Probability Theory and Telephone Transmission Engineering

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Part I of this paper contributes methods, theorems, formulas and graphs to meet a previously unfilled need in dealing with certain types of two-dimensional probability problems—especially those relating to alternating current transmission systems and networks, in which the variables occur naturally in complex form and thus are two-dimensional. The paper is concerned particularly with "normal" probability functions (distribution functions) in two dimensions, which are analogous to the familiar "normal" probability functions in one-dimensional probability problems. It supplies a comprehensive set of graphs for the probability that a "normal" complex chance-variable deviates from its mean value by an amount whose magnitude (absolute value) exceeds any stated value; in other words, the probability that the chance-variable lies without any specified circle centered at the mean value in the plane of its "scatter-diagram," that is, in the complex plane of the chance-variable. It gives a comprehensive treatment of the distribution-parameters of the "normal" complex chance-variable, and convenient formulas for the necessary evaluation of these parameters. For use in various portions of the paper, as well as for various possible outside uses, it supplies a considerable number of formulas and theorems on "mean values" ("expected values") of complex chance-variables.

Part II of the paper makes application of Part I to some important problems in telephone transmission systems and networks involving chance irregularities of structure and hence requiring the application of probability theory.

INTRODUCTION

IN telephone transmission engineering a frequent problem is that of determining the effects of random manufacturing variations upon the value of some characteristic (for instance, a transfer admittance, or a driving-point impedance, or a current-ratio) of a transmission system or network.[1] In certain cases, such effects may be of great or even controlling importance in the performance of the system and hence must be fully taken into account when designing the system and when making calculations for predicting its performance.

For example, in a multi-pair telephone cable the crosstalk between any two pairs is directly proportional to (strictly, a linear function

[1] Such problems have in the past been handled by various approximate methods the most satisfactory of which for many purposes was that described in a paper by George Crisson, entitled, "Irregularities in Loaded Telephone Circuits," published in this Journal for October, 1925. The method given in the present paper, while necessarily more involved than approximate methods, yields more precise results; and this additional precision is expected to be of importance in practice. Moreover, there has been an increasing need for a comprehensive paper covering the entire ground, and it is hoped that the present paper meets this need to a measurable extent.

In Crisson's paper references will be found to various engineers in the Bell System who had previously contributed to specific probability problems of the type dealt with in Part II of the present paper.

of) the deviations of certain internal parameters from their nominal values. Another example is furnished by two telephone lines connected by the usual type of two-way telephone repeater: If the two lines and their associated apparatus could be made identically alike, a state of perfect balance would exist at the repeater and there would be no tendency for the repeater to sing; however, as a result of manufacturing variations, perfect balance is unattainable and thus the practicable amplification obtainable from the repeater is limited by the manufacturing deviations of the lines and associated apparatus—particularly the deviations in the inductances and spacings of the loading coils, in case the lines are loaded.

Such examples may furnish at least the three types of probability problems described in the following three paragraphs:

Before the construction of the system there may arise the "direct" problem of calculating the characteristic to be expected, corresponding to the known (or assumed) ranges of the manufacturing variations in the elements. Before the elements are manufactured, the deviation of any element from its nominal value is of course unknown; moreover, such deviation is not completely predictable, since from its very nature it depends on chance. The deviation is a variable in the sense that it can take any value within a certain possible range. But it is a particular sort of variable, namely a chance-variable, in the sense that there exists a certain chance or probability that the deviation will lie within any stated range of values, with the chance depending of course on this range and on the specific probability law of deviation for the kind of element under consideration. Correspondingly the deviation of the contemplated transmission characteristic of the proposed system is a chance-variable, whose probability law depends of course on the probability laws for the deviations of the elements and on the functional formula connecting the contemplated transmission characteristic with the elements.

Before the elements of the system have been manufactured there may, on the other hand, arise the "inverse" problem of setting such restrictions on the manufacturing deviations of the elements as to insure that the contemplated characteristic of the proposed transmission system will have a preassigned probability of lying within a certain specified range. As might be expected, this "inverse" problem is more difficult than the "direct" problem, and often it can be solved only by successive tentative solutions of the corresponding "direct" problem.

Finally, after the system has been constructed and tested, there may arise the question as to whether its elements have been correctly connected together when installed. Assuming that the elements themselves are known, from previous individual measurements on them, to fall within their specified ranges of allowable variation, a comparison of the measured value of the contemplated characteristic with the calculated value to be expected on the basis of probability theory will give some indication as to whether some of the elements are incorrectly connected. Further, when there is present not merely a single system but a large number of systems which are nominally alike (for instance, the various pairs in a multi-pair telephone cable), measurement of the contemplated transmission characteristic of each of the systems and comparison of the statistical distribution of these measured values with their calculated theoretical distribution will give a more conclusive indication as to whether some of the elements are incorrectly connected.

Any particular problem to be solved can be handled most conveniently and advantageously if the general problem is first formulated analytically. Let us suppose, therefore, that H denotes the specific transmission characteristic under consideration (for instance, a transfer admittance, or a driving-point impedance, or a current-ratio), and $K_1, \cdots K_n$ the internal parameters on which H depends; and let the functional formula for H be

$$H = F(K_1, \cdots K_n), \tag{I}$$

where, of course, H and the K's are in general complex (on the supposition that the usual complex quantity method of treating alternating-current problems is being employed). As we shall be particularly concerned with the deviations of the various quantities from their nominal values it will be convenient to suppose that H and the K's denote the nominal values of the corresponding quantities, and that any actual set of values are denoted by $H + h$ and $K_1 + k_1, \cdots K_n + k_n$, so that h and $k_1, \cdots k_n$ will denote the corresponding complex deviations of these quantities from their nominal values. Then the general functional formula for h will of course be

$$h = F(K_1 + k_1, \cdots K_n + k_n) - F(K_1, \cdots K_n). \tag{II}$$

Since h may be regarded as causally dependent on the k's, it may naturally be called the "resulting" chance-variable.

Usually the k's will be so small compared with the K's that the right side of (II) can be replaced, as a good approximation, by the first-order terms of a Taylor expansion; thus, approximately,

$$h = D_1k_1 + \cdots + D_nk_n, \tag{III}$$

where

$$D_r = \partial F(K_1, \cdots K_n)/\partial K_r, \qquad (r = 1, 2, \cdots n). \qquad \text{(IV)}$$

Before the physical elements are manufactured the k's are chance-variables, in the sense already defined; for it is not possible to predict the value which any one, say k_r, will have, but only to state the chance that it will lie within any specified range, this chance being calculable from the known (or assumed) probability law $p_r(k_r)$. Hence h is also a chance-variable, whose probability law $p(h)$ depends on the functional formula for h and on the individual probability laws $p_1(k_1)$, $\cdots p_n(k_n)$. In the general case, the "direct" problem is to calculate from $p(h)$ the probability that h will have a value lying within any specified region of the h-plane.

In the types of problem contemplated in the present paper, the probability law $p(h)$ of h may usually be assumed to be approximately "normal" (Subsection 1.2). Moreover, the specified region in the h-plane will usually be a circle, since in such problems we are usually concerned only with the magnitude of h, not with its angle. For crosstalk, this is obviously true. For the usual type of two-way telephone repeater operating between lines whose impedances do not balance each other, it is true as a good approximation when the unbalance is not too large, since then the practicable amplification depends (approximately) only on the magnitude of the unbalance, not on its angle.

Unfortunately the complete solution of the problem for a circular region is sufficiently difficult and laborious, particularly as regards numerical evaluation, that apparently there has not heretofore been sufficient incentive to lead to its being carried through—at least so far as I am aware.[2] The present paper includes the needed solution, in convenient form for practical applications, by means of the comprehensive set of graphs described in Subsection 1.3, supplemented by Subsection 1.2 defining and formulating the "normal" complex chance-variable, and further supplemented by Section 2 giving general methods and formulas for evaluating the distribution-parameters of the "normal" complex chance-variable; and by Section 3, which applies Section 2 to the case where, as is usual, the contemplated "resulting" complex chance-variable is (at least approximately) a linear function of other complex chance-variables.

Section 4, which is somewhat in the nature of an appendix, supplies a considerable number of formulas and theorems on "mean values"

[2] As well-known to those familiar with the literature of the subject, the solution is quite easy for regions having certain other shapes, notably for an equiprobability ellipse and for a rectangle lying parallel to a principal axis of such an ellipse. However, those solutions are of no help in the case of a circular region.

("expected values") of complex chance-variables. These formulas and theorems find frequent and important uses in the present paper; and outside of the paper they may well find varied uses.

The method of treatment characterizing the present paper will now be very briefly indicated in the remainder of this Introduction.

As a preliminary step toward this objective we shall now return to the functional formulas (II) and (III) with the remark that, if the K's and k's were all real quantities and if these formulas were such that h also were a real quantity, then the "direct" problem would be to calculate the probability that h would lie within any stated linear range, say h_a to h_b; thus the probability problem would then be one-dimensional, and the well-known existing probability theory for real quantities would be immediately applicable, including the corresponding known methods and formulas for evaluation of the distribution-parameters.

When, as in the present paper, the K's and k's are in general complex quantities, the corresponding probability problem is inherently two-dimensional. The distribution-parameters, which naturally are more numerous than in the one-dimensional case, could be evaluated in a roundabout way by an extensive process of resolution into rectangular components; but it is believed that very superior advantages are possessed by the probability methods and formulas contributed by the present paper, for dealing with complex chance-variables in a more direct manner, as set forth at some length in Sections 2 and 3, extensively utilizing Section 4. The advantages of this method for evaluating the distribution-parameters are perhaps particularly marked whenever there is involved a summation of propagated effects, as in transmission lines; for then, as will appear more concretely in the applications in Part II, the necessary summations can be accomplished much more easily and the resulting expressions are much more compact and manageable than if a method employing rectangular resolutions were used.

Regardless of which method is used for evaluating the distribution-parameters, the new material contributed by Subsection 1.3 is necessary for the complete numerical solution of the problem in any specific case where the "resulting" complex chance-variable h is "normal." It may be recalled that this will be the case when h is a linear function of the k's and the k's themselves are "normal." Even when these two conditions are rather far from being fulfilled, however, it is known from certain rather broad theoretical considerations that in many practical problems h will be approximately "normal"; it may perhaps be recalled that one of the most important among a set of sufficient conditions for approximate "normality" is that the k's be numerous (n a large number).

As stated in the Synopsis, Part II makes application of Part I to some important problems in telephone transmission systems and networks involving chance irregularities in structure. One of these problems, namely that in Section 5, is the general problem already outlined in connection with the equations in this Introduction.

Part I: Theory

1. PROBABILITY OF THE DEVIATION OF A NORMAL COMPLEX CHANCE-VARIABLE FROM ITS MEAN VALUE

Toward the end of the Introduction it was stated that in many problems of the types contemplated in the present paper the distribution of the "resulting" complex chance-variable is approximately "normal."

To meet a previously unfilled need in the solution of such problems, this Section of the paper supplies (in Subsection 1.3) a comprehensive set of graphs for the probability that a "normal" complex chance-variable deviates from its mean value by an amount whose magnitude (absolute value) exceeds any stated value; that is, the probability that the chance-variable lies without any specified circle centered at the mean value in the plane of its "scatter-diagram." These graphs are accompanied by sufficient explanation to enable them to be understood and used without any necessity for studying the formulas from which they were computed—which, because of their length and complexity, have not been included in this paper.[3]

To furnish the necessary precise basis for the graphs, Subsection 1.3 describing them is preceded by Subsection 1.2 giving analytical definitions of the normal complex chance-variable and its distribution-parameters; and these quantities are discussed at moderate length there.

To lead up to the normal complex chance-variable, it is preceded by a brief review of the normal real chance-variable (Subsection 1.1), which is more familiar.

1.1. *The Normal Real Chance-Variable*

In order to lead up to the normal complex chance-variable (which is 2-dimensional) it will be recalled that a real chance-variable (which

[3] The formulas are given (with derivations) in an unpublished Appendix (Appendix A). Another unpublished Appendix (B) gives various concepts and definitions employed in two-dimensional probability theory, and also gives various analytical and graphical ways of representing probability. Still another (C) treats a problem of crosstalk in a telephone cable.

is 1-dimensional) is defined as "normal" if its probability law, or distribution function, can by the proper choice of origin be written in the form

$$P_u = \frac{1}{\sqrt{2\pi}S_u} \exp\left(-\frac{u^2}{2S_u^2}\right), \tag{1}$$

where, by definition of the term "probability law," $P_u du$ represents in general the probability that the unknown value u' of a random sample consisting of a single value of the chance-variable lies between u and $u + du$; or, what is ultimately equivalent, the probability that u' lies in the differential range $u \pm du/2$, namely in the differential range du containing the point u. S_u is a distribution-parameter called the "standard deviation" of u and defined by the equation

$$S_u^2 = \overline{(u - \bar{u})^2} = \overline{u^2} = \int_{-\infty}^{\infty} u^2 P_u du, \tag{2}$$

the superbar connoting the "mean value," or "mean," of any chance-variable to which it is applied. In this paper the term "mean value" is used as an alternative for "expected value," namely the "weighted average value" with the weighting in accordance with the probability of occurrence of each particular possible value of the variable. (Section 4 supplies a considerable number of formulas and theorems on mean values of complex chance-variables—and hence of real chance-variables, by specialization.)

From the foregoing definitions, it is easily verified that

$$\int_{-\infty}^{\infty} P_u du = 1,$$

which corresponds to taking unity as the measure of certainty.

It will be recognized that the chance-variable u in equation (1) is related to the original given chance-variable, which will be denoted by x, by the equation $u = x - \bar{x}$. Hence $\bar{u} = 0$, as has already appeared in equation (2); thus the origin is at the "center" c of the distribution, namely the point u_c with respect to which as origin the "mean value" of the chance-variable is zero, that is, such that $\overline{u - u_c} = 0$, whence $u_c = \bar{u} = 0$. If, in terms of the original variable x, the position of c is denoted by x_c, then $\overline{x - x_c} = 0$ and hence $x_c = \bar{x}$. Since $u = x - \bar{x}$, it is seen from (2) that

$$S_u^2 = \overline{(x - \bar{x})^2} = S_x^2. \tag{3}$$

The probability that the magnitude (absolute value) $|u'|$ of a random sample u' of u is less than any stated value r will be denoted by $p(|u'| < r)$. Then

$$p(|u'| < r) = \int_{-r}^{r} P_u du = \frac{2}{\sqrt{2\pi} S_u} \int_0^r \exp\left(-\frac{u^2}{2S_u^2}\right) du. \tag{4}$$

Evidently the number of parameters can be reduced from one (which is S_u) to none by taking as chance-variable the ratio u/S_u, which may be called the "reduced" chance-variable. Thus, with $|u'|$ denoted by r' and with r'/S_u and r/S_u denoted by R' and R respectively, equation (4) becomes

$$p(|u'| < r) = p(R' < R) = \text{erf}\ (R/\sqrt{2}), \tag{5}$$

where erf ( ) is the so-called "error function" defined, for any variable z, by the equation

$$\text{erf}\ (z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp\ (-\lambda^2) d\lambda \tag{6}$$

and extensively tabulated [4] for real values of z. For some purposes it is more convenient to employ the "error function complement," defined by the equation

$$\text{erfc}\ (z) = \frac{2}{\sqrt{\pi}} \int_z^\infty \exp\ (-\lambda^2) d\lambda \tag{7}$$

and hence related to erf (z) by the equation

$$\text{erf}\ (z) + \text{erfc}\ (z) = 1. \tag{8}$$

If u_3 denotes any fixed value of u, and if U_3 denotes u_3/S_u, then

$$p(u' > u_3) = \int_{u_3}^{\infty} P_u du = \frac{1}{2} \text{erfc} \frac{U_3}{\sqrt{2}}. \tag{9}$$

If u_1 and u_2 denote any two fixed values of u such that $u_1 < u_2$, and if U_1 and U_2 denote u_1/S_u and u_2/S_u respectively, then

$$p(u_1 < u' < u_2) = \frac{1}{2}\left(\text{erfc} \frac{U_1}{\sqrt{2}} - \text{erfc} \frac{U_2}{\sqrt{2}}\right). \tag{10}$$

[4] To avoid possible confusion, it may be well to remind the reader that there has also been extensively tabulated, for real values of z, the closely related function

$$\frac{1}{\sqrt{2\pi}} \int_0^z \exp\ (-\lambda^2/2) d\lambda,$$

which is more convenient for some purposes, though less convenient in the present paper.

If, with a view to generalizing (5), we inquire as to the probability $p(|u' - u_0| < r)$ that u' deviates from any fixed value u_0 of u by an amount whose magnitude is less than any stated value r, and if now we let r' and r_0 denote $|u' - u_0|$ and $|u_0|$ respectively and R, R', R_0 denote r/S_u, r'/S_u, r_0/S_u respectively, then

$$p(|u' - u_0| < r) = p(R' < R)$$
$$= \frac{1}{2}\left[\operatorname{erf}\left(\frac{R_0 + R}{\sqrt{2}}\right) - \operatorname{erf}\left(\frac{R_0 - R}{\sqrt{2}}\right)\right]. \tag{11}$$

When $u_0 = 0$ this formula correctly reduces to (5).

1.2. *The Normal Complex Chance-Variable*

Before proceeding to the "normal" complex chance-variable it should be remarked that, although any 2-dimensional chance-variable can be represented either as a complex chance-variable $z = x + iy = \mu \exp(i\eta)$ or as a pair of real chance-variables (x,y) or (μ,η), nevertheless the two modes of representation, though of course mutually equivalent, are not always equally advantageous. For the types of problems contemplated in the present paper, the complex representation has important advantages resulting from the fact that the chance-variable when so represented is formally a single entity and subject to the laws and transformations of complex algebra. In Sections 2, 3, 4 of Part I and also in Part II, the complex representation possesses very great advantages. In the present Subsection, however, which is mainly concerned with formulations of the 2-dimensional "normal" probability law (distribution function), the representation in terms of a pair of real variables is the more advantageous. In this Subsection, therefore, the complex representation is used only in those places where it is particularly conducive to brevity and sharpness of statement, and to simplicity and clearness of correlation with the remainder of the paper where the complex representation is mainly used.

The normal complex chance-variable (which of course is 2-dimensional) may be defined in several mutually-equivalent ways. Here a complex chance-variable z will be defined as "normal" if its probability law can, by the proper choice of a pair of rectangular axes u,v in the plane of the "scatter-diagram" of z, be written in the form

$$P_{u,v} = \frac{1}{2\pi S_u S_v} \exp\left(-\frac{u^2}{2S_u^2} - \frac{v^2}{2S_v^2}\right) = P_u P_v, \tag{12}$$

u and v being the pair of coordinates of any point of the scatter-diagram with respect to the u,v-axes. P_u and S_u have the values already defined by equations (1) and (2) respectively, and P_v and S_v are defined by those same two equations after changing u to v throughout; S_u and S_v are distribution-parameters called the "standard deviations" of u and v respectively.

It will be recognized that the u,v-axes are the "central principal axes," namely that pair of rectangular axes which have their origin at the "center" c of the scatter-diagram of z, and hence of $w = u + iv$, and are so oriented in the scatter-diagram that $\overline{uv} = 0$. By the "center" c of the scatter-diagram of any complex chance-variable z is meant that point z_c with respect to which as origin the "mean value" (Section 4) of the chance-variable is zero, that is, such that $\overline{z - z_c} = 0$; thus, $z_c = \bar{z}$. In the case of the chance-variable $w = u + iv$, whose origin is the center of the scatter-diagram, so that $w_c = 0$, it is thus seen that $\bar{w} = 0$; the fact that the u,v-axes have their origin at c may conveniently be indicated by designating them as the ucv-axes.

Instead of taking S_u and S_v as the distribution-parameters it will be found preferable to take b and S, defined by the equations [5]

$$b = \frac{S_u^2 - S_v^2}{S_u^2 + S_v^2} = \frac{1 - (S_v/S_u)^2}{1 + (S_v/S_u)^2}, \tag{13}$$

$$S^2 = S_u^2 + S_v^2 = \overline{u^2} + \overline{v^2} = \overline{|w|^2}. \tag{14}$$

It is convenient, and fairly natural, to call S the "resultant standard deviation" of [6] u and v. More explicit formulas for b and S^2 are (37) and (38) established in Section 2.

Equation (12) shows that the equiprobability curves of the complex chance-variable $w = u + iv$ are a set of similar ellipses centered at the center c of the scatter-diagram; and that the axes of these ellipses coincide with the principal axes of the scatter-diagram and have lengths proportional to S_u and S_v, and hence proportional to $\sqrt{1 + b}$ and $\sqrt{1 - b}$ respectively, since, from (13) and (14),

$$2S_u^2 = (1 + b)S^2, \qquad 2S_v^2 = (1 - b)S^2.$$

Thus, when $S_v = S_u$ and hence when $b = 0$, the ellipses degenerate to circles. When $S_v = 0$ or $S_u = 0$ and hence when $b = +1$ or

[5] A parameter which itself is simpler than b is $a = S_v/S_u$; but if a were used instead of b most of the formulas in the unpublished Appendix A, mentioned in footnote 3, would be rendered considerably longer and more complicated.

[6] It is to be noted that $\overline{|w|^2}$ is not equal to $S^2_{|w|}$ if, as is natural, this is defined by the equation

$$S^2_{|w|} = \overline{(|w| - \overline{|w|})^2} = \overline{|w|^2} - \overline{|w|}^2.$$

$b = -1$ respectively, the ellipses degenerate to superposed straight line segments coinciding with the u-axis or the v-axis respectively; owing to this superposition of the straight line segments the "probability density" on the resulting straight line locus is not constant but varies in accordance with the 1-dimensional normal law, as expressed by equation (1).

With the object of reducing the number of parameters from 2 to 1 and of dealing with variables that are independent of units, it will be preferable not to deal directly with the original chance-variable $w = u + iv$, which is referred to the central principal axes ucv, but rather to deal with the "reduced" chance-variable $W = U + iV$ defined by the equation

$$W = w/S = u/S + iv/S = U + iV, \tag{15}$$

which is referred to the central principal axes UCV coinciding with the central principal axes ucv (Fig. 1), so that the position of any point T

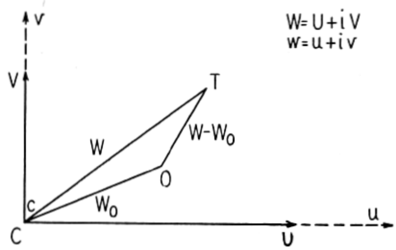


Fig. 1

in the W-plane will be represented by $W = U + iV$. Thus we shall be directly concerned with the scatter-diagram of $W = U + iV$ instead of with that of $w = u + iv$.

From (12) it is easily found that the probability law, say $P_{U,V}$, for $W = U + iV$ is

$$P_{U,V} = \frac{1}{\pi\sqrt{1-b^2}} \exp\left(-\frac{U^2}{1+b} - \frac{V^2}{1-b}\right), \tag{16}$$

which contains only the one parameter b, defined by (13), while moreover the variables U and V are independent of units. Thus the "reduced" complex chance-variable $W = U + iV$ given by (15) is defined as "normal" if its probability law can by the proper choice of a pair of rectangular axes UCV in the plane of its scatter-diagram be written in the form (16); the UCV-axes are the "central principal axes" of the scatter-diagram of $W = U + iV$; and the "mean value" of W is then zero, that is, $\overline{W} = 0$.

1.3. *Graphs for the Probability of the Deviation of a Normal Complex Chance-Variable from its Mean Value*

Before taking up the technical description of the graphs presented in this Subsection, some indication of their field for practical use will be furnished by the statement that the chance-variable $w = u + iv$ of the next paragraph may, for instance, be identified with the chance-variable h given by equation (II) of the Introduction, in case h is "normal" and is of zero "mean value," so that $\overline{h} = 0$; in case $\overline{h} \neq 0$, then w would be identified with $h - \overline{h}$. On referring to equation (II), it will be seen that h there denotes the deviation of any transmission characteristic from its nominal value; more generally, h may be any complex chance-variable which is "normal"—or approximately "normal."

The graphs here to be presented and described relate directly to the "reduced" complex chance-variable $W = U + iV$ given by equation (15) in terms of the original chance-variable $w = u + iv$ and the parameter S defined by equation (14). Assuming w to be "normal" and of zero "mean value" ($\bar{w} = 0$), it has the probability law formulated by equation (12); and hence $W = U + iV$ is normal and of zero mean value ($\overline{W} = 0$), and has the probability law formulated by (16), with the parameter b defined by (13).

With W' denoting the unknown value of a random sample consisting of a single value of the chance-variable W, the graphs herewith represent the probability that the magnitude $R' = |W'|$ of W' exceeds [7] any stated value R; that is, the probability that W' lies without a circle of radius R whose center coincides with the center C (Fig. 1) of the scatter-diagram of W, so that the center of the circle is at $W = 0$. This probability will be denoted by $p_b(R' > R)$, the subscript b implying dependence on the parameter b. The complementary probability will be denoted by $p_b(R' < R)$; this is of course the probability that R' is less than the stated value R; or, what is equivalent, the probability that W' lies within a circle of radius R centered at C. Of course the sum of the two foregoing probabilities is unity, that is,

$$p_b(R' > R) + p_b(R' < R) = 1. \tag{17}$$

[7] In engineering applications it is usually preferable to deal with the relatively small probability of exceeding, rather than with the complementary probability, nearly equal to unity, of being less than a preassigned rather large value of R.

Moreover,

$$p_b(R_1 < R' < R_2) = p_b(R' > R_1) - p_b(R' > R_2) \qquad (18)$$
$$= p_b(R' < R_2) - p_b(R' < R_1), \qquad (19)$$

where R_1 and R_2 denote any two stated values of R such that $R_1 < R_2$.

From (13) the total possible range of b is seen to be from -1 to $+1$, corresponding to the total possible range of S_v/S_u from ∞ to 0, with $b = 0$ corresponding to $S_v/S_u = 1$. However, it will evidently suffice to consider for b the range 0 to 1, corresponding to the range 1 to 0 for S_v/S_u, which will be secured by choosing S_u as the greater and hence S_v as the smaller of the two "standard deviations" (with the *ucv*-axes chosen correspondingly, of course).

The graphs in Figs. 2 and 3 show the relation between R and $p_b(R' > R)$ with b as parameter; similarly, Figs. 4 and 5 show the relation between R and the quantity $p_{b,0}(R' > R)$ defined by the equation

$$p_{b,0}(R' > R) = p_b(R' > R) - p_0(R' > R). \qquad (20)$$

Here $p_0(R' > R)$, being a particular value of $p_b(R' > R)$, plays the part of a reference value. It is a natural reference value, being the value for $b = 0$; and it can be evaluated immediately and accurately, since its exact formula is merely

$$p_0(R' > R) = \exp(-R^2). \qquad (21)$$

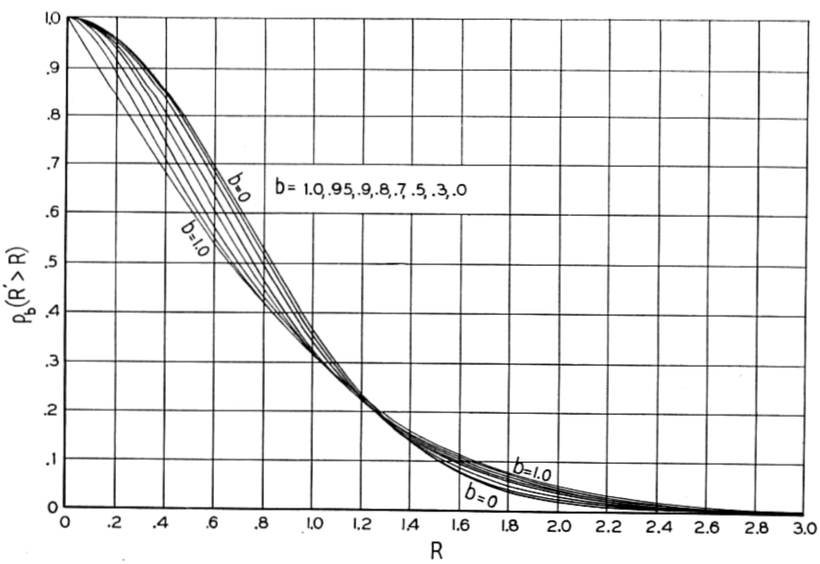


Fig. 2

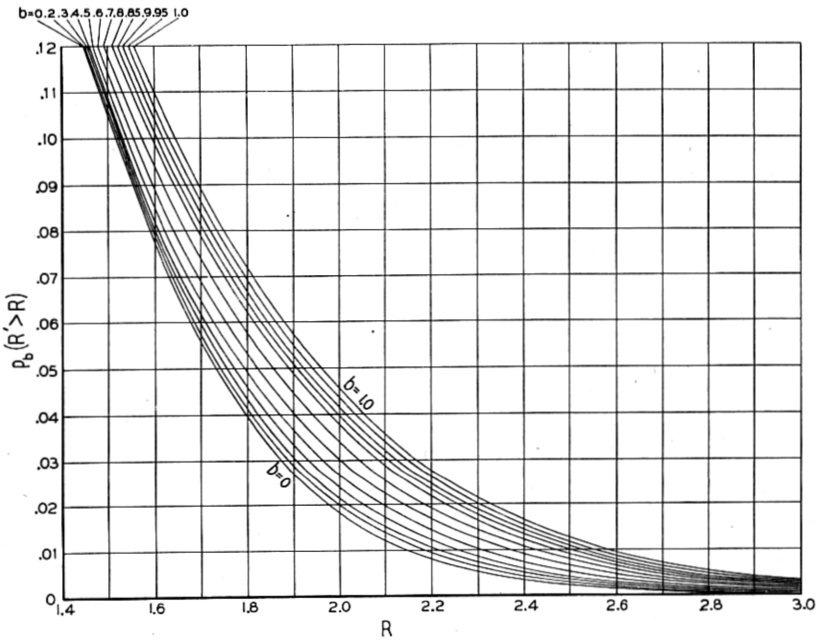

b=0 .2 .3 .4 .5 .6 .7 .8 .85 .9 .95 1.0
b=1.0
b=0
$p_b(R'>R)$
R

Fig. 3

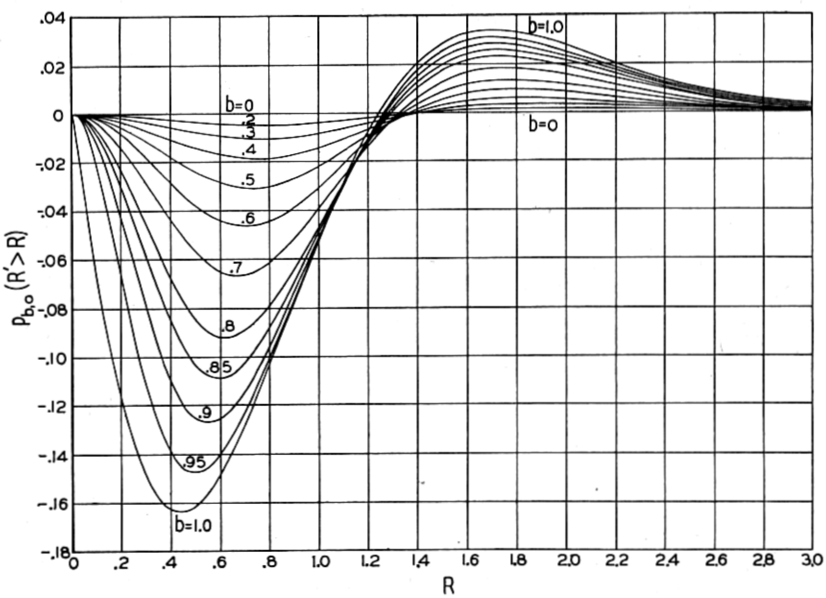

b=1.0
b=0
b=0
.2
.3
.4
.5
.6
.7
.8
.85
.9
.95
b=1.0
$p_{b,o}(R'>R)$
R

Fig. 4

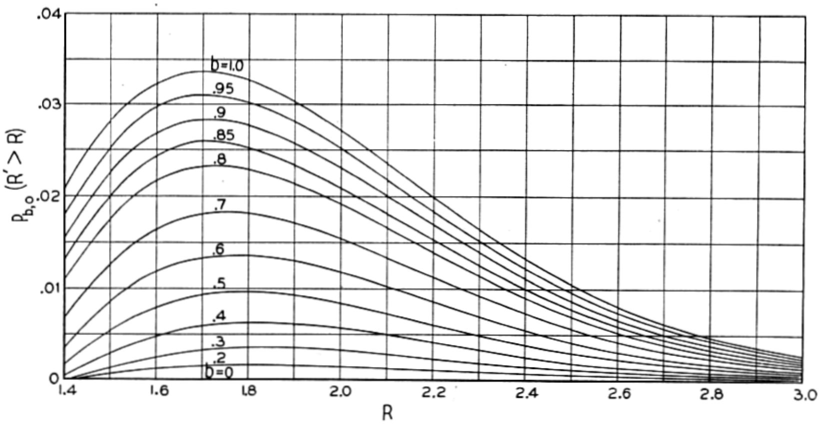


Fig. 5

The curves in Fig. 2 are chiefly useful for showing the form and range of the relations rather than for the reading-off of individual values; however, for the lower range of R ($R < 1$, say), they can be read with very fair accuracy. Fig. 3 is merely an enlarged plot of Fig. 2, over the R-range of about 1.5 to 3. The curves in Fig. 3 are accurately readable except in the upper part of this R-range; and the deficiency there is compensated by the curves of Fig. 5 described in the next paragraph.

The curves in Figs. 2 and 3 were plotted by aid of the much more accurately readable curves in Figs. 4 and 5, namely curves of R versus the quantity $p_{b,0}(R' > R)$ defined by equation (20); thus, by aid of (21),

$$p_b(R' > R) = p_{b,0}(R' > R) + \exp(-R^2). \tag{22}$$

Fig. 5 is merely an enlarged plot of Fig. 4, over the R-range of 1.4 to 3.0.

The material of Fig. 2 is represented in alternative forms, which are more convenient for some purposes, by Figs. 6 and 7, the former giving curves of $p_b(R' > R)$ versus b with R as parameter, the latter giving curves of b versus R with $p_b(R' > R)$ as parameter.

The material of Fig. 4 is represented in one alternative form by Figs. 8 and 9 each of which gives curves of $p_{b,0}(R' > R)$ versus b with R as parameter.

Returning to Fig. 2, it will be noted that the curves cross each other, but not at a common point; they cross rather diffusely in the neighborhood of $R = 1.2$. In the lower range of R, $p_b(R' > R)$ decreases with increasing b; while in the upper range of R, it increases with increasing b. Quantitatively these relations are shown more clearly and accurately by Figs. 6 and 7.

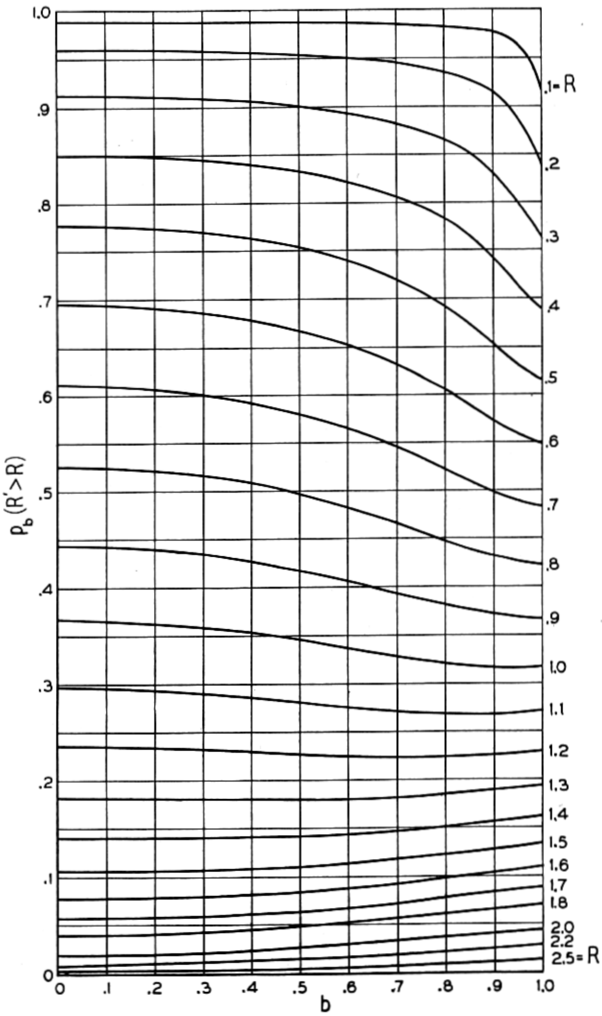


Fig. 6

Correspondingly in Fig. 4 the curves of $p_{b,0}(R' > R)$ cross each other rather diffusely in the neighborhood of[8] $R = 1.2$; thus, $p_{b,0}(R' > R)$ changes sign in this neighborhood. $p_{b,0}(R' > R)$ is nega-

[8] Except for values of b very nearly equal to 0; but in such cases $p_{b,0}(R' > R)$ is very small, so that the exception would be unimportant in most practical applications. A corresponding qualification applies, of course, to the discussion of Fig. in the preceding paragraph.

tive in the lower range of R and positive in the upper range; and the magnitude of $p_{b,0}(R' > R)$ always increases with increasing b. Since the value of R at which $p_{b,0}(R' > R)$ changes sign depends somewhat on b it will be denoted by R_b. Fig. 4 shows that R_1 is equal to about 1.24; and that R_b, when $1 > b > 0$, is greater than R_1 but only slightly greater except when b is very nearly zero. (See also Figs. 8 and 9.)

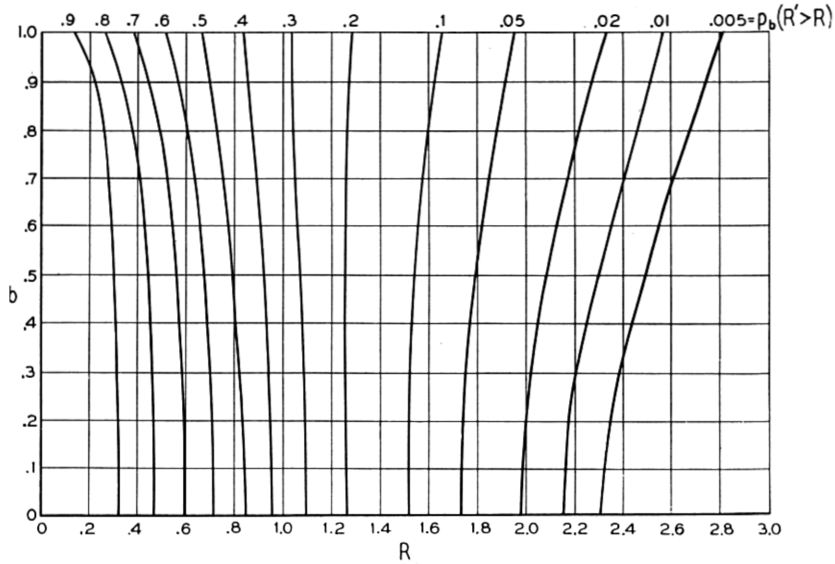


Fig. 7

Since the curves of $p_b(R' > R)$ in Fig. 2 cross each other (though somewhat diffusely) in the neighborhood of $R = 1.2$, it is unnecessary in approximate work to evaluate b when we are concerned only with values of R in this neighborhood; likewise when R is in the neighborhood of 0. Except in these two neighborhoods, however, a fairly accurate evaluation of b is necessary; for Fig. 2 shows that, in the upper R-range, $p_b(R' > R)$ depends very greatly on b, while even in the lower R-range the dependence on b is considerable. Thus the error resulting from assuming a value for b (in order to avoid the considerable labor of its actual evaluation) would usually be large. Quantitatively these facts are indicated more clearly and accurately by Figs. 6 and 7.

The computations underlying the graphs have proved to be so difficult and laborious that it has been deemed advisable to preserve he fundamental results in tabular form herewith (Table I), chiefly to enable the graphs to be replotted to a larger and more finely-divided scale by anybody so desiring. The values for $b = 0$ and $b = 1$ were omitted from the table, as being unnecessary because

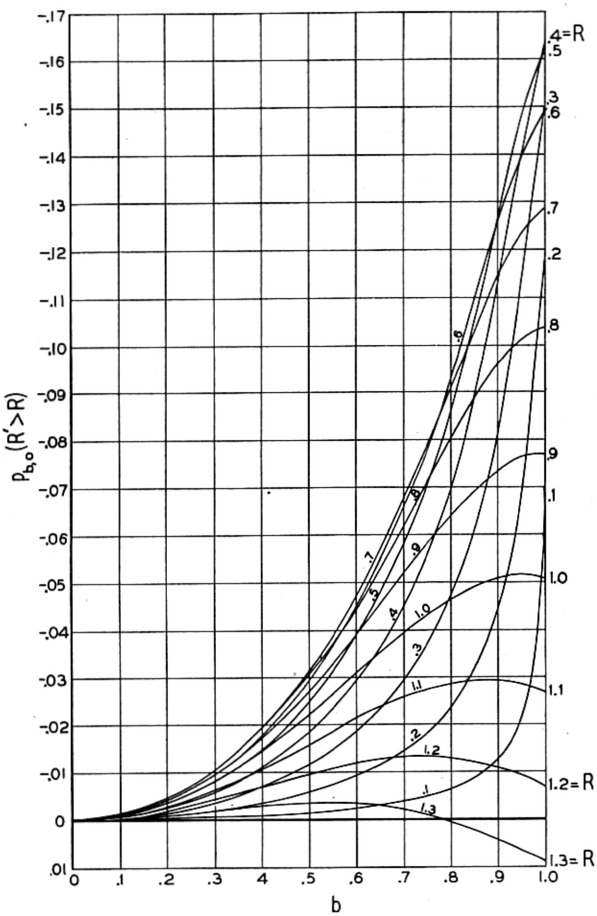


Fig. 8

$p_{b,0}(R' > R)$ is identically zero for $b = 0$, while for $b = 1$ it is given by the simple and exact formula

$$p_{1,0}(R' > R) = \text{erfc}\,(R/\sqrt{2}) - \exp\,(-R^2).$$

Although in many of the computed values in Table I the last digit (the third significant figure) cannot be regarded as reliable, it is thought that the tabulated values are accurate to about one per cen or better, which of course is quite adequate for all practical purpose

TABLE I

VALUES OF $p_{b,0}(R' > R)$

R	$b = .2$	$b = .3$	$b = .4$	$b = .5$	$b = .6$	$b = .7$	$b = .8$	$b = .85$	$b = .9$	$b = .95$
.1	−.000204	−.000469	−.000902	−.00154	−.00246	−.00391	−.00637	−.00862	−.0128	−.0217
.2	−.000773	−.00180	−.00340	−.00577	−.00922	−.0145	−.0236	−.0313	−.0440	−.0678
.3	−.00161	−.00377	−.00705	−.0119	−.0189	−.0295	−.0467	−.0604	−.0809	−.1120
.4	−.00257	−.00598	−.0111	−.0186	−.0293	−.0450	−.0691	−.0867	−.1101	−.1394
.5	−.00348	−.00808	−.0149	−.0247	−.0385	−.0577	−.0851	−.1036	−.1252	−.1471
.6	−.00419	−.00970	−.0178	−.0291	−.0446	−.0653	−.0922	−.1085	−.1256	−.1399
.7	−.00459	−.0106	−.0193	−.0311	−.0467	−.0665	−.0899	−.1025	−.1144	−.1235
.8	−.00464	−.0106	−.0192	−.0304	−.0447	−.0617	−.0796	−.0883	−.0957	−.1012
.9	−.00431	−.00978	−.0175	−.0272	−.0391	−.0520	−.0641	−.0692	−.0735	−.0765
1.0	−.00368	−.00829	−.0145	−.0221	−.0309	−.0395	−.0463	−.0486	−.0506	−.0518
1.1	−.00283	−.00630	−.0108	−.0160	−.0215	−.0260	−.0284	−.0291	−.0294	−.0287
1.2	−.00187	−.00411	−.00683	−.00953	−.0120	−.0132	−.0125	−.0117	−.0110	−.00920
1.3	−.000925	−.00196	−.00300	−.00353	−.00350	−.00214	.000771	.00241	.00451	.00670
1.4	−.0000796	−.0000620	.000327	.00152	.00342	.00654	.0108	.0130	.0154	.0178
1.5	.000603	.00147	.00295	.00536	.00853	.0127	.0176	.0200	.0227	.0252
1.6	.00110	.00256	.00477	.00792	.0118	.0165	.0217	.0242	.0269	.0297
1.7	.00141	.00320	.00581	.00933		.0181	.0232	.0258	.0284	.0309
1.818	.00157	.00353	.00631	.00969	.0137	.0181	.0227	.0250	.0275	.0299
2.0	.00144	.00321	.00562	.00849	.0118	.0154	.0192	.0210	.0231	.0252
2.222	.00102	.00225	.00392	.00588	.00814	.0107	.0133	.0146	.0162	.0176
2.5	.000507	.00112	.00197	.00296	.00413	.00554	.00702	.00782	.00872	.00963
2.857	.000145	.000329	.000612	.000915	.00134	.00188	.00245	.00282	.00320	.00362
3.333	.0000187	.0000432	.0000868	.000140	.000203	.000334	.000458	.000559	.000657	.000798

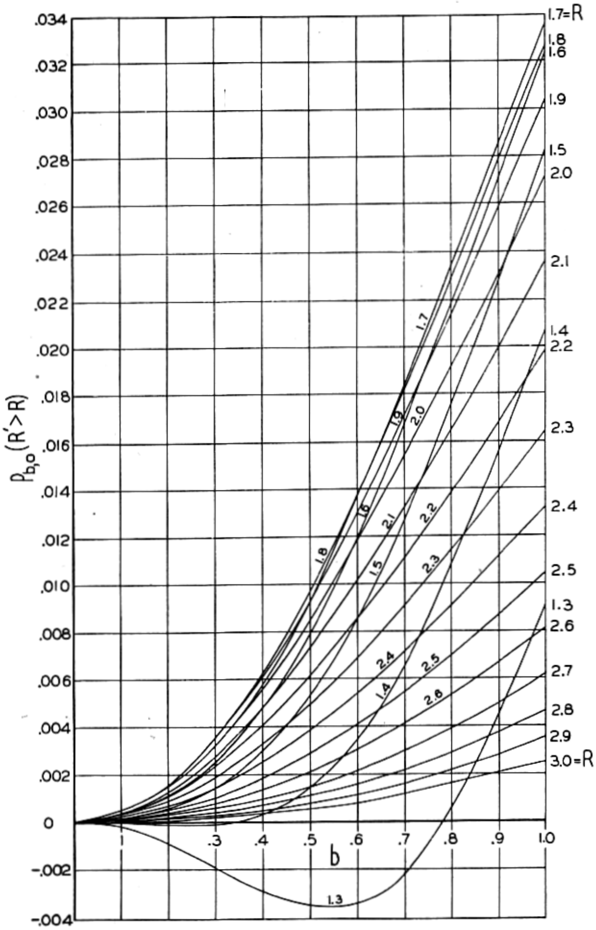


Fig. 9

2. THE LEADING DISTRIBUTION-PARAMETERS OF ANY COMPLEX CHANCE-VARIABLE

By the "leading distribution-parameters" of any complex chance-variable will here be meant a certain set of distribution-parameters (specified below) which would be sufficient for completely fixing the distribution if it were "normal." Even when the distribution is not "normal" these parameters are usually present among the other parameters in the distribution-function; indeed they are often th most important of the distribution-parameters.

In order to define and formulate the "leading distribution-para eters" of any complex chance-variable $Z = X + iY$ in an exp manner, conformably to the implicit definition in the preceding paragraph, we could proceed in a purely analytical manner, as outlined in Subsection 2.2 below. However, in recognition of the very substantial aid to thought and description furnished by the concept of the "scatter-diagram," for graphically representing any two-dimensional distribution, this concept will here be invoked in framing the definitions and in deriving the desired formulas.

Proceeding on this basis, it will be found that three of the "leading distribution-parameters" are certain "average values" pertaining to the scatter-diagram of the contemplated chance-variable; for any "average value" pertaining to the scatter-diagram is equal to the corresponding "mean value" ("expected value") pertaining to the chance-variable, when the "mean value" is defined as just after equation (2). It will be recalled that there a superbar applied to the symbol denoting any chance-variable was used to connote the "mean value" of the chance-variable. In the present Section (2), owing to the above-noted relation, the superbar may interchangeably be regarded as connoting either an "average value" pertaining to the scatter-diagram or the corresponding "mean value" pertaining to the chance-variable.

Having in mind the definition of the "scatter diagram" of any complex chance-variable $Z = X + iY$, let XAY (Fig. 10) designate

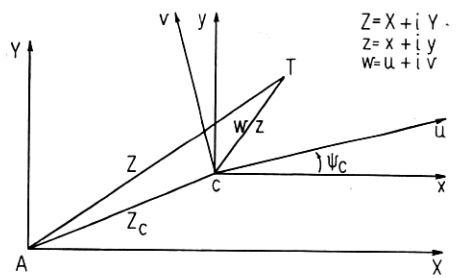


Fig. 10

the pair of rectangular axes with respect to which the scatter-diagram of Z is plotted, A designating the origin of the XAY-axes. Also, let T designate any plotted point in the scatter-diagram; and let c designate the "center" of the scatter-diagram, namely the point whose position Z_c with respect to the XAY-axes is such that $\overline{Z - Z_c} = 0$, whence $Z_c = \overline{Z}$. Further let xcy designate a pair of axes through c parallel to the XAY-axes, and ucv any other pair of rectangular axes through c; and let $w = u + iv$ represent the position of the point T with respect to the ucv-axes, the position of T with respect to the xcy-axes being represented by $z = x + iy$ and with respect to the

XAY-axes by $Z = X + iY$, whence $z = Z - Z_c$. Any pair of axes, such as ucv, through the center c are called "central axes"; ψ_c denotes their orientation-angle with respect to the xcy-axes, and hence with respect to the XAY-axes. When ψ_c has such a value ψ_c' that $\overline{uv} = 0$, the central axes ucv are called "principal central axes"; the corresponding values of $\overline{u^2}$ and $\overline{v^2}$ are denoted by S_u^2 and S_v^2 respectively, and S_u and S_v are called the "principal standard deviations" pertaining to the chance-variable $w = u + iv$.

Conformably to the implicit definition in the first paragraph of this Section, we may now state that the "leading distribution parameters" of any complex chance-variable $Z = X + iY$ are the four quantities Z_c, ψ_c', S_u, S_v defined and named in the preceding paragraph; it will be recognized that these four quantities would be sufficient for fixing the distribution if it were "normal."

(Still referring to Fig. 10, it may be noted that an alternative set of four parameters fixing the distribution of any "normal" complex chance-variable consists of Z_c, Π_{xy}, S_x, S_y, where $\Pi_{xy} = \overline{xy}$, $S_x^2 = \overline{x^2}$, $S_y^2 = \overline{y^2}$. The set Z_c, ψ_c', S_u, S_v was chosen as being much preferable for this paper.)

With a view to formulating precise definitions of the various additional technical terms needed, and to establishing general formulas from which to deduce the desired formulas for the last three of the "leading distribution parameters" Z_c, ψ_c', S_u, S_v, consider Fig. 11,

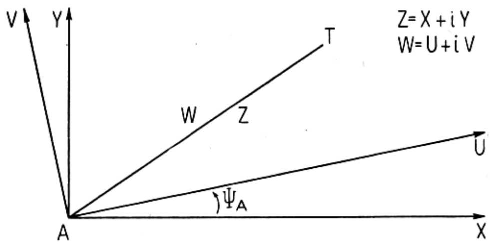


Fig. 11

which is a partial reproduction of Fig. 10, with the addition of the axes UAV, which are any pair of rectangular axes through A, so that $W = U + iV$ represents the position of any point T with respect to the UAV-axes, the position of T with respect to the XAY-axes being represented by $Z = X + iY$, of course. Then it can be shown (Subsection 2.1) that when the orientation-angle Ψ_A of the UAV-axes (Fig. 11) with respect to the XAY-axes has either of the values Ψ_A' given by the equation [9]

[9] In this paper, if Z denotes any complex quantity, then $ag\,Z$ denotes its angle, $|Z|$ its absolute value, $\hat{Z}$ its conjugate, $Re\,Z$ its real part, and $Im\,Z$ its imaginary part (that is, the cofactor of i when Z is written in rectangular form).

$$2\Psi_A' = ag(\pm \overline{Z^2}), \tag{23}$$

then the "mean" of the product UV vanishes, that is,

$$\overline{UV} = 0, \tag{24}$$

and the mean of $\overline{U^2}$ and the mean of $\overline{V^2}$ have the values expressed by the equations

$$2\overline{U^2} = \overline{|Z|^2} \pm |\overline{Z^2}|, \tag{25}$$

$$2\overline{V^2} = \overline{|Z|^2} \mp |\overline{Z^2}|, \tag{26}$$

and these values are extremum values in the sense that one is a maximum and the other a minimum when Ψ_A has either of the values Ψ_A' given by (23). Regarding the double signs in equations (23), (25), (26), it is hardly necessary to remark that the upper signs go together as one set, and the lower signs as another set. However, the presence of the double signs is a triviality; for the UAV-axes (Fig. 11) with respect to which equations (23), (24), (25), (26) are fulfilled are unique except merely as to their designations (U versus V, with signs), the values of Ψ_A' differing only by a multiple of $\pi/2$. (In numerical applications it will usually be convenient to choose the upper set of signs, so that $\overline{U^2}$ will be the maximum quantity and $\overline{V^2}$ the minimum.)

The particular UAV-axes (Fig. 11) for which equation (24) is fulfilled and for which Ψ_A therefore has a value Ψ_A' given by equation (23) are called the "principal axes" through A; and the corresponding mean squares $\overline{U^2}$ and $\overline{V^2}$ given by (25) and (26) are called the "principal mean squares." It will therefore be natural. and will be found convenient, to call Ψ_A', $\overline{U^2}$, $\overline{V^2}$ the "principal parameters" pertaining to the point A; they are seen to depend only on $\overline{Z^2}$ and $\overline{|Z|^2}$.

More generally, when the point A in Fig. 11 is not restricted to being the origin of the scatter-diagram of the given complex chance-variable but is any point in that scatter-diagram and when the XAY-axes and the UAV-axes are any two pairs of rectangular axes through A, it is readily seen that the formulas (23), (24), (25), (26) remain unchanged, although of course Z no longer represents the given chance-variable but now represents merely the position of any point T with respect to the XAY-axes, while W represents the position of T with respect to the UAV-axes. The quantities Ψ_A', $\overline{U^2}$, $\overline{V^2}$ given by equations (23), (25), (26) will naturally continue to be called the "principal parameters" relating to the point A, which is now any point. Thus the "principal parameters" are more general than the last three (ψ_c', S_u, S_v) of the "leading distribution-parameters," to which the "principal parameters" reduce when A coincides with the "center" c.

Continuing to regard A in Fig. 11 as any point in the scatter-diagram, it can be shown that in the degenerate case characterized by $\overline{Z^2} = 0$ all pairs of rectangular axes through A are "principal axes"; for when $\overline{Z^2} = 0$, equation (24) is fulfilled for all values of Ψ_A (as will be shown in the last paragraph of Subsection 2.1). Furthermore the mean squares with respect to all pairs of rectangular axes through A are then equal, as is shown by the fact that equations (25) and (26) reduce to

$$2\overline{U^2} = 2\overline{V^2} = \overline{|Z|^2} = \overline{X^2} + \overline{Y^2}. \tag{27}$$

Since A in Figs. 10 and 11 can be any point, the desired formulas for the last three of the "leading distribution-parameters" Z_c, ψ_c', S_u, S_v, relating to the point c in Fig. 10, are now seen to be immediately obtainable from formulas (23), (25), (26) for the "principal parameters" relating to the point A, by merely letting A coincide with c, the XAY-axes with the xcy-axes and the UAV-axes with the ucv-axes; for then Ψ_A', U, V, Z become ψ_c', u, v, z respectively; whence, after writing S_u^2 and S_v^2 for $\overline{u^2}$ and $\overline{v^2}$, the desired formulas are seen to be

$$2\psi_c' = ag(\pm \overline{z^2}), \tag{28}$$

$$2S_u^2 = \overline{|z|^2} \pm |\overline{z^2}|, \tag{29}$$

$$2S_v^2 = \overline{|z|^2} \mp |\overline{z^2}|, \tag{30}$$

where, as will be recalled, $z = Z - Z_c = Z - \overline{Z}$ represents (Fig. 10) the position of any point T of the scatter-diagram of Z with respect to the axes xcy through the center c parallel to the XAY-axes, which latter are there the axes of Z; thus $\bar{z} = 0$, though of course $\overline{Z} \neq 0$ in general. In accordance with (28), (29), (30) the last three of the leading distribution-parameters of $Z = z + Z_c = z + \overline{Z}$, which are the same as the last three of the leading distribution-parameters of z, are completely determined by the two mean values $\overline{z^2}$ and $\overline{|z|^2}$.

In order to represent explicitly the last three of the leading distribution-parameters of Z as depending on $Z - \overline{Z}$, it seems worth while to rewrite (28), (29), (30) in the following equivalent forms:

$$2\psi_c' = ag(\pm \overline{[Z - \overline{Z}]^2}), \tag{31}$$

$$2S_u^2 = \overline{|Z - \overline{Z}|^2} \pm |\overline{[Z - \overline{Z}]^2}|, \tag{32}$$

$$2S_v^2 = \overline{|Z - \overline{Z}|^2} \mp |\overline{[Z - \overline{Z}]^2}|, \tag{33}$$

which are completely determined by the two mean values $\overline{[Z - \overline{Z}}$

and $\overline{|Z - \overline{Z}|^2}$, though each of these depends on $\overline{Z}$, which plays the part of a reference value.

The foregoing formulas, by aid of (88) and (89) in Subsection 4.2, can be written also in the forms:

$$2\psi_c' = ag(\pm[\overline{Z^2} - \overline{Z}^2]), \tag{34}$$

$$2S_u^2 = \overline{|Z|^2} - |\overline{Z}|^2 \pm |\overline{Z^2} - \overline{Z}^2|, \tag{35}$$

$$2S_v^2 = \overline{|Z|^2} - |\overline{Z}|^2 \mp |\overline{Z^2} - \overline{Z}^2|, \tag{36}$$

which are completely determined by the three mean values $\overline{Z}$, $\overline{Z^2}$, $\overline{|Z|^2}$.

S_u and S_v are termed the "*principal* standard deviations," obviously because they relate to the "*principal* central axes," namely the particular *ucv*-axes corresponding to $\psi_c = \psi_c'$ (Fig. 10). They are special values of the "standard deviations" S_x and S_y, which latter relate to any specified central axes, *xcy*, and are defined by the equations $S_x^2 = \overline{x^2}$ and $S_y^2 = \overline{y^2}$.

By aid of the pairs of equations (29), (30) and (32), (33) and (35), (36), the parameters b and S defined by equations (13) and (14) can now be written in the following more explicit forms:

$$b = \frac{|\overline{z^2}|}{\overline{|z|^2}} = \frac{|\overline{(Z - \overline{Z})^2}|}{\overline{|Z - \overline{Z}|^2}} = \frac{|\overline{Z^2} - \overline{Z}^2|}{\overline{|Z|^2} - |\overline{Z}|^2}, \tag{37}$$

$$S^2 = \overline{|z|^2} = \overline{|Z - \overline{Z}|^2} = \overline{|Z|^2} - |\overline{Z}|^2. \tag{38}$$

Returning now to the general case in which point A in Fig. 11 is any point in the scatter-diagram of the given complex chance-variable, it will be recalled that formulas (23), (25), (26) give the values of the "principal parameters" relating to the point A. Let it now be required to formulate the principal parameters relating to any other point, a, in terms of quantities relating to the point A. With this purpose, consider Fig. 12. Here the *XAY*-axes are any rectangular axes through A; but the *UAV*-axes are the principal axes through A, as implied by the symbol Ψ_A' for their orientation-angle. The *xay*-axes are merely a pair of auxiliary axes through a drawn parallel to the *XAY*-axes; and the *uav*-axes are the principal axes through a. Z, W, z, w represent the position of any point T with respect to the axes *XAY*, *UAV*, *xay*, *uav* respectively; and Z_a represents the position of point a with respect to the *XAY*-axes. Then, corresponding to

(23), (25), (26), the formulas for the principal parameters relating to the point a are, of course,

$$2\psi_a' = ag(\pm \overline{z^2}), \tag{39}$$

$$2\overline{u^2} = \overline{|z|^2} \pm |\overline{z^2}|, \tag{40}$$

$$2\overline{v^2} = \overline{|z|^2} \mp |\overline{z^2}|. \tag{41}$$

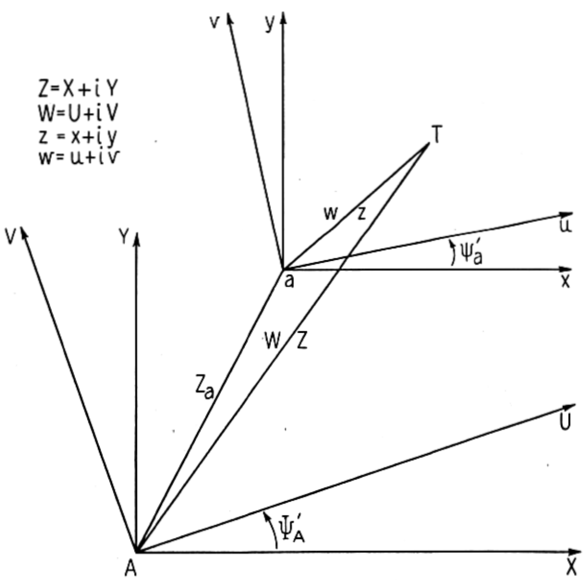


Fig. 12

But, since the xay-axes are parallel to the XAY-axes,

$$z = Z - Z_a. \tag{42}$$

Squaring (42) and taking the mean of the result gives

$$\overline{z^2} = \overline{Z^2} + Z_a^2 - 2\overline{Z}Z_a. \tag{43}$$

Multiplying (42) by its conjugate [9] and taking the mean of the result gives

$$\overline{|z|^2} = \overline{|Z|^2} + |Z_a|^2 - 2\text{Re}\,(\overline{Z}\hat{Z}_a). \tag{44}$$

Substituting (43) and (44) into (39), (40), (41) yields the desired formulas expressing the principal parameters relating to the point a (Fig. 12) in terms of quantities relating to the point A.

In particular, the formulas (34), (35), (36) for the last three of the "leading distribution parameters" of the original given chance-variable Z are immediately obtainable by merely letting the point a (Fig. 12) coincide with the center c; for then equations (42), (43), (44) reduce to

$$z = Z - Z_c = Z - \overline{Z}, \tag{45}$$

$$\overline{z^2} = \overline{Z^2} - \overline{Z}^2, \tag{46}$$

$$\overline{|z|^2} = \overline{|Z|^2} - |\overline{Z}|^2. \tag{47}$$

2.1. *Proofs of Formulas (23), (25), (26)*

With $W = U + iV$ here denoting any complex quantity,[9] formulas (23), (25), (26) will be proved by starting with the three identities [10]

$$2UV = \text{Im } W^2, \tag{48}$$

$$2U^2 = |W|^2 + \text{Re } W^2, \tag{49}$$

$$2V^2 = |W|^2 - \text{Re } W^2. \tag{50}$$

In order to apply these identities in proving formulas (23), (25), (26), which relate to Fig. 11, we evidently must identify the W appearing in these identities with the W in Fig. 11, and also must introduce the relation existing between W and Z in Fig. 11, namely

$$W = Z \exp(-i\Psi_A). \tag{51}$$

To prove (23) we substitute (51) into (48) and take the mean value of the result, thus getting

$$2\overline{UV} = |\overline{Z^2}| \sin(\text{ag } \overline{Z^2} - 2\Psi_A). \tag{52}$$

For the general case in which $|\overline{Z^2}|$ is not zero, this equation shows that the necessary and sufficient condition for $\overline{UV}$ to be zero is that Ψ_A shall have any of the special values Ψ_A' satisfying the following equation, in which n is real:

$$\text{ag } \overline{Z^2} - 2\Psi_A' = n\pi, \qquad (|n| = 0, 1, 2, 3, \cdots). \tag{53}$$

[10] These are equivalent to the identities

$$i4UV = W^2 - \hat{W}^2,$$
$$4U^2 = W^2 + \hat{W}^2 + 2W\hat{W},$$
$$-4V^2 = W^2 + \hat{W}^2 - 2W\hat{W},$$

ıich are immediately obtainable from the pair of simpler identities $2U = W + \hat{W}$ 1 $i2V = W - \hat{W}$. However, formulas (48), (49), (50) can be readily verified by ·ely substituting $W = U + iV$.

Hence

$$2\Psi_A' = \text{ag}\,\overline{Z^2} - n\pi = \text{ag}\,(\pm\overline{Z^2}), \tag{54}$$

which is (23). Evidently there are only two geometrically distinct values of Ψ_A', namely that for even n and that for odd n; and even this duality is a triviality, in the sense indicated in the latter part of the paragraph containing equations (25) and (26).

To prove (25) and (26) and at the same time to show that they are extrema, we substitute (51) into (49) and (50) and take the mean value of each result, thus getting

$$2\overline{U^2} = \overline{|Z|^2} + |\overline{Z^2}|\cos(\text{ag}\,\overline{Z^2} - 2\Psi_A), \tag{55}$$

$$2\overline{V^2} = \overline{|Z|^2} - |\overline{Z^2}|\cos(\text{ag}\,\overline{Z^2} - 2\Psi_A). \tag{56}$$

For the general case in which $|\overline{Z^2}|$ is not zero, these two equations show that when Ψ_A is varied, $\overline{U^2}$ and $\overline{V^2}$ have extremum values when Ψ_A has any of the special values Ψ_A' satisfying (53) and hence satisfying (23). Substitution of (53) into (55) and (56) gives (25) and (26), which are thus proved.

In the degenerate case characterized by $\overline{Z^2} = 0$, the unrestricted equation (52) shows that (24) will be fulfilled for all values of Ψ_A. This remark serves to prove the statement made in the paragraph containing equation (27).

2.2 Outline of a Purely Analytical Treatment of the Leading Distribution-Parameters

This Subsection is supplied, in accordance with the second paragraph of Section 2, in order to show that the leading distribution-parameters can be equivalently defined and formulated in a purely analytical manner, that is, without the aid of the "scatter-diagram" concept.

With $Z = X + iY$ denoting the given chance-variable, let Z_c denote that particular value of Z determined by the equation $\overline{Z - Z_c} = 0$, so that $Z_c = \overline{Z}$, the superbar connoting the "mean value" ("expected value") of Z, as defined just after equation (2). On account of the restriction of the present Subsection to pure analysis, Z_c cannot here be consistently called the "center of the scatter-diagram"; instead it will be called the "central value" of Z.

Next let $z = x + iy$ and $w = u + iv$ be the auxiliary chanc variables defined by the equations

$$z = Z - Z_c, \tag{57}$$ $$w = z\exp(-i\psi_c), \quad ($$

where, however, ψ_c is arbitrary, so that w is not determined until ψ_c is assigned. Also let ψ_c' be such a value of ψ_c that $\overline{uv} = 0$; and let S_u^2 and S_v^2 denote the corresponding values of $\overline{u^2}$ and $\overline{v^2}$ respectively, that is, the particular values taken by $\overline{u^2}$ and $\overline{v^2}$ when $\psi_c = \psi_c'$, so that $\overline{uv} = 0$.

The formulas (28), (29), (30) for ψ_c', S_u, S_v can now be established in a purely analytical manner in just the same way as the more general formulas (23), (25), (26) were established in Subsection 2.1.

3. FORMULAS FOR THE LEADING DISTRIBUTION-PARAMETERS OF A LINEAR FUNCTION OF COMPLEX CHANCE-VARIABLES

To meet the needs in dealing with problems of the type handled in Part II, namely problems involving linear functions of complex chance-variables, the present Section furnishes formulas for the "leading distribution-parameters" of any complex chance-variable Z which is a linear function of any number n of complex chance-variables $Z_1, \cdots Z_n$, so that

$$Z = a + b_1 Z_1 + \cdots + b_n Z_n, \tag{59}$$

where $a, b_1, \cdots b_n$ are any constants, complex in general.

It will be recalled that the "leading distribution-parameters" of any complex chance-variable Z are the quantities Z_c, ψ_c', S_u, S_v defined and formulated in Section 2.

Since, in general, $Z_c = \overline{Z}$, application of Theorem 3 of Subsection 4.2 to (59) gives

$$\overline{Z} = a + b_1 \overline{Z}_1 + \cdots + b_n \overline{Z}_n, \tag{60}$$

so that here $\overline{Z}$ is not zero even when $\overline{Z}_1, \cdots \overline{Z}_n$ are all zero.

The formulas for ψ_c', S_u, S_v are (28), (29), (30), where $z = Z - Z_c$; or the equivalent formulas (31), (32), (33) or (34), (35), (36).

With a view to using formulas (28), (29), (30), which have the advantage of compactness, we introduce the quantities z and z_r defined by the equations

$$z = Z - Z_c = Z - \overline{Z}, \tag{61}$$

$$z_r = Z_r - \overline{Z}_r, \qquad (r = 1, \cdots n), \tag{62}$$

which show that $\bar{z} = 0$ and that

$$\bar{z}_r = 0, \qquad (r = 1, \cdots n). \tag{63}$$

Subtracting (60) from (59) and then substituting (61) and (62) into he result gives

$$z = b_1 z_1 + \cdots + b_n z_n, \tag{64}$$

hich has the advantage of not involving a.

Formulas (28), (29), (30) involve $\overline{z^2}$ and $\overline{|z|^2}$. To evaluate $\overline{z^2}$ we square z and take the mean value of the result; to evaluate $\overline{|z|^2}$ we multiply z by its conjugate $\hat{z}$ and take the mean value of the result. We thus obtain from (64) the formulas

$$\overline{z^2} = b_1^2\overline{z_1^2} + \cdots + b_n^2\overline{z_n^2} + \cdots + 2b_sb_t\overline{z_sz_t} + \cdots, \tag{65}$$

$$\overline{|z|^2} = |b_1|^2\overline{|z_1|^2} + \cdots + |b_n|^2\overline{|z_n|^2} + \cdots + 2\text{Re}\, b_s\hat{b}_t\overline{z_s\hat{z}_t} + \cdots, \tag{66}$$

where $s = 1, \cdots n - 1$ and $t = s + 1, \cdots n$. These two formulas can also be written

$$\overline{z^2} = \sum_r^{1\cdots n} b_r^2\overline{z_r^2} + 2\sum_{s<t}^{1\cdots n} b_sb_t\overline{z_sz_t}, \tag{67}$$

$$\overline{|z|^2} = \sum_r^{1\cdots n} |b_r|^2\overline{|z_r|^2} + 2\text{Re}\sum_{s<t}^{1\cdots n} b_s\hat{b}_t\overline{z_s\hat{z}_t}, \tag{68}$$

corresponding respectively to formulas (94) and (95) in Subsection 4.3.

When the subscripted Z's are independent, and hence the subscripted z's are independent, equations (65) and (66) respectively reduce to

$$\overline{z^2} = b_1^2\overline{z_1^2} + \cdots + b_n^2\overline{z_n^2}, \tag{69}$$

$$\overline{|z|^2} = |b_1|^2\overline{|z_1|^2} + \cdots + |b_n|^2\overline{|z_n|^2}, \tag{70}$$

on account of Theorem 1 in Subsection 4.1 together with equation (63).

4. SOME FORMULAS AND THEOREMS ON MEAN VALUES OF COMPLEX CHANCE-VARIABLES

The present Section supplies a considerable number of formulas and theorems on "mean values" ("expected values") [11] of complex chance-variables. Many of these formulas and theorems have already been used in Part I, and further use for them will be found in Part II; while outside of this paper they may well find varied other uses.

The theorems are word-statements of the simpler and more frequently useful of the formulas; the remaining formulas are more general and are not simple enough to be profitably expressed as theorems.

Theorems 1 and 2 regarding the mean of a product of complex chance-variables and Theorem 3 regarding the mean of a sum are generalizations of the corresponding known theorems for real chance-variables, are formally the same as the latter, and are susceptible of the same sort of proofs. These three theorems furnish a natural basis for the remaining theorems, besides having extensive other uses.

[11] Defined just after equation (2).

4.1. *Mean of a Product of Independent Complex Chance-Variables*

The following Theorems 1 and 2 relating to the mean of a product of complex chance-variables are very important notwithstanding their limitation to chance-variables which are independent.

Two discrete chance-variables are said to be "independent" (or "uncorrelated" or "non-correlated") if the probability that either takes any given value is independent of the value taken by the other.

Two continuous chance-variables are said to be "independent" if the probability that either lies close to any given value is independent of the value taken by the other.

THEOREM 1. *If any number of complex chance-variables are independent, the mean of their product is equal to the product of their individual means.*

That is, if the Z's are independent,

$$\overline{Z_1 Z_2 \cdots Z_n} = \bar{Z}_1 \bar{Z}_2 \cdots \bar{Z}_n. \tag{71}$$

THEOREM 2. *If the magnitudes (absolute values) of any number of complex chance-variables are independent, the mean of the magnitude of the product of these complex chance-variables is equal to the product of the means of their individual magnitudes.*

That is, if the $|Z|$'s are independent,

$$\overline{|Z_1 Z_2 \cdots Z_n|} = \overline{|Z_1|}\,\overline{|Z_2|} \cdots \overline{|Z_n|}. \tag{72}$$

For the validity of Theorem 2 it is not necessary that the angles of the chance-variables be independent, but only their magnitudes. Moreover, if $\phi_1, \cdots \phi_n$ denote the angles of $Z_1, \cdots Z_n$ and Φ the angle of their product, then, by Theorem 3,

$$\bar{\Phi} = \overline{\phi_1} + \cdots + \overline{\phi_n}, \tag{72a}$$

whether or not the ϕ's are independent.

4.2. *Mean of a Sum of Complex Chance-Variables*

The following Theorem 3 is of unlimited scope, in the sense that it involves no assumption as to independence of the chance-variables.

THEOREM 3. *Given any number of complex chance-variables, which need not be independent, the mean of their sum is equal to the sum of their individual means.*

That is, whether or not the Z's are independent,

$$\overline{Z_1 + \cdots + Z_n} = \bar{Z}_1 + \cdots + \bar{Z}_n. \tag{73}$$

Since the Z's in Theorem 3 need not be independent, the theorem will continue to be valid when the Z's are any functions of any number of other chance-variables $w_1, \cdots w_m$.

The following six simple and useful equations, in which $Z = X + iY$ denotes any complex [9] chance-variable, are immediately obtainable by means of Theorem 3.

$$\bar{Z} = \bar{X} + i\bar{Y}, \qquad (74) \qquad\qquad \bar{\hat{Z}} = \bar{X} - i\bar{Y} = \hat{\bar{Z}}, \qquad (75)$$

$$\overline{Z^2} = \overline{X^2} - \overline{Y^2} + i2\overline{XY}, \qquad (76)$$

$$\overline{|Z|^2} = \overline{Z\hat{Z}} = X^2 + \overline{Y^2}, \qquad (77)$$

$$\bar{Z}^2 = \bar{X}^2 - \bar{Y}^2 + i\,2\bar{X}\bar{Y}, \qquad (78)$$

$$|\bar{Z}|^2 = \bar{Z}\,\hat{\bar{Z}} = \bar{X}^2 + \bar{Y}^2. \qquad (79)$$

The following eight equations can be obtained by solving the foregoing set of equations or by applying Theorem 3 to the appropriate identities.

$$\bar{X} = \overline{\mathrm{Re}\,Z} = \mathrm{Re}\,\bar{Z}, \qquad (80) \qquad\qquad \bar{Y} = \overline{\mathrm{Im}\,Z} = \mathrm{Im}\,\bar{Z}, \qquad (81)$$

$$2\overline{XY} = \mathrm{Im}\,\overline{Z^2}, \qquad (82)$$

$$2\overline{X^2} = \overline{|Z|^2} + \mathrm{Re}\,\overline{Z^2}, \qquad (83)$$

$$2\overline{Y^2} = \overline{|Z|^2} - \mathrm{Re}\,\overline{Z^2}, \qquad (84)$$

$$2\bar{X}\bar{Y} = \mathrm{Im}\,\bar{Z}^2, \qquad (85)$$

$$2\bar{X}^2 = |\bar{Z}|^2 + \mathrm{Re}\,\bar{Z}^2, \qquad (86)$$

$$2\bar{Y}^2 = |\bar{Z}|^2 - \mathrm{Re}\,\bar{Z}^2. \qquad (87)$$

Theorem 3 yields also the following two useful equations

$$\overline{(Z - \bar{Z})^2} = \overline{Z^2} - \bar{Z}^2, \qquad (88)$$

$$\overline{|Z - \bar{Z}|^2} = \overline{|Z|^2} - |\bar{Z}|^2. \qquad (89)$$

The first can be obtained immediately by squaring $Z - \bar{Z}$ and then applying Theorem 3; the second by expanding the product $(Z - \bar{Z})$ $(\hat{Z} - \hat{\bar{Z}})$ and then applying Theorem 3 together with equation (75).

When, instead of a single chance-variable Z, there are n chance variables $Z_1, \cdots Z_n$, not restricted to being independent, equatior

(88) and (89) become

$$\overline{\sum (Z_r - \bar{Z}_r)^2} = \sum (\overline{Z_r^2} - \overline{Z_r}^2), \tag{90}$$

$$\overline{\sum |Z_r - \bar{Z}_r|^2} = \sum (\overline{|Z_r|^2} - |\bar{Z}_r|^2), \tag{91}$$

where each summation $\sum$ covers the set $r = 1, \cdots n$.

4.3. *Mean of a Squared Sum of Complex Chance-Variables*

With a view to arriving at Theorems 4 and 5 below, and also several formulas which are more general than the theorems but are not simple enough to be profitably expressed as theorems, let $Z_1, \cdots Z_n$ denote any complex chance-variables; and for brevity let W denote their sum, so that

$$W = Z_1 + \cdots + Z_n. \tag{92}$$

As indicated by its title, this Subsection will be concerned particularly with formulas for $\overline{W^2}$ and $\overline{|W|^2}$, but it will also include formulas for $\overline{W}^2$ and $|\overline{W}|^2$.

Squaring W, given by (92), and then applying Theorem 3 gives

$$\overline{W^2} = \sum_{r=1}^{n} \overline{Z_r^2} + 2 \sum_{h=1}^{n-1} \sum_{k=h+1}^{n} \overline{Z_h Z_k}, \tag{93}$$

or, in a briefer notation,

$$\overline{W^2} = \sum_{r}^{1\cdots n} \overline{Z_r^2} + 2 \sum_{h<k}^{1\cdots n} \overline{Z_h Z_k}, \tag{94}$$

the second $\sum$ in (94) thus denoting double summation.[12]

Taking the product of W and its conjugate $\hat{W}$ and then applying Theorem 3 gives

$$\overline{|W|^2} = \sum \overline{|Z_r|^2} + 2\mathrm{Re} \sum \overline{Z_h \hat{Z}_k}. \tag{95}$$

Applying Theorem 3 to (92) and then squaring the result gives

$$\overline{W}^2 = \sum \overline{Z_r}^2 + 2 \sum \overline{Z_h}\,\overline{Z_k}. \tag{96}$$

Taking the product of $\overline{W}$ and $\hat{\overline{W}}$ gives

$$|\overline{W}|^2 = \sum |\overline{Z_r}|^2 + 2\mathrm{Re} \sum \overline{Z_h}\,\hat{\overline{Z}}_k. \tag{97}$$

[12] In (95), $\cdots$ (99) the summations evidently cover the same sets of values as in (94).

When the Z's are independent, so that Theorem 1 is applicable, equations (94) and (95) respectively reduce to

$$\overline{W^2} = \sum \overline{Z_r^2} + 2 \sum \overline{Z}_h \overline{Z}_k, \tag{98}$$

$$\overline{|W|^2} = \sum \overline{|Z_r|^2} + 2\mathrm{Re} \sum \overline{Z}_h \hat{\overline{Z}}_k, \tag{99}$$

although (96) and (97) remain unchanged. Thus, when the Z's are independent, the following relations exist:

$$\overline{W^2} - \overline{W}^2 = \sum (\overline{Z_r^2} - \overline{Z}_r^2), \tag{100}$$

$$\overline{|W|^2} - |\overline{W}|^2 = \sum (\overline{|Z_r|^2} - |\overline{Z}_r|^2). \tag{101}$$

It is of interest to compare these with (90) and (91), which do not require the Z's to be independent.

When, further, not more than one of the Z's is of non-zero mean value, so that at least $n - 1$ are of zero mean value, that is, when [13]

$$\overline{Z}_r = 0, \qquad (r = 1, \cdots, j-1, j+1, \cdots, n), \tag{102}$$

then (98) and (99) reduce to

$$\overline{W^2} = \sum \overline{Z_r^2}, \tag{103}$$

$$\overline{|W|^2} = \sum \overline{|Z_r|^2}. \tag{104}$$

After substitution of the value of W from the defining equation (92), and with due regard to (102), equations (103) and (104), on account of their importance and simplicity, may profitably be expressed in the form of two theorems, respectively, as follows:

THEOREM 4. *If any number of complex chance-variables are independent and if not more than one is of non-zero mean value, then the mean of the squared value of their sum is equal to the sum of the means of their individual squared values.*

That is,

$$\overline{(Z_1 + \cdots + Z_n)^2} = \overline{Z_1^2} + \cdots + \overline{Z_n^2}, \tag{105}$$

provided the Z's are independent and not more than one is of non-zero mean value, in accordance with (102).

THEOREM 5. *If any number of complex chance-variables are independent and if not more than one is of non-zero mean value, then the mean of the squared magnitude (absolute value) of their sum is equal to the sum of the means of their individual squared magnitudes.*

[13] An important practical instance in which one of the Z's is of non-zero mea value will be found in connection with equation (120) in the problem treated Section 6.

That is,

$$\overline{|Z_1 + \cdots + Z_n|^2} = \overline{|Z_1|^2} + \cdots + \overline{|Z_n|^2}, \tag{106}$$

provided the Z's are independent and not more than one is of non-zero mean value, in accordance with (102).

Part II: Applications

The methods, theorems and formulas presented in Part I will now be applied to two important problems in telephone transmission engineering.[14] However, in each of these problems the solution is carried no further than to formulate the "leading distribution-parameters" in a form suitable for numerical evaluation in any specific case, since Subsection 1.3 of Part I has furnished the means of solving such problems when once these parameters have been evaluated and when the distribution is known to be approximately "normal."

The two problems mentioned above are treated separately in the following Sections 5 and 6. Section 5 sketches the solution of the general problem which was outlined in the Introduction (in Part I) in connection with the equations there; Section 6 deals somewhat fully with another problem, which, though specific, is yet of a rather broad type.

The problem in Section 6 has heretofore been handled by various approximate and less comprehensive methods, as indicated in the first footnote of the Introduction. The relative simplicity of the method described by Crisson in his paper there cited is due to his simplifying assumption (made just after his equations 26 and 27) which amounts to assuming that the scatter-diagram is circular instead of, as actually, elliptical.

5. Deviation of any Characteristic of a Transmission System or of a Network

This Section sketches an approximate solution of the general problem outlined in the Introduction, in connection with equations (I) and (II), which are the general functional formulas for the contemplated characteristic H and its deviation h, respectively; in general H and h are complex.

The present Section relates chiefly to formulas for the "leading distribution-parameters" of h when this is regarded as a chance-variable.

In accordance with Section 2 (in Part I) the leading distribution-parameters of h are completely determined by $\bar{h}$, $\overline{h^2}$, $\overline{|h|^2}$. Evidently

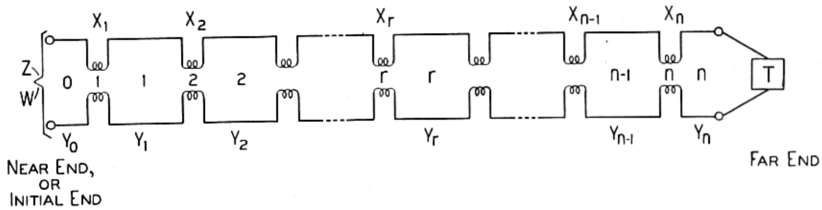
[14] An additional problem, crosstalk in a telephone cable, is treated in the unpublished Appendix C already mentioned in footnote 3.

the exact formulas for these three quantities must depend, in any specific case, on the corresponding specific form of the function F in equations (I) and (II) of the Introduction. However, general approximate formulas can be obtained when, as usual, the k's in (II) are small enough compared with the K's to enable the right side of (II) to be represented by the first-order terms of a Taylor expansion, so that h will be given by formula (III), as a good approximation. Since h, when so given, is a linear function of the chance-variables $k_1, \cdots k_n$, the formulas of Section 3 (in Part I) are directly applicable by setting $a = 0$ there, and identifying Z, b_r, Z_r there with h, D_r, k_r here, and hence z and z_r there with $h - \overline{h}$ and $k_r - \overline{k}_r$ here, respectively. Thus it is not necessary to write down here the formulas for $\overline{h}$, $\overline{h^2}$, $\overline{|h|^2}$.

When h is approximately "normal," the chance that the unknown value h' of a random sample consisting of a single value of h lies without a circle of specified radius centered at the mean value $\overline{h}$ of h can be found by application of the graphs presented and described in Subsection 1.3.

6. IMPEDANCE-DEVIATION AND REFLECTION COEFFICIENT OF A LOADED CABLE DUE TO LOADING IRREGULARITIES AND TERMINAL IRREGULARITY

As represented schematically by Fig. 13, the physical system considered in this problem consists of a periodically loaded cable whose loading-coil impedances and loading-section admittances, and also the



Z = IMPEDANCE OF SYSTEM: W = 1/Z = ADMITTANCE OF SYSTEM.
T = ADMITTANCE OF TERMINAL APPARATUS.
X_r = IMPEDANCE OF TYPICAL LOADING-COIL NO. r.
Y_r = ADMITTANCE OF TYPICAL WHOLE LOADING SECTION, NO. r.
X,Y = NOMINAL VALUES OF X_r, Y_r.
Y/2 = NOMINAL VALUE OF Y_0 AND Y_n.

Fig. 13

terminal admittance (T), deviate randomly from their nominal values, so that the deviations are complex chance-variables; however, the nominal value of the terminal admittance is not here restricted to equality with the iterative impedance of the loaded cable, since such a restriction would not correspond to the conditions usually existing in practice.

The resulting deviation in the impedance Z of the initial end of the system (Fig. 13) from the iterative impedance of the loaded cable is a complex chance-variable which is of much engineering importance in case the loaded cable is to constitute part of a transmission system containing a 2-way repeater, of the 22-type, connected between the initial end of the loaded cable and the remainder of the transmission system (not shown in Fig. 13); for, so far as the loaded cable is concerned, the practicable amplification obtainable from the repeater will depend approximately inversely on the impedance-deviation of the loaded cable; more precisely, it will depend inversely on the reflection coefficient defined, in terms of the impedance-deviation, by equation (107) below.

In Fig. 13 the loaded cable is represented as beginning with a half-section, and as ending with a half-section, and the latter as terminated with an admittance T. The formulas herein established are for this system. Analogous formulas for a system beginning and ending with half-coils, instead of with half-sections, can be obtained in an analogous manner, or even written down directly by analogy.

The important reflection coefficient mentioned at the end of the second paragraph, and to be denoted by ρ, is defined by the equation

$$\rho = -\frac{Z-h}{Z+h} = -\frac{(Z-h)}{2h+(Z-h)} = -\frac{(Z-h)/2h}{1+(Z-h)/2h}, \qquad (107)$$

Z denoting the impedance of the system in Fig. 13, and h the mid-section iterative impedance of the loaded cable. Each of the forms in (107) is useful and significant. However, if $W = 1/Z$ denotes the admittance of the system, and $H = 1/h$ the mid-section iterative admittance of the loaded cable, the equation for ρ can be written in the equivalent forms

$$\rho = \frac{W-H}{W+H} = \frac{(W-H)}{2H+(W-H)} = \frac{(W-H)/2H}{1+(W-H)/2H}, \qquad (108)$$

and these forms, instead of those in (107), will be the ones mostly used herein, because of their simpler and more direct relations to the corresponding current deviations. For, if an electromotive force E is impressed between the terminals of the system in Fig. 13, the current I there will be WE; and if I^0 denotes the value that I would have if W were equal to H, then $I^0 = HE$. Thus the reflection coefficient ρ defined in terms of W and H by equation (108) can be expressed in terms of I and I^0 by the equation

$$\rho = \frac{I - I^0}{I + I^0} = \frac{(I - I^0)}{2I^0 + (I - I^0)} = \frac{(I - I^0)/2I^0}{1 + (I - I^0)/2I^0}. \tag{109}$$

If the system contained no internal irregularities within the loaded cable itself and also no terminal irregularity at the far end, ρ would of course be zero. There are three types of irregularities here to be considered: section-irregularities, coil-irregularities, and the terminal-irregularity. Each of these types will be considered separately, with the ultimate object of constructing, by superposition, an approximate formula for ρ in terms of all of the existing irregularities.

First, consider the typical section-irregularity, situated in section No. r and consisting in the admittance-deviation [15] $y_r = Y_r - Y$ of the admittance Y_r of this section from its nominal value Y. The admittance-increment y_r may evidently be regarded as situated anywhere within the section. However, for the present purpose it is most conducive to simplicity of thought to regard y_r as situated just beyond the nominal mid-point of the section, namely the point which is at a distance of half a normal, or "regular," section from the initial end of the section; for then it is immediately evident that the admittance of the portion of the system beyond the nominal mid-point will deviate from the mid-section iterative admittance H by an amount approximately [16] equal to y_r, and hence that the corresponding reflection coefficient ζ_r pertaining to that mid-point will, in accordance with (108), be given (approximately) by the formula

$$\zeta_r = \frac{y_r}{2H + y_r} = \frac{y_r/2H}{1 + y_r/2H}. \tag{110}$$

Due to the presence of the internal admittance-increment y_r in section No. r, the admittance W of the whole system (Fig. 13) at its initial end will deviate somewhat from the mid-section iterative admittance H; the admittance-deviation W-H will be denoted by y_r', and the corresponding reflection coefficient of the system will be denoted by ζ_r', so that, in accordance with (108),

$$\zeta_r' = \frac{y_r'}{2H + y_r'} = \frac{y_r'/2H}{1 + y_r'/2H}. \tag{111}$$

[15] Here $r = 1, 2, \cdots n - 1$; for of course the nominal values of Y_0 and Y_n are each $Y/2$, and hence $y_0 = Y_0 - Y/2$ and $y_n = Y_n - Y/2$. With these qualifications duly observed, formula (110) is valid for $r = 0$ and $r = n$ as well as for $r = 1, 2, \cdots n - 1$. As seen below, y_0 is to be regarded as situated at the initial end of section No. 0, and y_n at the far end of section No. n.

[16] "Approximately," because y_r is distributed; "exactly," if y_r were localized.

Then it can rather easily be shown that ζ_r' is related to ζ_r in accordance with the simple but exact equation

$$\zeta_r' = \zeta_r e^{-2r\Gamma} = \zeta_r Q^{2r}, \tag{112}$$

where

$$Q = e^{-\Gamma} = e^{-A} e^{-iB}, \tag{113}$$

$\Gamma = A + iB$ denoting the propagation constant and Q the propagation factor of the loaded cable, each per periodic interval. It is sometimes convenient to call ζ_r' the "propagated value" of ζ_r, though it is to be observed that the apparent propagation constant of ζ_r is 2Γ not Γ. Alternatively, ζ_r' may be called the "apparent value" of ζ_r as viewed from the initial end of the system.

Second, consider the typical coil-irregularity, situated in coil No. r and consisting in the impedance-deviation $x_r = X_r - X$ of the impedance X_r of this coil from its nominal value X. The impedance-increment x_r will be regarded as situated just beyond the nominal mid-point of the coil; and the corresponding reflection coefficient ξ_r pertaining to that mid-point will, in accordance with (107), be given by the following formula, in which K denotes the mid-coil iterative impedance of the loaded cable:

$$\xi_r = -\frac{x_r}{2K + x_r} = -\frac{x_r/2K}{1 + x_r/2K}. \tag{114}$$

Since ξ_r is situated at a distance of $r - 1/2$ periodic intervals from the initial end, it appears at that end as a reflection coefficient ξ_r' such that

$$\xi_r' = \xi_r Q^{2r-1}. \tag{115}$$

Third, consider the terminal-irregularity situated at the junction of the loaded cable with the terminal-admittance T and consisting in the admittance-deviation $t = T\text{-}H$ of the admittance T from the mid-section iterative admittance H of the loaded cable. The corresponding reflection coefficient τ pertaining to that point will be given by the formula

$$\tau = \frac{t}{2H + t} = \frac{t/2H}{1 + t/2H}. \tag{116}$$

This will appear at the initial end as a reflection coefficient τ' given by the formula

$$\tau' = \tau Q^{2n}. \tag{117}$$

Finally let all of the loading-section admittances differ from their nominal values, all of the loading-coil impedances from their nominal values, and the terminal-admittance T from the mid-section iterative admittance H. Then, when these deviations are not too large, the resulting reflection coefficient ρ at the initial end of the system will be approximately equal to the sum of the "propagated" or "apparent" values of the reflection coefficients arising from all of the individual irregularities, that is,

$$\rho = \sum_{r=0}^{n} \zeta_r' + \sum_{r=1}^{n} \xi_r' + \tau', \tag{118}$$

whence, by substitution of (112), (115), (117),

$$\rho = \sum_{r=0}^{n} \zeta_r Q^{2r} + \sum_{r=1}^{n} \xi_r Q^{2r-1} + \tau Q^{2n}. \tag{119}$$

Since ζ_r, ξ_r, τ are chance-variables, ρ is a complex chance-variable. In accordance with Section 2 (in Part I) the leading distribution-parameters of ρ are completely determined by $\overline{\rho}$, $\overline{\rho^2}$, $\overline{|\rho|^2}$; and these will completely determine the distribution of ρ if it is "normal." In the present problem, owing to the presence of τ in equation (119), $\overline{\rho}$ is not to be taken as zero; for, in accordance with the second half of the first paragraph of this Section, $\overline{\tau}$ would usually not be zero in practice. However, $\overline{\zeta}_r$ and $\overline{\xi}_r$ would usually be zero and will here be so taken. Hence, from (119),

$$\overline{\rho} = \overline{\tau} Q^{2n}. \tag{120}$$

Since the chance-variables ζ_r, ξ_r, τ are independent, and since only one of them, namely τ, has a non-zero mean value, Theorems 4 and 5 of Subsection 4.3 (in Part I) are applicable to (119). Assuming all of the loading-section deviations to be statistically alike, so that [17]

$$\overline{\zeta_r^2} = \overline{\zeta^2}, \qquad \overline{|\zeta_r|^2} = \overline{|\zeta|^2}, \qquad (r = 0, 1, 2, \cdots n), \tag{121}$$

and all of the loading-coil deviations to be statistically alike, so that

$$\overline{\xi_r^2} = \overline{\xi^2}, \qquad \overline{|\xi_r|^2} = \overline{|\xi|^2}, \qquad (r = 1, 2, \cdots n), \tag{122}$$

application of Theorems 4 and 5 to (119), followed by the execution of the indicated summations, gives the formulas

$$\overline{\rho^2} = \overline{\zeta^2}\frac{1 - Q^{4(n+1)}}{1 - Q^4} + \overline{\xi^2}\frac{1 - Q^{4n}}{1 - Q^4} Q^2 + \overline{\tau^2} Q^{4n}, \tag{123}$$

$$\overline{|\rho|^2} = \overline{|\zeta|^2}\frac{1 - q^{4(n+1)}}{1 - q^4} + \overline{|\xi|^2}\frac{1 - q^{4n}}{1 - q^4} q^2 + \overline{|\tau|^2} q^{4n}, \tag{124}$$

where q denotes the attenuation factor of the loaded cable per peri-

[17] The assumption represented by (121) is an approximation to the extent that, statistically, ζ_0 and ζ_n would usually differ somewhat from ζ_j, where $j = 1, 2, \cdots n - 1$.

odic interval, that is,

$$q = |Q| = e^{-A}, \tag{125}$$

A denoting the attenuation constant of the loaded cable per periodic interval, in accordance with equation (113).

When q^{4n} is small compared to unity, formulas (123) and (124) reduce approximately to

$$\overline{\rho^2} = \frac{\overline{\zeta^2} + \overline{\xi^2}Q^2}{1 - Q^4} + \overline{\tau^2}Q^{4n}, \tag{126}$$

$$\overline{|\rho|^2} = \frac{\overline{|\zeta|^2} + \overline{|\xi|^2}q^2}{1 - q^4} + \overline{|\tau|^2}q^{4n}. \tag{127}$$

When, further, q is nearly equal to unity, which by (125) will be the case when $2A$ is small compared to unity, then formula (127) reduces approximately to

$$\overline{|\rho|^2} = \frac{\overline{|\zeta|^2} + \overline{|\xi|^2}q^2}{4A} + \overline{|\tau|^2}q^{4n}. \tag{128}$$

Returning to the formulas (110) and (114), which give ζ_r and ξ_r in terms of $y_r/2H$ and $x_r/2K$ respectively, it may be said that for practical applications it is more convenient to express ζ_r and ξ_r in terms of the fractional deviations δ_r and ϵ_r and the coefficients D and G, defined by the following four equations:

$$\delta_r = y_r/Y, \tag{129}$$

$$\epsilon_r = x_r/X, \tag{130}$$

$$D = Y/2H, \tag{131}$$

$$G = X/2K. \tag{132}$$

With these substitutions, formulas (110) and (114) become

$$\zeta_r = \frac{D\delta_r}{1 + D\delta_r}, \tag{133}$$

$$\xi_r = -\frac{G\epsilon_r}{1 + G\epsilon_r}. \tag{134}$$

It can be shown that D and G, defined by equations (131) and (132), are approximately equal and may be expressed approximately in each of the forms appearing in the equation

$$D = G = \sqrt{\frac{XY/4}{1 + XY/4}} = \sqrt{1 - 1/HK} = \tanh(\Gamma/2), \tag{135}$$

with H, K, Γ already defined in connection with equations (108), (114), (113) respectively. Equation (135) would be exact if the cable wires were perfectly conducting, since then each section-admittance Y could be regarded as effectively localized, so that the loaded cable would be effectively a ladder-type structure, for which equation (135) is known to be rigorously exact.