}(1 + i)r'. \end{aligned}\right\} \tag{7}$$

For actual computation we note that

$$Q_1^{(2)}(r') = \tfrac{1}{2}\left[\frac{i}{r'} - N_0(r')\right].$$

The real and imaginary parts of these four functions are shown in Figs. 8, 9, 10, and 11. The dominating terms in the asymptotic expansions of Q_1 and N_1 are thus given by those of $Q_1^{(1)}$ and $N_1^{(1)}$. For large values of r', $Q_1^{(1)}$ approaches zero as $(1 + i)/r'$, and $N_1^{(1)}$ as $(1 + i)/r'^3$.

For very large values of r' it is convenient to express the functions as follows:

$$\left.\begin{aligned} Q_0(r') + Q_1(r',s') &= Q_0(r')\left[1 + \frac{Q_1^{(1)}(r')}{Q_0(r')}s' + \cdots\right], \\ N_0(r') + N_1(r',s') &= N_0(r')\left[1 + \frac{N_1^{(1)}(r')}{N_0(r')}s' + \cdots\right]. \end{aligned}\right\} \quad (8)$$

The real and imaginary parts of these ratios of functions—the coefficients of s' in the above expansions—are shown in Figs. 12 and 13. We note that each of these ratios approaches the value $(1 + i)$ as r' increases without limit. Hence, as a rough approximation, we may say that the mutual impedance for wires at heights H and h, with separations large in comparison with these heights, is equal to the impedance for wires at zero heights multiplied by the factor:

$$1 + (1 + i)(H' + h') = 1 + \Gamma(H + h). \quad (9)$$

The mutual impedance formula (A) was originally derived from first principles, following the method used in the previous paper for

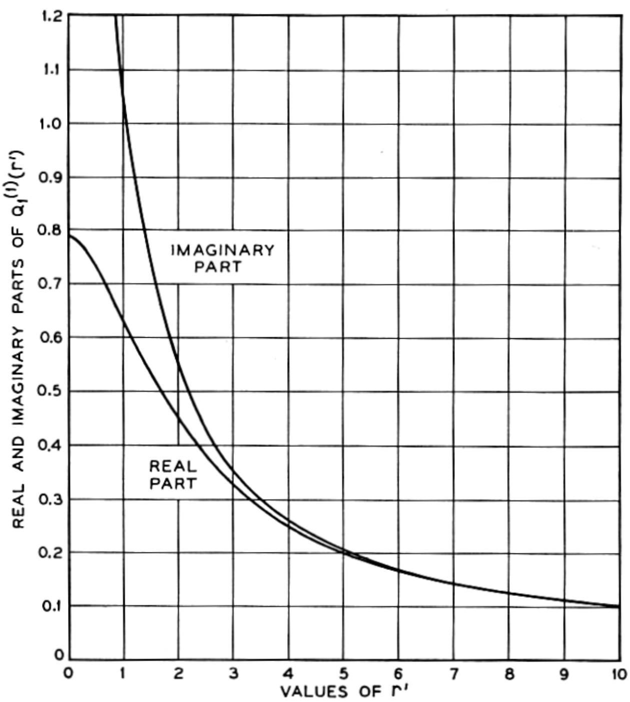

Fig. 8—Real and imaginary parts of $Q_1^{(1)}(r')$.

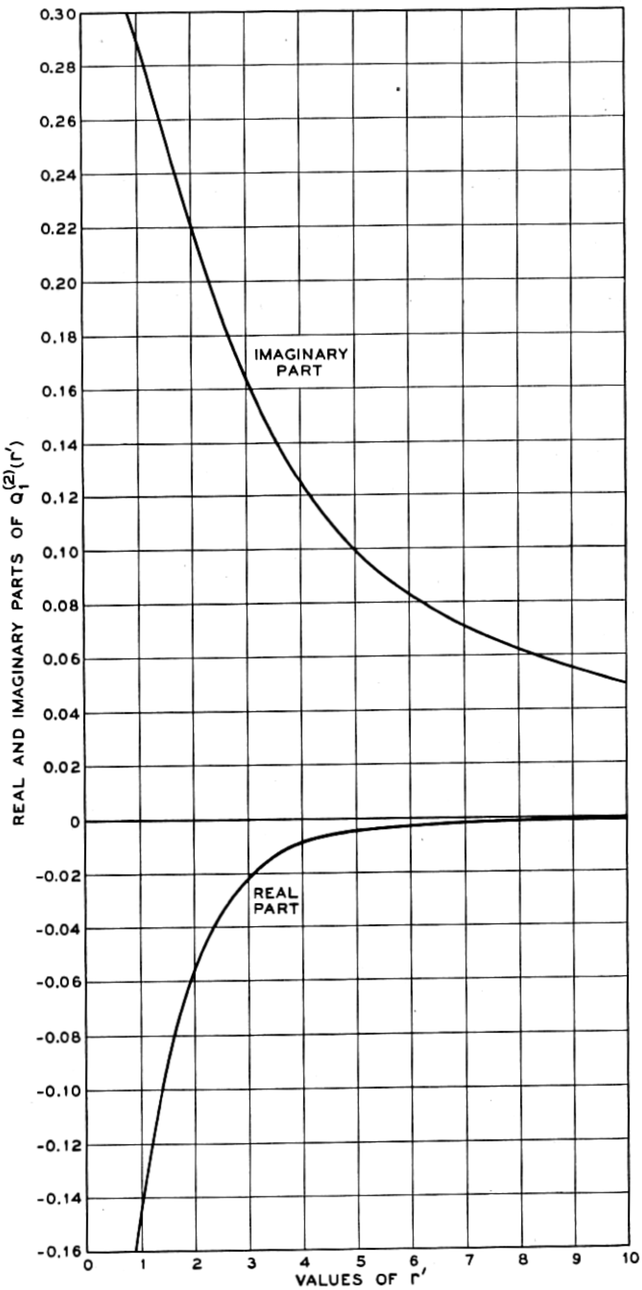

Fig. 9—Real and imaginary parts of $Q_1^{(2)}(r')$.

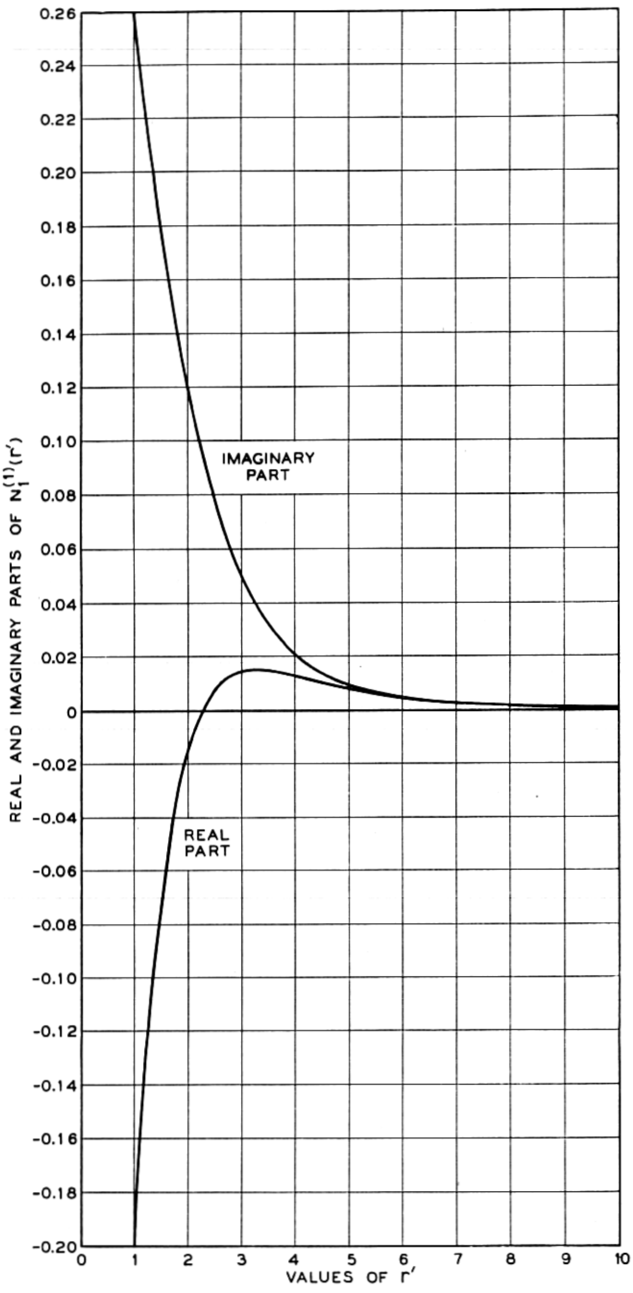

Fig. 10—Real and imaginary parts of $N_1^{(1)}(r')$.

wires on the surface. A brief outline of this derivation is given here. We first find the formulæ for the components of the electric field due to a current flowing in a straight wire of length $2a$ parallel to the surface of the earth and at the height H above it, assuming the air to be replaced by a medium of finite resistivity ρ_1. This part of the

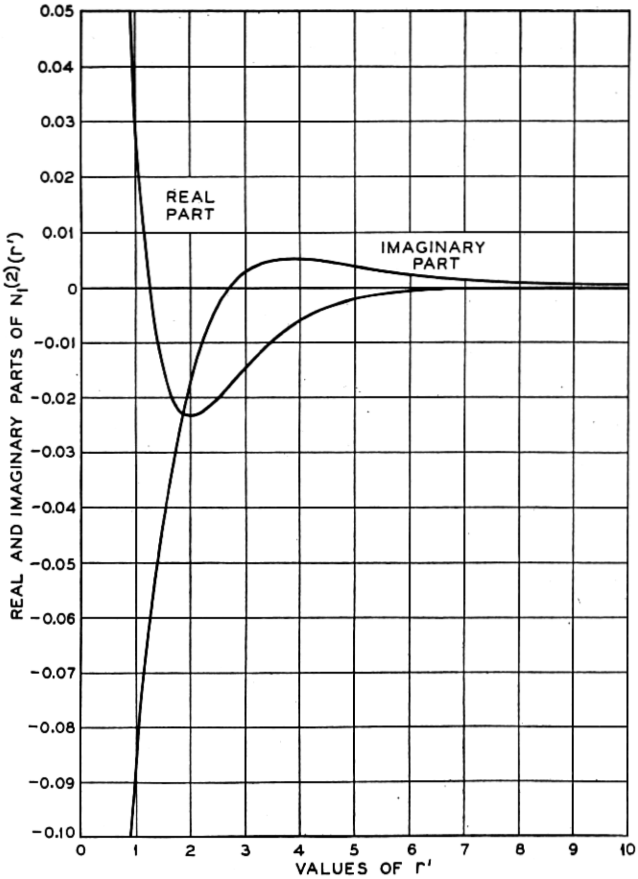

Fig. 11—Real and imaginary parts of $N_1^{(2)}(r')$.

derivation follows closely the work involved in the previous case of wires on the surface. We next find the electric field due to a current in a vertical wire extending from the surface of the earth up to the height H in the assumed medium. This part of the derivation is simpler since there is circular symmetry. Upon combining these two results, we obtain the field due to a current flowing through three sides of a rectangular circuit beginning and ending at the surface of the earth, extending up to the height H, and of width $2a$. We can now allow ρ_1 to become infinite, corresponding to the assumptions of our problem, since this circuit is completed through the earth. Upon allowing a to approach zero, such that $2a = dS$, we find the field corresponding to a rectangle of infinitesimal width. We then take the

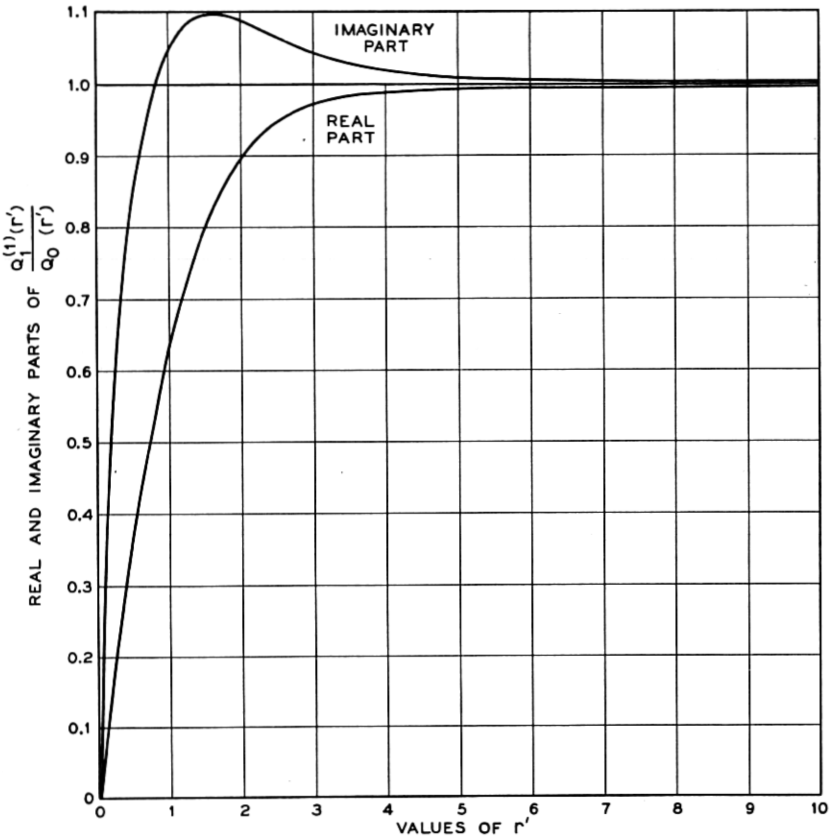

Fig. 12—Real and imaginary parts of $\dfrac{Q_1^{(1)}(r')}{Q_0(r')}$.

integral of this expression around a similar circuit consisting of a horizontal element of wire of length ds at the height h, grounded by wires at its end-points. Upon making various algebraic simplifications, we finally obtain the mutual impedance as given by formula (A).

It is perhaps more convenient to derive this formula from results obtained by H. von Hoerschelmann,[7] again following the method

[7] H. von Hoerschelmann, "Über die Wirkungsweise des geknickten Marconischen Senders der drahtlosen Telegraphie," *Jahrbuch der drahtlosen Telegraphie und Telephonie*, *5*, 14–34, 188–211 (September, November, 1911).

employed in the previous paper in a similar derivation for wires on the surface. For our present problem we use his formulæ for the Hertzian vectors due to horizontal and vertical electric antennæ above the surface of the earth. It is important, at first, to retain a non-

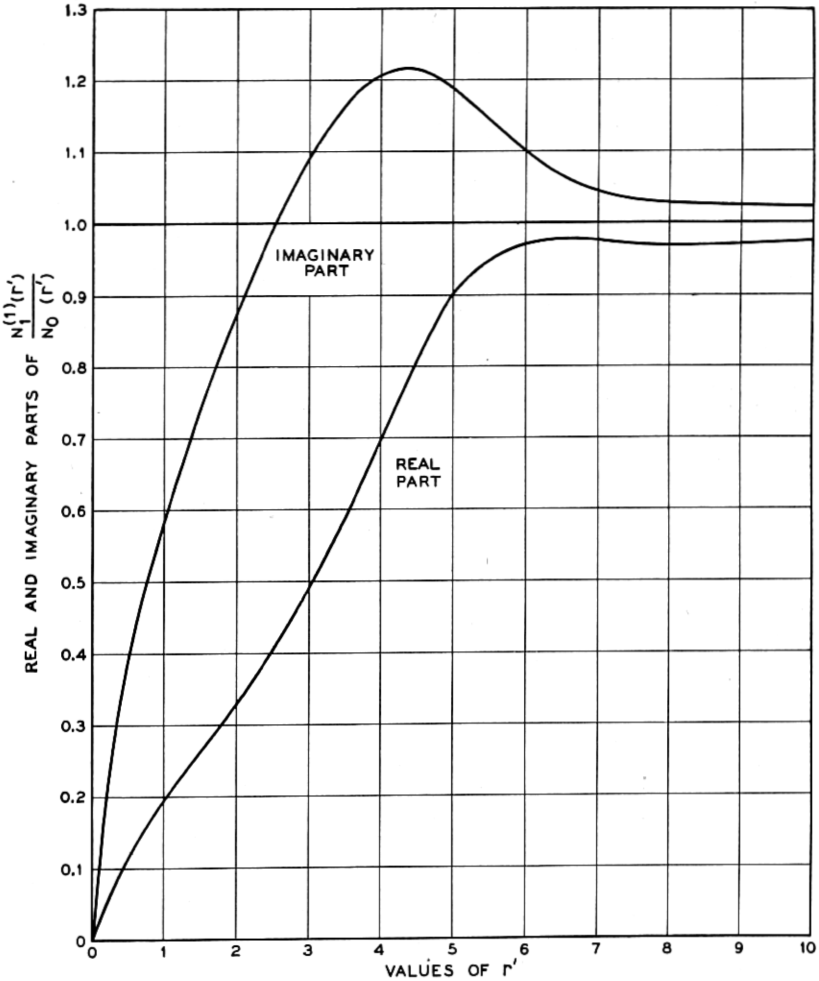

Fig. 13—Real and imaginary parts of $\dfrac{N_1^{(1)}(r')}{N_0(r')}$.

vanishing value of the capacitivity of the air. From these formulæ we obtain the vector Π, due to a current flowing through a horizontal element of wire of length dS at height H above the surface of the earth, grounded by vertical wires at its end-points. Next, we obtain the electric field **E** in the air by the relation:

$$\mathbf{E} = \text{grad div } \Pi - \Gamma_1^2 \Pi, \tag{10}$$

where Γ_1 is the propagation constant in the air. We can now allow Γ_1 to vanish, thus obtaining the expression for the field corresponding to the assumptions of our problem. We then proceed as before to find the expression for the mutual impedance.

I am greatly indebted to my colleagues, Dr. Marion C. Gray and Miss Helen M. Kammerer, for much valuable assistance in the preparation of this paper, particularly in the compilation of the tables and curves.