Mutual Impedance of Grounded Wires for Horizontally Stratified Two-Layer Earth *

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A general formula is derived for the mutual impedance of wires embedded in a conducting medium and lying in one of two parallel planes of discontinuity in the conductivity. The general formula is quite complicated, but simplifies in a number of important cases, and summarizes the published mutual impedance formulas relating to two-layer earth and its special cases. The two most important cases are obtained by making the conductivity of one or the other of the outer regions zero. The special case in which the conductivity of the outer region adjoining the wires is zero gives the mutual impedance of any thin grounded wires lying on the surface of a horizontally stratified earth having conductivities λ_1 and λ_2 at depths less than or greater than b, respectively. When the conductivity of the other outer region is zero, the formula gives the mutual impedance of wires lying in the plane of separation, at the depth b. The formulas in both cases involve integrals which apparently cannot be evaluated in closed form: for practical application the use of curves of the kind given is suggested for approximate numerical results. The formulas for the special cases, of which there are 11, together with some of their limiting forms, are tabulated together for ready reference.

I

THE general formula for the mutual impedance of wires embedded in a conducting medium and lying in one of two parallel planes of discontinuity in the conductivity with displacement currents neglected is of the following form:

$$Z_{Ss} = \int_a^b \int_A^B \left\{ \frac{d^2Q(r)}{dSds} + i\omega P(r) \cos \epsilon \right\} dSds. \tag{I}$$

The integrations are extended in the double integral over the two wires S and s extending between points A, B, and a, b, respectively, whose elements dS and ds are separated by distance r and include the angle ϵ between their directions. $Q(r)$ and $P(r)$ are functions of the frequency, the conductivities, and of the separation b between the planes of discontinuity in the conductivities as well as of r.

For wires on the surface of a horizontally stratified earth having conductivities λ_1 and λ_2 at depths less than or greater than b, respectively, $Q(r)$ and $P(r)$ are given by:

* A preliminary report of some of the results of this paper has been given in a letter to the Editor of the *Physical Review*, Vol. 37, No. 10, pp. 1369–1370 (May 15, 1931).

$$Q(r) = \frac{1}{2\pi\lambda_1}\int_0^\infty \left[1 + \frac{4\alpha_1^2(u+\alpha_2)(\lambda_1-\lambda_2)e^{-2b\alpha_1}}{\Delta[\alpha_1\lambda_2+\alpha_2\lambda_1+(\alpha_1\lambda_2-\alpha_2\lambda_1)e^{-2b\alpha_1}]}\right] \times J_0(ru)du,$$

$$P(r) = 2\int_0^\infty \frac{u}{\Delta}[\alpha_1+\alpha_2+(\alpha_1-\alpha_2)e^{-2b\alpha_1}]J_0(ru)du.$$

For wires lying in the plane of separation, at the depth b:

$$Q(r) = \frac{1}{2\pi}\int_0^\infty \frac{u[\alpha_1+u+(\alpha_1-u)e^{-2b\alpha_1}] \times [\alpha_1+\alpha_2+(\alpha_1-\alpha_2)e^{-2b\alpha_1}]+4\alpha_1^2\alpha_2 e^{-2b\alpha_1}}{\Delta[\alpha_1\lambda_2+\alpha_2\lambda_1+(\alpha_1\lambda_2-\alpha_2\lambda_1)e^{-2b\alpha_1}]} J_0(ru)du,$$

$$P(r) = 2\int_0^\infty \frac{u}{\Delta}[\alpha_1+u+(\alpha_1-u)e^{-2b\alpha_1}]J_0(ru)du.$$

In these formulas:

$$\begin{aligned} i &= \sqrt{-1} = \text{imaginary unit}, \\ \omega &= 2\pi f = \text{radian frequency}, \\ \Delta &= (\alpha_1+\alpha_2)(u+\alpha_1)+(\alpha_1-\alpha_2)(u-\alpha_1)e^{-2b\alpha_1}, \\ \alpha_j^2 &= u^2 + i4\pi\omega\lambda_j \qquad (j = 1 \text{ and } 2), \\ J_0 &= \text{Bessel function of the first kind, zero order.} \end{aligned}$$

Expression (I) is identical in general form with the formula for mutual impedances of grounded wires given by R. M. Foster [1] and the $Q(r)$ and $P(r)$ functions for the two cases above reduce to agreement with his formula, with appropriate changes in notation where necessary, in any of the cases resulting in wires on the surface of homogeneous earth, namely, for the first pair of functions, (i) $\lambda_1 = \lambda_2$, (ii) $b = 0$, (iii) $b = \infty$, and for the second, (i) $b = 0$, (ii) $\lambda_1 = 0$, (iii) $b = \infty$, $\lambda_2 = 0$.

It may be noted that the integrations involving the $Q(r)$ function are accomplished by inserting the four limits, which are the four distances between wire terminals, since each of the indicated integrations has a corresponding differentiation. Symbolically the result of carrying out the integrations in (I) may be written as follows:

$$Z_{Ss} = Q_{(A-B)(a-b)} + i\omega N_{Ss}, \qquad \text{(II)}$$

where

$$Q_{(A-B)(a-b)} = \int_a^b\int_A^B \frac{d^2Q(r)}{dSds}dSds = Q(Aa) - Q(Ab) + Q(Bb) - Q(Ba)$$

[1] R. M. Foster, "Mutual Impedances of Grounded Wires Lying on the Surface of the Earth," *Bulletin of the American Math. Soc.*, Vol. XXXVI, pages 367–368, May, 1930. *Bell System Technical Journal*, Vol. X, pages 408–419, July, 1931.

and

$$N_{Ss} = \int_a^b \int_A^B P(r) \cos \epsilon \, dSds.$$

Both $Q_{(A-B)(a-b)}$ and N_{Ss} are generally complex-valued and thus do not represent resistance and inductance, as ordinarily defined, as might be inferred from the similarity of expression (II) to the usual impedance expression. At zero frequency $i\omega N_{Ss}$ vanishes and $Q_{(A-B)(a-b)}$ becomes $R_{(A-B)(a-b)}$, a real number, the d.-c. mutual resistance of the circuits. For frequencies sufficiently low, such that terms involving higher powers of the frequency in the expansions of the functions in powers of the frequency are negligible, the mutual impedance can be expressed in the ordinary form; that is, in the formula

$$Z_{Ss} = R_{(A-B)(a-b)} + i\omega[N^\circ_{(A-B)(a-b)} + N^\circ_{Ss}] \qquad \text{(III)}$$

$R_{(A-B)(a-b)}$ is as above the d.-c. mutual resistance; $N^\circ_{(A-B)(a-b)}$ is the coefficient of $i\omega$ in the expansion of $Q_{(A-B)(a-b)}$, a real number, and generally equal to the sum of the Neumann integrals of the earth flows with the wires and with each other, the earth flows being those for direct current; N°_{Ss} is generally the Neumann integral of the wires.[2] The bracketed terms thus give the d.-c. mutual inductance of the wires with earth return.

For infinite distance between all terminal grounds A, B, a, b, taken in pairs, $Q_{(A-B)(a-b)}$ vanishes.

The physical distinction of $Q_{(A-B)(a-b)}$ and N_{Ss} may be illustrated by the following two cases: In the first, one wire is supposed straight and of arbitrary length; the second extends at right angles to it from two grounding points and is closed at infinity (that is, by a segment parallel to the first wire and at such distance that its mutual impedance with the first wire is negligibly small). In this case, in the perpendicular segments $\cos \epsilon = 0$, and in the parallel segment $P(r) = 0$, since $r = \infty$, so that $N_{Ss} = 0$ and the mutual impedance is given entirely by $Q_{(A-B)(a-b)}$; that is, the mutual impedance depends only on the grounding points. In the second case, the two perpendicular segments of the second wire extend away from a parallel segment to grounding points at infinity. Here the mutual impedance is given entirely by N_{Ss}, since $Q(r)$ and, therefore, $Q_{(A-B)(a-b)}$ vanishes for the limit $r = \infty$.

Table I is a summary of mutual impedance formulas obtained as special cases of the general formula. For each case the first column entries consist of the $Q(r)$ and $P(r)$ functions in the mutual impedance

[2] An ambiguity concerning this statement as well as that referring to $N^\circ_{(A-B)(a-b)}$, arising in certain particular cases, is discussed below.

Table I—Mutual Impedance Formulas

Condition	Mutual Impedance		D-C. Mutual Resistance	D-C. Mutual Inductance	Mut. Imp. Gradient Parallel to an Infinite Straight Wire	
Horizontally Stratified Earth the Conductivities of Which at Depths Less than and Greater than b Are Resp.	$Z_{Ss}=\int_a^b\int_A^B\left\{\frac{d^2Q(r)}{dSds}+i\omega P(r)\cos\epsilon\right\}dSds$		$R_{(A-B)(a-b)}=R(Aa)-R(Ab)+R(Bb)-R(Ba)$	$L^\circ_{Ss}=N^\circ_{Ss}+N^\circ(Aa)-N^\circ(Ab)+N^\circ(Bb)-N^\circ(Ba)$	$\frac{dZ_{Ss}}{ds}=i\omega\int_{-\infty}^{\infty}P(r)ds$	
	$Q(r)$	$P(r)$	$R(r)$	$N^\circ(r)$	Exact	Approximate
Wires on the Surface of the Earth						
1.1 λ_1 and λ_2	$\frac{1}{2\pi\lambda_1}\int_0^\infty\left\{1+\frac{4\alpha_1^2(u+\alpha_2)(\lambda_1-\lambda_2)e^{-2b\alpha_1}}{\Delta[\alpha_1\lambda_2+\alpha_2\lambda_1+(\alpha_1\lambda_2-\alpha_2\lambda_1)e^{-2b\alpha_1}]}\right\}J_0(ru)du$ $\Delta=(\alpha_1+\alpha_2)(u+\alpha_1)+(\alpha_1-\alpha_2)(u-\alpha_1)e^{-2b\alpha_1}$	$2\int_0^\infty\frac{u}{\Delta}[\alpha_1+\alpha_2+(\alpha_1-\alpha_2)e^{-2b\alpha_1}]J_0(ru)du$	$\frac{1}{2\pi\lambda_1 r}\left\{1+2\int_0^\infty\frac{\beta e^{-ku}}{1-\beta e^{-ku}}J_0(u)du\right\}$ $=\frac{1}{2\pi\lambda_1 r}\left\{1+2\sum_{n=1}^{\infty}\frac{\beta^n}{(1+n^2k^2)^{1/2}}\right\}$ (1)	$2\beta r\int_0^\infty e^{-ku}\frac{\beta(1-e^{-ku})-ku}{(1-\beta e^{-ku})^2}\frac{J_0(u)-1}{u^2}du$	$4i\omega\int_0^\infty\frac{1}{\Delta}[\alpha_1+\alpha_2+(\alpha_1-\alpha_2)e^{-2b\alpha_1}]\cos yudu$ $\Delta=(\alpha_1+\alpha_2)(u+\alpha_1)+(\alpha_1-\alpha_2)(u-\alpha_1)e^{-2b\alpha_1}$ (2)	
1.2 λ and 0	$\frac{1}{2\pi\lambda}\int_0^\infty\left\{1+\frac{8\alpha^3}{\Delta(e^{2b\alpha}-1)}\right\}J_0(ru)du$. $\Delta=(u+\alpha)^2-(u-\alpha)^2e^{-2b\alpha}$	$2\int_0^\infty\frac{u}{\Delta}[u+\alpha-(u-\alpha)e^{-2b\alpha}]J_0(ru)du$	$\frac{1}{2\pi\lambda}\int_0^\infty\coth(bu)[J_0(ru)-1]du$	$2r\int_0^\infty e^{-ku}\frac{1-e^{-ku}-ku}{(1-e^{-ku})^2}\frac{J_0(u)-1}{u^2}du$	$4i\omega\int_0^\infty\frac{1}{\Delta}[u+\alpha-(u-\alpha)e^{-2b\alpha}]\cos yudu$ $\Delta=(u+\alpha)^2-(u-\alpha)^2e^{-2b\alpha}$	
1.3 0 and λ	∞	$\frac{1}{r}-\frac{1}{R}+2\int_0^\infty\frac{ue^{-2bu}}{u+\alpha}J_0(ru)du$	∞	$2r\int_0^\infty e^{-ku}\frac{1-e^{-ku}+ku}{(1+e^{-ku})^2}\frac{J_0(u)-1}{u^2}du$	$4i\omega\left[\log\frac{y^2+4b^2}{y^2}+\int_0^\infty\frac{e^{-2bu}\cos yudu}{u+\alpha}\right]$ (3) (4) (5)	$\frac{\pi\omega}{2}+i\omega\log\frac{e^{[1+2\psi(1)]}}{\pi\lambda\omega y^2}$ $\sqrt{y^2+4b^2}\cdot\sqrt{4\pi\lambda\omega}\ll 1$
1.4 λ and ∞	$\frac{1}{2\pi\lambda}\int_0^\infty\frac{u+\alpha+(u-\alpha)e^{-2b\alpha}}{u+\alpha-(u-\alpha)e^{-2b\alpha}}\frac{1-e^{-2b\alpha}}{1+e^{-2b\alpha}}J_0(ru)du$	$2\int_0^\infty\frac{1-e^{-2b\alpha}}{u+\alpha-(u-\alpha)e^{-2b\alpha}}J_0(ru)udu$	$\frac{1}{2\pi\lambda r}\left\{1+2\sum_{n=1}^{\infty}\frac{(-1)^n}{(1+k^2n^2)^{1/2}}\right\}$	$2r\int_0^\infty e^{-ku}\frac{1-e^{-ku}+ku}{(1+e^{-ku})^2}\frac{J_0(u)-1}{u^2}du$	$4i\omega\int_0^\infty\frac{(1-e^{-2b\alpha})}{u+\alpha-(u-\alpha)e^{-2b\alpha}}\cos yudu$	
1.5 $\sigma=\lim_{b\to 0}b\lambda_1$ and λ	$\frac{1}{2\pi\sigma}\int_0^\infty\frac{(u+\alpha)J_0(ru)du}{(u+\alpha+i4\pi\sigma\omega)(\alpha+\lambda\sigma^{-1})}$	$2\int_0^\infty\frac{J_0(ru)udu}{u+\alpha+i4\pi\sigma\omega}$	$\frac{1}{4\sigma}[\mathbf{H}_0(\lambda\sigma^{-1}r)-Y_0(\lambda\sigma^{-1}r)]$	$\frac{2\sigma}{\lambda}\left[\log r+\frac{\pi}{2}\mathbf{H}_0(\lambda\sigma^{-1}r)-\frac{\pi}{2}Y_0(\lambda\sigma^{-1}r)\right]$ $-r\left[1-\frac{\pi}{2}\mathbf{H}_1(\lambda\sigma^{-1}r)+\frac{\pi}{2}Y_1(\lambda\sigma^{-1}r)\right]$	$4i\omega\int_0^\infty\frac{\cos yudu}{u+\alpha+i4\pi\sigma\omega}$ (5)	
1.6 $\sigma=\lim_{b\to 0}b\lambda_1$ and 0	$\frac{1}{4\sigma}[\mathbf{H}_0(i2\pi\sigma\omega r)-Y_0(i2\pi\sigma\omega r)]$	$\frac{1}{r}\{1-i\pi^2\sigma\omega r[\mathbf{H}_0(i2\pi\sigma\omega r)-Y_0(i2\pi\sigma\omega r)]\}$	$\frac{1}{2\pi\sigma}\log\left(\frac{1}{r}\right)$ (6)	r (7)	$\omega\pi e^{-u}-i\omega[e^{u}\mathrm{Ei}(-u)+e^{-u}\mathrm{Ei}(u)]$ $u=2\pi\sigma\omega y$ (8)	$\omega\pi+i\omega\log\frac{e^{2\psi(1)}}{(2\pi\sigma\omega y)^2}$ $2\pi\sigma\omega y\ll 1$
1.7 λ and λ	$\frac{1}{2\pi\lambda r}$ (9)	$\frac{2}{\gamma^2r^3}[1-(1+\gamma r)e^{-\gamma r}]$ (9)	$\frac{1}{2\pi\lambda r}$ (7)	0 (7)	$\frac{1}{\pi\lambda y^2}[1-\gamma yK_1(\gamma y)]$ (3) (5)	$\frac{\pi\omega}{2}+i\omega\log\frac{e^{[1+2\psi(1)]}}{\pi\lambda\omega y^2}$ $y\sqrt{4\pi\lambda\omega}\ll 1$
Wires in the Plane of Separation, at the Depth b						
2.1 λ_1 and λ_2	$\frac{1}{2\pi}\int_0^\infty\frac{u[\alpha_1+u+(\alpha_1-u)e^{-2b\alpha_1}][\alpha_1+\alpha_2+(\alpha_1-\alpha_2)e^{-2b\alpha_1}]+4\alpha_1^2\alpha_2e^{-2b\alpha_1}}{\Delta[\alpha_1\lambda_2+\alpha_2\lambda_1+(\alpha_1\lambda_2-\alpha_2\lambda_1)e^{-2b\alpha_1}]}J_0(ru)du$ $\Delta=(\alpha_1+\alpha_2)(u+\alpha_1)+(\alpha_1-\alpha_2)(u-\alpha_1)e^{-2b\alpha_1}$	$2\int_0^\infty\frac{u}{\Delta}[\alpha_1+u+(\alpha_1-u)e^{-2b\alpha_1}]J_0(ru)du$	$\frac{1}{2\pi(\lambda_1+\lambda_2)r}\int_0^\infty\frac{1+e^{-ku}}{1-\beta e^{-ku}}J_0(u)du$ $=\frac{\lambda_1}{\pi(\lambda_1^2-\lambda_2^2)r}\sum_0^\infty\frac{\beta^n}{(1+n^2k^2)^{1/2}}$	$\frac{2\lambda_1}{(\lambda_1+\lambda_2)^2}\int_0^\infty\frac{\lambda_1(1-e^{-ku}-ku)e^{-ku}-\lambda_2(1-e^{-2ku})}{(1-\beta e^{-ku})^2}\frac{J_0(u)-1}{u^2}du$	$4i\omega\int_0^\infty\frac{1}{\Delta}[\alpha_1+u+(\alpha_1-u)e^{-2b\alpha_1}]\cos yudu$ $\Delta=(\alpha_1+\alpha_2)(u+\alpha_1)+(\alpha_1-\alpha_2)(u-\alpha_1)e^{-2b\alpha_1}$	
2.11 λ_1 and λ_2 $b\to\infty$	$\frac{1}{2\pi}\int_0^\infty\frac{u}{\alpha_1\lambda_2+\alpha_2\lambda_1}J_0(ru)du$	$\frac{2}{(\gamma_1^2-\gamma_2^2)r^3}[(1+\gamma_2 r)e^{-\gamma_2 r}-(1+\gamma_1 r)e^{-\gamma_1 r}]$	$\frac{1}{2\pi(\lambda_1+\lambda_2)r}$	$\frac{2\lambda_1\lambda_2}{(\lambda_1+\lambda_2)^2}r$	$\frac{1}{\pi(\lambda_1-\lambda_2)y}[\gamma_2K_1(\gamma_2 y)-\gamma_1K_1(\gamma_1 y)]$	$\frac{\pi\omega}{2}+i\omega\left[\log\frac{e^{[1+2\psi(1)]}}{\pi\omega y^2}-\frac{\lambda_1\log\lambda_1-\lambda_2\log\lambda_2}{\lambda_1-\lambda_2}\right]$ $y\sqrt{4\pi\lambda_1\omega}\ll 1$ $y\sqrt{4\pi\lambda_2\omega}\ll 1$
2.2 λ and λ	$\frac{1}{4\pi\lambda}\left\{\frac{e^{-\gamma r}}{r}+\frac{2}{R}\left[Z_2I_0(Z_1)K_1(Z_2)+Z_1I_1(Z_1)K_0(Z_2)-\frac{e^{-\gamma R}}{2}\right]\right\}$ $Z_1=\frac{1}{2}\gamma(R-2b)$ $Z_2=\frac{1}{2}\gamma(R+2b)$	$\frac{e^{-\gamma r}}{r}-\frac{e^{-\gamma R}}{R}+2\int_0^\infty\frac{ue^{-2b\alpha}}{u+\alpha}J_0(ru)du$	$\frac{1}{4\pi\lambda}\left[\frac{1}{r}+\frac{1}{R}\right]$	$\frac{1}{2}(r-R)+2b\log\frac{R+2b}{4b}$	$2i\omega\left[K_0(\gamma y)-K_0(\gamma\sqrt{y^2+4b^2})+2\int_0^\infty e^{-2b\alpha}\frac{\cos yudu}{u+\alpha}\right]$ (10)	
2.21 λ and λ $b\to\infty$	$\frac{1}{4\pi\lambda}\frac{e^{-\gamma r}}{r}$	$\frac{e^{-\gamma r}}{r}$	$\frac{1}{4\pi\lambda r}$	$\frac{1}{2}r$ (7)	$2i\omega K_0(\gamma y)$ (11)	$\frac{\pi\omega}{2}+i\omega\log\frac{e^{2\psi(1)}}{\pi\lambda\omega y^2}$ $y\sqrt{4\pi\lambda\omega}\ll 1$

Notation

The integrations in the formula for Z_{Ss} are extended over the wires S and s; whose elements dS and ds are separated by distance r, and include the angle ϵ between their directions; the wires extend from A to B and from a to b, respectively. Distances are in cm.; conductivity in abmhos per cm.

$i=\sqrt{-1}$

$\omega=2\pi f=$ Radian frequency

$y=$ Horizontal separation between wires

$\gamma^2=i4\pi\omega\lambda$ $\alpha^2=u^2+\gamma^2$ $\gamma_j^2=i4\pi\omega\lambda_j$ $\alpha_j^2=u^2+\gamma_j^2$ $j=1$ and 2

$R^2=r^2+4b^2$

$\beta=\frac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2}$

$k=\frac{2b}{r}$

$\mathbf{H}_0=$ Struve's associated Bessel function, zero order.*
$\mathbf{H}_1=$ " " " " , first order.*
$I_0=$ Bessel function of first kind for imaginary argument, zero order.*
$I_1=$ " " " " " " " " , first order.*
$J_0=$ " " " " " , zero order.*
$K_0=$ " " " second kind for imaginary argument, zero order.*
$K_1=$ " " " " " " " " , first order.*
$Y_0=$ " " " " " , zero order.*
$Y_1=$ " " " " " , first order.*
Ei = Exponential integral, as defined by Jahnke and Emde, Funktionentafeln.

$N^\circ_{Ss}=$ Mutual Neumann integral of wires S and $s=\int\int\frac{\cos\epsilon}{r}dSds$

$-\psi(1)=.5772=$ Euler's Constant.

* As defined in G. N. Watson's "Theory of Bessel Functions," Cambridge, 1922.

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formula for arbitrary paths; then follow the d.-c. mutual resistance and inductance, and in the last columns exact and approximate expressions for the mutual impedance gradient parallel to a straight wire of infinite length.

The first entry in each group is the general case of two-layer earth. In the first group, the next three entries are those in which one of the conductivities is given the special value zero or infinity, one of the four possible cases being trivial. The fifth and sixth entries involve finite surface conductivity which is defined by $\sigma = \lim_{b \to 0} b\lambda_1$; in the first of these the surface conductivity differs from the conductivity of the homogeneous earth below it, in the second the earth below is abolished. The latter may serve as a convenient approximation to the case in which the earth consists of a thin upper layer of high conductivity relative to the layer below. The final entry of this group is the case of homogeneous ground. In the second group the second entry is the limiting case for $b = \infty$, which places the wires at the plane of separation of two semi-infinite media of conductivities λ_1 and λ_2; the general formula for this case has been independently obtained by R. M. Foster. With either conductivity zero this case reduces to the case of homogeneous earth; with equal conductivities the case of an infinite medium is obtained, which is the final entry of this group. The third entry is the case of wires at depth b in homogeneous earth; for sufficiently large depths the formulas approach those of an infinite medium.

Further information regarding these special cases may be obtained from the papers referred to in Table I.

In case 1.4 where the conductivity λ_2 approaches an infinite limit, an ambiguity arises concerning the d.-c. mutual inductance, two cases appearing according as the approach of λ_2 to infinity is assumed faster or slower, respectively, than the approach of the frequency to zero, that is, according as the limits are taken $\lambda_2 \to \infty$, $\omega \to 0$ or $\omega \to 0$, $\lambda_2 \to \infty$. The entry in the table corresponds to the latter limit and also to d.-c. distribution of earth current. The alternate limit gives:

$$L^\circ_{Ss} = N^\circ_{Ss} - N^\circ_{Ss'} + N^\circ_{(A-B)(a-b)},$$

where

$N^\circ_{Ss'}$ = Mutual Neumann integral of one wire and the image of the other wire, the image plane being the plane of separation of the media.

$$N^\circ(r) = -r\int_0^\infty e^{-ku}\frac{1 - e^{-2ku} - 2ku}{(1 + e^{-ku})^2}\,\frac{J_0(u) - 1}{u^2}\,du$$

$$(0 > N^\circ(r) > -.2r).$$

The ambiguous cases arise only in the limits $\lambda \to \infty$, $\omega \to 0$ and $\lambda \to 0$, $\omega \to \infty$, the product $\lambda\omega$ appearing in the expressions then being strictly indeterminate, until the order of the limits is defined.

II

Different problems are encountered in obtaining numerical results for the two functions $Q_{(A-B)(a-b)}$ and N_{Ss}; $Q_{(A-B)(a-b)}$ is determined in terms of four values, with proper sign, as given in equation (II), of $Q(r)$; while $Q(r)$ apparently is not generally expressible in terms of known functions it may always be evaluated by numerical integration. The case of N_{Ss} is different because of the necessity of double integration over the wires; general numerical results involve carrying out at least one of these integrations in addition to that required in evaluating $P(r)$, involving a considerable amount of labor and complexity of results.

However, without carrying out either of the evaluations completely, the formulas for the limiting cases may be used to obtain results approximating certain practical conditions. The important limiting cases are (i) one wire infinite, and (ii) zero frequency. Curves for these cases for wires on the surface of the ground and for special values of the parameters are given in Figs. 1–6, as described below.

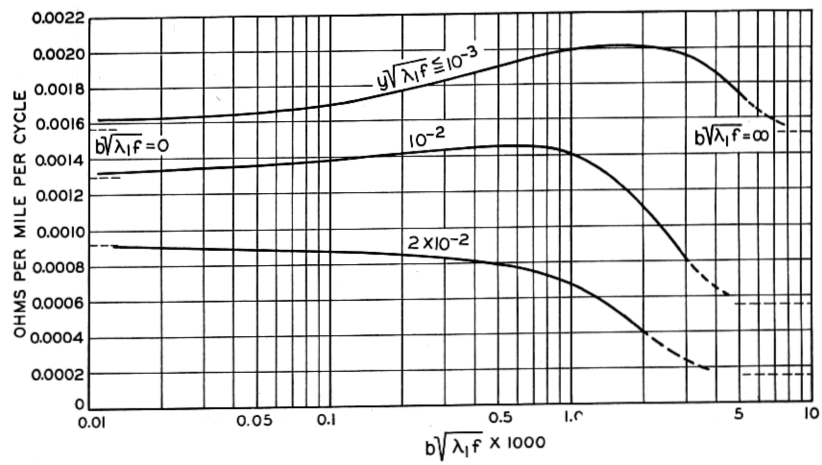

Fig. 1—Mutual impedance gradient at earth's surface parallel to an infinite straight wire on the earth's surface; real component; two-layer earth $\lambda_1 = 10\lambda_2$; b and y in feet, frequency in cycles per second, conductivities in abmhos per cm.

Figures 1 and 2 show, respectively, the real and imaginary parts of the mutual impedance gradient parallel to an infinite straight wire for the conductivity ratio $\lambda_2/\lambda_1 = 0.1$; Figs. 3 and 4 show the same quantities for $\lambda_2/\lambda_1 = 10$. When the depth b approaches the limits zero and infinity, the ground condition approaches the limits of homogeneous ground of conductivities λ_2 and λ_1, respectively. By reference to Figs. 2 and 4 it will be seen that there is a wide range in which the curves for other values of the depth are parallel to these

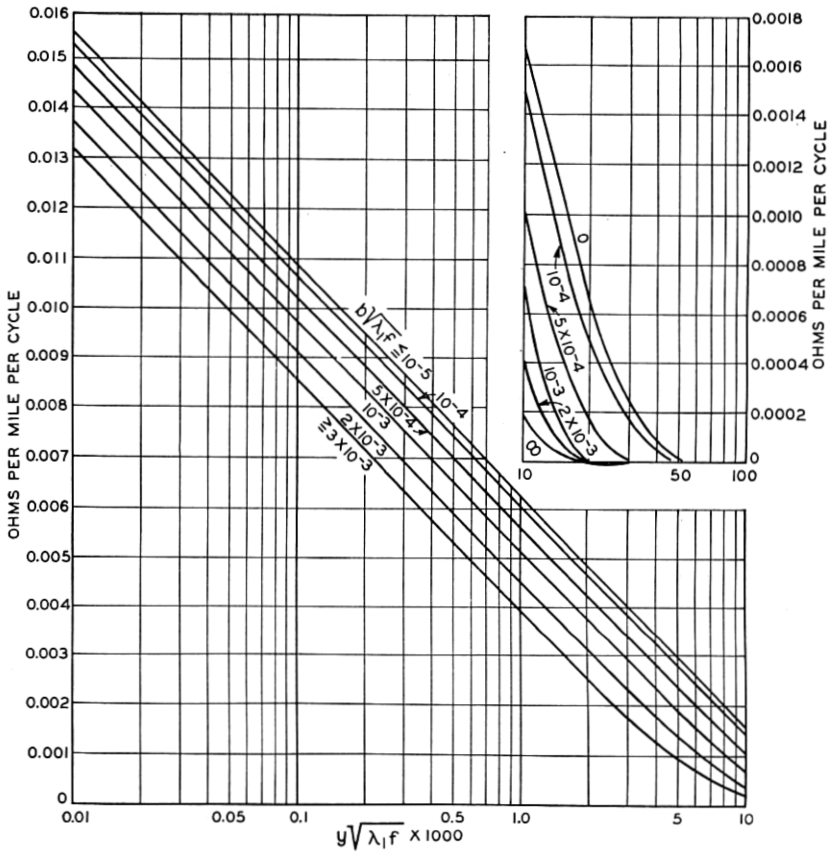

Fig. 2—Imaginary component of mutual impedance gradient for conditions of Fig. 1.

limiting curves so that for a given frequency a properly chosen homogeneous ground conductivity leads to equivalent results. The equivalent conductivity varies with the frequency, increasing or decreasing with increasing frequency according as λ_1 is greater or less than λ_2; this variation of apparent homogeneous conductivity with frequency has been frequently observed in results obtained from mutual im-

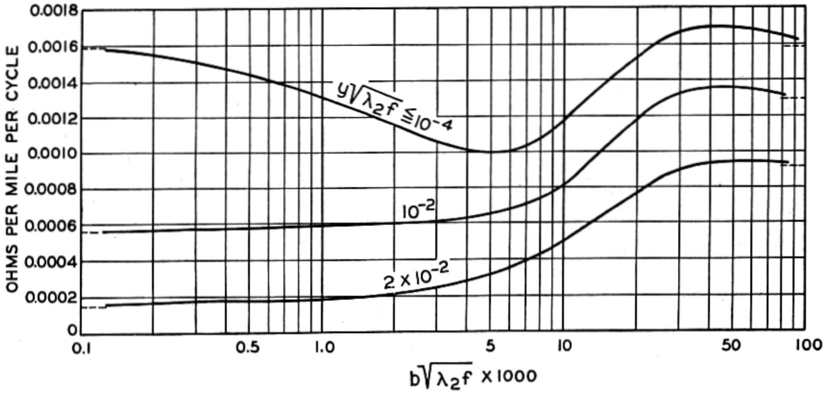

Fig. 3—Two-layer earth $\lambda_2 = 10\lambda_1$; real component of mutual impedance gradient.

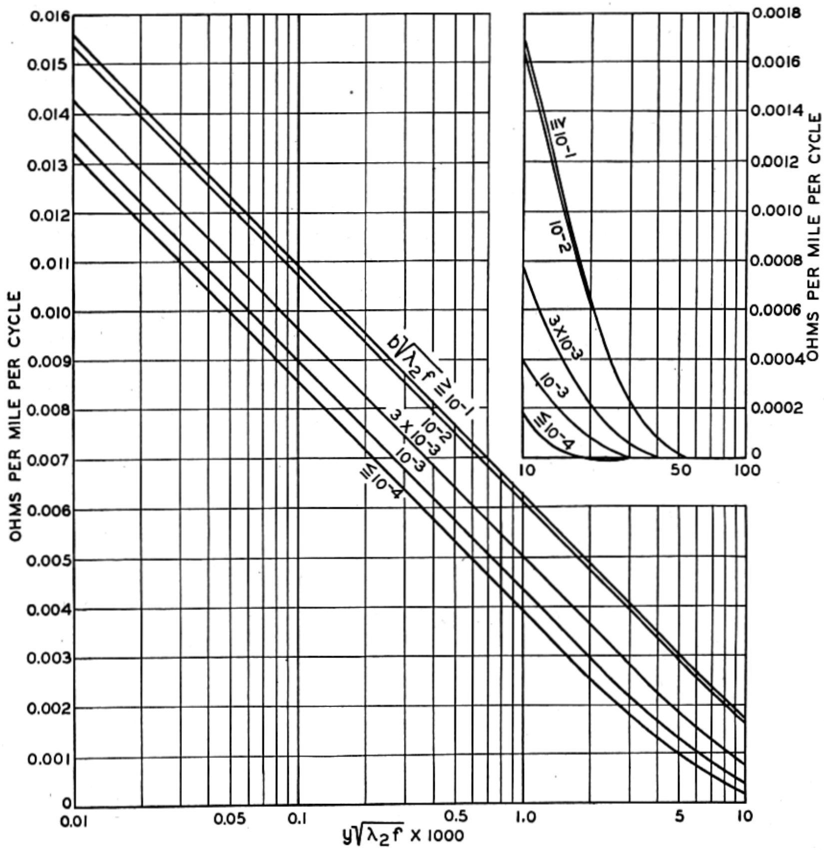

Fig. 4—Imaginary component of mutual impedance gradient for conditions of Fig. 3.

pedance measurements.[3] For a given frequency the complete group of curves of which Figs. 2 and 4 are examples may be put in more convenient form by plotting the equivalent conductivity as dependent on the other parameters of the problem.

The d.-c. mutual inductance of wires S and s, as given at the head of the corresponding column in Table I, is:

$$L^\circ_{Ss} = N^\circ_{Ss} + N^\circ(Aa) - N^\circ(Ab) - N^\circ(Ba) + N^\circ(Bb),$$

where N°_{Ss} is the mutual Neumann integral of the wires and is the main term in the formula.[4] The small contribution arising from the remaining terms is as shown on Fig. 5 always less than $\Delta = -Aa$

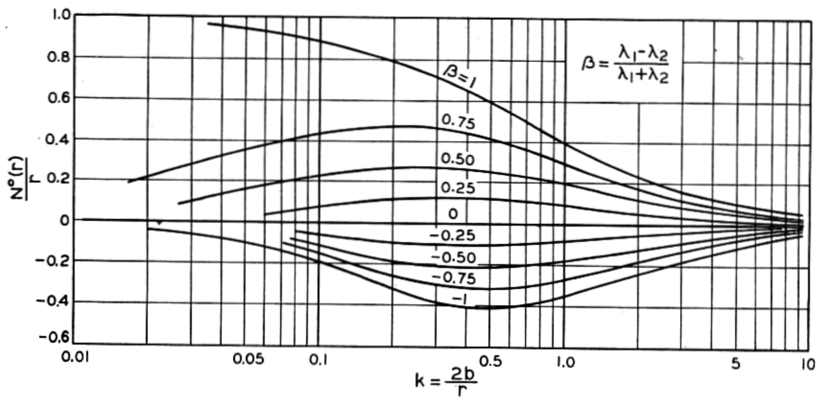

Fig. 5—D.-C. mutual inductance, exclusive of mutual Neumann integral of wire paths, of wires on surface of two-layer earth;

$$N^\circ_{(A-B)(a-b)} = N^\circ(Aa) - N^\circ(Ab) + N^\circ(Bb) - N^\circ(Ba).$$

$+ Ab + Ba - Bb$. Figure 5 shows values of $N^\circ(r)/r$ for values of $k = 2b/r$ from .01 to 10 and for a range of values of $\beta = (\lambda_1 - \lambda_2)/(\lambda_1 + \lambda_2)$ from -1 to $+1$.

The d.-c. mutual resistance, of course, also varies between the

[3] Extensive results of such measurements are published in the following papers:

A. E. Bowen and C. L. Gilkeson: "Mutual Impedances of Ground Return Circuits," *Trans. A.I.E.E.*, *49*, 1370–1383 (Oct. 1930), and *Bell System Technical Journal*, *9*, 628–651, Oct. 1930.

G. Swedenborg: "Investigations Regarding Mutual Induction in Parallel Conductors Earthed at the Ends." The L. M. Ericsson Review, English Ed. No. 7–9, 1931, pages 189–204.

H. Klewe: "Gegeninductivitäts Messungen an Leitungen mit Erdruckleitung," *Elektrische Nachrichten Technik*, 1929, page 467, and 1931, page 533.

J. Collard: "Measurement of Mutual Impedance of Circuits with Earth Return," *The Journal of The Institution of Electrical Engineers*, Vol. 71, No. 430 (Oct. 1932), pages 674–682.

[4] The only formal result in the evaluation of the Neumann integral known to us is that for arbitrary straight paths published by G. A. Campbell, "Mutual Inductances of Circuits Composed of Straight Wires," *Phys. Rev.*, *5*, pp. 452–458 (June 1915); see also his "Mutual Impedance of Grounded Circuits," *Bell System Technical Journal*, *2*, pp. 1–30 (Oct. 1923).

limiting cases of conductivities λ_2 and λ_1 corresponding to the limiting depths zero and infinity. The curves showing the dependence of mutual resistance on the depth and conductivities are put in the simplest form in terms of the two quantities $2b/r$ and the conductivity ratio λ_1/λ_2. Curves of this kind are shown on Fig. 6.[5]

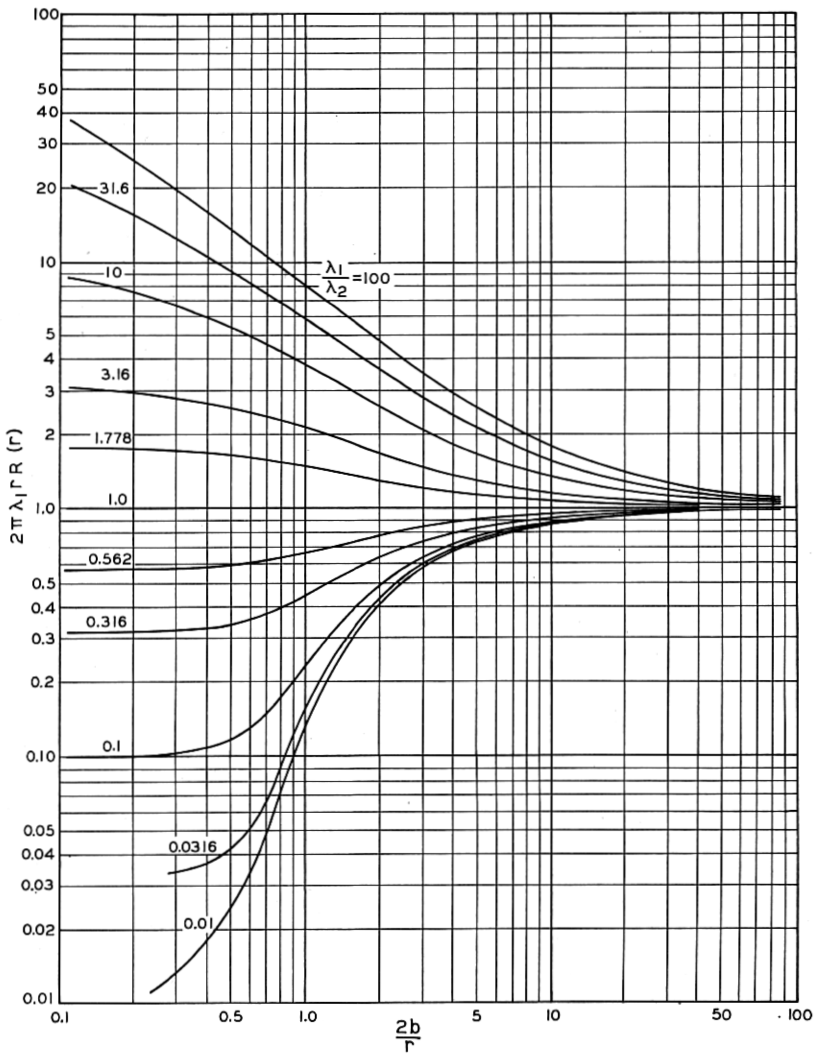

Fig. 6—D.-C. mutual resistance of wires on surface of two-layer earth; $R_{(A-B)(a-b)} = R(Aa) - R(Ab) + R(Bb) - R(Ba)$.

[5] We are indebted to our colleague Mr. L. L. Lockrow for these curves. Tables from which the function $R(r)$ can be evaluated are published in the Bureau of Mines Technical Paper No. 502.

Thus for low frequencies and short wires the main effect of two-layer earth will consist of the effect on the d.-c. mutual resistance.

III

The mutual impedance of wires in a medium having two parallel planes of discontinuity in the conductivity may be derived by extension of certain results published by A. Sommerfeld,[6] who has obtained the electric and magnetic fields of a horizontal electric doublet in a medium having one plane of discontinuity; the doublet may be regarded as an element dS of a wire of negligible diameter carrying a finite current and the mutual impedance of wires obtained by double integration over their lengths. The general formula may also be derived by extension of the second method of derivation given by R. M. Foster (loc. cit.), but for brevity this derivation is omitted here.

In the following both rectangular coordinates (x, y, z) and cylindrical coordinates (r, ϕ, z) are employed, with the origin in the upper horizontal plane of discontinuity, z in the vertical direction and x in the direction of the doublet. Electromagnetic c.g.s. units are used, and the field variation with time taken as $e^{i\omega t}$, this factor being omitted throughout. The fields are defined through "Hertzian Vectors,"[7] the rectangular components of which must individually satisfy the wave equation:

$$\frac{\partial^2 \Pi}{\partial x^2} + \frac{\partial^2 \Pi}{\partial y^2} + \frac{\partial^2 \Pi}{\partial z^2} - \gamma^2 \Pi = 0, \tag{1}$$

and in terms of which

$$E = c \text{ grad div } \Pi - c\gamma^2 \Pi, \tag{2}$$

$$H = -\frac{ic}{\omega} \gamma^2 \text{ curl } \Pi, \tag{3}$$

where

$$\Pi = \text{Hertzian vector} = \Pi_x, \Pi_y, \Pi_z,$$

$$\gamma^2 = 4\pi\lambda i\omega - \epsilon\omega^2,$$

$$\lambda, \epsilon = \text{conductivity and dielectric constant},$$

$$\omega = 2\pi f = \text{radian frequency},$$

$$c = \text{velocity of light}.$$

[6] A. Sommerfeld, "Über die Ausbreitung der Wellen in der drahtlosen Telegraphie," *Annalen der Physik* (4), *81*, 1135–1153, December, 1926.

[7] Abraham and Föppl, "Theorie der Elektrizität," 7th ed., Leipzig and Berlin, 1922, Vol. I, § 79, page 322.

The conductivities and dielectric constants are taken as:

$$\begin{aligned} &\lambda_0,\ \epsilon_0 \quad \text{for} \qquad z > 0 \\ &\lambda_1,\ \epsilon_1 \quad \text{for} - b < z < 0 \\ &\lambda_2,\ \epsilon_2 \quad \text{for} \qquad z < -b \end{aligned}$$

where b is the distance between the parallel planes of discontinuity.

The primary field of a doublet in the direction of the x-axis at $z = h$ is given by

$$\Pi_{0x}' = A\,\frac{e^{-\gamma_0 R}}{R} = A \int_0^\infty \frac{e^{\alpha_0(z-h)}}{\alpha_0} u J_0(ru)du \qquad 0 > z > h, \tag{4}$$

where

$$\begin{aligned} r^2 &= x^2 + y^2, \\ R^2 &= r^2 + (z - h)^2, \\ \gamma_0^2 &= 4\pi i \omega \lambda_0 - \omega^2 \epsilon_0, \\ \alpha_0^2 &= u^2 + \gamma_0^2. \end{aligned}$$

The constant A which is the moment of the doublet is as yet undetermined. From (3):

$$H_0' = -A\,\frac{ic}{\omega}\gamma_0^2 \operatorname{curl} \Pi_{0x}',$$

and for $\gamma_0 = 0$:

$$H_0' = \left[-A\,\frac{ic}{\omega}\gamma_0^2 \right]_{\gamma_0 = 0} \operatorname{curl} \frac{1}{R}.$$

For this case the magnetic force due to unit current in an element dS is given by Biot-Savart's law [8] as

$$H = dS \operatorname{curl} \frac{1}{R},$$

so that

$$A = \frac{i\omega dS}{\gamma_0^2}.$$

The secondary fields are derived from Hertzian vectors having components in the direction of the x and z axes. Components in the direction of the y-axis are eliminated due to symmetry with respect to the $x - z$ plane. The resulting fields are then composed as follows:

$$\begin{aligned} \Pi_{0x} = \Pi_{0x}' + \Pi_{0x}'', \quad & \Pi_{0z} = \Pi_{0z}'' & z \geqslant 0, \\ \Pi_{1x}, \quad & \Pi_{1z} & -b \leqslant z \leqslant 0, \\ \Pi_{2x}, \quad & \Pi_{2z} & z \leqslant -b, \end{aligned} \tag{5}$$

[8] Abraham and Föppl, "Theorie der Elektrizität," 7th ed., Vol. I, § 55, page 189.

in the first of which the double primes indicate the components of the secondary field.

The expressions for the field intensities in terms of the Hertzian vector components may now be written out by equations (2) and (3) and are as follows:

$$
\begin{aligned}
H_x &= -\frac{ic}{\omega}\gamma^2\frac{\partial \Pi_z}{\partial y},\\
H_y &= -\frac{ic}{\omega}\gamma^2\left[\frac{\partial \Pi_x}{\partial z}-\frac{\partial \Pi_z}{\partial x}\right],\\
H_z &= \frac{ic}{\omega}\gamma^2\frac{\partial \Pi_x}{\partial y},\\
E_x &= -c\gamma^2\Pi_x + c\frac{\partial}{\partial x}\left(\frac{\partial \Pi_x}{\partial x}+\frac{\partial \Pi_z}{\partial z}\right),\\
E_y &= \qquad\qquad c\frac{\partial}{\partial y}\left(\frac{\partial \Pi_x}{\partial x}+\frac{\partial \Pi_z}{\partial z}\right),\\
E_z &= -c\gamma^2\Pi_z + c\frac{\partial}{\partial z}\left(\frac{\partial \Pi_x}{\partial x}+\frac{\partial \Pi_z}{\partial z}\right).
\end{aligned}
\tag{6}
$$

The proper general solution of (1) for the components of the secondary fields is of the following form:

$$
\Pi = \cos n\phi \int_0^{\infty} (f(u)e^{\alpha z} + g(u)e^{-\alpha z})J_n(ru)du. \tag{7}
$$

where $\alpha^2 = u^2 + \gamma^2$ and $\cos\phi = \dfrac{x}{r}$.

The boundary conditions at $z = 0$ and $z = -b$ consist in the continuity of the tangential (x, y) components of H and E. The equations arising from the boundary conditions can be simplified by differentiation or integration with respect to x or y (which is possible by virtue of (7)), and are taken in the following convenient form:

$z = 0$:

$$
\gamma_0^2\Pi_{0z} = \gamma_1^2\Pi_{1z}, \tag{8}
$$

$$
\gamma_0^2\frac{\partial \Pi_{0x}}{\partial z} = \gamma_1^2\frac{\partial \Pi_{1x}}{\partial z}, \tag{9}
$$

$$
\frac{\partial \Pi_{0x}}{\partial x}+\frac{\partial \Pi_{0z}}{\partial z} = \frac{\partial \Pi_{1x}}{\partial x}+\frac{\partial \Pi_{1z}}{\partial z}, \tag{10}
$$

$$
\gamma_0^2\Pi_{0x} = \gamma_0^2\Pi_{1x}; \tag{11}
$$

$z = -b$:

$$\gamma_1^2 \Pi_{1z} = \gamma_2^2 \Pi_{2z}, \tag{12}$$

$$\gamma_1^2 \frac{\partial \Pi_{1x}}{\partial z} = \gamma_2^2 \frac{\partial \Pi_{2x}}{\partial z}, \tag{13}$$

$$\frac{\partial \Pi_{1x}}{\partial x} + \frac{\partial \Pi_{1z}}{\partial z} = \frac{\partial \Pi_{2x}}{\partial x} + \frac{\partial \Pi_{2z}}{\partial z}, \tag{14}$$

$$\gamma_1^2 \Pi_{1x} = \gamma_2^2 \Pi_{2x}. \tag{15}$$

From boundary conditions (9), (11), (13), and (15), the x-components of the Hertzian vectors can be determined separately and then used in finding the z-components.

For the x-components the arbitrary functions $f(\lambda)$ and $g(\lambda)$ can be determined from the boundary conditions if in (7) $n = 0$. These components are therefore taken as follows:

$$\Pi_{0x}'' = \int_0^\infty f_0(u) e^{-\alpha_0 z} J_0(ru)\, du \qquad z \geqslant 0, \tag{16}$$

$$\Pi_{1x} = \int_0^\infty (f_1(u) e^{\alpha_1 z} + g_1(u) e^{-\alpha_1 z}) J_0(ru)\, du \qquad -b \leqslant z \leqslant 0, \tag{17}$$

$$\Pi_{2x} = \int_0^\infty f_2(u) e^{\alpha_2 z} J_0(ru)\, du \qquad z \leqslant -b. \tag{18}$$

The arbitrary functions $f(u)$ and $g(u)$ are then determined by the following equations, obtained from (9), (11), (13), and (15):

$$\begin{aligned} \gamma_1^2 \alpha_1 (f_1 - g_1) &= A \gamma_0^2 u e^{-\alpha_0 h} - \gamma_0^2 \alpha_0 f_0, \\ \gamma_1^2 \alpha_0 (f_1 + g_1) &= A \gamma_0^2 u e^{-\alpha_0 h} + \gamma_0^2 \alpha_0 f_0, \\ \gamma_1^2 \alpha_1 (f_1 e^{-\alpha_1 b} - g_1 e^{\alpha_1 b}) &= \gamma_2^2 \alpha_2 f_2 e^{-\alpha_2 b}, \\ \gamma_1^2 (f_1 e^{-\alpha_1 b} + g_1 e^{\alpha_1 b}) &= \gamma_2^2 f_2 e^{-\alpha_2 b}, \end{aligned} \tag{19}$$

where for convenience the argument u of the arbitrary functions has been omitted.

The solutions of (19) for f_1 and g_1 are

$$\begin{aligned} f_1 &= \frac{2Au(\alpha_1 + \alpha_2)}{\Delta} e^{-h\alpha_0}, \\ g_1 &= \frac{2Au(\alpha_1 - \alpha_2)}{\Delta} e^{-h\alpha_0 - 2b\alpha_1}, \end{aligned} \tag{20}$$

where

$$\Delta = {}^{0}\varpi) + \alpha_1)(\alpha_1 + \alpha_2) + (\alpha_0 - \alpha_1)(\alpha_1 - \alpha_2) e^{-2b\alpha_1}.$$

The boundary conditions for the z-components may be satisfied by taking $n = 1$ in equation (7); the resulting expressions are as follows:

$$\Pi_{0z} = \cos\phi \int_0^\infty p_0(u) e^{-\alpha_0 z} J_1(ru) du \qquad z \geqslant 0, \tag{21}$$

$$\Pi_{1z} = \cos\phi \int_0^\infty (p_1(u) e^{-\alpha_0 z} + q_1(u) e^{-\alpha_0 z}) J_1(ru) du \quad -b \leqslant z \leqslant 0, \tag{22}$$

$$\Pi_{2z} = \cos\phi \int_0^\infty p_2(u) e^{\alpha_0 z} J_1(ru) du \qquad z \leqslant -b. \tag{23}$$

From boundary conditions (8), (10), (12), and (14) and the values of the x-components as now determined, Π_{1x} being given by equation (17), Π_{0x} and Π_{2x} being given in terms of it by equations (11) and (15), the following equations are obtained:

$$\begin{aligned}
\gamma_1^2(p_1 + q_1) &= \gamma_0^2 p_0, \\
\alpha_1\gamma_0^2(p_1 - q_1) &= -\alpha_0\gamma_0^2 p_0 + u(\gamma_0^2 - \gamma_1^2)(f_1 - g_1), \\
\gamma_1^2(p_1 e^{-\alpha_1 b} + q_1 e^{\alpha_1 b}) &= \gamma_2^2 p_2 e^{-\alpha_2 b}, \\
\alpha_1\gamma_1^2(p_1 e^{-\alpha_1 b} - q_1 e^{\alpha_1 b}) &= \alpha_2\gamma_2^2 p_2 e^{-\alpha_2 b} \\
&\quad + u(\gamma_2^2 - \gamma_1^2)(f_1 e^{-\alpha_1 b} + g_1 e^{\alpha_1 b}),
\end{aligned} \tag{24}$$

the arguments of the functions being omitted as before.

From (24), p_1 and q_1 are obtained as

$$\begin{aligned}
p_1 &= u\,\frac{(\gamma_0^2 - \gamma_1^2)(\alpha_1\gamma_2^2 + \alpha_2\gamma_1^2)(f_1 + g_1) - (\gamma_1^2 - \gamma_2^2)(\alpha_0\gamma_1^2 - \alpha_1\gamma_0^2)(f_1 e^{-\alpha_1 b} + g_1 e^{\alpha_1 b}) e^{-\alpha_1 b}}{(\alpha_0\gamma_1^2 + \alpha_1\gamma_0^2)(\alpha_1\gamma_2^2 + \alpha_2\gamma_1^2) + (\alpha_0\gamma_1^2 - \alpha_1\gamma_0^2)(\alpha_1\gamma_2^2 - \alpha_2\gamma_1^2) e^{-2b\alpha_1}}, \\
q_1 &= u\,\frac{(\gamma_0^2 - \gamma_1^2)(\alpha_1\gamma_2^2 - \alpha_2\gamma_1^2)(f_1 + g_1) e^{-2b\alpha_1} + (\gamma_1^2 - \gamma_2^2)(\alpha_0\gamma_1^2 + \alpha_1\gamma_0^2)(f_1 e^{-\alpha_1 b} + g_1 e^{\alpha_1 b}) e^{-\alpha_1 b}}{(\alpha_0\gamma_1^2 + \alpha_1\gamma_0^2)(\alpha_1\gamma_2^2 + \alpha_2\gamma_1^2) + (\alpha_0\gamma_1^2 - \alpha_1\gamma_0^2)(\alpha_1\gamma_2^2 - \alpha_2\gamma_1^2) e^{-2b\alpha_1}}.
\end{aligned} \tag{25}$$

The tangential components of the electric force at $z = 0$ are by (6), (17), and (22):

$$\begin{aligned}
E_x = &- c\gamma_1^2 \int_0^\infty (f_1(u) + g_1(u)) J_0(ru) du \\
&+ c\frac{\partial}{\partial x}\cos\phi \int_0^\infty [-u(f_1(u) + g_1(u)) \\
&\qquad + \alpha_1(p_1(u) - q_1(u))] J_1(ru) du,
\end{aligned} \tag{26}$$

$$E_y = + c\frac{\partial}{\partial y}\cos\phi\int_0^\infty[-u(f_1(u) + g_1(u)) + \alpha_1(p_1(u) - q_1(u))]J_1(ru)du.$$

Or since $\cos\phi\int_0^\infty J_1(ru)du = -\frac{\partial}{\partial x}\int_0^\infty\frac{J_0(ru)}{u}$,

$$E_x = - c\gamma_1^2\int_0^\infty(f_1(u) + g_1(u))J_0(ru)du - c\frac{\partial^2}{\partial x^2}\int_0^\infty\left[-(f_1(u) + g_1(u)) + \frac{\alpha_1}{u}(p_1(u) - q_1(u))\right]J_0(ru)du, \quad (26a)$$

$$E_y = - c\frac{\partial^2}{\partial x\partial y}\int_0^\infty\left[-(f_1(u) + g_1(u)) + \frac{\alpha_1}{u}(p_1(u) - q_1(u))\right]J_0(ru)du.$$

Inserting the values of $f_1(u)$, $g_1(u)$, $p_1(u)$, and $q_1(u)$ for $h = 0$ and neglecting all displacement currents ($\epsilon_0 = \epsilon_1 = \epsilon_2 = 0$), the following expression is obtained:

$$E_x, E_y = dS\left[-i\omega P(r) + \frac{\partial^2 Q(r)}{\partial x^2}, \frac{\partial^2 Q(r)}{\partial x\partial y}\right], \quad (27)$$

where

$$P(r) = 2\int_0^\infty\frac{u}{\Delta}[\alpha_1 + \alpha_2 + (\alpha_1 - \alpha_2)e^{-2b\alpha_1}]J_0(ru)du,$$

$$Q(r) = \frac{1}{2\pi}\int_0^\infty\frac{u}{\Delta\Delta_1}\{4(\lambda_1 - \lambda_2)\alpha_0\alpha_1^2e^{-2b\alpha_1} + [\alpha_1 + \alpha_2 + (\alpha_1 - \alpha_2)e^{-2b\alpha_1}]\Delta_2\}J_0(ru)du,$$

where as before

$$\Delta = (\alpha_0 + \alpha_1)(\alpha_1 + \alpha_2) + (\alpha_0 - \alpha_1)(\alpha_1 - \alpha_2)e^{-2b\alpha_1}$$

and

$$\Delta_1 = (\alpha_0\lambda_1 + \alpha_1\lambda_0)(\alpha_1\lambda_2 + \alpha_2\lambda_1) + (\alpha_0\lambda_1 - \alpha_1\lambda_0)(\alpha_1\lambda_2 - \alpha_2\lambda_1)e^{-2b\alpha_1},$$
$$\Delta_2 = (\alpha_0 + \alpha_1)(\alpha_1\lambda_2 + \alpha_2\lambda_1) + (\alpha_0 - \alpha_1)(\alpha_1\lambda_2 - \alpha_2\lambda_1)e^{-2b\alpha_1}.$$

The mutual impedance of wire elements dS and ds lying in the plane $z = 0$, and including the angle ϵ between their directions, is:

$$
\begin{aligned}
dZ_{Ss} &= -\,ds[E_x \cos \epsilon + E_y \sin \epsilon] \\
&= dSds\left[i\omega P(r) \cos \epsilon - \frac{\partial^2 Q(r)}{\partial x^2} \cos \epsilon - \frac{\partial^2 Q(r)}{\partial x \partial y} \sin \epsilon \right] \qquad (28) \\
&= \left\{ \frac{d^2 Q(r)}{dSds} + i\omega P(r) \cos \epsilon \right\} dSds.
\end{aligned}
$$

Integration over the two wires S and s extending from A to B and from a to b, respectively, gives their mutual impedance:

$$
Z_{Ss} = \int_a^b \int_A^B \left\{ \frac{dQ(r)}{dSds} + i\omega P(r) \cos \epsilon \right\} dSds. \qquad (29)
$$

With $\lambda_0 = 0$, the expressions following (27) for $P(r)$ and $Q(r)$ reduce to those given in Part I for wires on the surface of the earth. The expressions given in Part I for wires in the plane of separation at the depth b below the earth's surface are obtained by putting $\lambda_2 = 0$ and changing λ_0 to λ_2 in the resulting expression.

We are indebted to Mr. R. M. Foster for evaluation of some of the integrals as well as for general suggestions and counsel.